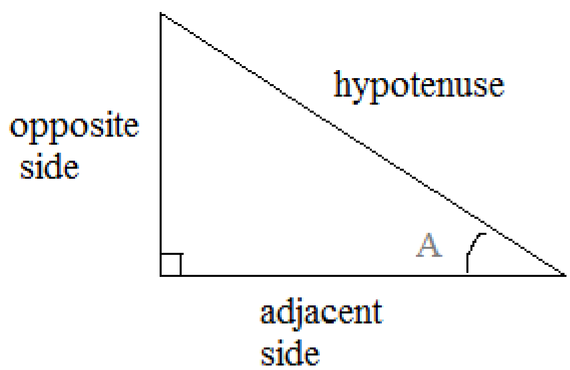
Trigonometry Introduction

Formulas, examples, and practice exercises (and solutions)



Topics include SohCahToa, finding inverse trig values, reciprocals, principal values, and more.

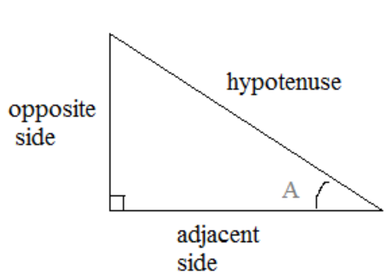
Trigonometry Introduction: Sine, Cosine, Tangent

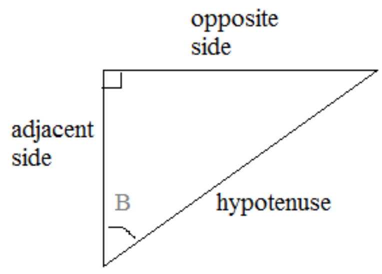
Sine: $\frac{\text{opposite}}{\text{hypotenuse}}$ Cosine: $\frac{\text{adjacent}}{\text{hypotenuse}}$ Tangent: $\frac{\text{opposite}}{\text{adjacent}}$

> Some use "Soh Cah Toa"
> to memorize the ratios
> (sine: opp/hyp cos: adj/hyp toa: opp/adj)

Identifying the sides of a (right) triangle:

Examples:





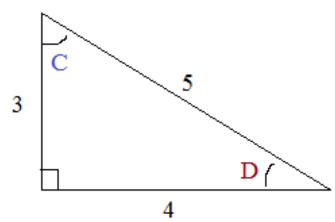
Applying the formulas:

Examples:

Sine ∠C $\frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5}$

Cosine ∠C $\frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5}$

Tangent ∠C $\frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3}$



Sine ∠D $\frac{\text{opposite}}{\text{hypotenuse}} = \frac{3}{5}$

Cosine ∠D $\frac{\text{adjacent}}{\text{hypotenuse}} = \frac{4}{5}$

Tangent ∠D $\frac{\text{opposite}}{\text{adjacent}} = \frac{3}{4}$

> Note the relationship between C and D:
> Sine D = Cosine C = 3/5
> Sine C = Cosine D = 4/5
> and, Tangent C is the reciprocal of Tangent D

Trigonometry Reciprocals: Cotangent, Secant, Cosecant :

Cotangent = $\frac{1}{\text{Tangent}} = \frac{\text{adjacent}}{\text{opposite}}$

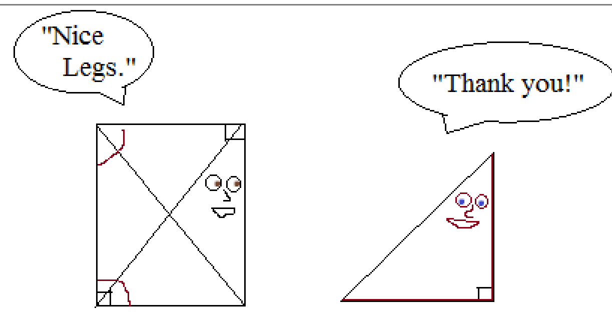
Secant = $\frac{1}{\text{Cosine}} = \frac{\text{hypotenuse}}{\text{adjacent}}$

Cosecant = $\frac{1}{\text{Sine}} = \frac{\text{hypotenuse}}{\text{opposite}}$

> A helpful memory trick to remember the pairs:
>
> cotangent -- tangent ("tangent")
> secant -- cosine ("2 syllable words")
> sine -- cosecant ("the others")

Hidden Message

Solve the following trig equations to reveal the answer!



What did the polite geometry figure have?

1) $\sin B = \dfrac{\boxed{l}}{m}$

2) $\sin D = \dfrac{p}{\square}$

3) $\cos E = \dfrac{s}{\square}$

4) $\tan F = \dfrac{\square}{r}$

5) $\sin C = \dfrac{q}{\square}$

6) $\sin \square = \dfrac{s}{t}$

7) $\cos \square = \dfrac{p}{o}$

8) $\cos D = \dfrac{q}{\square}$

9) $\cos B = \dfrac{n}{\square}$

10) $\tan D = \dfrac{\square}{q}$

11) $\tan B = \dfrac{\square}{n}$

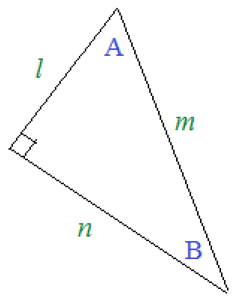
12) $\cos A = \dfrac{l}{\square}$

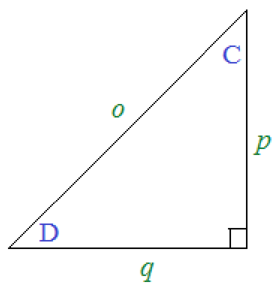
13) $\tan \square = \dfrac{r}{s}$

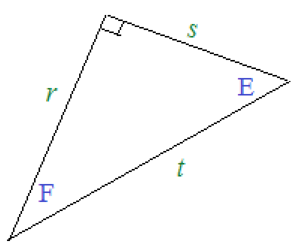
14) $\sin A = \dfrac{\square}{m}$

15) $\cos F = \dfrac{r}{\square}$

16) $\sin F = \dfrac{\square}{t}$





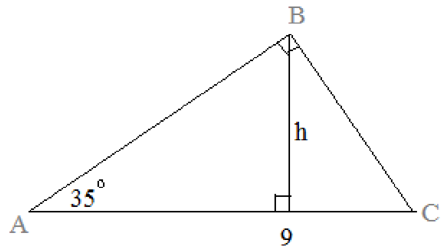


1) L
2) ______
3) ______
4) ______
5) ______
6) ______
7) ______
8) ______
9) ______
10) ______
11) ______

E or I

12) ______
13) ______
14) ______
15) ______
16) ______

Example: Can you find h?

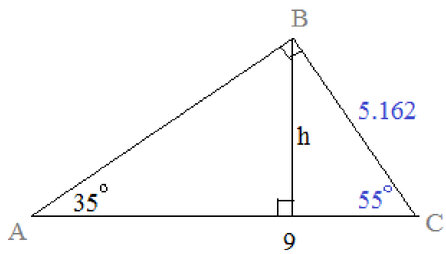


Method 1:

Since ABC is a right triangle, we can find BC using sine function.

$$\sin(35^\circ) = \frac{BC}{9}$$

$$BC = 5.162$$



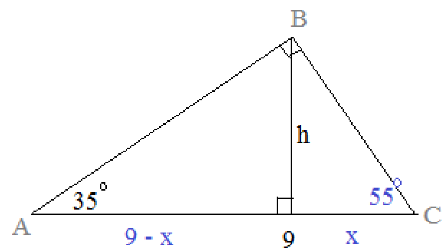
Then, we can use sine function to find h in the small right triangle:

$$\text{sine}(55^\circ) = \frac{h}{5.162}$$

$$h = 4.23$$

Method 2:

Label the parts of the triangle(s). Then, construct 2 equations containing h:



$$\tan(35^\circ) = \frac{h}{(9-x)} \qquad \tan(55^\circ) = \frac{h}{x}$$

$$h = (9-x)(\tan 35) \qquad h = x(\tan 55)$$

using substitution,

$$(9-x)(\tan 35) = x(\tan 55)$$

$$.70(9-x) = 1.428x$$

$$6.3 = 2.128x$$

$$x = 2.96$$

Finally, $\tan(55^\circ) = \frac{h}{2.96}$

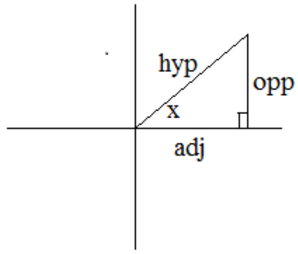
$$h = 4.23$$

Inverse Trig Functions: "Why does Sin^{-1} have a capital 'S'? "

Review: $\sin x = \dfrac{\text{opposite}}{\text{hypotenuse}}$

$\cos x = \dfrac{\text{adjacent}}{\text{hypotenuse}}$

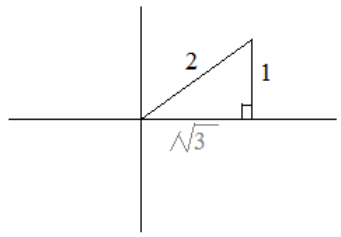
$\tan x = \dfrac{\text{opposite}}{\text{adjacent}}$



Inverse Trig Functions:

Suppose I ask, "what angle has a sine of $\frac{1}{2}$? "

$\sin X = \frac{1}{2}$



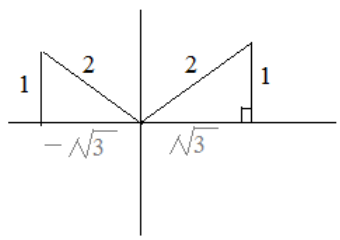
Pythagorean Theorem confirms that the adjacent side is $\sqrt{3}$.

We know the triangle is 30-60-90

therefore, $X = 30^\circ$ or $\frac{\pi}{6}$

But, WAIT!!

$\sin X = \frac{1}{2}$



$X = 30^\circ$ AND 150°

or $\frac{\pi}{6}$ and $\frac{5\pi}{6}$

There are multiple values!!

Uniquely defined values:

Now, suppose I ask, find $\tan(\sin^{-1}\frac{1}{2})$

$(\sin^{-1}\frac{1}{2}) = 30^\circ$ or 150°

$\tan(30) = \dfrac{1}{\sqrt{3}}$ $\tan(150) = \dfrac{1}{-\sqrt{3}}$

There are 2 solutions!!

** What if we had wanted to specify a <u>unique</u> solution?

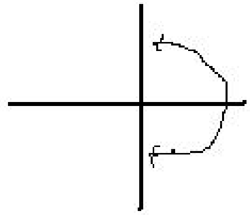
We must use *principal values*..

"Principal Values" defined:

y = arctan x
x = tan y

domain: all real numbers
range: $-90^\circ \le y \le 90^\circ$

$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



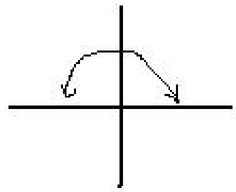
principal values of tangent and sine fall in quadrants I and IV

note: every trig value -- positive and negative -- is covered ONE time only

y = arcsin x
x = sin y

domain: $-1 \le x \le 1$
range: $-90^\circ \le y \le 90^\circ$

$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



principal values of cosine fall in quadrants I and II

y = arccos x
x = cos y

domain: $-1 \le x \le 1$
range: $0 \le y \le 180$

$0 \le y \le \pi$

To specify principal values:

1) use a capital letter *example:* ArcSin instead of arcsin

Sin^{-1} instead of $\sin^{-1}$

2) assume principal values when seeking a unique solution.

$\tan(\sin^{-1} 1/2) \longrightarrow$ to find unique solution, $\tan(\text{Sin}^{-1} 1/2) = \tan(30) = \frac{\sqrt{3}}{3}$

(consider principal values in quads I & IV only)

Study Break:
Math Snacks

Preferable to ordinary computer cookies...

Essential part of a well-rounded, academic diet.

Try with (t), or any beverage...

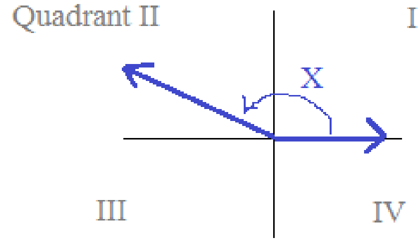
Also, look for Honey Graham Squares
in the geometry section of your local store...

Finding Trig values from given information

Example: $\text{Sin}X = \frac{5}{13}$ and, angle X lies in *Quadrant II*

Find the other 5 trig values of X.

Step 1: Draw a sketch

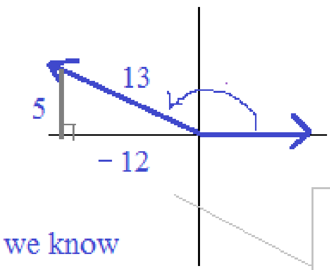


Step 2: "determine the triangle"

Since $\sin X = \frac{5}{13}$

5 is the opposite side
13 is the hypotenuse

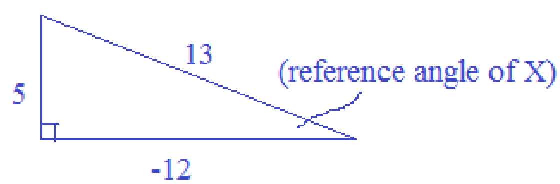
Using Pythagorean theorem, we know the adjacent side is 12



$\text{Sin} = \frac{\text{opposite}}{\text{hypotenuse}}$

**Important: note the adjacent side is *negative* 12, because it lies on the left side of the "y-axis" (between 90° and 270°)

Step 3: Find the trig values



$\text{Sin}X = \frac{5}{13}$ $\qquad$ $\text{Csc}X = \frac{13}{5}$

$\text{Cos}X = \frac{-12}{13}$ $\qquad$ $\text{Sec}X = \frac{13}{-12}$

$\text{Tan}X = \frac{5}{-12}$ $\qquad$ $\text{Cot}X = \frac{-12}{5}$

One more thing: What is the measure of angle X?

$\text{Sin}X = \frac{5}{13} = .3846$

$\arcsin(.3846) = 22.62^\circ$

$\sin^{-1}(.3846) = 22.62^\circ$

(using inverse trig values on a calculator)

since the reference angle lies in Quadrant II,
angle $X = 180 - 22.62 = 157.38^\circ$

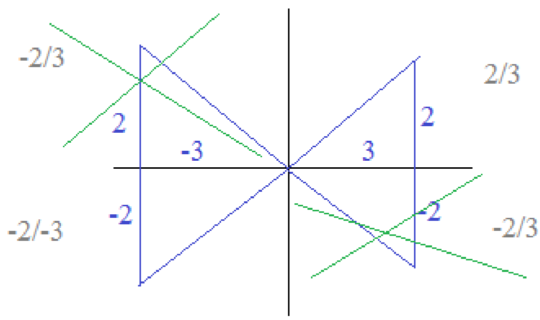
Example: $\tan\theta = \frac{2}{3}$

Finding Trig values from given information

$\sec\theta < 0$ Find the 5 other trig values of θ.

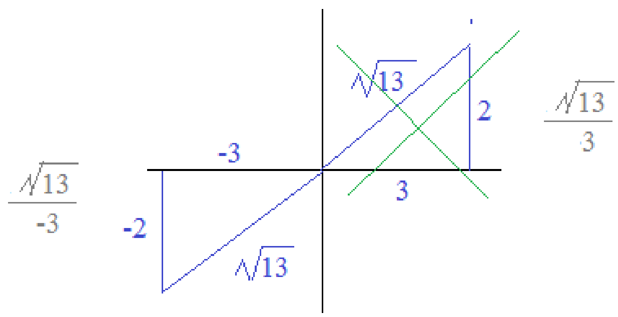
then, determine the measure of angle θ

Step 1: Draw a sketch



Since $\tan\theta = \frac{2}{3}$

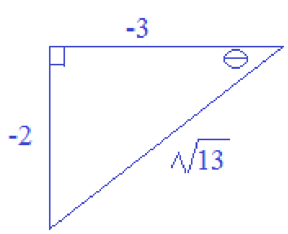
we can eliminate quadrants II and IV



Since $\sec\theta < 0$,

we can eliminate quadrant I

Step 2: Draw the triangle and determine the trig values



$\tan\theta = \frac{2}{3}$ $\cot\theta = \frac{3}{2}$

$\sin\theta = \frac{-2}{\sqrt{13}}$ $\csc\theta = \frac{\sqrt{13}}{-2}$

$\cos\theta = \frac{-3}{\sqrt{13}}$ $\sec\theta = \frac{\sqrt{13}}{-3}$

Step 3: Find the angle

since $\tan\theta = \frac{2}{3}$

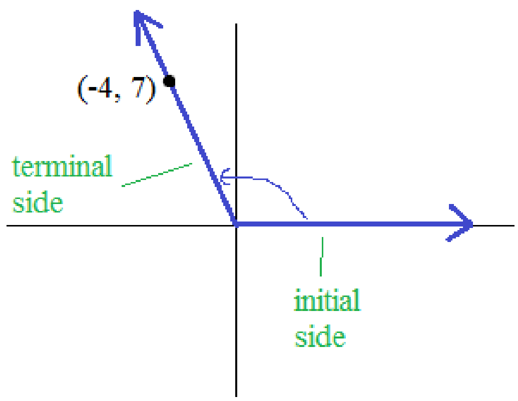
(using a calculator) we can find the arctan(.667) or $\tan^{-1}(.667)$: 33.7 degrees..

then, recognizing that the angle is in quadrant III, angle $\theta = 180 + 33.7 = 213.7^{\circ}$

Finding Trig values from given information

The *terminal side* of an angle in *standard position* passes through the point (-4, 7).
What are the 6 trigonometric values of the angle?

Step 1: Draw a sketch

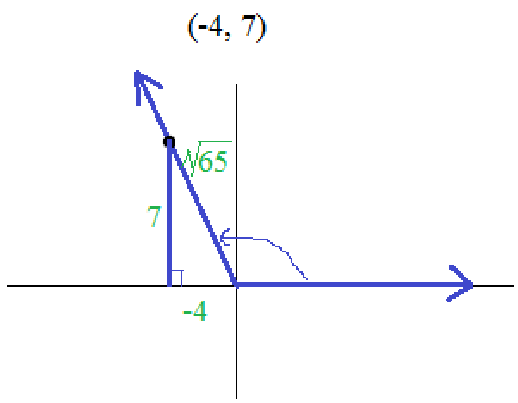


Step 2: "Determine the triangle"

Draw a vertical line segment from the point to the x-axis.

Then, label the sides...

(use pythagorean theorem to find measure of hypotenuse)



Step 3: Find 6 trig values

$\sin\theta = \dfrac{7}{\sqrt{65}}$

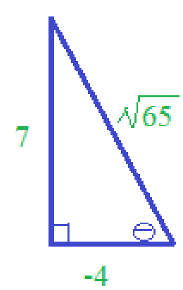
$\cos\theta = \dfrac{-4}{\sqrt{65}}$

$\tan\theta = \dfrac{7}{-4}$

$\csc\theta = \dfrac{\sqrt{65}}{7}$

$\sec\theta = \dfrac{\sqrt{65}}{-4}$

$\cot\theta = \dfrac{-4}{7}$

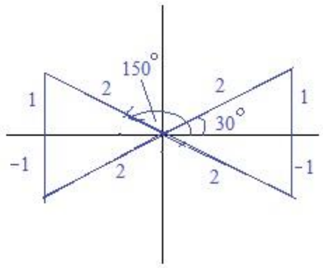


"Notes on Finding Inverse Trig Values"

Find exact value of $X = \sin^{-1}\left(\frac{1}{2}\right)$ for $0 \le X \le 2\pi$

Arcsin X $\longleftrightarrow$ $\sin^{-1}$X
(Equivalent expressions)

1) "Triangle Method"



Therefore, for $0 \le X \le 2\pi$

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}, \frac{5\pi}{6}$$

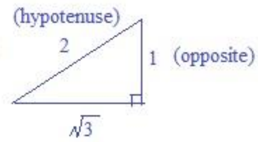
$\text{Sin } X = \text{Sin}(\sin^{-1} 1/2)$

$\text{Sin } X = \frac{1}{2}$

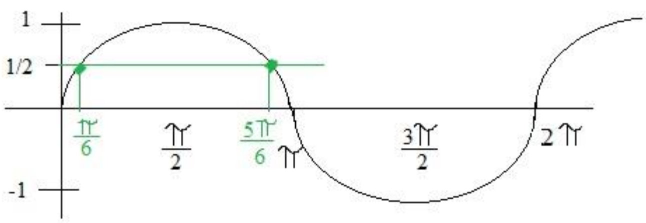
"Sine of what angle equals 1/2?"

$$\text{Sine} = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2}$$

30-60-90 triangle



2) Sine Function Graph



The diagram illustrates the points:

$$\left(\frac{\pi}{6}, \frac{1}{2}\right) \quad \left(\frac{5\pi}{6}, \frac{1}{2}\right)$$

3) Calculator

(Depending on the calculator),

Enter (1/2) or .5
Press "2nd" "Sine" (This should produce Sin^{-1} or arcsin)
The output will be 30 (degrees).. (**This calculator provided only the principal value!)
Finally, adjust your answer:
--- convert to radians if necessary
--- add values (**Since the above question wants values between 0 and 2π, you must find any other solutions)

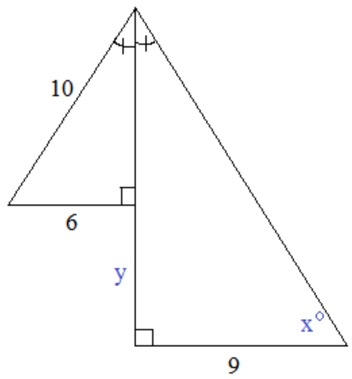
4) Ask a math teacher!

"Principal" or "Restricted" Inverse Trigonometry Functions have the following ranges:

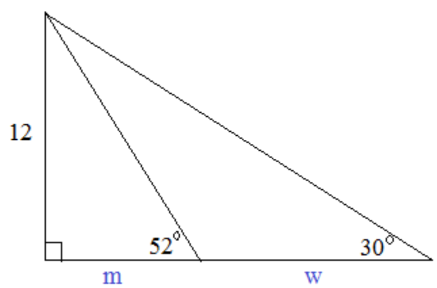
arcsin θ	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
arccos θ	$0 \le \theta \le \pi$
arctan θ	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\csc^{-1} x$	$\left[-\frac{\pi}{2}, 0\right)$ or $\left(0, \frac{\pi}{2}\right]$
$\sec^{-1} x$	$\left[0, \frac{\pi}{2}\right)$ or $\left(\frac{\pi}{2}, \pi\right]$
$\cot^{-1} x$	$\left[-\frac{\pi}{2}, 0\right)$ or $\left(0, \frac{\pi}{2}\right]$

Use Trigonometry and Geometry concepts to solve.

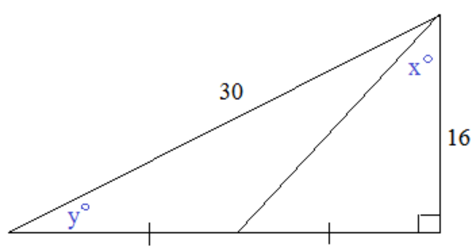
1) Find x and y:



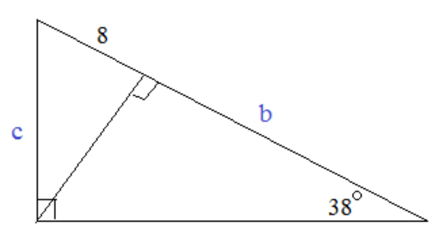
2) Find m and w:



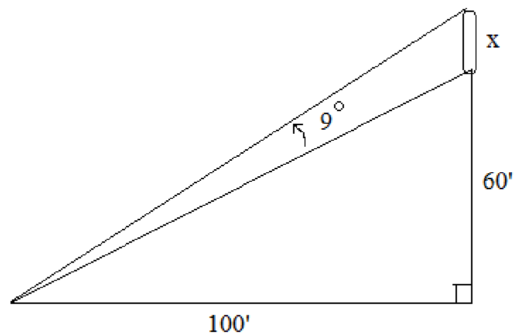
3) Find x and y:



4) Find b and c:

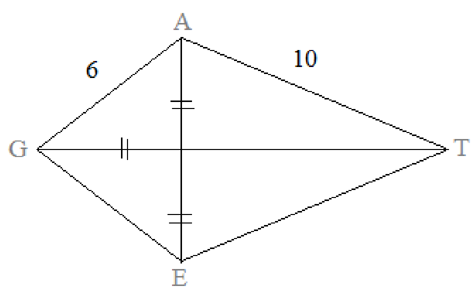


5) What is x ?



6) GATE is a kite.

What is the measure of angle $\angle ATG$?



7) sin(x) = .3 What are the other 5 trig values?

tan(x) =

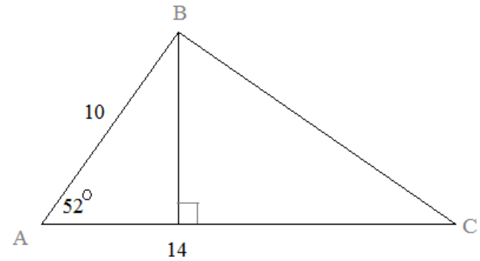
cos(x) =

csc(x) =

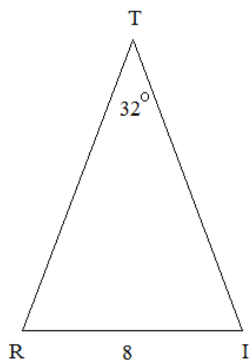
sec(x) =

cot(x) =

8) Find the area of triangle ABC

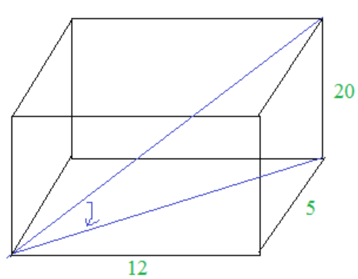


9) Find the perimeter of the isosceles triangle TRI

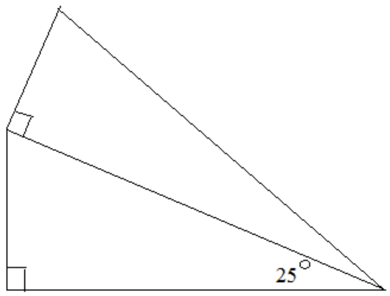


10) A 25 foot ladder extends from the ground to the window of a building.
If the window is 22 feet high, find the angle of depression from the window to the bottom of the ladder.

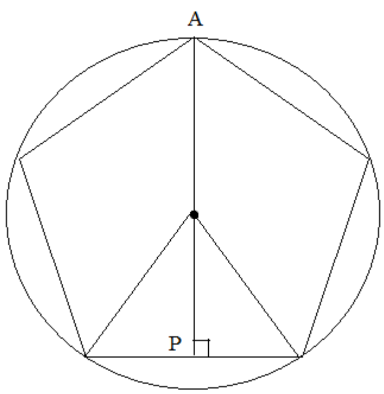
11) A rectangular box has dimensions 5, 12, and 20 inches.
Find the angle between the diagonal of the box and the diagonal of the 5 x 12 face.



12) If the triangles have the same perimeters,
what are the angle measures of the triangles?



13) If the regular pentagon with perimeter 60 is inscribed in a circle,
what is the length of $\overline{AP}$?

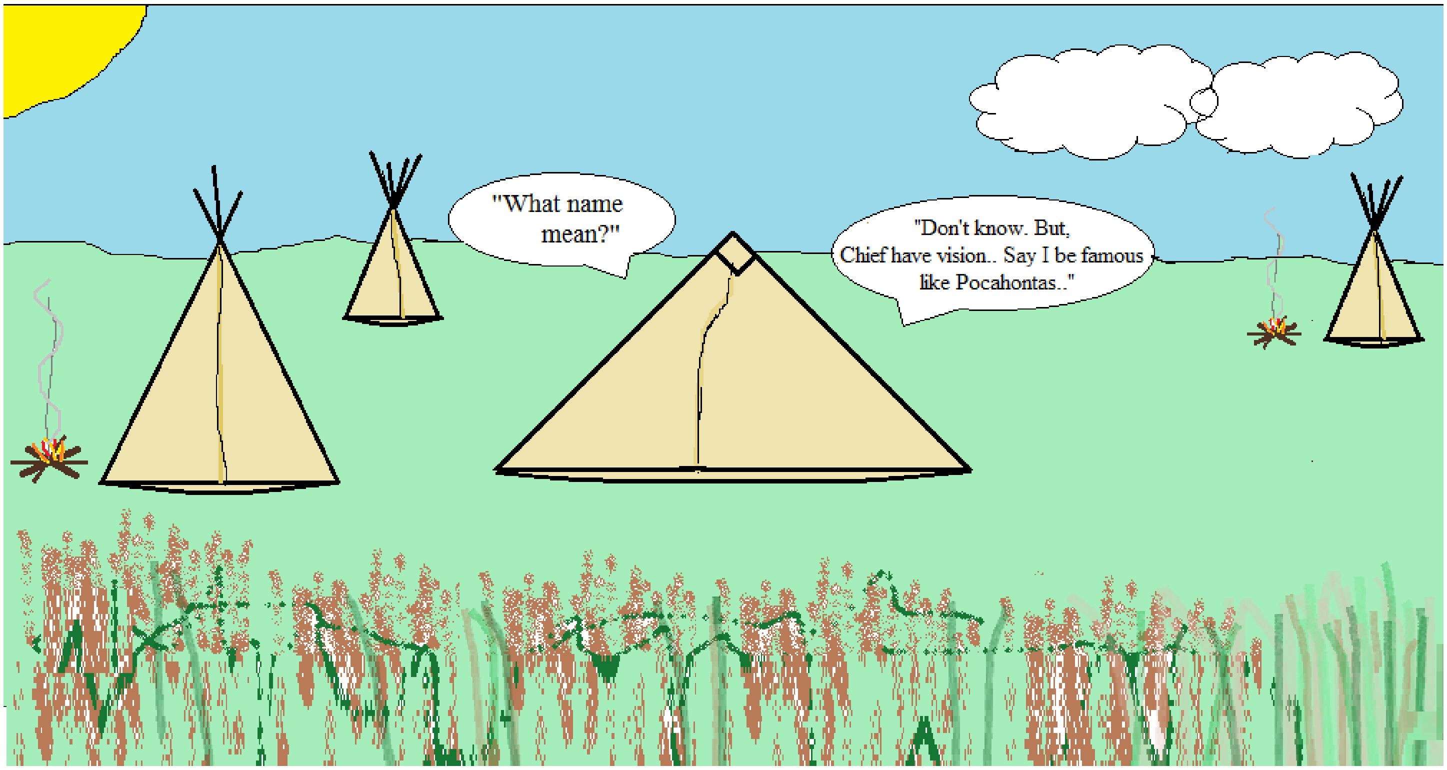


14) A 6-foot man looks at a mural that rises from the ground.
The angle of depression from his eyes to the ground is 14 degrees;
and, the angle of elevation from his eyes to the top of the mural is 27 degrees.

Approximately, how high is the mural?

(Hint: how far from the mural is the man standing?)

In(side) a Name



Conversation with Princess Sohcahtoa

Answer Keys -→

Hidden Message

Solve the following trig equations to reveal the answer!

SOLUTIONS

1) $\sin B = \frac{\boxed{l}}{m}$

2) $\sin D = \frac{p}{\boxed{o}}$ $\quad \frac{\text{Opposite}}{\text{Hypotenuse}}$

3) $\cos E = \frac{s}{\boxed{t}}$ $\quad \frac{\text{Adjacent}}{\text{Hypotenuse}}$

4) $\tan F = \frac{\boxed{s}}{r}$ $\quad \frac{\text{Opposite}}{\text{Adjacent}}$

5) $\sin C = \frac{q}{\boxed{o}}$ $\quad \frac{\text{Opposite}}{\text{Hypotenuse}}$

6) $\sin \boxed{F} = \frac{s}{t}$ $\quad \frac{\text{Opposite}}{\text{Hypotenuse}}$

7) $\cos \boxed{C} = \frac{p}{o}$ $\quad \frac{\text{Adjacent}}{\text{Hypotenuse}}$

8) $\cos D = \frac{q}{\boxed{o}}$ $\quad \frac{\text{Adjacent}}{\text{Hypotenuse}}$

9) $\cos B = \frac{n}{\boxed{m}}$ $\quad \frac{\text{Adjacent}}{\text{Hypotenuse}}$

10) $\tan D = \frac{\boxed{p}}{q}$ $\quad \frac{\text{Opposite}}{\text{Adjacent}}$

11) $\tan B = \frac{\boxed{l}}{n}$ $\quad \frac{\text{Opposite}}{\text{Adjacent}}$

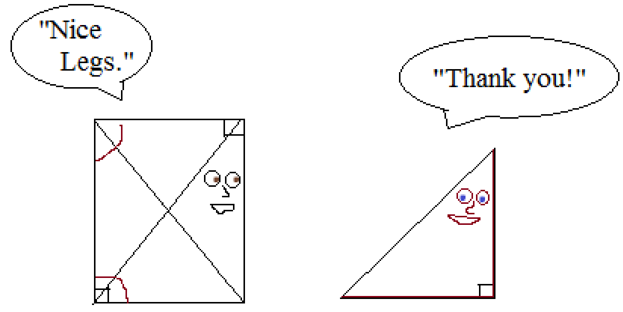
12) $\cos A = \frac{l}{\boxed{m}}$ $\quad \frac{\text{Adjacent}}{\text{Hypotenuse}}$

13) $\tan \boxed{E} = \frac{r}{s}$ $\quad \frac{\text{Opposite}}{\text{Adjacent}}$

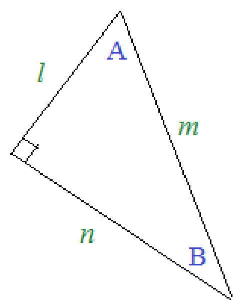
14) $\sin A = \frac{\boxed{n}}{m}$ $\quad \frac{\text{Opposite}}{\text{Hypotenuse}}$

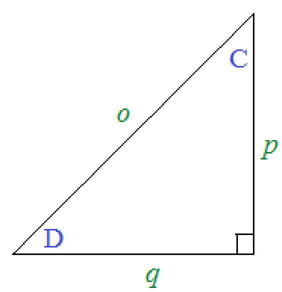
15) $\cos F = \frac{r}{\boxed{t}}$ $\quad \frac{\text{Adjacent}}{\text{Hypotenuse}}$

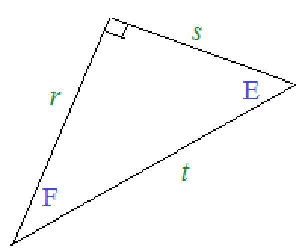
16) $\sin F = \frac{\boxed{s}}{t}$ $\quad \frac{\text{Opposite}}{\text{Hypotenuse}}$



What did the polite geometry figure have?







1) L
2) O
3) T
4) S
5) O
6) F
7) C
8) O
9) M
10) P
11) L

E or I

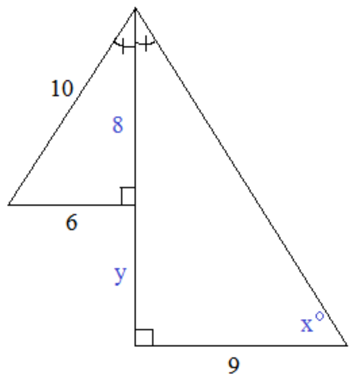
12) M
13) E
14) N
15) T
16) S

"Lots of Complements"
or "Lots of Compliments"

Use Trigonometry and Geometry concepts to solve.

SOLUTIONS

1) Find x and y:



The small (left) triangle is a 6-8-10 Pythagorean Triple

Since acute angles are congruent, the right triangles are similar. (angle-angle similarity)

$\cos(x) = \frac{6}{10}$

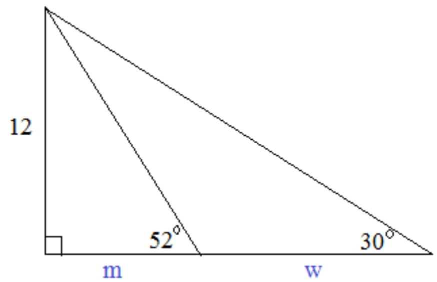
$x = \cos^{-1}(.6) = \boxed{53.1°}$

$\frac{8 + y}{9} = \frac{8}{6}$

$72 = 48 + 6y$

$\boxed{y = 4}$

2) Find m and w:



$\tan(52) = \frac{12}{m}$

$m = \frac{12}{1.28}$

$\boxed{m = 9.38}$

30-60-90 triangle, so

$(m + w) = 12\sqrt{3}$

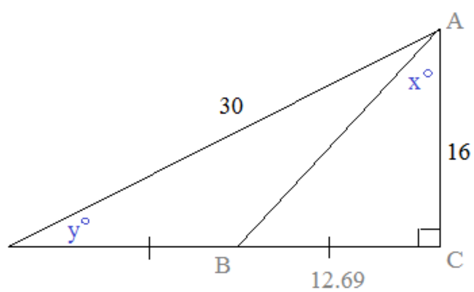
or, $\tan(30) = \frac{12}{(m + w)}$

$m + w = \frac{12}{.577}$

$9.38 + w = 20.8$

$\boxed{w = 11.42}$

3) Find x and y:



$\sin(y) = \frac{16}{30}$

$y = \sin^{-1}(16/30)$

$\boxed{y = 32.23°}$

Using Pythagorean Theorem,

$30^2 = 16^2 + (\text{leg})^2$

$\text{leg} = 25.38$

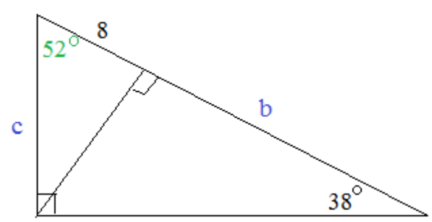
Since AB is a median,

$BC = (1/2)(25.38) = 12.69$

$\tan(x) = \frac{12.69}{16}$

$\boxed{x = 38.42°}$

4) Find b and c:



$\cos(52) = \frac{8}{c}$

$c = \frac{8}{\cos(52)}$

$\boxed{c = 12.99}$

then,

$\sin(38) = \frac{c}{(8 + b)}$

$.616 = \frac{12.99}{(8 + b)}$

$8 + b = 21.1$

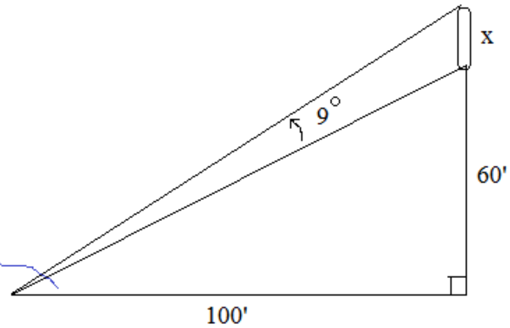
$\boxed{b = 13.1}$

5) What is x ? SOLUTIONS

$\tan(Y) = \frac{60}{100}$

$Y = \tan^{-1}(.6)$

$Y = 30.96^\circ$



Therefore, the "big triangle" has angle 39.96 and sides 100' and (60' + x)

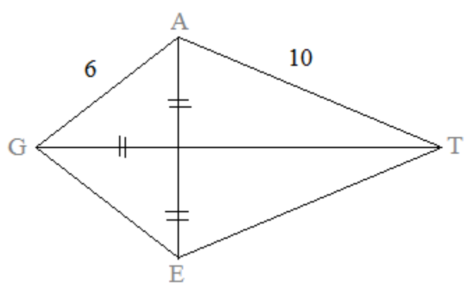
$\tan(39.96) = \frac{(60' + x)}{100'}$

$.838 \times (100') = 60' + x$

$x = 23.8$ feet

6) GATE is a kite.

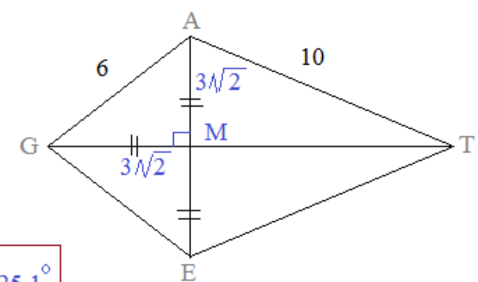
What is the measure of angle $\angle ATG$?



Since GATE is a kite, GT is a perpendicular bisector of AE....

Also, $\overline{GM} \cong \overline{AM} \cong \overline{EM}$

Using Pythagorean Theorem (or 45-45-90 properties) $\overline{AM}$ is $3\sqrt{2}$



$\text{Sin(ATG)} = \frac{3\sqrt{2}}{10} \longrightarrow \angle ATG = 25.1^\circ$

7) $\sin(x) = .3$ What are the other 5 trig values?

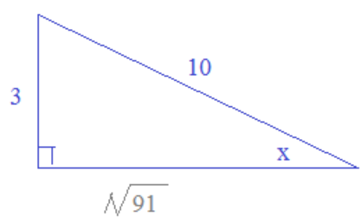
$\tan(x) = \frac{3\sqrt{91}}{91}$

$\cos(x) = \frac{\sqrt{91}}{10}$

$\csc(x) = \frac{10}{3}$

$\sec(x) = \frac{10\sqrt{91}}{91}$

$\cot(x) = \frac{\sqrt{91}}{3}$



The ratio of the 'opposite' over the 'hypotenuse' is .3

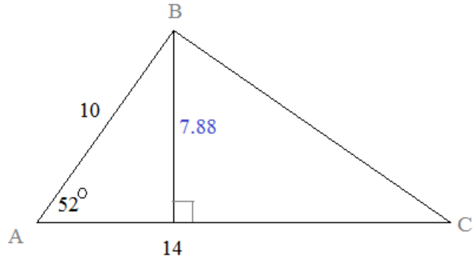
(so, any 2 numbers equal to 3/10 or .3 will work)

then, use pythagorean theorem:

$3^2 + b^2 = 10^2$

$b = \sqrt{91}$

8) Find the area of triangle ABC



SOLUTIONS

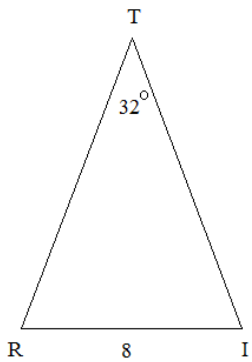
$\sin(52) = \frac{\text{height}}{10}$

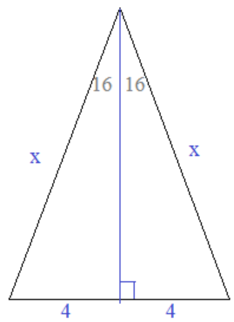
height = 7.88

area of ABC $= \frac{1}{2}(14)(7.88)$

$= 55.16$

9) Find the perimeter of the isosceles triangle TRI





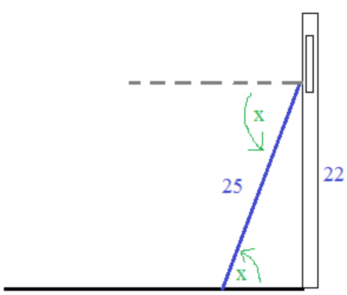
$\sin(16) = \frac{4}{x}$

$x = 14.51$

so, perimeter of TRI is

$14.51 + 14.51 + 8 = 37.02$

10) A 25 foot ladder extends from the ground to the window of a building.
If the window is 22 feet high, find the angle of depression from the window to the bottom of the ladder.



$\sin(x) = \frac{22}{25}$

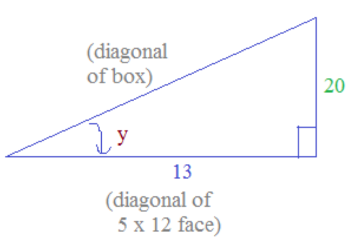
$x = 61.65$ degrees

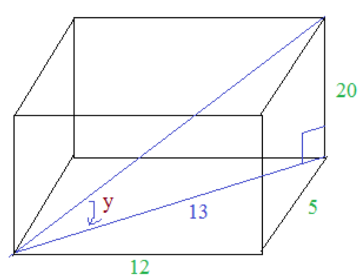
11) A rectangular box has dimensions 5, 12, and 20 inches.
Find the angle between the diagonal of the box and the diagonal of the 5 x 12 face.

$\tan(y) = \frac{20}{13}$

$y = \tan^{-1}(1.538)$

$y = 57$ degrees





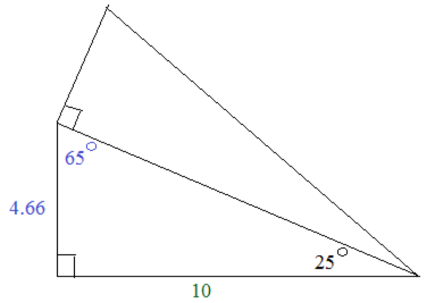
12) If the triangles have the same perimeters, what are the angle measures of the triangles?

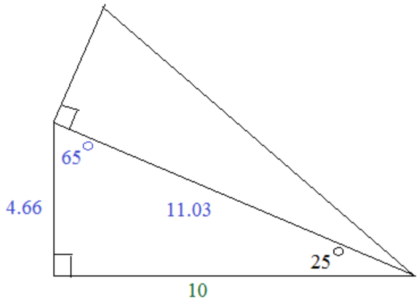
SOLUTIONS

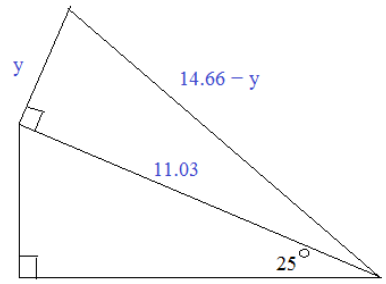
Let's assign a value and see what happens...
Assume the base of the lower triangle is 10 units...

$\tan(25) = \frac{x}{10}$

$x = 4.66$







$4.66^2 + 10^2 = 11.03^2$

so, the perimeters are 25.69

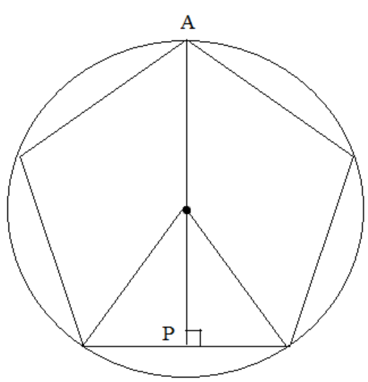
$11.03^2 + y^2 = (14.66 - y)^2$

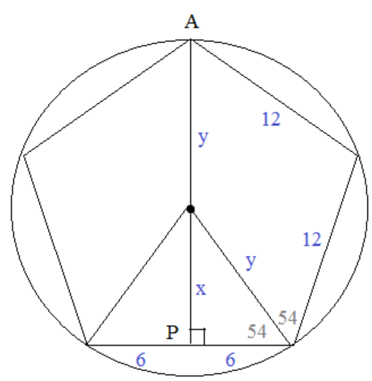
$y = 3.18$ $\tan(z) = 3.18/11.03$

$z = 16.08°$

and, $73.92°$

13) If the regular pentagon with perimeter 60 is inscribed in a circle, what is the length of $\overline{AP}$?





sum of interior angles of a pentagon are 540 degrees...
so, each interior angle of regular pentagon is 108 degrees

$\tan(54) = \frac{x}{6}$ $x = 8.26$

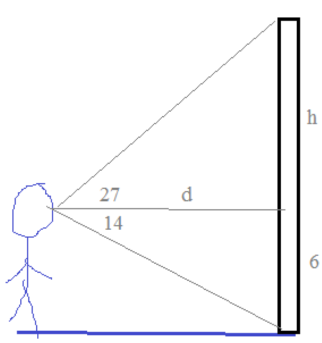
$\cos(54) = \frac{6}{y}$ $y = 10.21$

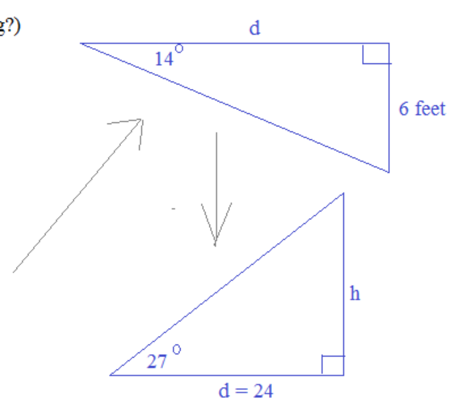
$AP = x + y =$ 18.47

14) A 6-foot man looks at a mural that rises from the ground.
The angle of depression from his eyes to the ground is 14 degrees;
and, the angle of elevation from his eyes to the top of the mural is 27 degrees.

Approximately, how high is the mural?

(Hint: how far from the mural is the man standing?)





to get d,

$\tan(14) = \frac{6}{d}$ distance to mural: 24 feet

then, to get h,

$\tan(27) = \frac{h}{24}$ $h = 12.24$

the mural is approximately 18.24 feet

Thanks for visiting.

If you have questions, suggestions, or requests, let us know.

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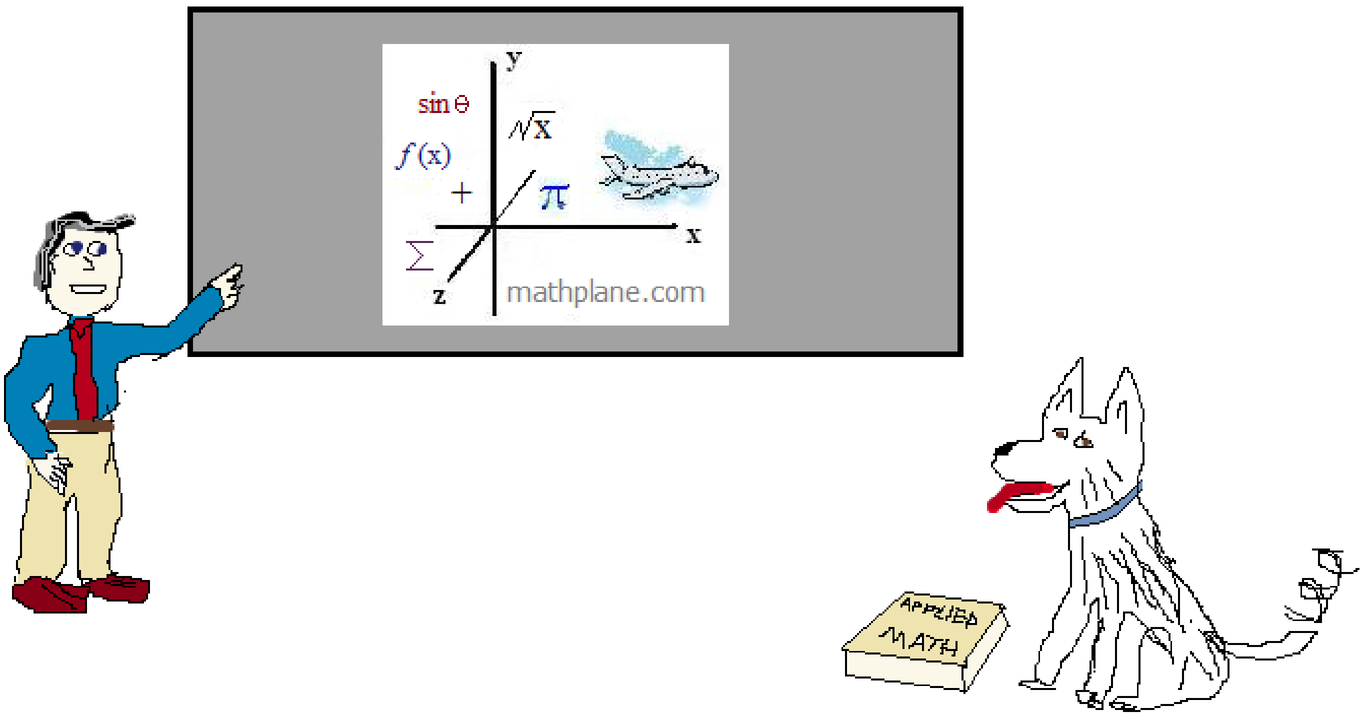

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