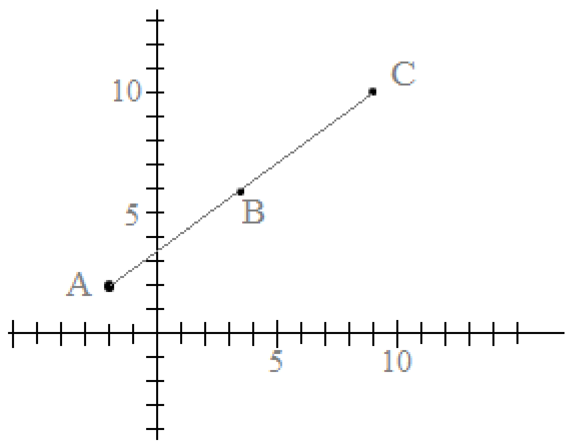
Geometry

Midpoint and Distance

Notes, Applications, and Practice Quiz (& Solutions)



Topics include number lines, cartesian plane, formulas, triangles, circles, and more.

Midpoint and Distance: Notes, Examples, and Formulas

Midpoint

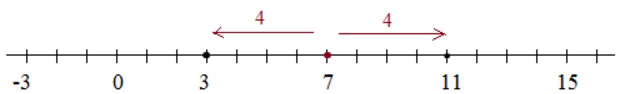
What is it? The "half-way point between two locations". It is equidistant to each point.

Number line: The midpoint between 3 and 11 is 7. 7 is four units from both 3 and 11.

The midpoint is similar to the "average"

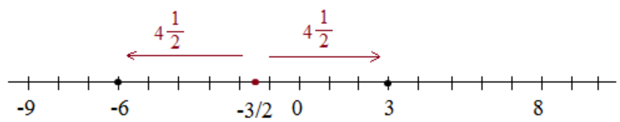
$$\frac{P_1 + P_2}{2} = \text{Midpoint}$$

$$\frac{3+11}{2} = 7$$



The midpoint between -6 and 3 is $\frac{-3}{2}$

$$\frac{-6+3}{2} = \frac{-3}{2}$$

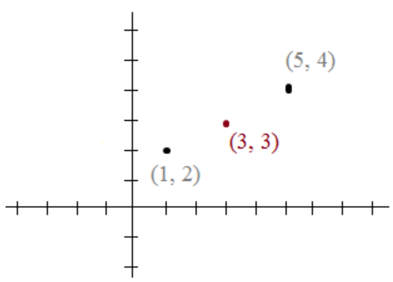


The midpoint extends to the Cartesian Plane:

Simply find the midpoint of the X values. And, the midpoint of the Y values.

$$\left(\frac{X_1 + X_2}{2}, \frac{Y_1 + Y_2}{2}\right)$$

Midpoint Formula



The midpoint of the X Values:

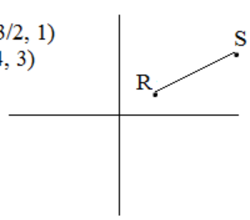
$$\frac{1+5}{2} = 3$$

The midpoint of the Y Values:

$$\frac{2+4}{2} = 3$$

Where does the perpendicular bisector pass through $\overline{RS}$?

R = (3/2, 1)
S = (4, 3)



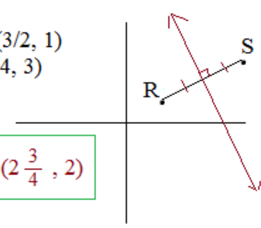
Find the midpoint of $\overline{RS}$:

X coordinate: $\frac{3/2 + 4}{2} = \frac{11/2}{2} = \frac{11}{4}$

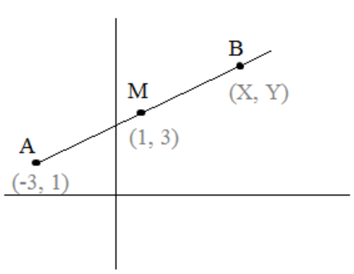
Y coordinate: $\frac{1+3}{2} = 2$

R = (3/2, 1)
S = (4, 3)

$M = (2\frac{3}{4}, 2)$



Given AB with midpoint M:
A = (-3, 1) M = (1, 3) What is B?



"Formula" Method

$$\frac{X_A + X_B}{2} = X_M \qquad \frac{Y_A + Y_B}{2} = Y_M$$

$$\frac{-3 + X_B}{2} = 1 \qquad \frac{1 + Y_B}{2} = 3$$

$$X_B = 5 \qquad Y_B = 5$$

(5, 5)

"Travel" Method

Start at the endpoint. Determine how far you "travel" to the midpoint. Then, add the same amount.

A (-3, 1) ⟶ M (1, 3)

X value increased 4 units..
Y value increased 2 units..

M (1, 3) ⟶ B (1 + 4, 3 + 2)

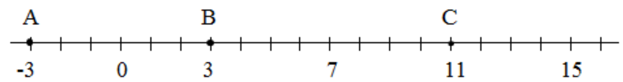
(5, 5)

Midpoint and Distance: Notes, Examples, and Formulas

Distance

What is it? The space between 2 points.
The length of the line segment connecting two points.

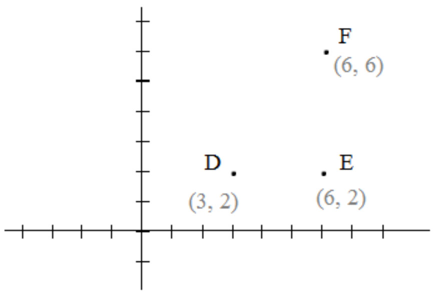
Number Line:



Length of $\overline{AB}$ = 6 units
$\overline{AC}$ = 14 units

Distance between A and B is 6
between A and C is 14

Cartesian Plane:

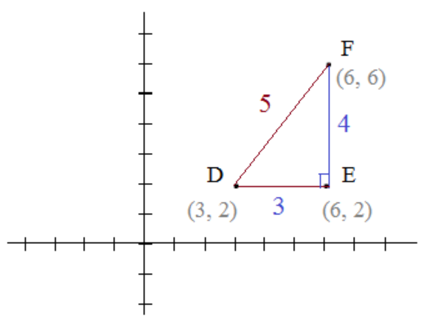


The distance between D and E is 3 units...
(3, 2), (4, 2), (5, 2), and (6, 2)

And, the distance between E and F is 4 units...
(6, 2), (6, 3), (6, 4), (6, 5), (6, 6)

So, what is the distance between D and F?
(And, it is not 7!!)

Pythagorean Theorem
$a^2 + b^2 = c^2$



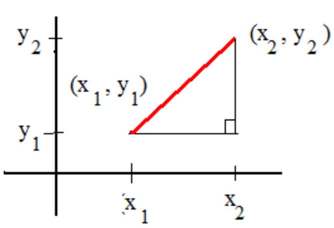
Notice, in this case, that the points can be vertices of a right triangle..

So, $\overline{DE}^2 + \overline{EF}^2 = \overline{DF}^2$
$9 + 16 = 25$

Therefore, the length of $\overline{DF}$
(i.e. distance between D and F)
= 5

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Distance Formula



Find the distance between (-2, 5) and (4, 7).

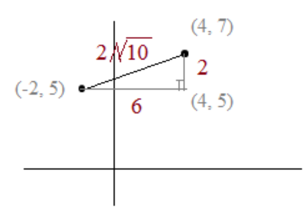
Using Distance Formula:

$$d = \sqrt{(-2-4)^2 + (5-7)^2}$$

$$= \sqrt{(-2-4)^2 + (5-7)^2}$$

$$= \sqrt{36 + 4} = 2\sqrt{10}$$

Using Pythagorean Theorem:



A vertical line drawn from (4, 7) intersects a horizontal line from (-2, 5) at (4, 5).. These form a right triangle!

Then, using the pythagorean theorem, the hypotenuse is $2\sqrt{10}$

Distance Formula and Pythagorean Theorem

Example: The distance between (3, 4) and (x, 7) is 5 units. Find x.

Using the distance formula:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$5 = \sqrt{(3 - x)^2 + (4 - 7)^2}$$

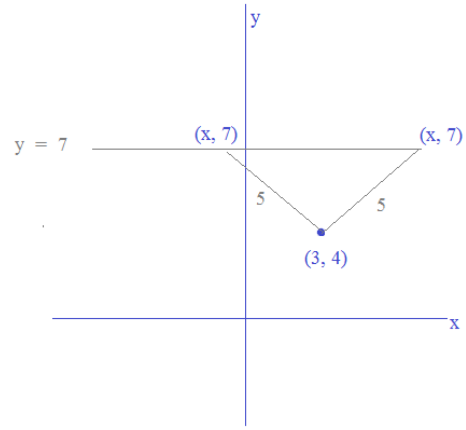
$$25 = (3 - x)^2 + 9$$

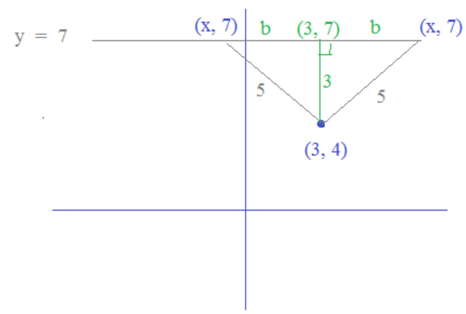
$$16 = (3 - x)^2$$

$$\pm 4 = 3 - x$$

$$\boxed{x = -1 \text{ or } 7}$$

Solving Graphically (Pythagorean Theorem)





$$3^2 + b^2 = 5^2$$

$$b^2 = 16$$

$$b = 4 \text{ or } -4$$

Example: The length of segment $\overline{AB}$ is 20. If the coordinate of A is (5, 1), and the coordinate of B is (-6, y), what is b?

Using the distance formula:

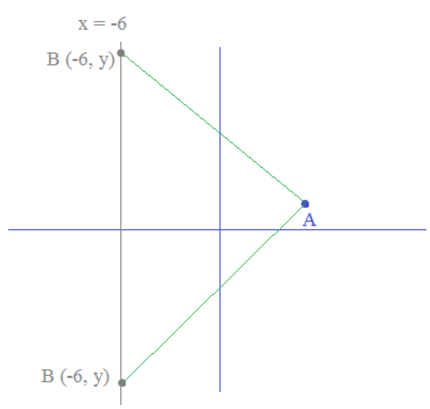
$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

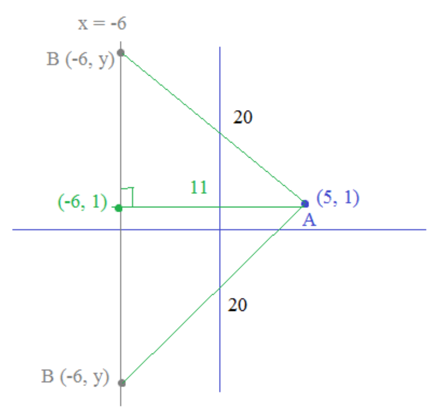
$$20 = \sqrt{(-6 - 5)^2 + (y - 1)^2}$$

$$400 = 121 + (y - 1)^2$$

$$\pm\sqrt{279} = y - 1$$

$$\boxed{y = 1 \pm \sqrt{279}}$$

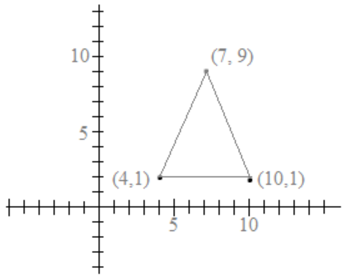




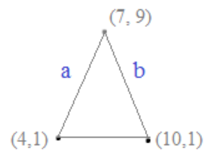
Distance and Midpoint Applications

I. Verify the following

1) The triangle is isosceles



Def. of isosceles: triangle with 2 congruent sides.



$a = \sqrt{(7-4)^2 + (9-1)^2}$

$= \sqrt{9 + 64} = \boxed{\sqrt{73}}$

$b = \sqrt{(7-10)^2 + (9-1)^2}$

$= \sqrt{9 + 64} = \boxed{\sqrt{73}}$

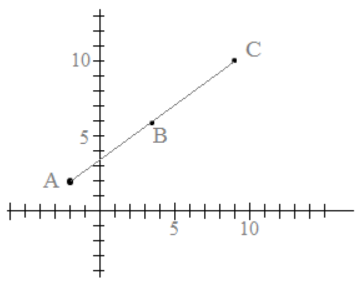
a = b, therefore the triangle is isosceles...

2) The length of AB equals the length of BC

A = (-2, 2)

B = (3.5, 6)

C = (9, 10)



Midpoint:

Midpoint of $\overline{AC}$

$\left(\frac{-2 + 9}{2}, \frac{2 + 10}{2} \right)$

$(3.5, 6)$

since B is the midpoint of $\overline{AC}$, $\overline{AB} = \overline{BC}$

Distance:

$d\overline{AB} = \sqrt{(-2 - 3.5)^2 + (2 - 6)^2}$

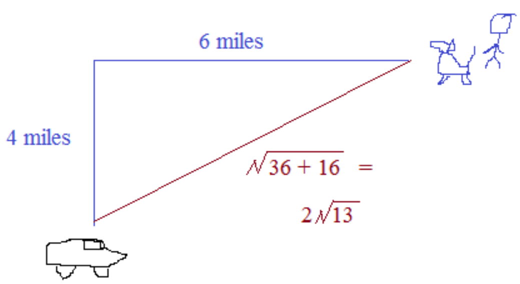
$= \sqrt{30.25 + 16} = \boxed{6.80}$

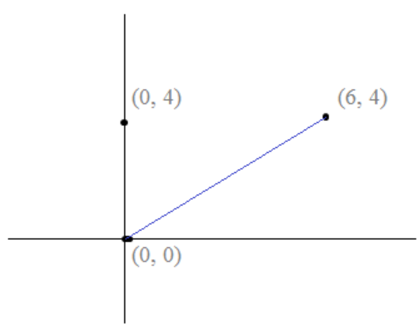
$d\overline{BC} = \sqrt{(3.5 - 9)^2 + (6 - 10)^2}$

$= \sqrt{30.25 + 16} = \boxed{6.80}$

$d\overline{AB} = d\overline{BC}$

II. My dog and I go for a hike in a field. We leave the car and walk due north 4 miles. Then, we turn 90° to the right and continue 6 miles due east. We get hungry and decide to go straight back to the car. How far must we go?





distance of $\overline{AB}$

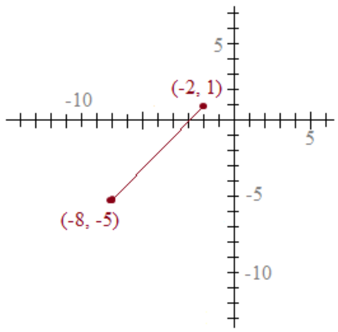
$\sqrt{(6 - 0)^2 + (4 - 0)^2}$

$= \sqrt{52} = 2\sqrt{13}$

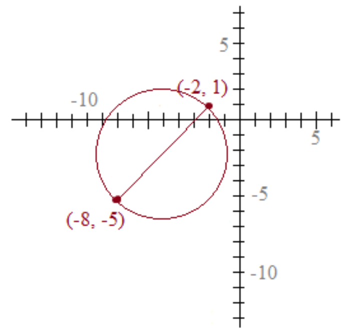
Distance and Midpoint Applications

III. Write the standard form of a circle with endpoints $(-2, 1)$ and $(-8, -5)$

Step 1: Sketch the figure



Step 2: Establish the strategy



The standard form of a circle is

$$(x-h)^2 + (y-k)^2 = r^2$$

where r is the radius and (h, k) is the center.

We need the center: *midpoint* of the diameter..

And, radius: *distance* between center and endpoint (or, 1/2 *distance* of diameter)

Step 3: Solve

Center: Find the midpoint of endpoints (-2, 1) and (-8, -5)

$$\left(\frac{-2+(-8)}{2}, \frac{1+(-5)}{2}\right) = (-5, -2)$$

Diameter: Distance between endpoints (-2, 1) and (-8, -5)

$$\text{distance} = \sqrt{(-2-(-8))^2 + (1-(-5))^2} = \sqrt{36 + 36} = 6\sqrt{2}$$

Or, Radius: Distance between center (-5, -2) and endpoint (-8, -5)

$$\text{distance} = \sqrt{(-5-(-8))^2 + (-2-(-5))^2} = \sqrt{9 + 9} = 3\sqrt{2}$$

$$(x-h)^2 + (y-k)^2 = r^2$$

$$(x-(-5))^2 + (y-(-2))^2 = \left(3\sqrt{2}\right)^2$$

$$(x+5)^2 + (y+2)^2 = 18$$

Eventually, Noah realizes that this assignment was NOT a geometry construction

Practice Review Quiz

Distance and Midpoint Review Quiz

Answer the following Questions. (Suggestion: Plot points and use graphs to verify solutions)

I. Midpoint

1) Find the midpoint between:

A) (0, 1) and (8, 3)

B) (11, -4) and (-6, -4)

C) (-17, -7) and (-7, -6)

2) Answer the following:

A) The midpoint of AB is (3, -3). If point A = (-2, -4), what is point B?

B) The endpoint of a segment is (5, -5). The midpoint of the segment is (9, -5). What is the other endpoint?

II. Distance

1) What is the distance between:

A) (3, 6) and (7, 9)

B) (7, -1) and (7, 7)

C) (-4, 5) and (1, 12)

2) The distance d between two points is given. Find the value(s) of b:

A) $(0, b)$ and $(3, 1)$; $d = 5$

B) $(b, -7)$ and $(-5, 1)$; $d = 10$

C) $(-9, -2)$ and $(b, 5)$; $d = 7$

Distance and Midpoint Review Quiz (continued)

III. Geometry application

A) Using the distance formula, determine whether the following are vertices of a right triangle
(i.e. Distances and converse of Pythagorean Theorem)

1) (5, 8) (5, 2) and (0, 2)

2) (3, -1) (1, 4) and (-3, 0)

3) (-1, 1) (2, 4) and (3, -3)

B) Find the perpendicular bisectors of the following line segments:
(express your answer in point slope form)

1) Line segment $\overline{AB}$, where A = (4, 7) and B = (11, 6)

2) Line segment $\overline{CD}$, where C = (3, -9) and D = (-6, -9)

Distance and Midpoint Review Quiz

SOLUTIONS

Answer the following Questions. (Suggestion: Plot points and use graphs to verify solutions)

I. Midpoint

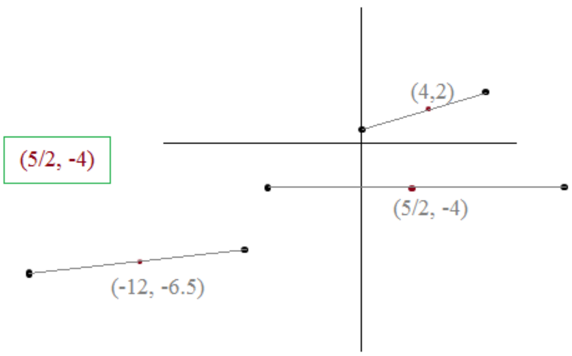
1) Find the midpoint between:

A) (0, 1) and (8, 3) $\left(\frac{0+8}{2}, \frac{1+3}{2}\right) = $ **(4, 2)**

B) (11, -4) and (-6, -4) average of x terms: $(11 + (-6))/2 = \frac{5}{2}$ **(5/2, -4)**

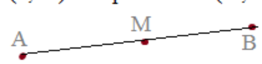
y terms are the same (no vertical change)

C) (-17, -7) and (-7, -6) **(-12, -6.5)**



2) Answer the following:

A) The midpoint of AB is (3, -3). If point A = (-2, -4), what is point B?

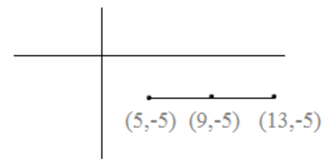


(-2, -4) ---> (3, -3) ---> **(8, -2)**

x add 5; y add 1

B) The endpoint of a segment is (5, -5). The midpoint of the segment is (9, -5). What is the other endpoint?

$(9, -5) = \left(\frac{5+x}{2}, \frac{-5+y}{2}\right)$ x = 13 y = -5 **(13, -5)**



II. Distance

1) What is the distance between:

A) (3, 6) and (7, 9)

$d = \sqrt{(7-3)^2 + (9-6)^2} = \sqrt{16+9} = $ **5**

B) (7, -1) and (7, 7)

Vertical line connecting both points: **8 units** from -1 to 7

C) (-4, 5) and (1, 12)

$d = \sqrt{(-4-1)^2 + (5-12)^2} = \sqrt{25+49} = \sqrt{74}$

$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

2) The distance d between two points is given. Find the value(s) of b:

A) (0, b) and (3, 1); $d = 5$

$5 = \sqrt{(0-3)^2 + (b-1)^2}$ (square both sides and solve)

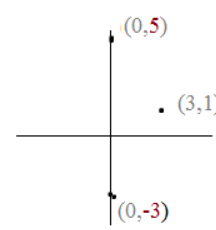
$25 = 9 + (b-1)^2$

$16 = b^2 - 2b + 1$

$b^2 - 2b - 15 = 0$

$(b - 5)(b + 3) = 0$

b = -3 and 5



B) (b, -7) and (-5, 1); $d = 10$

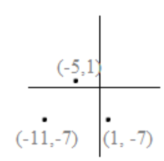
$10 = \sqrt{(b+5)^2 + (-7-1)^2}$

$100 = b^2 + 10b + 25 + 64$

$b^2 + 10b - 11 = 0$

$(b + 11)(b - 1) = 0$

b = -11 and 1



C) (-9, -2) and (b, 5); $d = 7$

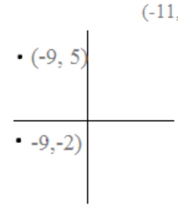
$7 = \sqrt{(-9-b)^2 + (-2-5)^2}$

$49 = 81 + 18b + b^2 + 49$

$0 = 81 + 18b + b^2$

$(b + 9)(b + 9) = 0$

b = −9



There is only one point that has a y coordinate 5 AND is 7 units away

Distance and Midpoint Review Quiz (continued)

SOLUTIONS

III. Geometry application

A) Using the distance formula, determine whether the following are vertices of a right triangle (i.e. Distances and converse of Pythagorean Theorem)

1) (5, 8) (5, 2) and (0, 2)
A B C

$dAB = 6$

$dBC = 5$

$dAC = \sqrt{(5-0)^2 + (8-2)^2}$

$= \sqrt{25 + 36} = \sqrt{61}$

AB = 6
BC = 5
AC = $\sqrt{61}$

$AB^2 + BC^2 = AC^2$

36 + 25 = 61

61 = 61 ✓
vertices of right triangle

2) (3, -1) (1, 4) and (-3, 0)
E F G

$dEF = \sqrt{(3-1)^2 + (-1-4)^2}$

$= \sqrt{29}$

$dFG = \sqrt{(1+3)^2 + (4-0)^2}$

$= \sqrt{32}$

$dEG = \sqrt{(3+3)^2 + (-1-0)^2}$

$= \sqrt{37}$

$EF^2 + FG^2 = EG^2$

29 + 32 = 37

61 ≠ 37 ✗
Vertices are not a right triangle

3) (-1, 1) (2, 4) and (3, -3)
M N P

$dMN = \sqrt{18}$

$dNP = \sqrt{50}$

$dMP = \sqrt{32}$

$MN^2 + MP^2 = NP^2$

18 + 32 = 50

50 = 50 ✓
Yes! Vertices of a right triangle

B) Find the perpendicular bisectors of the following line segments: (express your answer in point slope form)

1) Line segment $\overline{AB}$, where A = (4, 7) and B = (11, 6)

Find midpoint of $\overline{AB}$:

$$\left(\frac{4+11}{2}, \frac{7+6}{2}\right) = (15/2, 13/2)$$

To find perpendicular line, find slope of $\overline{AB}$:

$$m = \frac{7-6}{4-11} = \frac{-1}{7}$$

slope of perpendicular line is 7.
(opposite reciprocal)

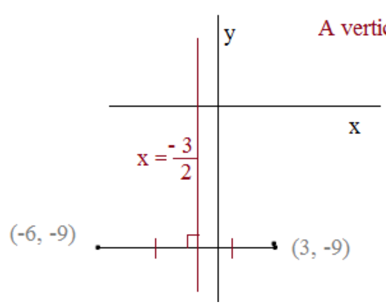
$$y - \frac{13}{2} = 7\left(x - \frac{15}{2}\right)$$

2) Line segment $\overline{CD}$, where C = (3, -9) and D = (-6, -9)

Midpoint of $\overline{CD}$ is (-3/2, -9)

Segment $\overline{CD}$ is horizontal!

Therefore, the perpendicular bisector will be vertical...



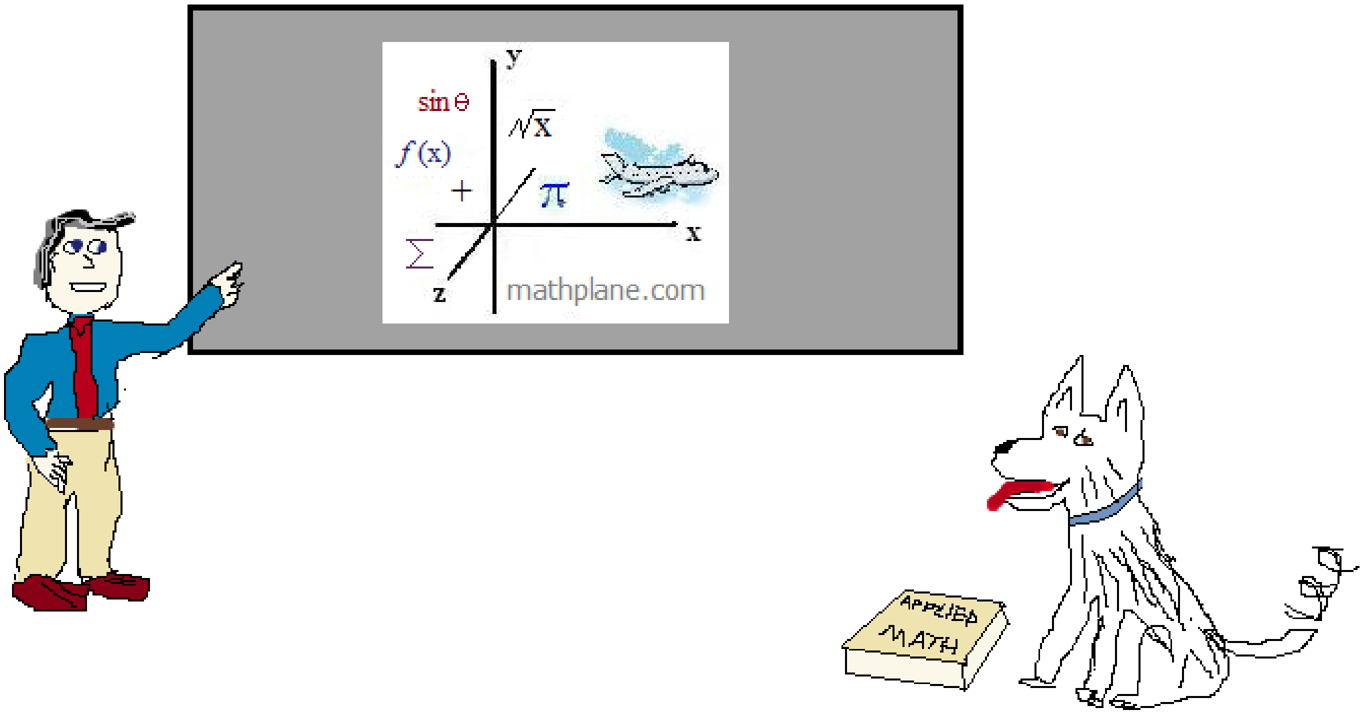
A vertical line going through $\left(\frac{-3}{2}, -9\right)$

$$x = \frac{-3}{2}$$

Thanks for visiting the site. (Hope it helped!)

If you have questions, suggestions, or requests, let us know!

Cheers.



One more question

The distance between A and B is 10 units.

If A is (3, 11) and B is (x, 5), then what is x?

Answer-→

The distance between A and B is 10.

If A is (3, 11) and B is (x, 5), what is x?

$$\text{distance} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$10 = \sqrt{(3 - x)^2 + (11 - 5)^2}$$

$$100 = (3 - x)^2 + 36$$

$$64 = (3 - x)^2$$

$$\pm 8 = 3 - x$$

$$x = -5 \text{ or } 11$$