Analytic Geometry: Translations, Transformations, and Tangents

Topics include conics, extrema, dilations, and more.

Example: For the given polynomial, $y = x^4 - 8x^3 + 24x^2 - 28x + 7$

eliminate the constant and x-term to identify the horizontal and vertical translations.

horizontal $x = x' + h$

vertical $y = y' + k$

$$y' + k = (x' + h)^4 - 8(x' + h)^3 + 24(x' + h)^2 - 28(x' + h) + 7$$

$$y' + k = x'^4 + 4hx'^3 + 6h^2x'^2 + 4h^3x' + h^4 - 8(x'^3 + 3x'^2h + 3x'h^2 + h^3) + 24(x'^2 + 2x'h + h^2) - 28(x' + h) + 7$$

$$y' + k = x'^4 + 4hx'^3 + 6h^2x'^2 + 4h^3x' + h^4 - 8(x'^3 + 3x'^2h + 3x'h^2 + h^3) + 24(x'^2 + 2x'h + h^2) - 28(x' + h) + 7$$

(coefficients of x') $4h^3x' - 24h^2x' + 48hx' - 28x'$

$$4x'(h^3 - 6h^2 + 12h - 7) = 0 \qquad \boxed{h = 1}$$

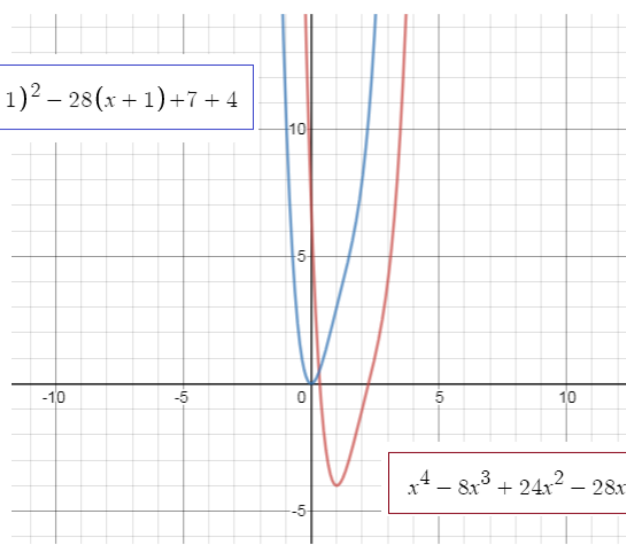
plug in (x' + 1) into original equation....

$(x' + 1)^4 - 8(x' + 1)^3 + 24(x' + 1)^2 - 28(x' + 1) + 7 = x'^4 - 4x'^3 + 6x'^2 - 4 \Rightarrow$ then, add 4 to y to eliminate the constant... $\boxed{k = 4}$

$\boxed{(x+1)^4 - 8(x+1)^3 + 24(x+1)^2 - 28(x+1) + 7 + 4}$

function after removing the (1, -4) shift!

$\Rightarrow x^4 - 4x^3 + 6x^2$



Example: In the following equation, $x^2y - 2x^2 + 2xy + y - 4x - 6 = 0$ eliminate the 2nd degree terms

$$x = x' + h$$

$$y = y' + k$$

$$(x' + h)^2(y' + k) - 2(x' + h)^2 + 2(x' + h)(y' + k) + (y' + k) - 4(x' + h) - 6 = 0$$

$$(x'^2 + 2x'h + h^2)(y' + k) - (2x'^2 + 4x'h + 2h^2) + 2x'y' + 2x'k + 2hy' + 2hk + y' + k - (4x' + 4h) - 6 = 0$$

$$x'^2y' + x'^2k + 2x'y'h + 2x'kh + y'h^2 + h^2k - (2x'^2 + 4x'h + 2h^2) + 2x'y' + 2x'k + 2hy' + 2hk + y' + k - (4x' + 4h) - 6 = 0$$

$$kx'^2 + 2hx'y' - 2x'^2 + 2x'y'$$

collect the 'like terms'

$$(k - 2)x'^2 \Rightarrow k = 2 \text{ (to eliminate } x'^2\text{)}$$

$$(2h + 2)x'y' \Rightarrow h = -1 \text{ (to eliminate } x'y'\text{)}$$

(-1, 2)

Plug in (h, k) to eliminate:

$$x^2y - 2x^2 + 2xy + y - 4x - 6 = 0$$

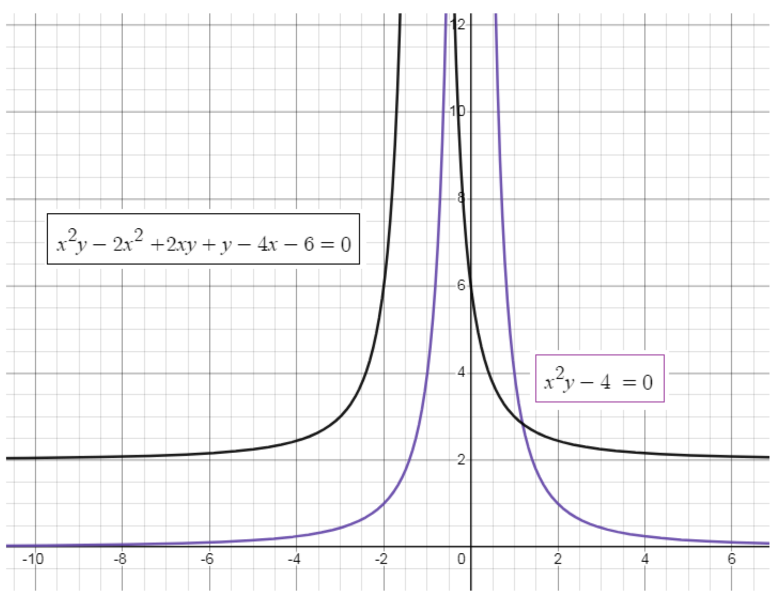
$$(x' - 1)^2(y' + 2) - 2(x' - 1)^2 + 2(x' - 1)(y' + 2) + (y' + 2) - 4(x' - 1) - 6 = 0$$

$$(x'^2 - 2x' + 1)(y' + 2) - 2(x'^2 - 2x' + 1) + 2(x'y' + 2x' - y' - 2) + (y' + 2) - 4(x' - 1) - 6 = 0$$

$$x'^2y' + \cancel{2x'^2} - \cancel{2x'y'} - 4x' + y' + 2 - \cancel{2x'^2} + 4x' - 2 + \cancel{2x'y'} + 4x' - 2y' - 4 + y' + 2 - 4x' + 4 - 6 = 0$$

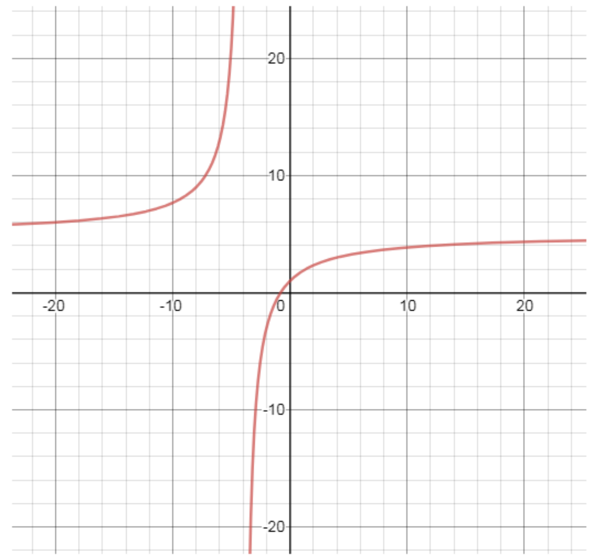
$$x'^2y' - \cancel{4x'} + \cancel{y'} + \cancel{4x'} + \cancel{4x'} - \cancel{2y'} + \cancel{y'} + 2 - \cancel{4x'} - 6 = 0$$

$$\boxed{x'^2y' - 4 = 0}$$



Example: Translate the equation, eliminating the linear terms

$$xy - 5x + 4y - 4 = 0$$



For the translation, let $x = x' + h$ and $y = y' + k$

Then, substitute....

$$(x' + h)(y' + k) - 5(x' + h) + 4(y' + k) - 4 = 0$$

$$x'y' + x'k + hy' + hk - 5x' - 5h + 4y' + 4k - 4 = 0$$

$$x'y' + (k - 5)x' + (h + 4)y' + hk - 5h + 4k - 4 = 0$$

xy term	x' linear term	y' linear term	constant

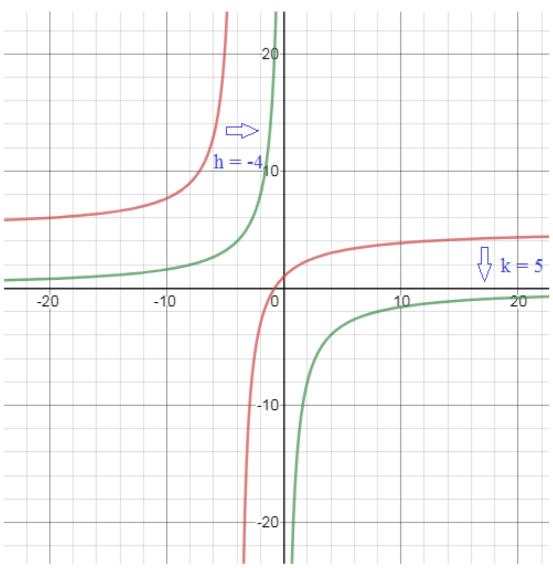
Since we're eliminating the linear terms,

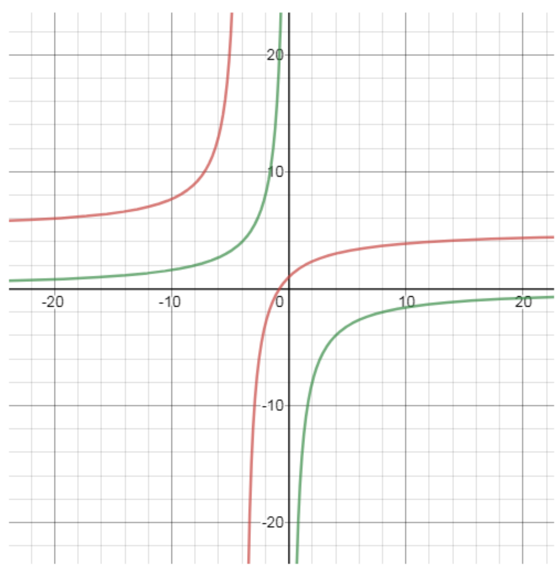
$(k - 5)x' = 0 \qquad k = 5$

$(h + 4)y' = 0 \qquad h = -4$

Substitute into the equation.... $x'y' + (5 - 5)x' + (-4 + 4)y' + (-4)(5) - 5(-4) + 4(5) - 4 = 0$

$$x'y' + 16 = 0$$

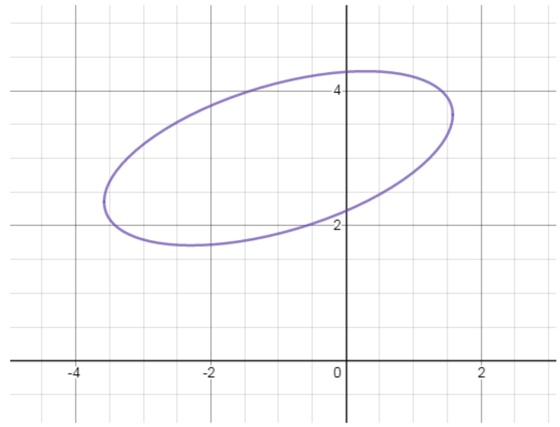




$xy - 5x + 4y - 4 = 0$

$x'y' + 16 = 0$

Example: Translate the following equation, eliminating the 1st degrees....



$$x^2 - 2xy + 4y^2 + 8x - 26y + 38 = 0$$

For the translation, let $x = x' + h$ and $y = y' + k$

Then, substitute....

$$(x' + h)^2 - 2(x' + h)(y' + k) + 4(y' + k)^2 + 8(x' + h) - 26(y' + k) + 38 = 0$$

$$x'^2 + 2x'h + h^2 - 2x'y' - 2x'k - 2hy' - 2hk + 4y'^2 + 8y'k + 4k^2 + 8x' + 8h - 26y' - 26k + 38 = 0$$

Rearrange into "like" terms...

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

$$x'^2 - 2x'y' + 4y'^2 + (2h - 2k + 8)x' + (8k - 2h - 26)y' + h^2 - 2hk + 4k^2 - 26k + 38 + 8h = 0$$

Since we're eliminating the 1st degrees,

$$2h - 2k + 8 = 0$$

$$8k - 2h - 26 = 0$$

solve by elimination/combination...

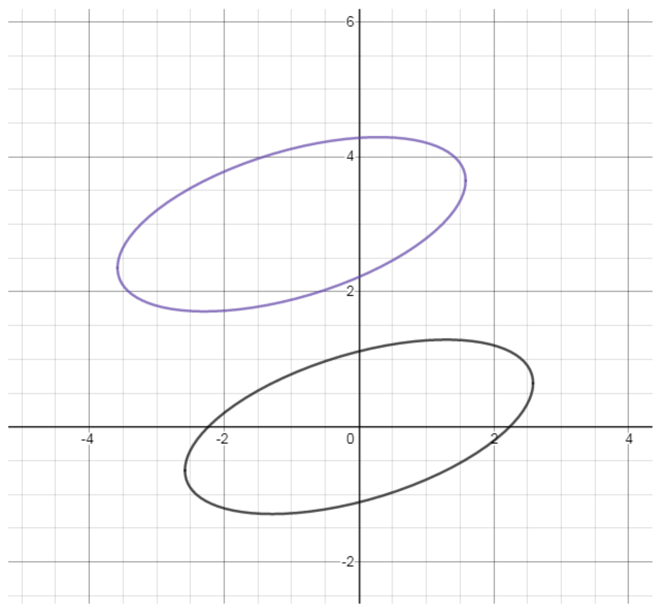
$$6k - 18 = 0$$

$$k = 3$$

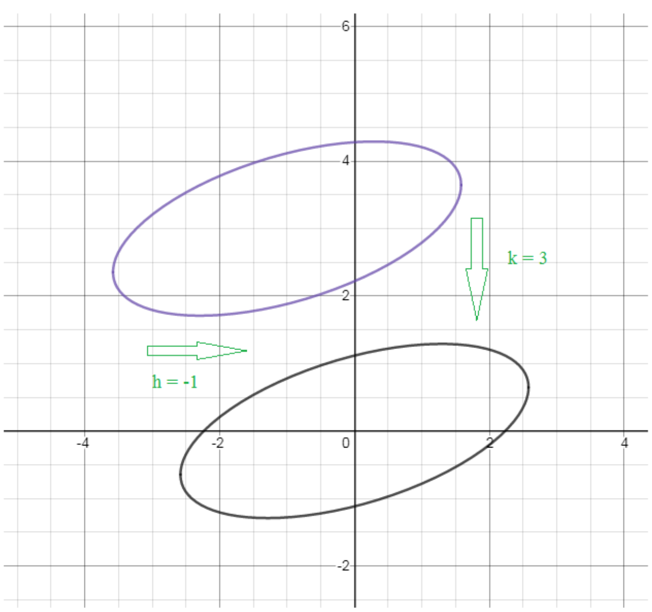
$$h = -1$$

Substitute back into the translated equation....

$$x'^2 - 2x'y' + 4y'^2 + 0x' + 0y' - 5 = 0$$



The rotated ellipse has been shifted down 3 units and to the right 1 unit.. Now, the 1st degree terms have been eliminated (i.e the ellipse is centered at the origin!)



Example: For the given polynomial $x^4 + 4x^3 - 20x^2 - 8y + 16 = 0$

translate and find the relative extrema and points of inflection....

the translation will be $< h, k >$

$(x + h)^4 + 4(x + h)^3 - 20(x + h)^2 - 8(y + k) + 16$

multiply out...

$$x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 + 4x^3 + 12x^2h + 12xh^2 + 4h^3 - 20x^2 - 40xh - 20h^2 - 8y - 8k + 16$$

rearrange...

$$x^4 + (4h + 4)x^3 + (6h^2 + 12h - 20)x^2 + (4h^3 + 12h^2 - 40h)x - 8y + (h^4 + 4h^3 - 20h^2 - 8k + 16)$$

4th degree	3rd	2nd	1st	constant

To find the inflection point(s), we set the constant and 2nd degree coefficient equal to zero...

$$(6h^2 + 12h - 20) = 0$$
$$(h^4 + 4h^3 - 20h^2 - 8k + 16) = 0$$

solve the system: (h, k) -------> $\boxed{(-3.08, -25.1) \quad \text{and} \quad (1.08, -.12)}$

To find the extrema, set the constant and first degree coefficient equal to zero....

$$(4h^3 + 12h^2 - 40h) = 0$$
$$(h^4 + 4h^3 - 20h^2 - 8k + 16) = 0$$

solve the system: (h, k) -------> $\boxed{(-5, -359/8) \quad \text{or} \quad (0, 2) \quad \text{or} \quad (2, -2)}$

Using calculus:

$8y = x^4 + 4x^3 - 20x^2 + 16$

$y = \frac{1}{8}x^4 + \frac{1}{2}x^3 - \frac{5}{2}x^2 + 2$

first derivative

$y' = \frac{1}{2}x^3 + \frac{3}{2}x^2 - 5x$

set equal to 0 to find critical values

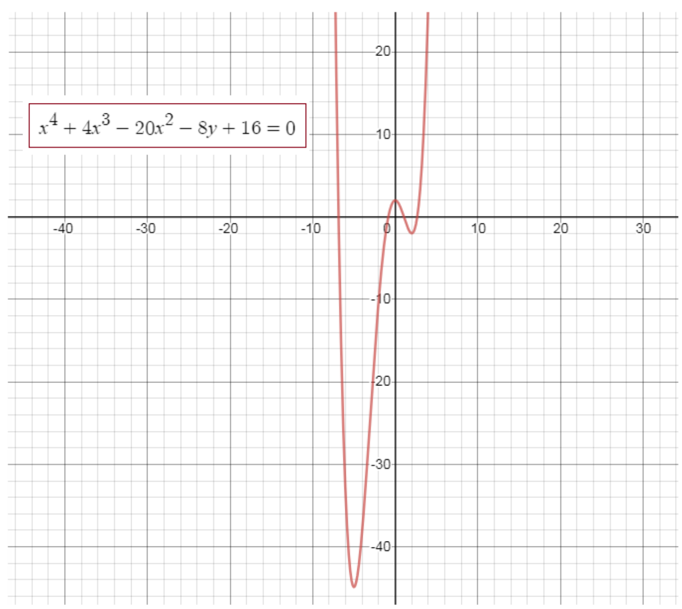
$0 = \frac{1}{2}x^3 + \frac{3}{2}x^2 - 5x \qquad x = -5, 0, 2$

second derivative

$y'' = \frac{3}{2}x^2 + 3x - 5$

set equal to 0 to find points of inflection

$0 = \frac{3}{2}x^2 + 3x - 5 \qquad x = -3.08, 1.08$



Example: Find the relative maximum of the function $x^2 - 4xy + 3x - 2y + 4 = 0$

Step 1: translate the function by $< h, k >$

$$(x+h)^2 - 4(x+h)(y+k) + 3(x+h) - 2(y+k) + 4 = 0$$

Step 2: expand and then regroup

$$x^2 + 2xh + h^2 - 4xy - 4hy - 4xk - 4hk + 3x + 3h - 2y - 2k + 4 = 0$$

$$x^2 + \underbrace{(2h - 4k + 3)x}_{\text{x-term}} - 4xy + (-4h - 2)y + \underbrace{h^2 - 4hk + 3h - 2k + 4}_{\text{constant term}} = 0$$

eliminate x-term $\quad (2h - 4k + 3) = 0$

eliminate constant term $\quad h^2 - 4hk + 3h - 2k + 4 = 0$

$\Rightarrow$

relative maximum $\quad$ relative minimum

$\boxed{(-2.158, -0.329) \text{ or } \cancel{(1.158, 1.328)}}$

$x^2 - 4xy + 3x - 2y + 4 = 0$

Using calculus: $\quad 2x - 4y - 4x\frac{dy}{dx} + 3 - 2\frac{dy}{dx} = 0$

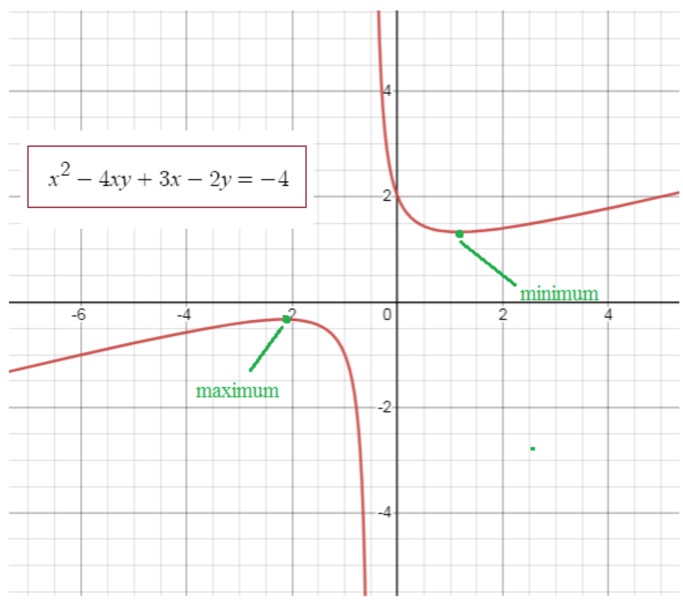
implicit differentiation

$$\frac{dy}{dx}(-4x - 2) = 4y - 2x - 3$$

$$\frac{dy}{dx} = \frac{4y - 2x - 3}{(-4x - 2)}$$

$$\left.\begin{array}{l} 4y - 2x - 3 = 0 \\ x^2 - 4xy + 3x - 2y + 4 = 0 \end{array}\right\}$$ any point in the system will satisfy the derivative = 0 AND will lie on the curve...

(-2.158, -0.329) ✓



Example: Find domain and range of $x^2 - 6xy + y^2 + 16 = 0$

First, translate $< h, k >$

$$(x + h)^2 - 6(x + h)(y + k) + (y + k)^2 + 16 = 0$$

Expand...

$$x^2 + 2xh + h^2 - 6xy - 6xk - 6hy - 6hk + y^2 + 2yk + k^2 + 16 = 0$$

Sort to 'like' variables..

$$x^2 + (2h - 6k)x - 6xy + y^2 + (-6h + 2k)y + (h^2 - 6hk + k^2) + 16 = 0$$

x-terms

constant terms

For range, we need to find the maximum and minimum of the function...

set the constant and x coefficients equal to zero.. Then, solve the system..

$$2h - 6k = 0$$

$$h^2 - 6hk + k^2 + 16 = 0$$

$(-3\sqrt{2}, -\sqrt{2})$ and $(3\sqrt{2}, \sqrt{2})$

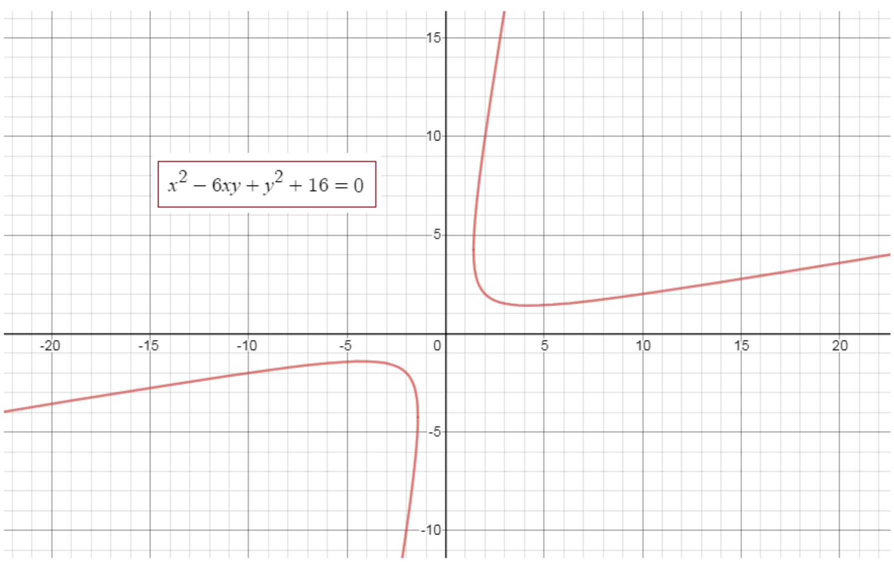
For the domain, set the constant and y coefficients equal to zero... Then, solve the system...

$$(-6h + 2k) = 0$$

$$h^2 - 6hk + k^2 + 16 = 0$$

$(-\sqrt{2}, -3\sqrt{2})$ and $(\sqrt{2}, 3\sqrt{2})$

(maximum and minimum with respect to y)



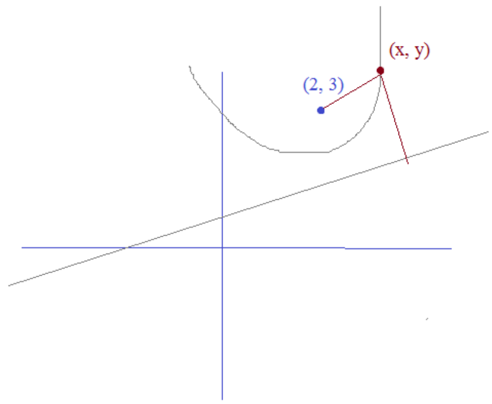
Domain: $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

Range: $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

Example: The focus is (2, 3) and directrix is $x - 4y + 3 = 0$

What is the equation of the parabola?

Parabola Properties



distance from (x, y) to directrix:

$$d = \left| \frac{x - 4y + 3}{\sqrt{1 + 16}} \right|$$

distance from (x, y) to the focus:

$$d = \sqrt{(2 - x)^2 + (3 - y)^2}$$

Set distances equal to each other...

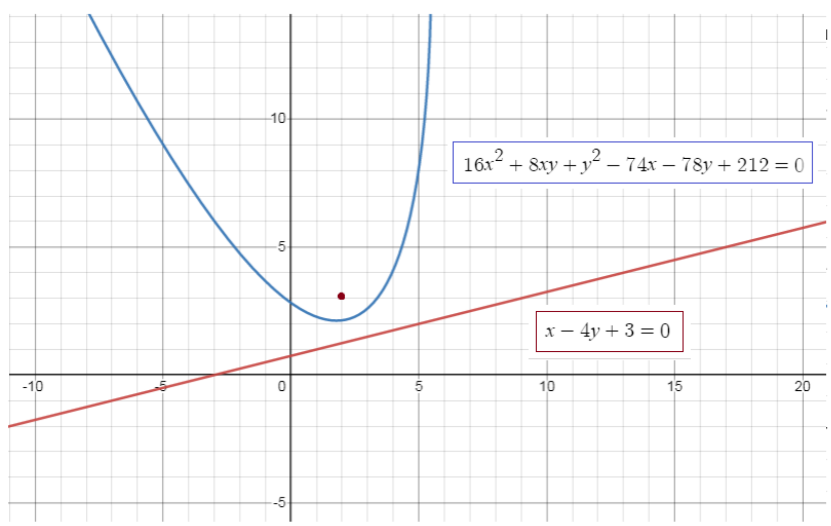
$$\frac{(x - 4y + 3)^2}{17} = (2 - x)^2 + (3 - y)^2$$

$$\begin{array}{rrrr}
 & (x - 4y + 3) & & \\
 & (x - 4y + 3) & & \\
\hline
x^2 - 4xy + 3x & & & \\
 - 4xy & + 16y^2 - 12y & & \\
 3x & - 12y + 9 & & \\
\hline
x^2 - 8xy + 6x + 16y^2 - 24y + 9 & & &
\end{array}$$

$$x^2 - 8xy + 6x + 16y^2 - 24y + 9 = 17(4 - 4x + x^2 + 9 - 6y + y^2)$$

$$x^2 - 8xy + 6x + 16y^2 - 24y + 9 = 68 - 68x + 17x^2 + 153 - 102y + 17y^2$$

$$\boxed{0 = 16x^2 + 8xy + y^2 - 74x - 78y + 212}$$



Example: What is the equation of the line tangent to $x^2 - y^2 = 144$ at $(13, 5)$?

The line will be $y - 5 = m(x - 13)$ $\Longrightarrow$ $y = mx - 13m + 5$ intersecting $x^2 - y^2 - 144 = 0$

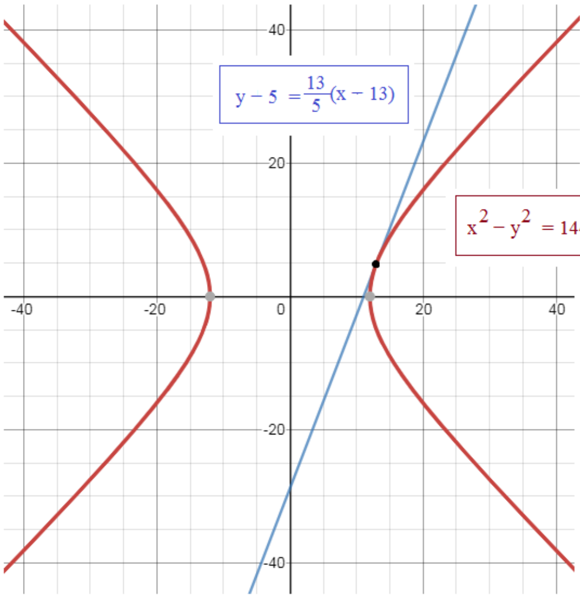
$$x^2 - (mx - 13m + 5)^2 - 144 = 0$$

$$x^2 - (m^2x^2 - 13m^2x + 5mx - 13m^2x + 169m^2 - 65m + 5mx - 65m + 25) - 144 = 0$$

$$x^2 - m^2x^2 + 26m^2x - 10mx - 169m^2 + 130m - 169 = 0$$

$$(1 - m^2)x^2 + (26m^2 - 10m)x - 169m^2 + 130m - 169 = 0$$

A B C Quadratic



$$B^2 - 4AC = 0$$

$$(26m^2 - 10m)^2 - 4(1 - m^2)(-169m^2 + 130m - 169) = 0$$

$$m = 13/5$$

derivative:

$$2x - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{-2y} = \frac{x}{y}$$

slope at $(13, 5)$ is $13/5$

$$y - 5 = \frac{13}{5}(x - 13) \checkmark$$

Example: What lines containing the point $(9, 9)$ are tangent to $x^2 - y^2 = 9$

Any line will be $y - 9 = m(x - 9)$ $\Longrightarrow$ $y = mx - 9m + 9$

intersecting $x^2 - y^2 - 9 = 0$

$$x^2 - (mx - 9m + 9)^2 - 9 = 0$$

$$x^2 - [m^2x^2 - 18m^2x + 18mx + 81m^2 - 162m + 81] - 9 = 0$$

$$(1 - m^2)x^2 + (18m^2 - 18m)x + (-81m^2 + 162m - 90) = 0$$

A B C Quadratic

$$\begin{array}{r} mx - 9m + 9 \\ mx - 9m + 9 \\ \hline m^2x^2 - 9m^2x + 9mx \qquad\qquad \\ - 9m^2x \qquad + 81m^2 - 81m \\ 9mx \qquad - 81m + 81 \\ \hline [\,m^2x^2 - 18m^2x + 18mx + 81m^2 - 162m + 81\,] \end{array}$$

$$B^2 - 4AC = 0$$

$$(18m^2 - 18m)^2 - 4(1 - m^2)(-81m^2 + 162m - 90) = 0$$

$$m = 1 \text{ or } 5/4$$

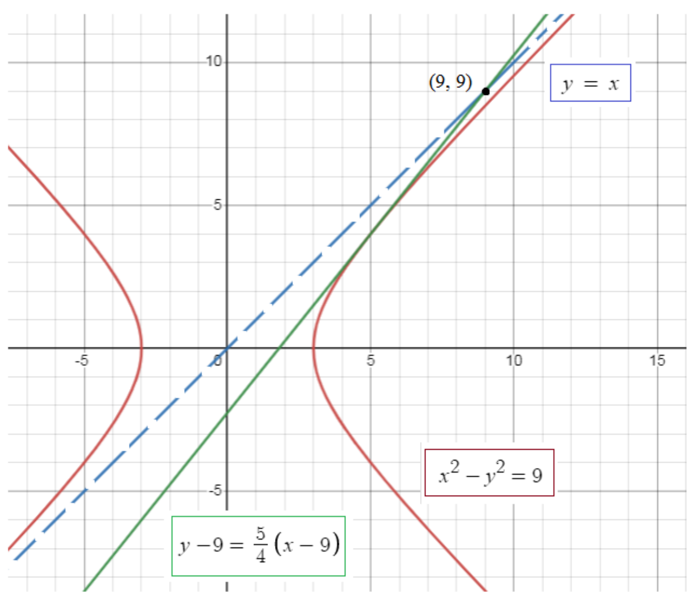
$$\boxed{y - 9 = \frac{5}{4}(x - 9)}$$

Tangent line

$$y - 9 = (x - 9)$$

$$y = x$$

Asymptote
(where limit approaches
the hyperbola)



1) For the given equation $4x^2 + y^2 + 24x - 2y + 21 = 0$

Translate the graph so the center is at the origin.

2) For the rotated conic, eliminate the 1st degree (centering the figure at the origin).

$$x^2 - 2xy + 4y^2 + 8x - 26y + 38 = 0$$

3) For the equation $y = x^3 + 6x^2 + 11x + 8$, eliminate the constant and the x^2 terms

4) Translate/reorient the following hyperbola where the center is on the origin.

$$x^2 + 4xy - y^2 - 2x - 14y - 3 = 0$$

1) For the given equation $4x^2 + y^2 + 24x - 2y + 21 = 0$

Translate the graph so the center is at the origin.

SOLUTIONS

Method 1: Put equation in standard form.
Then, remove the (h, k) values

$$4(x^2 + 6x \quad) + (y^2 - 2y \quad) = -21$$

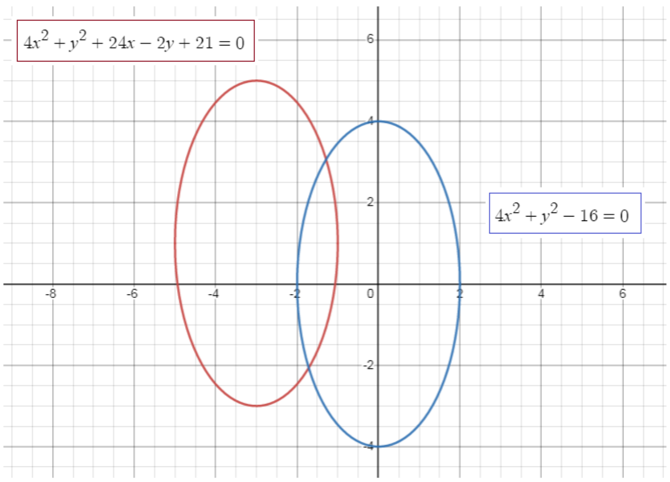
Complete the square

$$4(x^2 + 6x + 9) + (y^2 - 2y + 1) = -21 + 36 + 1$$

$$4(x+3)^2 + (y-1)^2 = 16$$

$$\frac{(x+3)^2}{4} + \frac{(y-1)^2}{16} = 1$$ Ellipse centered at (-3, 1)

$$\boxed{\frac{x^2}{4} + \frac{y^2}{16} = 1 \quad \text{Ellipse centered at (0, 0)}}$$



Method 2: Find h and k... then, remove linear terms

$x = x' + h$

$y = y' + k$ Substitute into equation...

$$4(x'+h)^2 + (y'+k)^2 + 24(x'+h) - 2(y'+k) + 21 = 0$$

Expand...

$$4x'^2 + 8x'h + 4h^2 + y'^2 + 2y'k + k^2 + 24x' + 24h - 2y' - 2k + 21 = 0$$

Pull out the linear terms x' and y' and set equal to 0...

$8x'h + 24x' = 0 \Rightarrow 8x'(h+3)$ $h = -3$

$2y'k - 2y' = 0$ $2y'(k-1)$ $k = 1$

substitute into original equation...

$x = x' - 3$ $y = y' + 1$

$$4x^2 + y^2 + 24x - 2y + 21 = 0 \Rightarrow 4(x'-3)^2 + (y'+1)^2 + 24(x'-3) - 2(y'+1) + 21 = 0$$

$$4x'^2 - \cancel{24x'} + 36 + y'^2 + \cancel{2y'} + 1 + \cancel{24x'} - 72 - \cancel{2y'} - 2 + 21 = 0$$

$$\boxed{4x'^2 + y'^2 - 16 = 0 \quad \text{Ellipse centered at (0, 0)}}$$

2) For the rotated conic, eliminate the 1st degree (centering the figure at the origin).

$$x^2 - 2xy + 4y^2 + 8x - 26y + 38 = 0$$

We need to find the amount of shift (h, k)

$x = x' + h$

$y = y' + k$

$$(x'+h)^2 - 2(x'+h)(y'+k) + 4(y'+k)^2 + 8(x'+h) - 26(y'+k) + 38 = 0$$

$$x'^2 + 2x'h + h - 2x'y' - 2x'k - 2y'h - 2hk + 4y'^2 + 8y'k + 4k^2 + 8x' + 8h - 26y' - 26k + 38 = 0$$

Extract the linear terms x' and y'... then, set equal to 0 (to eliminate the shifts)

$$x'^2 + \boxed{2x'h} + h - 2x'y' - \boxed{2x'k} - 2y'h - 2hk + 4y'^2 + 8y'k + 4k^2 + \boxed{8x'} + 8h - 26y' - 26k + 38 = 0$$

$2x'(h - k + 4) = 0$ linear system $h - k = -4$ $h = -1$ $\Rightarrow$ $x = x' - 1$

$2y'(-h + 4k - 13) = 0$ $-h + 4k = 13$ $k = 3$ $y = y' + 3$

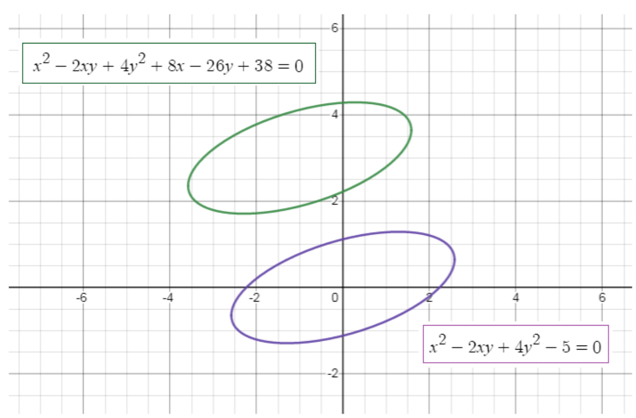
Substitute into original equation...

$$x^2 - 2xy + 4y^2 + 8x - 26y + 38 = 0$$

$$(x'-1)^2 - 2(x'-1)(y'+3) + 4(y'+3)^2 + 8(x'-1) - 26(y'+3) + 38 = 0$$

$$x'^2 - \cancel{2x'} + 1 - 2x'y' - \cancel{6x'} + \cancel{2y'} + 6 + 4y'^2 + \cancel{24y'} + 36 + \cancel{8x'} - 8 - \cancel{26y'} - 78 + 38 = 0$$

note: the x' and y' terms should cancel! $\boxed{x'^2 - 2x'y' + 4y'^2 - 5 = 0}$



3) For the equation $y = x^3 + 6x^2 + 11x + 8$, eliminate the constant and the x^2 terms

SOLUTIONS

$x = x' + h$

$y = y' + k$

$x^3 + 6x^2 + 11x - y + 8 = 0$

Substitute the translations...

$(x'+h)^3 + 6(x'+h)^2 + 11(x'+h) - (y'+k) + 8 = 0$

$x'^3 + 3x'^2h + 3x'h^2 + h^3 + 6x'^2 + 12x'h + 6h^2 + 11x' + 11h - y' - k + 8 = 0$

Set x^2 and constants equal to 0

$3x'^2h + 6x'^2 = 0 \Longrightarrow 3x'^2(h+2) = 0 \qquad h = -2$

$h^3 + 6h^2 + 11h - k + 8 = 0 \Longrightarrow$ since $h = -2$, $-8 + 24 - 22 - k + 8 = 0 \qquad k = 2$

$x = x' - 2$

$y = y' + 2$ Substitute into original equation...

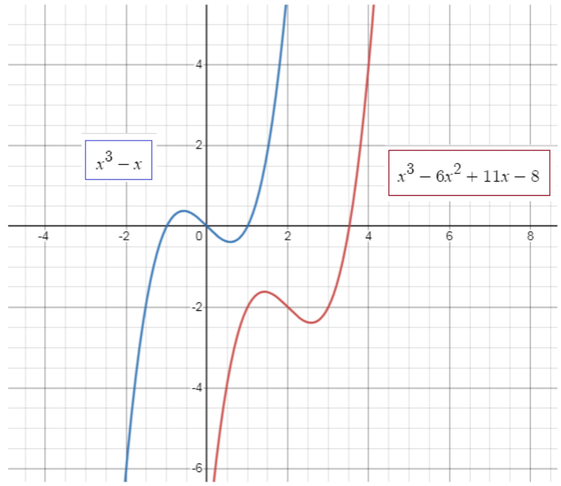
$y = x^3 + 6x^2 + 11x + 8$

$y' + 2 = (x'-2)^3 + 6(x'-2)^2 + 11(x'-2) + 8$

$x'^3 - \cancel{6x'^2} + 12x' - \cancel{8} + \cancel{6x'^2} - 24x' + \cancel{24} + 11x' - \cancel{22} + \cancel{8} - y' - \cancel{2} = 0$

note: the x'^2 and constants cancel!

$\boxed{x'^3 - x' - y' = 0 \quad \text{or} \quad y' = x'^3 - x'}$



4) Translate/reorient the following hyperbola where the center is on the origin.

$x^2 + 4xy - y^2 - 2x - 14y - 3 = 0$

translations: $x = x' + h$, $y = y' + k$

Substitute into equation:

$(x'+h)^2 + 4(x'+h)(y'+k) - (y'+k)^2 - 2(x'+h) - 14(y'+k) - 3 = 0$

Expand:

$x'^2 + 2x'h + h^2 + 4x'y' + 4x'k + 4y'h + 4hk - y'^2 - 2y'k - k^2 - 2x' - 2h - 14y' - 14k - 3 = 0$

Since we want to eliminate the translations, we set the linear x and y coefficients equal to zero...

$x'^2 \boxed{+2x'h} + h^2 + 4x'y' + \boxed{4x'k} + \boxed{4y'h} + 4hk - y'^2 \boxed{-2y'k} - k^2 \boxed{-2x'} - 2h \boxed{-14y'} - 14k - 3 = 0$

$2x'h + 4x'k - 2x' = 0 \Longrightarrow 2x'(h + 2k - 1) = 0 \Longrightarrow h + 2k = 1$

$4y'h - 2y'k - 14y' = 0 \Longrightarrow 2y'(2h - k - 7) = 0 \Longrightarrow 2h - k = 7$

$\boxed{h = 3 \quad k = -1}$

$x = x' + 3$

$y = y' + (-1)$

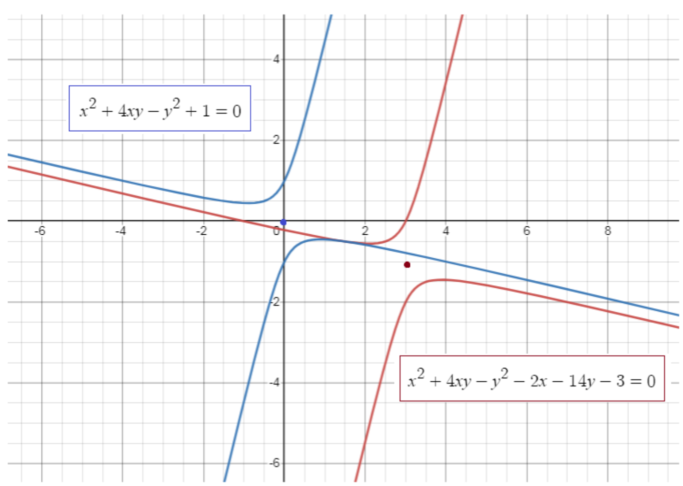
Now, we substitute the translations into the original equation!

$(x'+3)^2 + 4(x'+3)(y'-1) - (y'-1)^2 - 2(x'+3) - 14(y'-1) - 3 = 0$

$x'^2 \boxed{+6x'} + 9 + 4x'y' \boxed{-4x'} \boxed{+12y'} - 12 - y'^2 \boxed{+2y'} - 1 \boxed{-2x'} - 6 \boxed{-14y'} + 14 - 3 = 0$

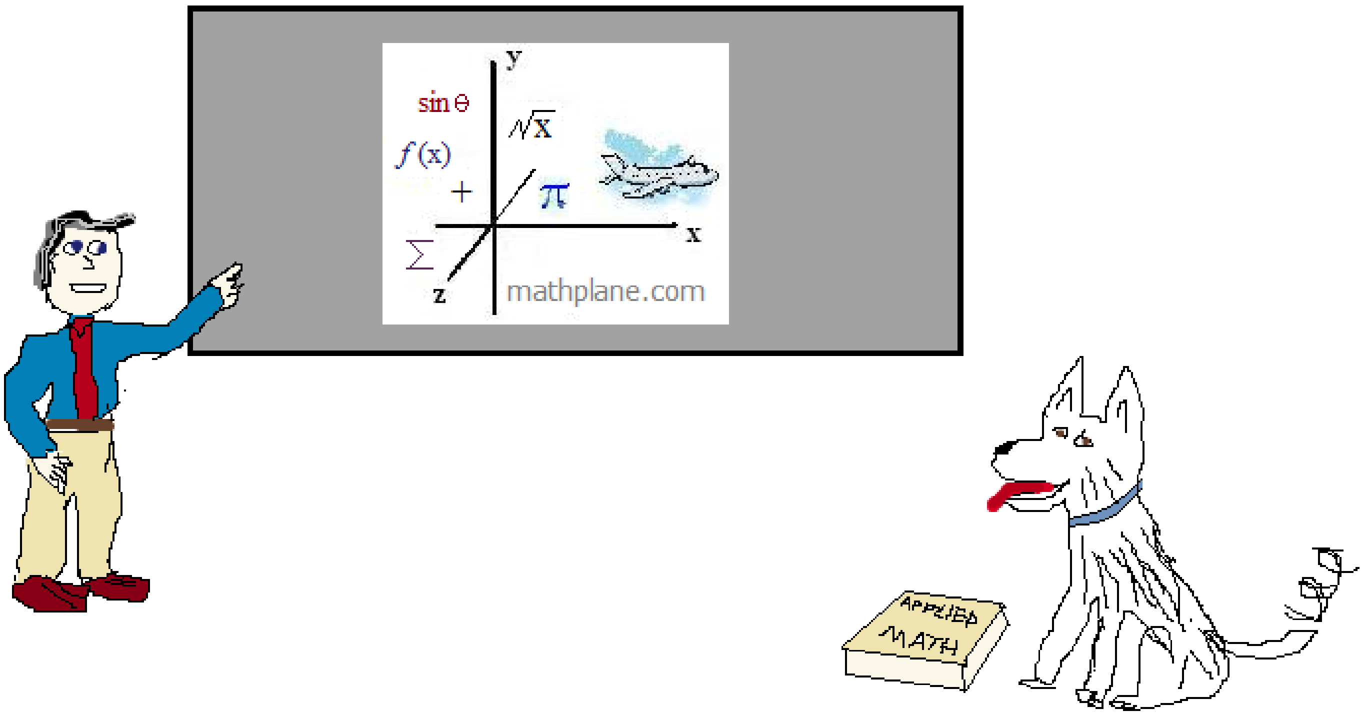
Note: the x' and y' terms cancel! (this checks our calculations...)

$\boxed{x'^2 + 4x'y' - y'^2 + 1 = 0}$



Thanks for visiting. Hope it helped!

If you have questions, suggestions, or requests, let us know.



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