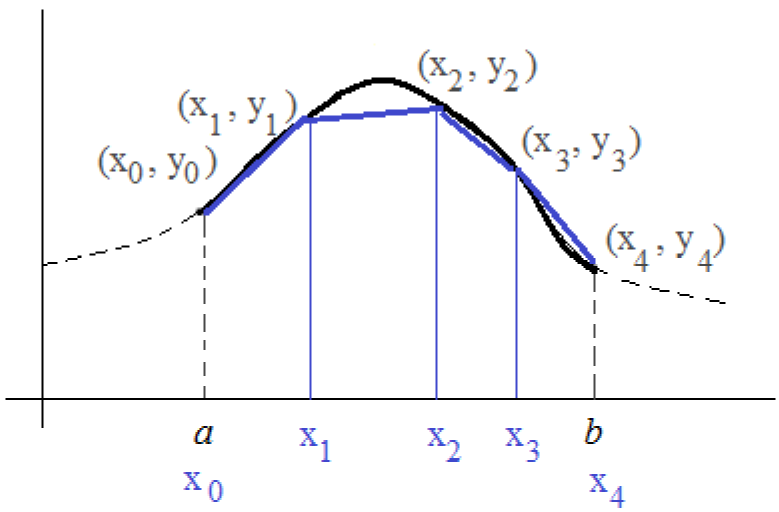
Calculus: Arc Length

(Notes, Formulas, Examples, and practice w/solutions)



Topics include derivatives, integrals, parametric equations, conics, limits, trig, and more..

Calculus: Integrals and Arc Length

A terrific application of integrals is computing the *arc length* of a function.

> If $y = f(x)$ has a continuous derivative f' on the interval [a, b], then the arc length of f between a and b is
>
> $$\int_a^b \sqrt{1 + [f'(x)]^2}\; dx$$

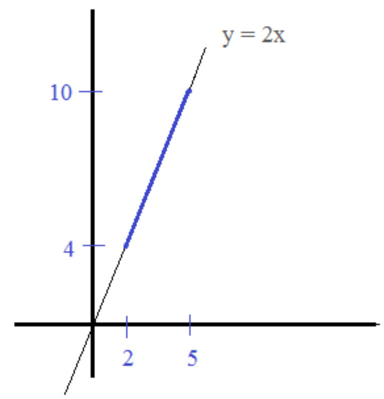
Here is a simple example that we can verify:

Example: Find the arc length of the line y = 2x over the interval [2, 5]

This arc is a line segment from (2, 4) to (5, 10).

$f'(x) = 2$

(and, the derivative is continuous)



$$\int_2^5 \sqrt{1 + [2]^2}\; dx = \sqrt{5}\, x \Big|_2^5 = 5\sqrt{5} - 2\sqrt{5} = \boxed{3\sqrt{5}}$$

Distance Formula:
(line segment between 2 points)

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \longrightarrow \sqrt{(2-5)^2 + (4-10)^2} = \sqrt{45} = 3\sqrt{5}\ \checkmark$$

> Similarly, if $x = g(y)$ has a continous derivative g' on the interval [c, d], then the arc length of g between c and d is
>
> $$\int_c^d \sqrt{1 + [g'(x)]^2}\; dy$$

Rewrite y = 2x in terms of y $\longrightarrow$ $x = \frac{y}{2}$

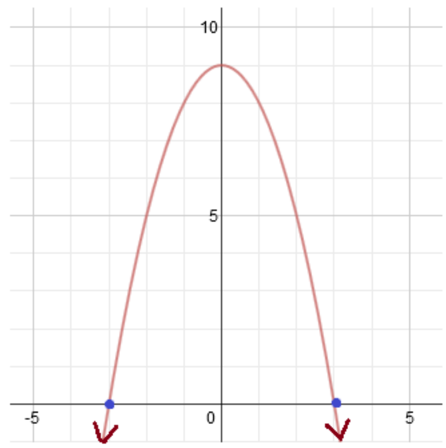
$$g'(y)\ \left(\text{or } \frac{dx}{dy}\right) = \frac{1}{2}$$

$$\int_4^{10} \sqrt{1 + \left(\frac{1}{2}\right)^2}\; dy = \sqrt{\frac{5}{4}}\, y \Big|_4^{10} = \frac{10\sqrt{5}}{2} - \frac{4\sqrt{5}}{2} = \boxed{3\sqrt{5}}$$

Example: Find the arc length of the curve $y = -x^2 + 9$ *above the x-axis.*

Step 1: Sketch the graph

The curve is a parabola that faces down.



Step 2: Determine arc length boundaries

$$-x^2 + 9 = 0$$

$$x^2 = 9$$

$$x = -3,\ 3$$

The curve is above the x-axis between -3 and 3

Step 3: Apply arc length formula

$$\frac{dy}{dx} = -2x$$

$$\text{Arc Length} = \int_{-3}^{3} \sqrt{1 + (-2x)^2}\ dx$$

$$\int_{-3}^{3} \sqrt{1 + 4x^2}\ dx$$

$$\text{Arc Length} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$\left. \frac{\ln(\sqrt{4x^2+1} + 2x)}{4} + \frac{x\sqrt{4x^2+1}}{2} \right|_{-3}^{3}$$

$$\frac{\ln(\sqrt{37}+6)}{4} + \frac{3\sqrt{37}}{2} - \left(\frac{\ln(\sqrt{37}-6)}{4} + \frac{-3\sqrt{37}}{2} \right)$$

$$.623 \quad + 9.12 \quad + .623 \quad + 9.12$$

$$= 19.49$$

$$\text{Arc Length} = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\ dy$$

$$y = -x^2 + 9$$

$$x^2 = 9 - y$$

$$x = \pm\sqrt{9 - y}$$

$$\frac{dx}{dy} = \frac{1}{2}(9 - y)^{\frac{-1}{2}}(-1)$$

and, the interval (along the *y-axis*) is [0, 9]

$$\text{Arc Length} = \int_0^9 \sqrt{1 + \left(\frac{-1}{2\sqrt{(9-y)}}\right)^2}\ dy$$

Arc length in Quadrant I

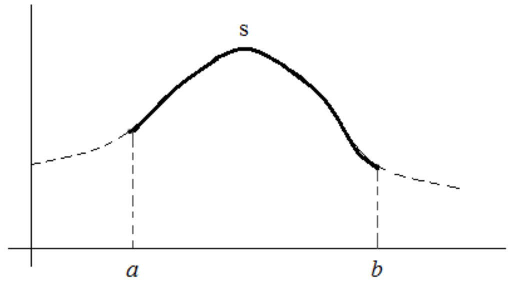
$$\int_0^9 \sqrt{1 + \frac{1}{36 - 4y}}\ dy = 9.747$$

And, the arc length in Quadrant II will be identical.

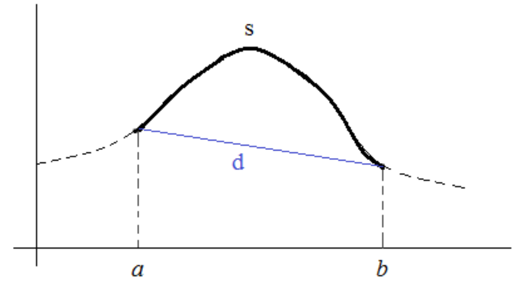
So, total arc length is $9.747 + 9.747 = 9.49$ ✓

Understanding the Arc Length Formula:

Calculus Arc Length



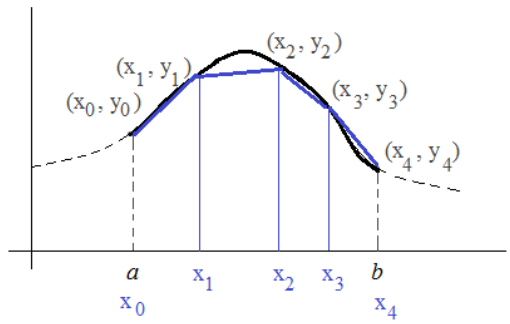
s is the arc length from *a* to *b*



The distance between endpoints:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Suppose we add multiple segments:



We get a more accurate measurement...

$$d = \sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2} + \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} + \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2} + \sqrt{(x_3 - x_4)^2 + (y_3 - y_4)^2}$$

Then, suppose we could add an infinite number of segments between *a* and *b*:

$$\lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{(\triangle x_i)^2 + (\triangle y_i)^2}$$

where $\triangle x_i = x_i - x_{i-1}$

e.g. $\triangle x_1$ would equal $x_1 - x_0$

$$\lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{(\triangle x_i)^2 \left(1 + \frac{(\triangle y_i)^2}{(\triangle x_i)^2}\right)}$$

$$\lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{\left(1 + \frac{(\triangle y_i)^2}{(\triangle x_i)^2}\right)} (\triangle x_i)$$

$\frac{\triangle y_i}{\triangle x_i} \longrightarrow$ rate of change $f'(x)$

$(\triangle x_i) \longrightarrow dx$

$$S = \lim_{n \to \infty} \sum_{i=1}^{n} \sqrt{1 + [f'(x)]^2}\, dx$$

$$\longrightarrow \quad S = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Calculus Arc Length

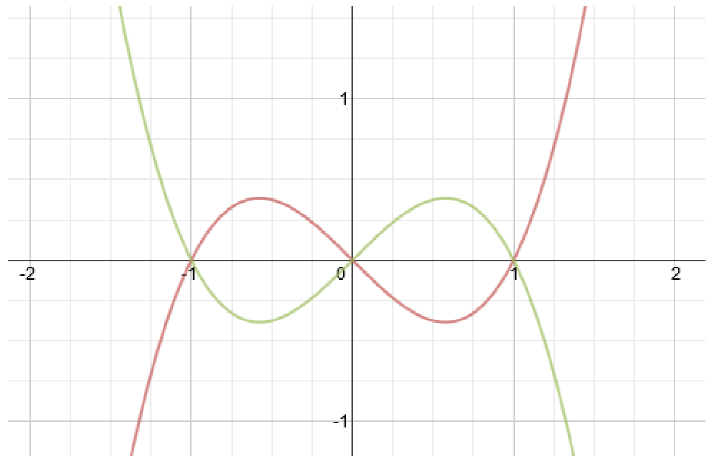
$$\text{Arc Length} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Example. Find the perimeter of the "bow tie" created by the intersection of

$x^3 - x$ and $x - x^3$

Step 1: Sketch the graph

The interlinking cubics form a "bow tie" (or, figure 8)



Step 2: Determine the boundaries for integration

Since the curves are reflections, we can integrate one of them, and then double the arc length to get the entire perimeter...

The curves intersect at -1, 0, and 1...

Step 3: Apply formula and solve

The boundaries for integration will be -1 to 1

red curve: $y = x^3 - x$

$$\frac{dy}{dx} = 3x^2 - 1$$

Arc Length (red curve) $= \displaystyle\int_{-1}^{1} \sqrt{1 + (3x^2 - 1)^2}\,dx$

$$\int_{-1}^{1} \sqrt{1 + 9x^4 - 6x^2 + 1}\,dx$$

$$= 2.62$$

And, the green curve has an arc length of 2.62 between -1 and 1...

Total perimeter of bow tie is 5.24

Arc Length for Parametric Equations

Example: Find the arc length of the parametric curve

$$x = 1 + 3t^2 \qquad y = 4 + 2t^3 \qquad \text{between } 0 \le t \le 1$$

If $x = f(t)$ and $y = g(t)$ for $a \le t \le b$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Note: This assumes that
1) as t goes from a to b, it only traces the curve once.
and
2) as t increases, x increases (to prevent reversing direction)

Find the derivatives with respect to t.

$$\frac{dx}{dt} = 6t$$

$$\frac{dy}{dt} = 6t^2$$

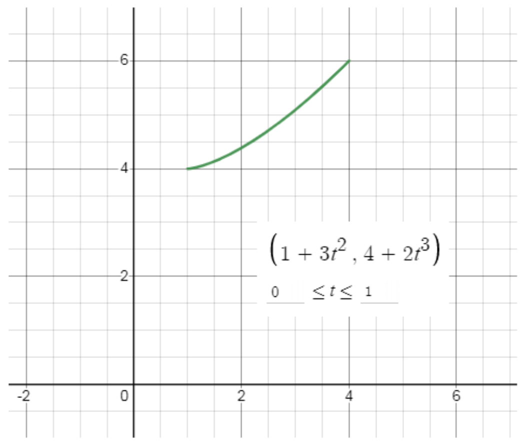
Apply the arc length formula.

$$L = \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\, dt$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\int_0^1 \sqrt{36t^2 + 36t^4}\, dt$$

$$\int_0^1 \sqrt{36t^2\,(1 + t^2)}\, dt$$



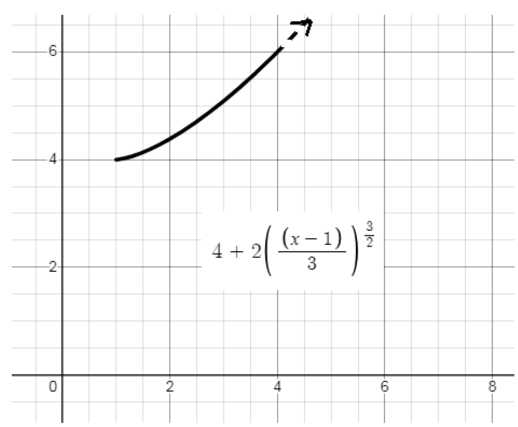
$$\int_0^1 6t\sqrt{(1 + t^2)}\, dt = \left.\frac{3\,[1 + t^2]^{3/2}}{3/2}\right|_0^1 \Rightarrow \left. 2\,[1 + t^2]^{3/2}\right|_0^1 = 4\sqrt{2} - 2$$

Now, let's remove the parameter t and find the arc length....

$$x = 1 + 3t^2 \qquad y = 4 + 2t^3 \qquad \text{between } 0 \le t \le 1$$

$$t^2 = \frac{x - 1}{3} \quad \text{rearrange the x equation}$$

$$y = 4 + 2\left(\frac{x - 1}{3}\right)^{3/2} \quad \text{substitute into y equation}$$



$$\frac{dy}{dx} = 0 + 3\left(\frac{x - 1}{3}\right)^{1/2} \cdot \frac{1}{3}$$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

$$\frac{dy}{dx} = \left(\frac{x - 1}{3}\right)^{1/2}$$

and, the boundaries $0 < t < 1$ become $1 < x < 4$

$$\int_1^4 \sqrt{1 + \left[\left(\frac{x - 1}{3}\right)^{1/2}\right]^2} \Rightarrow \int_1^4 \sqrt{1 + \left(\frac{x - 1}{3}\right)}\, dx \Rightarrow \int_1^4 \sqrt{\frac{2}{3} + \frac{x}{3}}\, dx = \left.\frac{3\left(\frac{2}{3} + \frac{x}{3}\right)^{3/2}}{3/2}\right|_1^4 = 4\sqrt{2} - 2$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example: Find the arc length on the curve

$x = t + e^{-t}$

$y = t - e^{-t}$ between $0 \le t \le 2$

Step 1: find the separate derivatives...

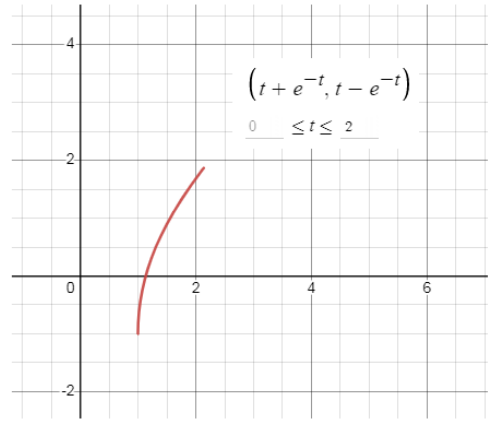
$\frac{dx}{dt} = 1 - e^{-t}$ $\quad\quad$ $\frac{dy}{dt} = 1 + e^{-t}$

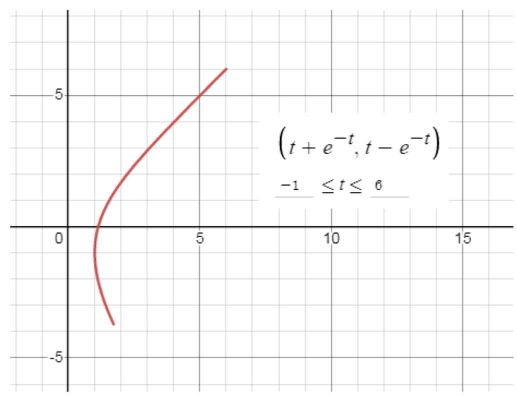
Step 2: apply the arc length formula

$$L = \int_0^2 \sqrt{(1 - e^{-t})^2 + (1 + e^{-t})^2}\, dt$$

$$\int_0^2 \sqrt{1 - 2e^{-t} + e^{-2t} + 1 + 2e^{-t} + e^{-2t}}\, dt$$

$$\int_0^2 \sqrt{2 + 2e^{-2t}}\, dt = 3.1416 \quad \text{(very close to pi)}$$





Example: Find the arc length on the curve

$x = 4\sin(t)$

$y = 4\cos(t)$ between $0 \le t \le \frac{2\pi}{3}$

$\frac{dx}{dt} = 4\cos(t)$

$\frac{dy}{dt} = -4\sin(t)$

trigonometry identity

$\sin^2 + \cos^2 = 1$

$$L = \int_0^{\frac{2\pi}{3}} \sqrt{(4\cos(t))^2 + (-4\sin(t))^2}\, dt$$

$$\int_0^{\frac{2\pi}{3}} \sqrt{16\cos^2 t + 16\sin^2 t}\, dt$$

$$\int_0^{\frac{2\pi}{3}} 4\, dt = 4t \Big|_0^{\frac{2\pi}{3}} \Rightarrow \frac{8\pi}{3}$$

$\sin(t) = \frac{x}{4}$

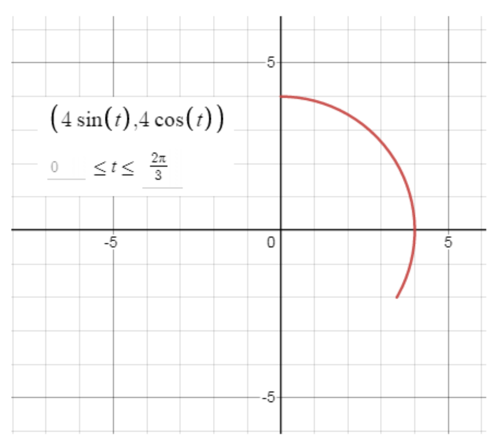
$\cos(t) = \frac{y}{4}$

$\sin^2 + \cos^2 = 1$

$\frac{x^2}{16} + \frac{y^2}{16} = 1$

$x^2 + y^2 = 16$

The arc is 1/3 of a circle with radius 4...



Practice questions→

Find the arc lengths of the curves:

Integral & Arc Length Practice

1) $y = \frac{2}{3}x^{\frac{3}{2}} + 4$ between $x = 0$ and $x = 1$

2) $y = \frac{x^3}{6} + \frac{1}{2x}$ between $x = 2$ and $x = 4$

3) $f(x) = \frac{\sqrt{x}\,(x-3)}{3}$ from $x = 0$ to $x = 3$

Find the arc lengths of the curves:

SOLUTIONS

Integral & Arc Length Practice

1) $y = \frac{2}{3}x^{\frac{3}{2}} + 4$ between $x = 0$ and $x = 1$

$\frac{dy}{dx} = x^{\frac{1}{2}} + 0$

$$\int_0^1 \sqrt{1 + \left(x^{\frac{1}{2}}\right)^2}\, dx = \int_0^1 \sqrt{1 + x}\, dx$$

$$\frac{2}{3}(1+x)^{\frac{3}{2}} \Big|_0^1 = \frac{4\sqrt{2}}{3} - \frac{2}{3} \quad \boxed{\approx 1.219}$$

2) $y = \frac{x^3}{6} + \frac{1}{2x}$ between $x = 2$ and $x = 4$

$$\frac{dy}{dx} = \frac{3x^2}{6} - \frac{1}{2x^2} = \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)$$

$$\text{Arc Length} = \int_2^4 \sqrt{1 + \left(\frac{1}{2}\left(x^2 - \frac{1}{x^2}\right)\right)^2}\, dx = \int_2^4 \sqrt{1 + \frac{1}{4}\left(x^4 - 2 + \frac{1}{x^4}\right)}\, dx$$

$1 \to \frac{1}{4}(4)$

$$\int_2^4 \sqrt{\frac{1}{4}\left(x^4 + 2 + \frac{1}{x^4}\right)}\, dx = \int_2^4 \sqrt{\frac{1}{4}\left(x^2 + \frac{1}{x^2}\right)^2}\, dx$$

$$\int_2^4 \frac{1}{2}\left(x^2 + \frac{1}{x^2}\right) dx = \frac{1}{2}\left(\frac{x^3}{3} - \frac{1}{x}\right)\Big|_2^4 = \frac{1}{2}\left(\frac{64}{3} - \frac{1}{4} - \frac{8}{3} + \frac{1}{2}\right) \quad \boxed{\approx 9.46}$$

3) $f(x) = \frac{\sqrt{x}\,(x-3)}{3}$ from $x = 0$ to $x = 3$

Step 1: Find $f'(x)$

$$f(x) = \frac{1}{3}\left(x^{\frac{3}{2}} - 3x^{\frac{1}{2}}\right) \qquad f'(x) = \frac{1}{3}\left(\frac{3}{2}x^{\frac{1}{2}} - \frac{3}{2}x^{-\frac{1}{2}}\right) = \frac{1}{2}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)$$

Step 2: Identify definite integral values and substitute

endpoints of integral are $x = 0$ and 3

$$\text{Arc Length} = \int_0^3 \sqrt{1 + \left(\frac{1}{2}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)\right)^2}\, dx$$

Step 3: Solve

$$\int_0^3 \sqrt{1 + \frac{1}{4}\left(x - 2 + \frac{1}{x}\right)}\, dx = \int_0^3 \sqrt{\frac{1}{4}\left(x + 2 + \frac{1}{x}\right)}\, dx$$

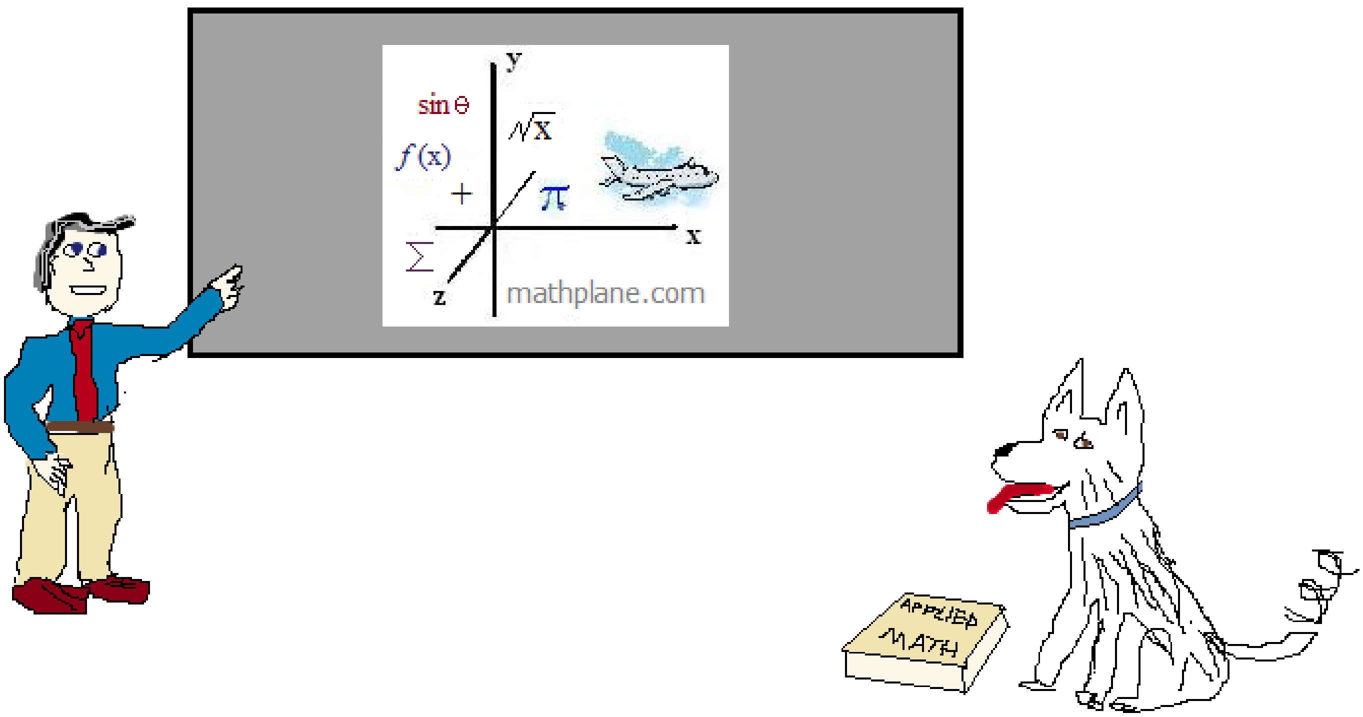
$$\frac{1}{2}\int_0^3 \sqrt{\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2}\, dx = \frac{1}{2}\int_0^3 \sqrt{x} + \frac{1}{\sqrt{x}}\, dx$$

$$\frac{1}{2}\left(\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}}\right)\Big|_0^3 = \frac{1}{2}\left(2\sqrt{3} + 2\sqrt{3}\right) = \boxed{2\sqrt{3}}$$

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers



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