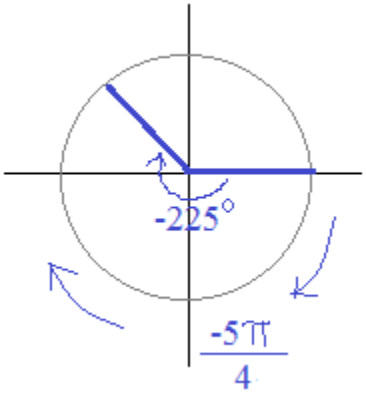
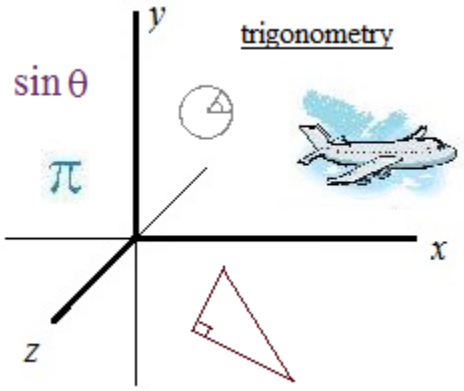
Unit Circle and Trig Measures



Examples and Practice Exercises (with solutions)

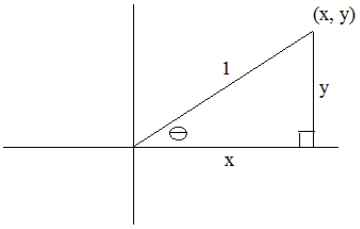


The Unit Circle

What is it? A circle with a radius of one unit...

It's also a visual representation of special angles that give exact trig values...

Here is a triangle with hypotenuse length 1...



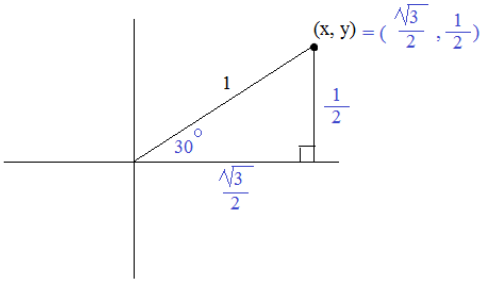
$\sin\ominus = \frac{y}{1} = y$ $\quad \csc\ominus = \frac{1}{y}$

$\cos\ominus = \frac{x}{1} = x$ $\quad \sec\ominus = \frac{1}{x}$

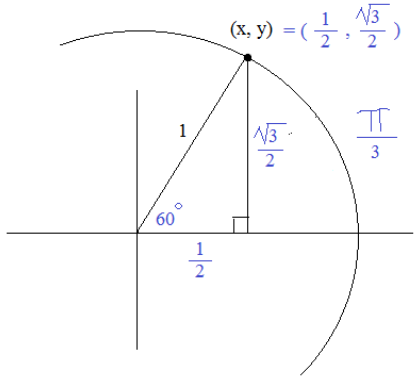
$\tan\ominus = \frac{y}{x}$ $\quad \cot\ominus = \frac{x}{y}$

We can determine points on a circle with radius 1...

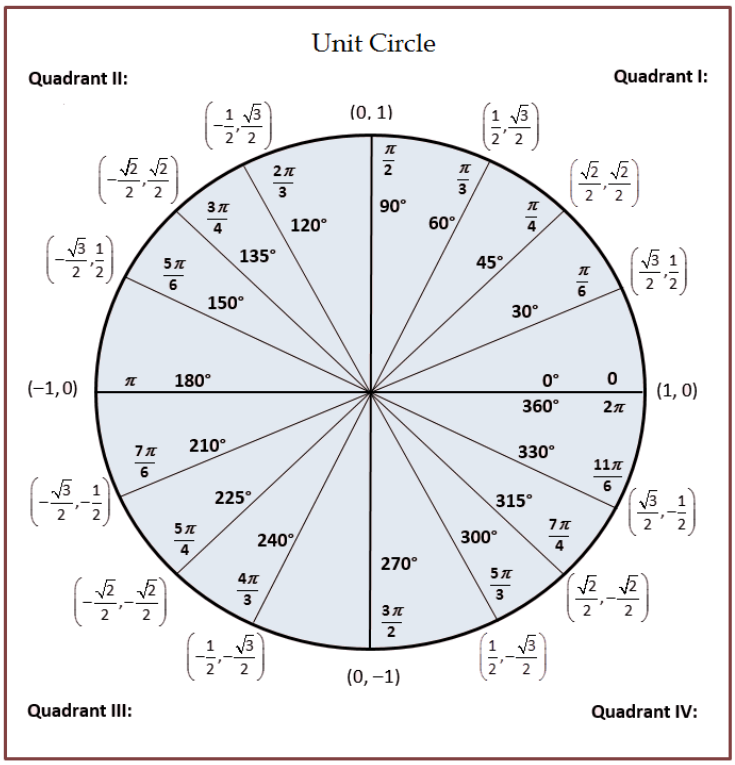
Degrees



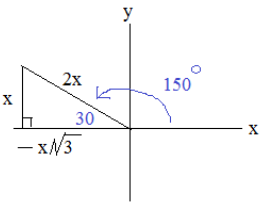
Radians



Using the angles related to 45-45-90 and 30-60-90 right triangles, using the hypotenuse of 1, and applying it to a coordinate plane, we create a "unit circle centered on the origin"....



Example: What is $\sin(150^\circ)$? What is $\cos(\frac{5\pi}{6})$?



"TRIANGLES"

$\text{sine} = \frac{\text{opposite}}{\text{hypotenuse}}$ $\quad \sin(150) = \frac{1}{2}$

$\text{cosine} = \frac{\text{adjacent}}{\text{hypotenuse}}$ $\quad \cos(\frac{5\pi}{6}) = \frac{-\sqrt{3}}{2}$

Now, look at the unit circle....

The point that corresponds to 150 degrees and $\frac{5\pi}{6}$ radians is

$$\left(\frac{-\sqrt{3}}{2}, \frac{1}{2}\right)$$

since $\sin = y$, $\quad \sin(150) = \frac{1}{2}$

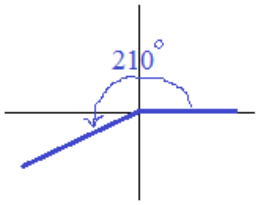
"UNIT CIRCLE"

$\cos = x$, $\quad \cos(\frac{5\pi}{6}) = \frac{-\sqrt{3}}{2}$

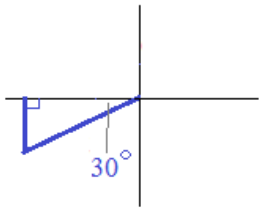
Unit Circle: Finding Trig Values

Example: find $\sin 210^\circ$

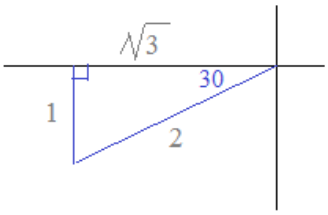
Step 1: Draw angle in standard position



Step 2: Find reference angle

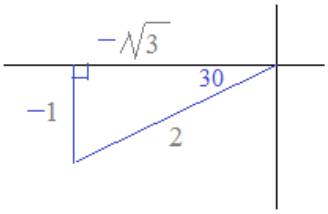


Step 3: Identify the triangle and label



30-60-90 right triangle

Step 4: distinguish 'negatives' and 'positives'



Quadrant III

Step 5: Find Trigonometry Value

$$\text{Sine} = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{-1}{2}$$

$$\text{Sin}(210) = \frac{-1}{2}$$

Note: --- If the angle is given in radians, convert to degrees and begin.
--- Step 4 is important! Don't forget the negatives.
Also, the hypotenuse is always positive.

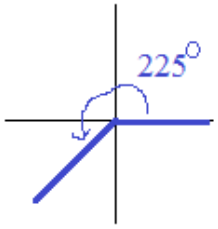
Unit Circle: Finding Trig Values

Example: Tan $\frac{5\pi}{4}$

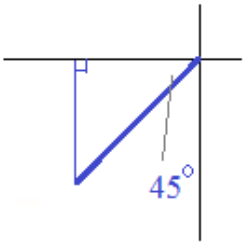
(Convert to Degrees)

$$\frac{5\pi \text{ radians}}{4} \cdot \frac{180^\circ}{\pi \text{ radians}} = 225^\circ$$

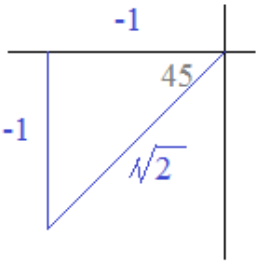
Step 1: Draw Angle in standard position



Step 2: Find Reference Angle



Step 3: Identify Triangle and label



45-45-90 right triangle

Step 4: Negatives and Positives

Quadrant III

Step 5: Find Trigonometry Value

$$\text{tangent} = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{-1}{-1} = 1$$

Note: $180^\circ = \pi$ radians

$$\text{Tan } \frac{5\pi}{4} = 1$$

What is a quadrantal angle?

Evaluating Trig Functions of Quadrantal Angles

An angle in standard position whose terminal side lies on the x or y axis.
These angles include:

$$0,\ \pm 90,\ \pm 180,\ \pm 270,\ \pm 360\ldots$$

$$\pm \frac{\sqrt{}}{2} \quad \pm\sqrt{} \quad \pm \frac{3\sqrt{}}{2} \quad \pm 2\sqrt{}$$

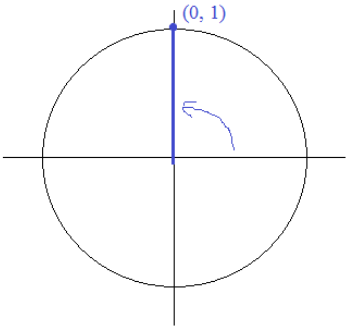
The trig values of a quadrantal angle will be 0, -1, 1, or undefined

Using the unit circle or utilizing a "fictional triangle" and 'Soh Cah Toa'

Here are 2 approaches to finding the trig values of a quadrantal angle...

Example: Find $\sin(90^\circ)$

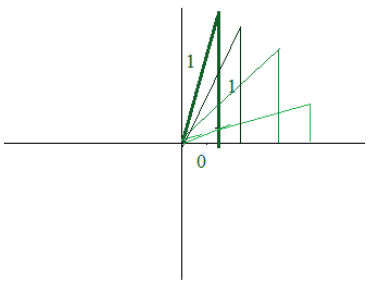
Using the Unit Circle

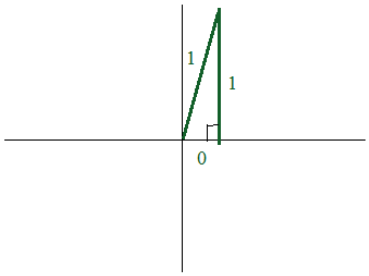


We know that $\sin = \frac{y}{r}$

so, $\sin(90) = \frac{1}{1} = 1$

Creating an (imaginary) Triangle





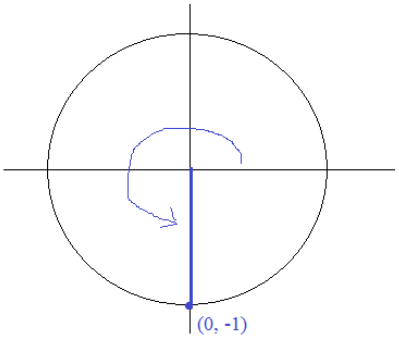
This triangle has a reference angle of 90..

It's opposite side is 1,
adjacent side is 0,
and, hypotenuse is 1

$$\sin = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{1} = 1$$

Example: Find the value of $\cot(\frac{3\sqrt{}}{2})$

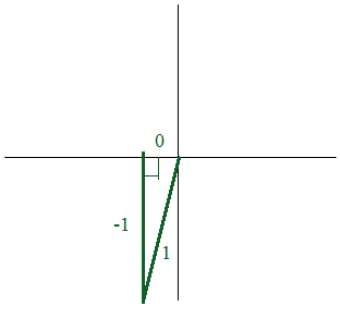
Unit Circle



We know that $\cot = \frac{x}{y}$

Therefore, $\cot(\frac{3\sqrt{}}{2}) = \frac{0}{-1} = 0$

Creative Triangle



('reference angle' is 90 degrees)

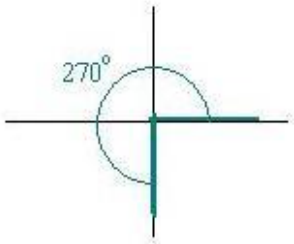
Using an imaginary triangle, we know that

$$\cot = \frac{\text{adjacent}}{\text{opposite}} = \frac{0}{-1} = 0$$

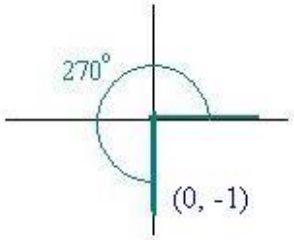
Unit Circle: Finding Trig Values -- *Quadrantals*

Example: Sin 270°

Step 1: Draw angle in standard position



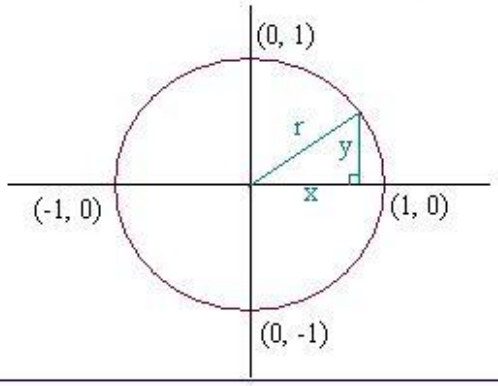
Step 2: "Label the point"
(Reminder: it is a UNIT circle; the radius is 1)



Step 3: Apply the trig function

$$\text{Sine} = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{y}{r} = \frac{-1}{1} = -1$$

Note: ---If the angle is given in radians, convert to degrees and begin.
---Observe the points and trig functions on the unit circle:



$r = 1$

$$\sin = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}$$

$$\cos = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}$$

$$\tan = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$$

Sin(270) = -1

Finding Trig Values: Coterminal and "Negative" Angles

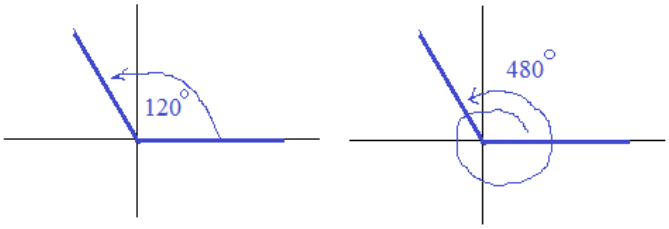
"Coterminal" Example: Find the exact trig value of $\sin(480^\circ)$

Step 1: Find Coterminal angle (between 0 and 360 degrees)

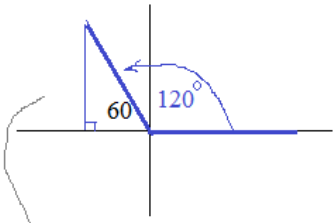
$480^\circ - 360^\circ = 120^\circ$

Since 480 and 120 are coterminal angles, their trig values are equal.

Step 2: Draw the angle in standard position

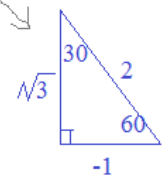


Step 3: Find Reference Angle



The reference angle is 60°

Step 4: "Identify the triangle and label"



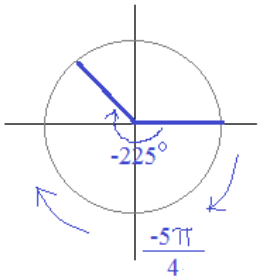
Step 5: Find the trigonometry value

30-60-90 triangle...

$$\text{Sine} = \frac{\text{opposite side}}{\text{hypotenuse}} = \boxed{\frac{\sqrt{3}}{2}}$$

"Negative Angle" Example: Evaluate $\tan(-5\pi/4)$

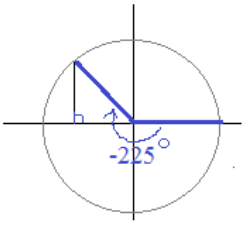
Step 1: Draw the angle in standard position

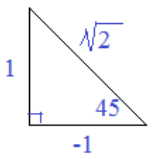


Since the angle is negative, move *clock-wise* along the unit circle!

Step 2: Find the reference angle

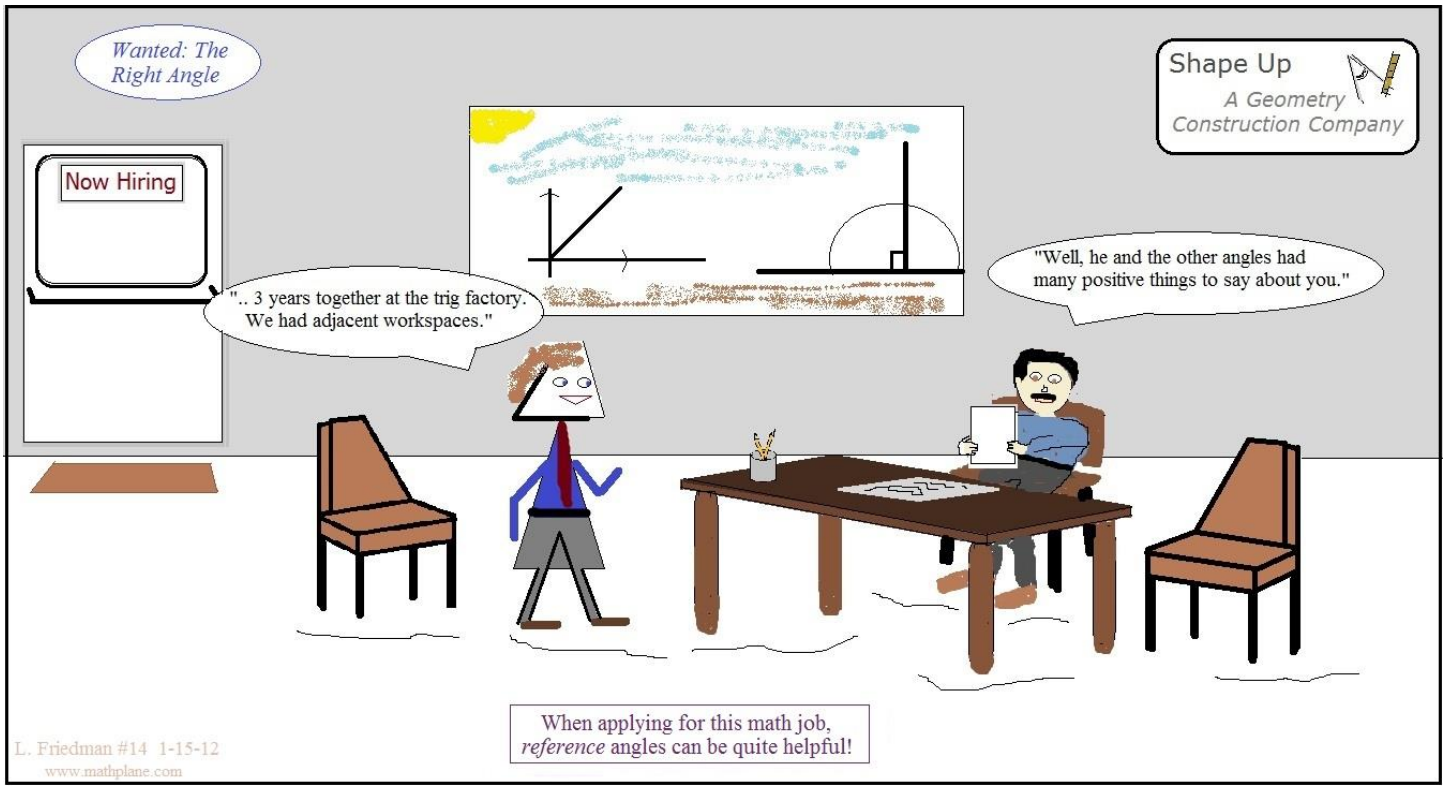
Step 3: Identify the triangle and label





Step 4: find the trig value

$$\text{Tan} = \frac{\text{Opposite side}}{\text{Adjacent side}} = \frac{1}{-1} = \boxed{-1}$$


Wanted: The Right Angle
Shape Up
A Geometry Construction Company
Now Hiring
".. 3 years together at the trig factory. We had adjacent workspaces."
"Well, he and the other angles had many positive things to say about you."
When applying for this math job, *reference* angles can be quite helpful!
L. Friedman #14 1-15-12
www.mathplane.com

Practice -→

Trig Values & Unit Circle: Practice worksheet

Evaluate the following (NO tables! NO calculators!)

1) $\sin 30$

2) $\tan 210$

3) $\csc 120$

4) $\sec 0$

5) $\sin 225$

6) $\cot 315$

7) $\cos -45$

8) $\sin 270$

9) $\cot 180$

10) $\cos \frac{\pi}{3}$

11) $\sin \frac{3\pi}{4}$

12) $\tan \frac{7\pi}{4}$

13) $\csc \frac{5\pi}{6}$

14) $\cos 3\pi$

15) $\cos \frac{-\pi}{3}$

Trig Values and Unit Circle Worksheet: Coterminal and Negative Angles

Evaluate the following (without using a calculator or table)

1) $\sin(420^\circ)$

2) $\cos(\frac{9\pi}{4})$

3) $-\tan(135^\circ)$

4) $\tan(-135^\circ)$

5) $\sec(\frac{11\pi}{3})$

6) $\csc(\frac{-11\pi}{3})$

7) $\cos(7\pi)$

8) $-\cot(\frac{14\pi}{3})$

9) $\csc(900^\circ)$

10) $-\sin(450^\circ)$

11) $\csc(-450^\circ)$

12) $\cot(\frac{-29\pi}{6})$

Evaluating Trig Functions of Quadrantal Angles

1) $\cos(0^\circ)$

2) $\tan(\sqrt{\ })$

3) $\sin(90^\circ)$

4) $\sec(-\frac{\sqrt{\ }}{2})$

5) $\cot(180^\circ)$

6) $\sin(\frac{3\sqrt{\ }}{2})$

7) $\csc(2\sqrt{\ })$

Find point (x, y) on the unit circle that corresponds to the real number t

1) $t = \frac{7\pi}{4}$

2) $t = \frac{-\pi}{2}$

3) $t = \frac{5\pi}{6}$

4) $t = \frac{-2\pi}{3}$

5) $t = \frac{2\pi}{7}$ (calculator)

6) $t = 3.5$ radians (calculator)

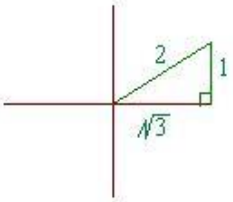
Trig Values & Unit Circle: Practice Worksheet

SOLUTIONS

Evaluate the following:

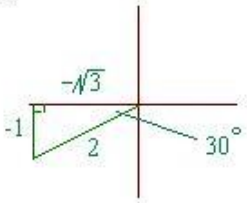
1) sin 30

$\frac{1}{2}$



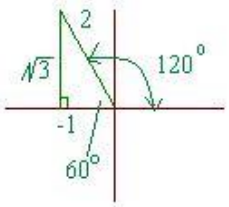
2) tan 210

$\frac{-1}{-\sqrt{3}} = \frac{\sqrt{3}}{3}$



3) csc 120

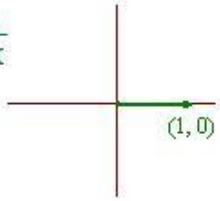
$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$



4) sec 0

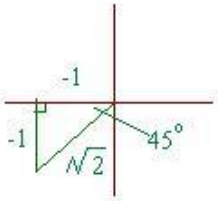
$\sec = \frac{r}{x}$

$\frac{1}{1} = 1$



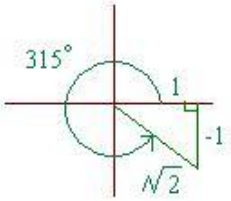
5) sin 225

$\frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$



6) cot 315

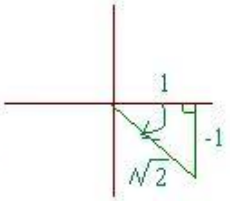
$\frac{adj}{opp} = -1$



7) cos -45

$\frac{1}{\sqrt{2}}$

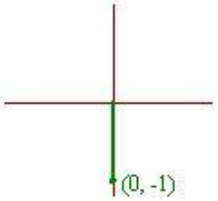
$= \frac{\sqrt{2}}{2}$



8) sin 270

$\sin = \frac{y}{r}$

$\frac{-1}{1} = -1$

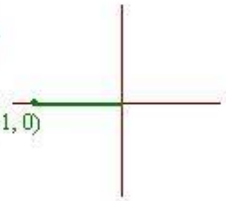


9) cot 180

$\cot = \frac{x}{y} = \frac{-1}{0}$

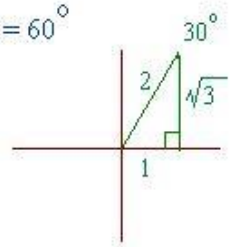
ϕ

Undefined



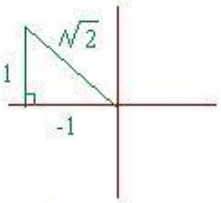
10) $\cos \frac{\pi}{3} = 60^\circ$

$\frac{1}{2}$



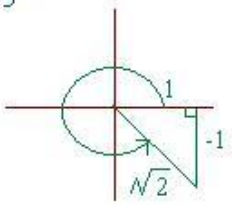
11) $\sin \frac{3\pi}{4} = 135^\circ$

$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$



12) $\tan \frac{7\pi}{4} = 315^\circ$

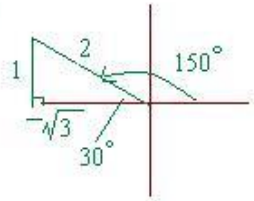
$\frac{-1}{1} = -1$



13) $\csc \frac{5\pi}{6} = 150^\circ$

$\csc = \frac{hyp}{opp}$

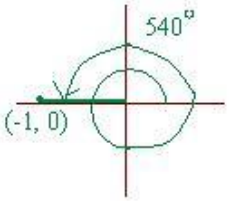
$\frac{2}{1} = 2$



14) $\cos 3\pi = 540^\circ$

$\cos = \frac{x}{r}$

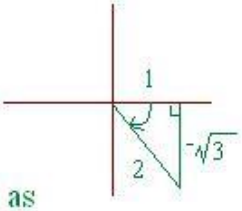
$\frac{-1}{1} = -1$



15) $\cos \frac{-\pi}{3} = -60^\circ$

$\frac{1}{2}$

(same solution as question 10!!)



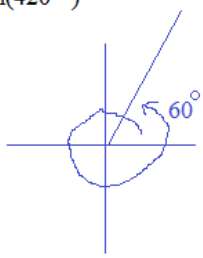
Trig Values and Unit Circle Worksheet: Coterminal and Negative Angles

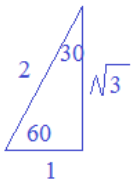
SOLUTIONS

Evaluate the following (without using a calculator or table)

1) $\sin(420^\circ)$

$\frac{\sqrt{3}}{2}$



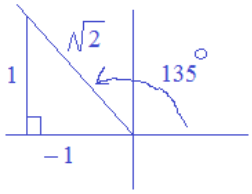


2) $\cos(\frac{9\pi}{4})$ coterminal angle

$\frac{9\pi}{4} - \frac{8\pi}{4} = \frac{\pi}{4}$

$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

3) $-\tan(135^\circ)$

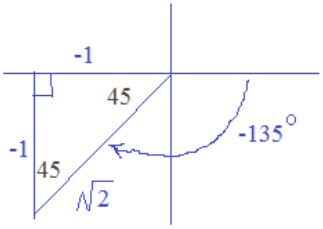


$\tan(135) = \frac{1}{-1} = -1$

so, $-\tan(135) = 1$

4) $\tan(-135^\circ)$

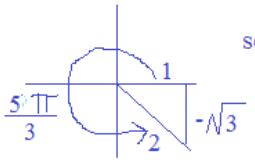
$\tan(-135) = \frac{-1}{-1}$



$= 1$

5) $\sec(\frac{11\pi}{3})$ coterminal angles

$\frac{11\pi}{3} - \frac{6\pi}{3} = \frac{5\pi}{3}$



$\sec = \frac{\text{hypo}}{\text{adj}} = 2$

6) $\csc(\frac{-11\pi}{3})$ coterminal angle

$-\frac{11\pi}{3} + \frac{12\pi}{3} = \frac{\pi}{3}$

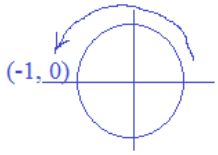
$\csc(60^\circ)$ or $\csc(\frac{\pi}{3})$

$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

7) $\cos(7\pi)$

$7\pi - 2\pi - 2\pi - 2\pi$

$\cos(7\pi) = \cos(\pi)$



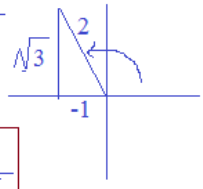
$\cos = \frac{x}{1}$

$\cos(7\pi) = -1$

8) $-\cot(\frac{14\pi}{3})$

$\frac{14\pi}{3} - \frac{12\pi}{3} = \frac{2\pi}{3}$

$\cot(\frac{2\pi}{3}) = \frac{-1}{\sqrt{3}}$



so, $-\cot(\frac{2\pi}{3}) = \frac{1}{\sqrt{3}}$

9) $\csc(900^\circ)$

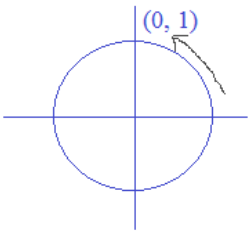
900 - 360 = 540...
540 - 360 = 180

$\csc(180)$ is undefined

10) $-\sin(450^\circ)$

coterminal angles

$-\sin(90^\circ)$



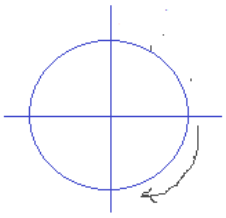
$\sin = \frac{y}{1}$

since $\sin(90) = 1$,

then $-\sin(90) = -1$

11) $\csc(-450^\circ)$

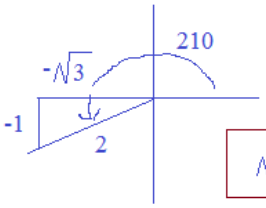
$\csc(-90) = -1$



$\csc = \frac{1}{y}$

12) $\cot(\frac{-29\pi}{6})$

$\frac{-29\pi}{6} + \frac{36\pi}{6} = \frac{7\pi}{6}$



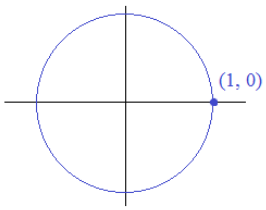
$\sqrt{3}$

Evaluating Trig Functions of Quadrantal Angles

mathplane.com

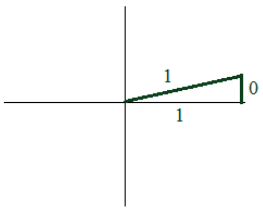
SOLUTIONS

1) $\cos(0^\circ)$



$\cos = x$

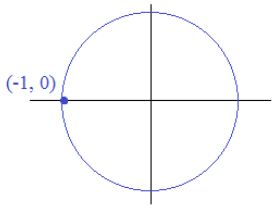
1



reference angle is 0

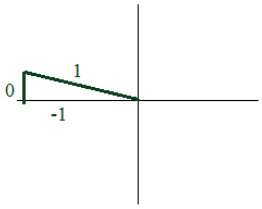
$\cos = \frac{\text{adjacent}}{\text{hypotenuse}} = 1$

2) $\tan(\sqrt{\pi})$



$\tan = \frac{y}{x}$

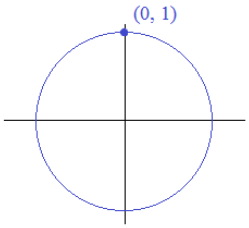
0



reference angle is 0

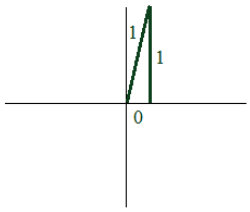
$\tan = \frac{\text{opposite}}{\text{adjacent}} = 0$

3) $\sin(90^\circ)$



$\sin = y$

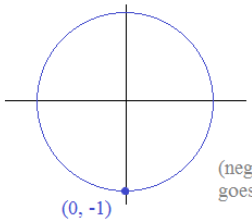
1



reference angle is 90

$\sin = \frac{\text{opposite}}{\text{hypotenuse}} = 1$

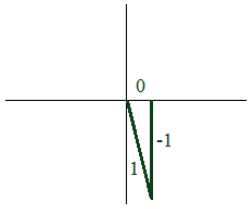
4) $\sec(-\frac{\sqrt{\pi}}{2})$



$\sec = \frac{1}{x}$

undefined

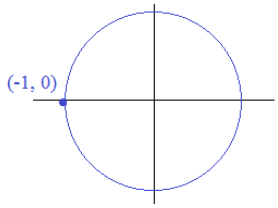
(negative angle goes clockwise)



reference angle is 90

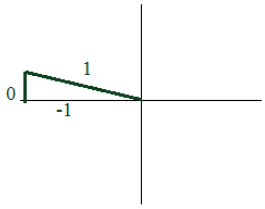
$\sec = \frac{\text{hypotenuse}}{\text{adjacent}} = \text{undefined}$

5) $\cot(180^\circ)$



$\cot = \frac{x}{y}$

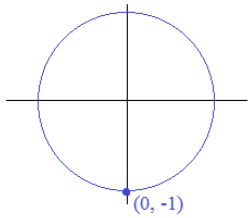
undefined



reference angle is 0

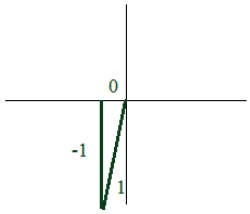
$\cot = \frac{\text{adjacent}}{\text{opposite}} = \text{undefined}$

6) $\sin(\frac{3\sqrt{\pi}}{2})$



$\sin = y$

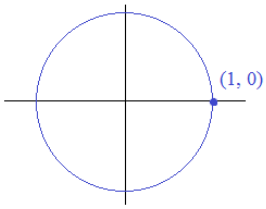
-1



reference angle is 90

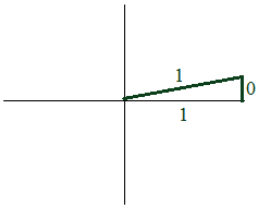
$\sin = \frac{\text{opposite}}{\text{hypotenuse}} = -1$

7) $\csc(2\sqrt{\pi})$



$\csc = \frac{1}{y}$

undefined



reference angle is 0

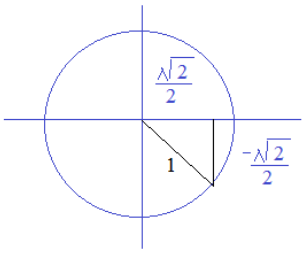
$\csc = \frac{\text{hypotenuse}}{\text{opposite}} = \text{undefined}$

Find point (x, y) on the unit circle that corresponds to the real number t

SOLUTIONS

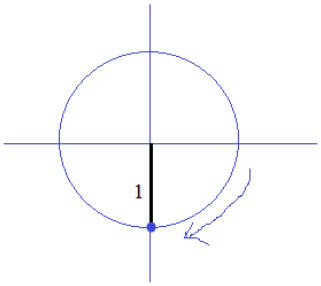
1) $t = \frac{7\pi}{4}$

$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$



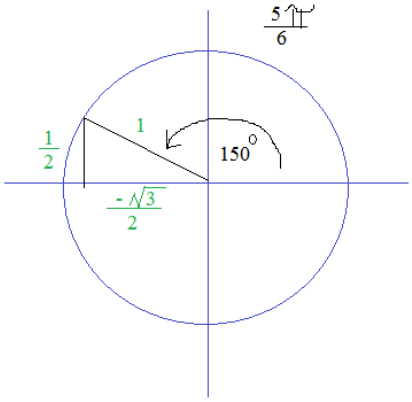
2) $t = \frac{-\pi}{2}$

(0, -1)



3) $t = \frac{5\pi}{6}$

$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

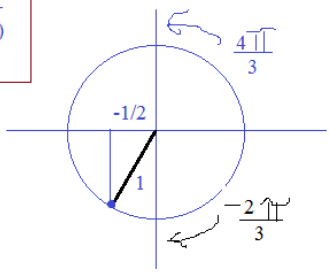


4) $t = \frac{-2\pi}{3}$

Option: add 2π

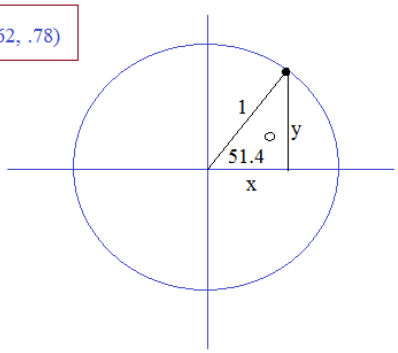
↳ coterminal angle $\frac{4\pi}{3}$

$\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$



5) $t = \frac{2\pi}{7}$ (calculator)

(.62, .78)



$\sin\left(\frac{2\pi}{7}\right)$ ↘

$\sin(51.4) = \frac{y}{1}$

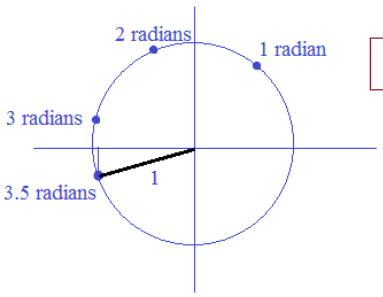
$y = .78$

$\cos(51.4) = \frac{x}{1}$

$x = .62$

6) $t = 3.5$ radians (calculator)

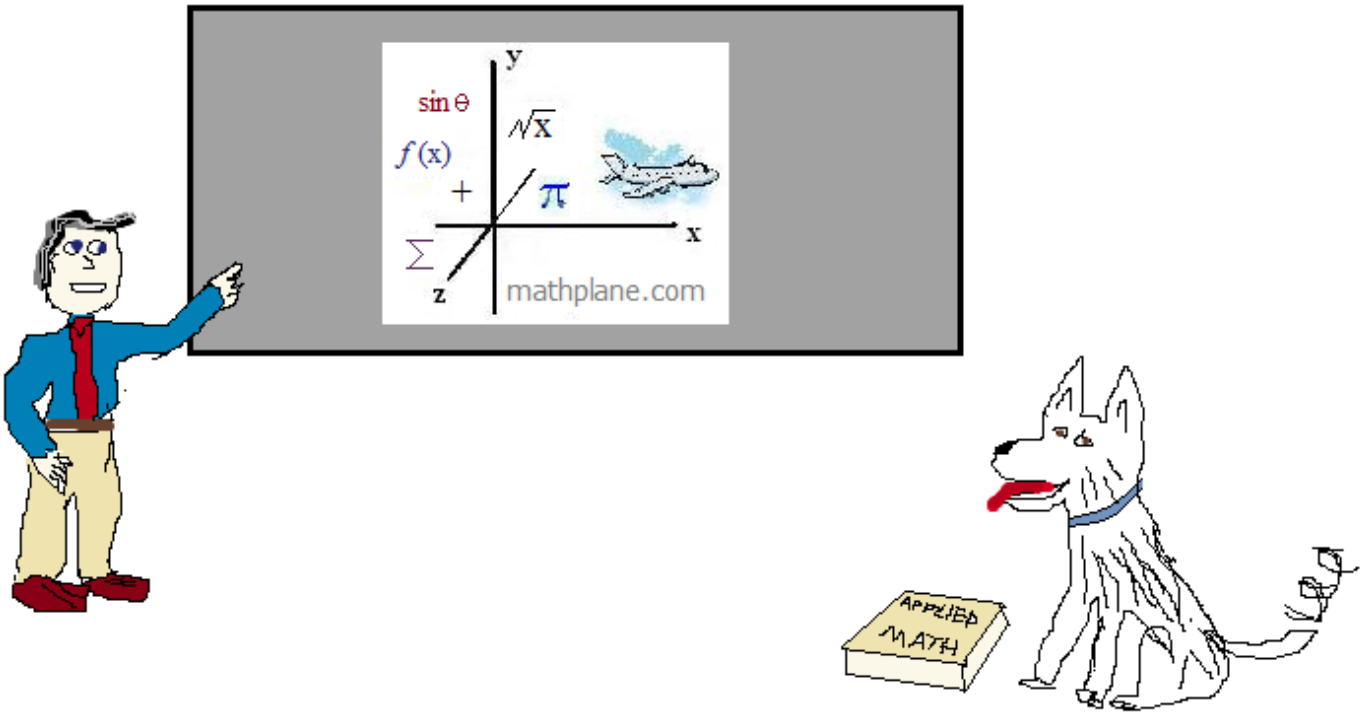
(-.94, -.35)



Hope this introduction helps.

If you have questions, suggestions, or requests, let us know.

Good luck!



Also, more trig and math resources at Mathplane *Express* for mobile at Mathplane.ORG

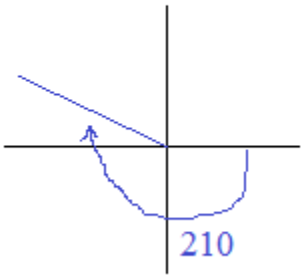
One more question….

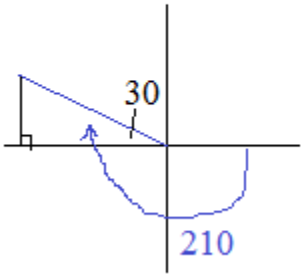
What is the exact value of sin(-210)?

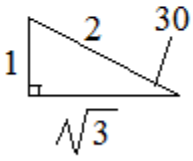
Answer on the next page…

What is the exact value of $\sin(-210^\circ)$?

$\frac{1}{2}$







Note: the 'direction' of the angle is clock-wise…