Introduction to Integrals

Examples and practice questions (with solutions)

Topics include U-Substitution, logarithms, definite and indefinite antiderivatives, trigonometry, partial fractions, tabular integration, and more.

Strategies for finding Antiderivatives

1) "Simplify first"

Example: $\int \frac{x^2 + 3x + 2}{\sqrt{x}} \, dx$

$$\int \frac{x^2}{\sqrt{x}} + \frac{3x}{\sqrt{x}} + \frac{2}{\sqrt{x}} \, dx$$

Splitting up the trinomial in the numerator creates 3 easier terms to integrate

$$\int \frac{x^2}{\sqrt{x}} \, dx + \int \frac{3x}{\sqrt{x}} \, dx + \int \frac{2}{\sqrt{x}} \, dx$$

$$\int x^{3/2} \, dx + \int 3x^{1/2} \, dx + \int 2x^{-1/2} \, dx$$

$$\frac{2}{5} x^{5/2} + 3 \cdot \frac{2}{3} x^{3/2} + 2 \cdot \frac{2}{1} x^{1/2}$$

$$\frac{2}{5} x^{5/2} + 2x^{3/2} + 4x^{1/2} + C$$

Example: $\int \sin^2(3x) + \cos^2(3x) \, dx$

$$\int 1 \, dx$$

trigonometry identity

$\sin^2 + \cos^2 = 1$

$$x + C$$

Example: $\int (x+7)^2 (x^2 + 3x + 2) \, dx$

(expand and combine)

$(x + 7)(x + 7) = x^2 + 14x + 49$

Then, $(x^2 + 3x + 2)(x^2 + 14x + 49)$

$$\begin{array}{r} x^4 + 14x^3 + 49x^2 \\ 3x^3 + 42x^2 + 147x \\ 2x^2 + 28x + 98 \\ \hline x^4 + 17x^3 + 93x^2 + 175x + 98 \end{array}$$

$$\int x^4 + 17x^3 + 93x^2 + 175x + 98 \, dx$$

$$\frac{x^5}{5} + \frac{17x^4}{4} + \frac{93x^3}{3} + \frac{175x^2}{2} + 98x + C$$

$$\frac{x^5}{5} + \frac{17x^4}{4} + 31x^3 + \frac{175x^2}{2} + 98x + C$$

Strategies for finding Antiderivatives

2) "Derivative beside the function"

Note: This technique is similar to *integration by substitution* (or, *U-substitution*)

Example: $\displaystyle\int \frac{2x}{\sqrt{3x^2+5}}\,dx$

$\displaystyle\int 2x \cdot (3x^2+5)^{-\frac{1}{2}}\,dx$

The 'function' is $3x^2 + 5$...
and, its derivative is $6x$

$\displaystyle\frac{1}{3}\int 3 \cdot 2x \cdot (3x^2+5)^{-\frac{1}{2}}\,dx$

Multiply 3 to get 6x....
(and, multiply 1/3 to keep the same equation)

$\displaystyle\frac{1}{3}\int 6x \cdot (3x^2+5)^{-\frac{1}{2}}\,dx$

Derivative beside the function
then, use power rule...

'derivative' 'function'

$\displaystyle\frac{1}{3} \cdot \frac{2}{1}(3x^2+5)^{\frac{1}{2}} = \frac{2}{3}\sqrt{(3x^2+5)} + C$

'function'
('derivative' disappears)

check the answer by taking the derivative!!

$\displaystyle f(x) = \frac{2}{3}\sqrt{(3x^2+5)} + C$

$\displaystyle = \frac{2}{3}(3x^2+5)^{\frac{1}{2}} + C$

use power rule to find derivative

$\displaystyle f'(x) = \frac{2}{6}(3x^2+5)^{-\frac{1}{2}}(6x) + 0$

('derivative' appears)

$\displaystyle = \frac{2x}{\sqrt{3x^2+5}}$

Example: $\displaystyle\int x\cos(3x^2)\,dx$

There are two terms: x and $3x^2$

Since the derivative of $3x^2$ is 6x,
we need a 6x beside it!

"derivative beside the function"

$\displaystyle\frac{1}{6}\int 6\,x\cos(3x^2)\,dx$

$\displaystyle\frac{1}{6}\sin(3x^2)$

When you take the integral of the trig function, the 6x 'goes away'

CHECK: $\displaystyle f(x) = \frac{1}{6}\sin(3x^2)$

$\displaystyle f'(x) = \frac{1}{6}\cos(3x^2)\cdot(6x)$

When you take the derivative of $\cos(3x^2)$, the 6x 'appears'
(because of the chain rule)

$f'(x) = x\cos(3x^2)$ ✓

Strategies for finding Antiderivatives

2a) "U-Substitution"

Example: $\int 4(8x+3)^3\,dx$

Expanding the expression $(8x+3)^3$ is one approach.
But, $(8x+3)(8x+3)(8x+3)$ can get messy...

Using "U-Substitution" is another approach:

identify the 'main function' and label u — Let $u = (8x+3)$

determine du — then, $\frac{du}{dx} = 8 \qquad dx = \frac{du}{8}$

substitute $\quad \int 4(8x+3)^3\,dx \longrightarrow \int 4(u)^3 \cdot \frac{du}{8} =$

solve $\quad \frac{4}{8}\int (u)^3\,du = \frac{1}{2}\cdot\frac{u^4}{4} + C = \frac{u^4}{8} + C$

substitute $\quad \frac{(8x+3)^4}{8} + C$.

(check) derivative of $\frac{(8x+3)^4}{8} + C$. $= 4\,\frac{(8x+3)^3}{8}\cdot 8 + 0 = 4(8x+3)^3$ ✓

using chain rule

Compare "u-substitution" and "derivative beside function"

Example: $\int \frac{3x}{\sqrt{x^2+8}}\,dx$

"Derivative beside the function"

$$\int 3x(x^2+8)^{-\frac{1}{2}}\,dx$$

the 'function' is (x^2+8), so we need 2x beside it...

$$\frac{3}{2}\int \frac{2}{3}\,3x(x^2+8)^{-\frac{1}{2}}\,dx$$

$$\frac{3}{2}\int 2x(x^2+8)^{-\frac{1}{2}}\,dx$$

derivative (2x) — function (x^2+8)

now, integrate using power rule...

$$\frac{3}{2}\,\frac{(x^2+8)^{\frac{1}{2}}}{\frac{1}{2}} = 3(x^2+8)^{\frac{1}{2}}$$

$$= 3\sqrt{x^2+8} + C$$

"U-Substitution"

$$\int 3x(x^2+8)^{-\frac{1}{2}}\,dx$$

let $u = (x^2+8)$

$\frac{du}{dx} = 2x \longrightarrow 2x\,dx = du \longrightarrow x\,dx = \frac{du}{2}$

$$3\int (x^2+8)^{-\frac{1}{2}}\,x\,dx$$

substitute the terms

$$3\int u^{-\frac{1}{2}}\,\frac{du}{2} = \frac{3}{2}\int u^{-\frac{1}{2}}\,du$$

$$\frac{3}{2}\,\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = 3u^{\frac{1}{2}} + C$$

substitute back to x $\quad 3\sqrt{x^2+8} + C$

Strategies for finding Antiderivatives

3) Integration by Parts

Review: Product Rule (derivatives)

If $u = u(x)$ and $v = v(x)$ and, therefore

$u' = du = u'(x)dx \quad v' = dv = v'(x)dx$

Then, derivative of $u(x) \cdot v(x)$ is $u'(x)dx \cdot v(x) + v'(x)dx \cdot u(x)$

or, derivative of $u \cdot v$ is $u'v + v'u$

This leads to integration by parts:

Since $u \cdot v = u'v + v'u$

$$\int u\,dv = u\,v - \int v\,du$$

Since the *derivative* of $uv = u'v + v'u$,

then the *antiderivative* of $u'v + v'u = uv$

$$\int (u'v + v'u) = uv$$

$$\int u'v + \int v'u = uv$$

so, $$\int v'u = uv - \int u'v$$

"If you have 2 continous functions -- where *one function's derivative is related to the other function* -- try integration by parts."

Example: $\int 2xe^x \, dx$

Since $2x$ and e^x are 'not related' (i.e. neither is the derivative of the other), we'll try integration by parts...

let $u = 2x$, $v' = e^x dx$

then, $u' = 2\,dx$, $v = e^x$

the derivative of 2x is 2...
And '2 is related to x'...
(i.e. *the derivative of x can lead to 2*)

$$\underset{u\ \ v'}{\int 2xe^x\,dx} = \underset{u\ \ v}{2x\ e^x} - \int \underset{v\ \ u'}{e^x\ 2\,dx}$$

$$= 2x\,e^x - 2\,e^x + C$$

$$= 2e^x(x-1) + C$$

Example: $\int x \sin x \, dx$

let $u = x$, $v' = \sin x \, dx$

then, $u' = 1\,dx$, $v = -\cos(x)$

$$\int x\sin x\,dx = x(-\cos(x)) - \int (-\cos(x))\,dx$$

$$= -x\cos(x) - (-\sin(x)) + C$$

$$= \sin(x) - x\cos(x) + C$$

'derivative of x is related to (x)'

Now, suppose we let $u = \sin x$, $v' = x\,dx$

then, $u' = \cos x\,dx$, $v = \frac{x^2}{2}$

$$\int x\sin x\,dx = \sin x \cdot \frac{x^2}{2} - \int \frac{x^2}{2}\cos x\,dx$$

'derivative of sin(x) is not related to x'
(the derivative of sin(x) cannot lead to x)

In this case, we would have to use integration by parts again!
***Note: selecting the correct "u" and "v" is important!

Integration by Parts: Tabular Integration

Example: $\int x^3 e^{-2x}\,dx$

$u = x^3 \qquad v' = e^{-2x}\,dx$

$u' = 3x^2\,dx \qquad v = \frac{-1}{2} e^{-2x}$

Integration by Parts Formula

$$\int v'u = uv - \int u'v$$

$$\int x^3 e^{-2x}\,dx = x^3 \cdot \frac{-1}{2} e^{-2x} - \int 3x^2\,dx \cdot \frac{-1}{2} e^{-2x}$$

$$= \frac{-1}{2} \cdot x^3 e^{-2x} - \frac{-1}{2} \int 3x^2 e^{-2x}\,dx$$

$$= \frac{-1}{2} \left[x^3 e^{-2x} - \int 3x^2 e^{-2x}\,dx \right]$$

Now, we'll use integration by parts again...

$u = 3x^2 \qquad v' = e^{-2x}\,dx$

$u' = 6x\,dx \qquad v = \frac{-1}{2} e^{-2x}$

$$= \frac{-1}{2} \left[x^3 e^{-2x} - \left(3x^2 \cdot \frac{-1}{2} e^{-2x} - \int 6x\,dx \cdot \frac{-1}{2} e^{-2x} \right) \right]$$

$$= \frac{-1}{2} \left[x^3 e^{-2x} - \left(\frac{-1}{2} \cdot 3x^2 e^{-2x} - \frac{-1}{2} \int 6x\, e^{-2x}\,dx \right) \right]$$

$$= \frac{-1}{2} \left[x^3 e^{-2x} - \frac{-1}{2} \left(3x^2 e^{-2x} - \int 6x\, e^{-2x}\,dx \right) \right]$$

Now, we'll use integration by parts again...

$u = 6x \qquad v' = e^{-2x}\,dx$

$u' = 6\,dx \qquad v = \frac{-1}{2} e^{-2x}$

$$= \frac{-1}{2} \left[x^3 e^{-2x} - \frac{-1}{2} \left(3x^2 e^{-2x} - \left(6x \cdot \frac{-1}{2} e^{-2x} - \int 6\,dx \cdot \frac{-1}{2} e^{-2x} \right) \right) \right]$$

$$= \frac{-1}{2} \left[x^3 e^{-2x} - \frac{-1}{2} \left(3x^2 e^{-2x} - \left(\frac{-1}{2} \cdot 6x\, e^{-2x} - \frac{-1}{2} \int 6\, e^{-2x}\,dx \right) \right) \right]$$

$$= \frac{-1}{2} \left[x^3 e^{-2x} - \frac{-1}{2} \left(3x^2 e^{-2x} - \frac{-1}{2} \left(6x\, e^{-2x} - \int 6\, e^{-2x}\,dx \right) \right) \right]$$

At last, we can determine the final integral!

$$= \frac{-1}{2} \left[x^3 e^{-2x} - \frac{-1}{2} \left(3x^2 e^{-2x} - \frac{-1}{2} \left(6x\, e^{-2x} - \left(\frac{-1}{2} \cdot 6\, e^{-2x} \right) \right) \right) \right]$$

$$= \frac{-1}{2} \left[x^3 e^{-2x} - \frac{-1}{2} \left(3x^2 e^{-2x} - \frac{-1}{2} \left(6x\, e^{-2x} - \frac{-1}{2} \left(6\, e^{-2x} \right) \right) \right) \right]$$

$$\frac{-1}{2} x^3 e^{-2x} - \frac{1}{4} \cdot 3x^2 e^{-2x} + \frac{-1}{8} \cdot 6x\, e^{-2x} - \frac{1}{16} \cdot 6\, e^{-2x}$$

Using Tabular Integration:

Observe: there is a pattern!

Derivative		Integral
x^3	+	e^{-2x}
$3x^2$	—	$\frac{-1}{2} e^{-2x}$
$6x$	+	$\frac{1}{4} e^{-2x}$
6	—	$\frac{-1}{8} e^{-2x}$
0		$\frac{1}{16} e^{-2x}$

Strategies for finding Antiderivatives

4) Utilizing Partial Fractions

Example: $\int \frac{8x+5}{x^2+3x-10}\,dx$

$(x+5)(x-2)$

$$\frac{A}{(x+5)} + \frac{B}{(x-2)} = \frac{8x+5}{x^2+3x-10}$$

$$\frac{A(x-2)}{(x+5)(x-2)} + \frac{B(x+5)}{(x+5)(x-2)} = \frac{8x+5}{x^2+3x-10}$$

$$A(x-2) + B(x+5) = 8x+5$$

$$Ax - 2A + Bx + 5B = 8x + 5$$

$$(A+B)(x) - 2A + 5B = 8x + 5$$

Then, we know $A + B = 8$

$-2A + 5B = 5$

$A = 5$
$B = 3$

$2A + 2B = 16$
$-2A + 5B = 5$
$7B = 21$
$B = 3$

then, $A = 5$

$$\frac{5}{(x+5)} + \frac{3}{(x-2)}$$

$$\int \frac{8x+5}{x^2+3x-10}\,dx = \int \frac{5}{(x+5)}\,dx + \int \frac{3}{(x-2)}\,dx$$

$$5\ln|x+5| + 3\ln|x-2| + C$$

Example: $\int \frac{-6x^2+3x+5}{x^3-x}\,dx$

$x(x^2-1) = x(x+1)(x-1)$

$$\frac{-6x^2+3x+5}{x^3-x} = \frac{A}{x} + \frac{B}{(x+1)} + \frac{C}{(x-1)}$$

$$= \frac{(x+1)(x-1)A}{x(x+1)(x-1)} + \frac{x(x-1)B}{x(x+1)(x-1)} + \frac{x(x+1)C}{x(x+1)(x-1)}$$

$$-6x^2+3x+5 = (x+1)(x-1)A + x(x-1)B + x(x+1)C$$

$$Ax^2 - 1A + Bx^2 - Bx + Cx^2 + Cx$$

(regroup the terms)

$$(A+B+C)x^2 + (-B+C)x + A(-1)$$

$$-6x^2 \quad + \quad 3x \quad + 5$$

"Express method"

Let x = 0 (to eliminate B and C)

$0 + 0 + 5 = -1A + 0B + 0C$

$A = -5$

Let x = -1 (to eliminate A and C)

$-6 + (-3) + 5 = 0A + 2B + 0C$

$B = -2$

Let x = 1 (to eliminate A and B)

$-6 + 3 + 5 = 0A + 0B + 2C$

$C = 1$

$A + B + C = -6$
$-B + C = 3$
$-A = 5$

$A = -5$

$-5 + B + C = -6$
$-B + C = 3$
$2C = 2$
$C = 1$

$-B + C = 3$
$-B + 1 = 3$
$B = -2$

$$\frac{-6x^2+3x+5}{x^3-x} = \frac{-5}{x} + \frac{-2}{(x+1)} + \frac{1}{(x-1)}$$

$$\int \frac{-5}{x} + \frac{-2}{(x+1)} + \frac{1}{(x-1)}\,dx$$

$$-5\ln|x| + -2\ln|x+1| + \ln|x-1| + C$$

Double U-Substitution

Example: $\int_{13}^{18} x^2\sqrt{x-9}\, dx$

Let $U = x - 9$ ⟹ NOTE: the boundaries for x are 13 - 18

$x = U + 9$ so, the boundaries for U are 4 - 9

$\frac{dU}{dx} = 1$ so, $dU = dx$

$$\int_4^9 (U+9)^2 (U)^{\frac{1}{2}}\, dU$$

$$\int_4^9 (U^2 + 18U + 81)(U)^{\frac{1}{2}}\, dU$$

$$\int_4^9 U^{\frac{5}{2}} + 18U^{\frac{3}{2}} + 81U^{\frac{1}{2}}\, dU$$

$$\frac{2}{7}U^{\frac{7}{2}} + \frac{36}{5}U^{\frac{5}{2}} + 54U^{\frac{3}{2}}\Bigg|_4^9$$

Also,

$$\int U^{\frac{5}{2}} + 18U^{\frac{3}{2}} + 81U^{\frac{1}{2}}\, dU$$

$$\frac{2}{7}U^{\frac{7}{2}} + \frac{36}{5}U^{\frac{5}{2}} + 54U^{\frac{3}{2}}$$

put back into terms of x

$$\frac{2}{7}(x-9)^{\frac{7}{2}} + \frac{36}{5}(x-9)^{\frac{5}{2}} + 54(x-9)^{\frac{3}{2}}\Bigg|_{13}^{18}$$

$$\frac{2}{7}(3)^7 + \frac{36}{5}(3)^5 + 54(3)^3 - \left(\frac{2}{7}(2)^7 + \frac{36}{5}(2)^5 + 54(2)^3\right) = \boxed{\frac{109672}{35}}$$

Example: $\int_5^9 t^3\sqrt{t-4}\, dt$

Let $U = t - 4$

$t = U + 4$

$\frac{dU}{dt} = 1$ $dU = dt$

$$\int_1^5 (U+4)^3 (U)^{\frac{1}{2}}\, dU$$

$$\int_1^5 (U^3 + 12U^2 + 48U + 64)(U)^{\frac{1}{2}}\, dU$$

$$\int_1^5 U^{\frac{7}{2}} + 12U^{\frac{5}{2}} + 48U^{\frac{3}{2}} + 64(U)^{\frac{1}{2}}\, dU \qquad \frac{2U^{\frac{9}{2}}}{9} + \frac{24U^{\frac{7}{2}}}{7} + \frac{96U^{\frac{5}{2}}}{5} + \frac{128(U)^{\frac{3}{2}}}{3}\Bigg|_1^5 = \boxed{\frac{79430\sqrt{5}}{63} - \frac{20638}{315}}$$

Example: $\int x\sqrt{2x+1}\, dx$

Let $U = 2x + 1$ $\frac{dU}{dx} = 2$

$x = \frac{U-1}{2}$ $dx = \frac{dU}{2}$

$$\int \frac{U-1}{2} \cdot U^{\frac{1}{2}} \cdot \frac{dU}{2}$$

$$\frac{1}{4}\int U^{\frac{3}{2}} - U^{\frac{1}{2}}\, dU \qquad \frac{1}{4}\left(\frac{2U^{\frac{5}{2}}}{5} - \frac{2U^{\frac{3}{2}}}{3}\right) + C \implies \frac{1}{10}(2x+1)^{\frac{5}{2}} - \frac{1}{6}(2x+1)^{\frac{3}{2}} + C$$

$$(2x+1)^{\frac{3}{2}}\left[\frac{1}{10}(2x+1) - \frac{1}{6}\right] + C$$

$$(2x+1)^{\frac{3}{2}}\left[\frac{3}{30}(2x+1) - \frac{5}{30}\right] + C$$

$$\boxed{(2x+1)^{\frac{3}{2}}\left[\frac{3x-1}{15}\right] + C}$$

Integration by Parts

Example: $\int y\ln(y)\, dy$

$\int y \cdot \ln(y)\, dy$

since the derivative of $\ln y$ is $1/y$,
we won't use u-substitution...

Instead, we'll try integration by parts...

Step 1: Identify "u" and "v'"

Since it is easier to take the derivative of ln(y),

$u = \ln(y)$

$v' = y\, dy$

Step 2: Find other parts

$u' = \frac{1}{y}\, dy$

$v = \int y\, dy = \frac{y^2}{2}$

Step 3: Apply the formula

$$\int v'u = uv - \int u'v$$

$$\int y\, dy \cdot \ln(y) = \ln(y) \cdot \frac{y^2}{2} - \int \frac{1}{y}\, dy \cdot \frac{y^2}{2}$$

$$\ln(y) \cdot \frac{y^2}{2} - \int \frac{y}{2}\, dy$$

$$\boxed{\ln(y) \cdot \frac{y^2}{2} - \frac{y^2}{4} + C}$$

Example: $\int \frac{2x^2 + 7x - 3}{x - 2}\, dx$

Divide using synthetic division

Synthetic Division (or, polynomial division)

2	2	7	-3
		4	22
	2	11	19

$2x + 11 + \frac{19}{x-2}$

Then, integrate...

$\int 2x + 11 + \frac{19}{x-2}\, dx$

$$\boxed{x^2 + 11x + \ln|x-2| + C}$$

Example: $\int (3x+5)^2\, dx$ Evaluate the indefinite integral using a) U-substitution and b) Expanding the term... then, compare your results...

Let $u = 3x + 5$

so, $\frac{du}{dx} = 3 \qquad dx = \frac{1}{3}\, du$

$\int u^2 \cdot \frac{1}{3}\, du$

$\frac{1}{3} \cdot \frac{u^3}{3} = \frac{1}{9}(3x+3)^3 + C$

$\frac{1}{9}(27x^3 + 135x^2 + 225x + 125) + C$

$3x^3 + 15x^2 + 25x + \frac{125}{9} + C$ ✓

$\int 9x^2 + 30x + 25\, dx$

$3x^3 + 15x^2 + 25x + C$ ✓

Since C is any constant...
C is equivalent to $\frac{125}{9} + C$

Example: $\int \frac{10^x}{\ln 10}\, dx$

Let $U = 10^x$

$\frac{dU}{dx} = (\ln 10)(10^x)$

$(\ln 10)(10^x)dx = dU$

$dx = \frac{dU}{(\ln 10)(10^x)}$

(substitute the dx)

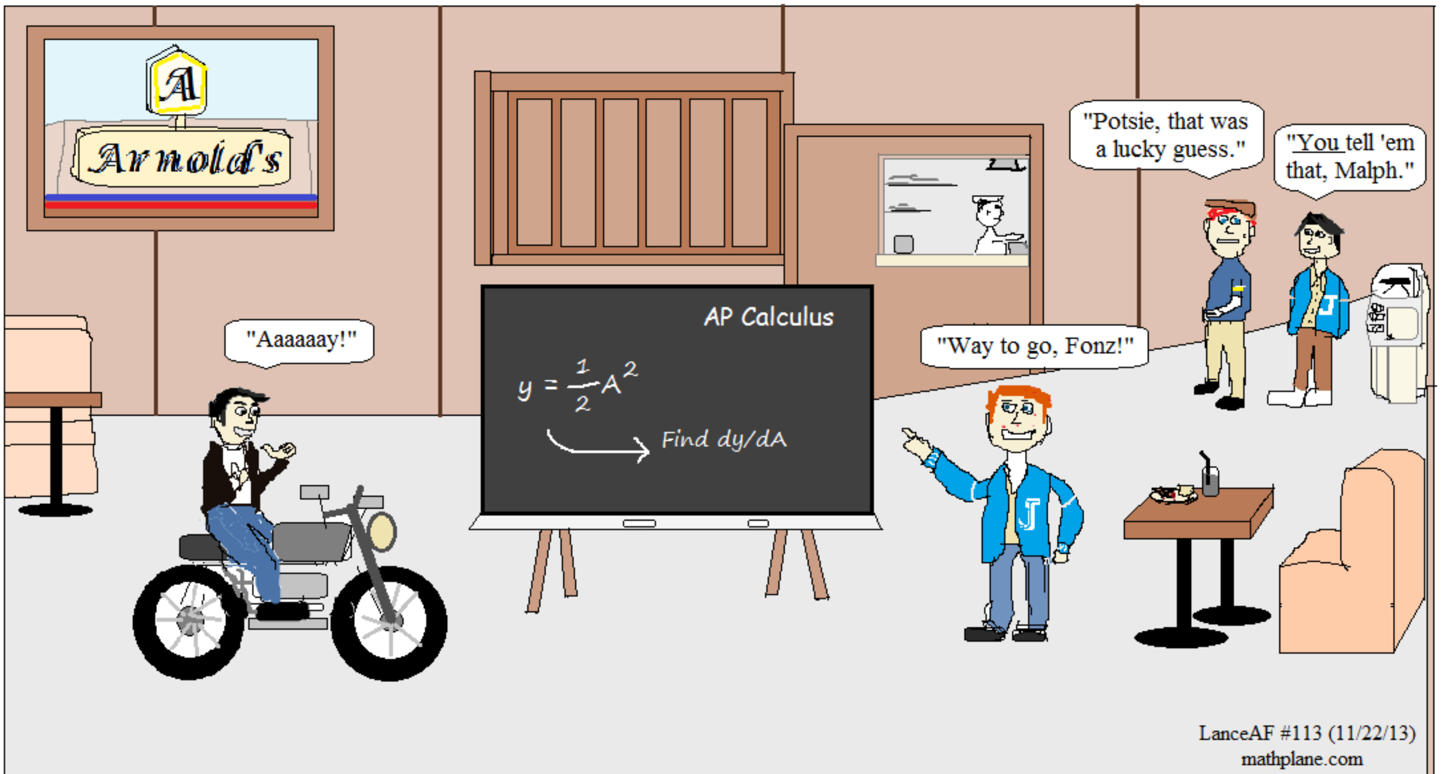
$\int \frac{10^x}{\ln 10} \frac{dU}{(\ln 10)(10^x)}$

$\int \frac{1}{\ln 10} \frac{dU}{(\ln 10)}$

(Remember ln10 is a constant/number... So, we're ready to integrate)

$\frac{1}{(\ln 10)^2} \int dU = \frac{1}{(\ln 10)^2} U + C$

$$\boxed{\frac{1}{(\ln 10)^2} 10^x + C}$$



Happy Days

Despite Richie's help, Fonzie dropped out of Calculus.
(... although he did have some success with velocity and acceleration!)

Practice -→

Indefinite Integrals Exercise

Evaluate the indefinite integrals. (Check your results by differentiation!)

1) $\int (x^3 + 4)\, dx$

2) $\int \sqrt{x^3}\, dx$

3) $\int \frac{1}{6x^2}\, dx$

4) $\int \frac{(x+2)^2}{x}\, dx$

5) $\int dx$

6) $\int t^2 (t^3 + 1)^4\, dt$

7) $\int \frac{2x}{\sqrt{x^2 - 3}}\, dx$

8) $\int 8(3 + 4x^2)^2\, dx$

9) $\int \cos 2t\, dt$

For the following, find the specific or general functions:

CHALLENGE

Antiderivatives

1) $f'(x) = 8x^3 + 10x + 5 \quad f(1) = 6$

2) $f''(x) = 3 + x^2 + x^5$

3) $f''(x) = 2 - 12x \quad f(0) = 9 \quad f(2) = 15$

4) $f'(x) = \dfrac{1}{\cos^2 x} + 3^x$

1) $\displaystyle\int \frac{3x^2 - 2x + 12}{(x^2 + 4)^2}\, dx$

2) $\displaystyle\int \frac{x^2 - 2x - 2}{x^3 - 1}\, dx$

Indefinite Integrals Exercise

Evaluate the indefinite integrals. (Check your results by differentiation!)

SOLUTIONS

1) $\int (x^3 + 4)\, dx$

$\int x^3 dx + \int 4\, dx$

$\boxed{\frac{x^4}{4} + 4x + C}$

the derivative of the solution:

$\frac{4x^3}{4} + 4 + 0 = x^3 + 4$ ✓

2) $\int \sqrt{x^3}\, dx$

$\int (x^3)^{\frac{1}{2}} dx = \int x^{\frac{3}{2}} dx$

$\frac{x^{\frac{5}{2}}}{\frac{5}{2}} = \boxed{\frac{2x^{\frac{5}{2}}}{5} + C}$

(derivative, using power rule)

$\frac{5}{2} \cdot \frac{2x^{3/2}}{5} + 0 = x^{3/2}$ ✓

3) $\int \frac{1}{6x^2}\, dx$

$\int \frac{1}{6} x^{-2} dx = \frac{1}{6} \int x^{-2} dx$

$= \frac{1}{6}\left(\frac{x^{-1}}{-1}\right) = \boxed{\frac{-1}{6x} + C}$

derivative of -1/6x (using quotient rule)

$\frac{0(6x) - 6(-1)}{(6x)^2} = \frac{6}{36x^2} = \frac{1}{6x^2}$ ✓

4) $\int \frac{(x+2)^2}{x}\, dx$

(expand numerator)

$\int \frac{x^2 + 4x + 4}{x}\, dx$

$\int x + 4 + 4\left(\frac{1}{x}\right) dx$

$\boxed{\frac{x^2}{2} + 4x + 4\ln x + C}$

5) $\int dx$

$\int 1\, dx$

$\boxed{x + C}$

6) $\int t^2 (t^3 + 1)^4\, dt$

Let $u = (t^3 + 1)$ then, we need $u' = 3t^2$

$\frac{1}{3} \int \underset{u'}{3t^2} \underset{u}{(t^3 + 1)^4}\, dt = \frac{1}{3}\left(\frac{(t^3 + 1)^5}{5}\right)$

$= \boxed{\frac{(t^3 + 1)^5}{15} + C}$

derivative:

$\frac{\not{5}(t^3 + 1)^4 \cdot (\not{3}t^2 + 0)}{\not{15}} + 0 = (t^3 + 1)^4 \cdot t^2$ ✓

7) $\int \frac{2x}{\sqrt{x^2 - 3}}\, dx$

$\int 2x(x^2 - 3)^{\frac{-1}{2}} dx$

$u = x^2 - 3$

$u' = 2x$

so, we can use power rule of integration

$\frac{(x^2 - 3)^{\frac{1}{2}}}{\frac{1}{2}}$

$\boxed{2\sqrt{x^2 - 3} + C}$

8) $\int 8(3 + 4x^2)^2\, dx$

If $u = (3 + 4x^2)$, then $u' = 8x$

since there is no x term, the power rule of integration cannot be used. instead expand the entire equation.

$\int 8(9 + 24x^2 + 16x^4)\, dx$

$\int 72 + 192x^2 + 128x^4\, dx$

$\boxed{72x + 64x^3 + \frac{128x^5}{5} + C}$

9) $\int \cos 2t\, dt$

The derivative of $\sin t$ is $\cos t$, and the derivative of $\sin 2t$ is $2\cos 2t$.

$\frac{1}{2} \int 2 \cos 2t\, dt$

$\boxed{\frac{1}{2} \sin 2t + C}$

For the following, find the specific or general functions:

CHALLENGE

Antiderivatives

SOLUTIONS

1) $f'(x) = 8x^3 + 10x + 5 \quad f(1) = 6$

Find the indefinite integral:

$$2x^4 + 5x^2 + 5x + C$$

Then, use the given value to determine C:

$$2(1)^4 + 5(1)^2 + 5(1) + C = 6$$

$$2 + 5 + 5 + C = 6$$

$$C = -6$$

$$f(x) = 2x^4 + 5x^2 + 5x - 6$$

2) $f''(x) = 3 + x^2 + x^5$

Find the antiderivative:

$$f'(x) = 3x + \frac{x^3}{3} + \frac{x^6}{6} + C$$ where C is a constant

Then, find the antiderivative again:

$$f(x) = \frac{3x^2}{2} + \frac{x^4}{12} + \frac{x^7}{42} + Cx + D$$

where C and D are constants...

3) $f''(x) = 2 - 12x \quad f(0) = 9 \quad f(2) = 15$

Take the antiderivative twice to determine the function:

$$f'(x) = 2x - 6x^2 + C$$

$$f(x) = x^2 - 2x^3 + Cx + D$$

then, use the given values to find the specific function:

$$f(0) = (0)^2 - 2(0)^3 + C(0) + D$$

$$D = 9$$

so, $f(x) = x^2 - 2x^3 + Cx + 9$

$$f(2) = (2)^2 - 2(2)^3 + C(2) + 9$$

$$15 = 4 - 16 + 2C + 9$$

$$C = 9$$

$$f(x) = -2x^3 + x^2 + 9x + 9$$

4) $f'(x) = \frac{1}{\cos^2 x} + 3^x$

to find the antiderivative (integral), separate the terms and rewrite...

$$\int \sec^2 x \, dx + \int 3^x \, dx$$

The derivative of what term equals $\sec^2 x$?

the answer: tanx

$$\tan x + \int 3^x \, dx$$

The derivative of 3^x is $3^x \cdot \ln 3$

so, the antiderivative of $(\ln 3)3^x$ would be 3^x

$$\tan x + \frac{1}{\ln 3}\int \ln 3 \cdot 3^x \, dx$$

$$\tan x + \frac{1}{\ln 3} \cdot 3^x = \tan x + \frac{3^x}{\ln 3} + C$$

SOLUTIONS

1) $$\int \frac{3x^2 - 2x + 12}{(x^2+4)^2}\,dx$$

Decompose using partial fractions...

$$\frac{Ax+B}{(x^2+4)} + \frac{Cx+D}{(x^2+4)^2} = \frac{3x^2 - 2x + 12}{(x^2+4)^2}$$

We want common denominators...

$$\frac{(Ax+B)(x^2+4)}{(x^2+4)^2} + \frac{Cx+D}{(x^2+4)^2} = \frac{3x^2 - 2x + 12}{(x^2+4)^2}$$

$$(Ax+B)(x^2+4) + Cx + D = 3x^2 - 2x + 12$$

Expand and regroup...

$$Ax^3 + 4Ax + Bx^2 + 4B + Cx + D = 3x^2 - 2x + 12$$

$Ax^3 = 0x^3$ — $A = 0$

$Bx^2 = 3x^2$ — $B = 3$

$4Ax + Cx = -2x$ — Since $A = 0$, then $C = -2$

$4B + D = 12$ — Since $B = 3$, then $D = 0$

$$\frac{3}{(x^2+4)} + \frac{-2x}{(x^2+4)^2} = \frac{3x^2 - 2x + 12}{(x^2+4)^2}$$

$$\int \frac{3}{(x^2+4)}\,dx + \int \frac{-2x}{(x^2+4)^2}\,dx$$

$$\frac{3}{2}\arctan\left(\frac{x}{2}\right) + (x^2+4)^{-1} + C$$

$$\boxed{\frac{3}{2}\arctan\left(\frac{x}{2}\right) + \frac{1}{(x^2+4)} + C}$$

2) $$\int \frac{x^2 - 2x - 2}{x^3 - 1}\,dx$$

Factor the denominator, then decompose into partial fractions

$$\frac{x^2 - 2x - 2}{(x-1)(x^2+x+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+x+1)}$$

Solve the rational equation with common denominators...

$$\frac{x^2 - 2x - 2}{(x-1)(x^2+x+1)} = \frac{A\,(x^2+x+1)}{(x-1)\,(x^2+x+1)} + \frac{(Bx+C)\,(x-1)}{(x^2+x+1)\,(x-1)}$$

$$x^2 - 2x - 2 = A\,(x^2+x+1) + (Bx+C)(x-1)$$

$$x^2 - 2x - 2 = Ax^2 + Ax + A + Bx^2 - Bx + Cx - C$$

$x^2 = (A+B)x^2$

$-2x = (A-B+C)x$

$-2 = (A - C)$

Solve the system:

$A + B = 1$

$A - B + C = -2$

$A - C = -2$

$2A - B = -4$

$3A = -3$

$A = -1$

$B = 2$

$C = 1$

$$\frac{x^2 - 2x - 2}{(x-1)(x^2+x+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+x+1)}$$

$$= \frac{-1}{(x-1)} + \frac{2x+1}{(x^2+x+1)}$$

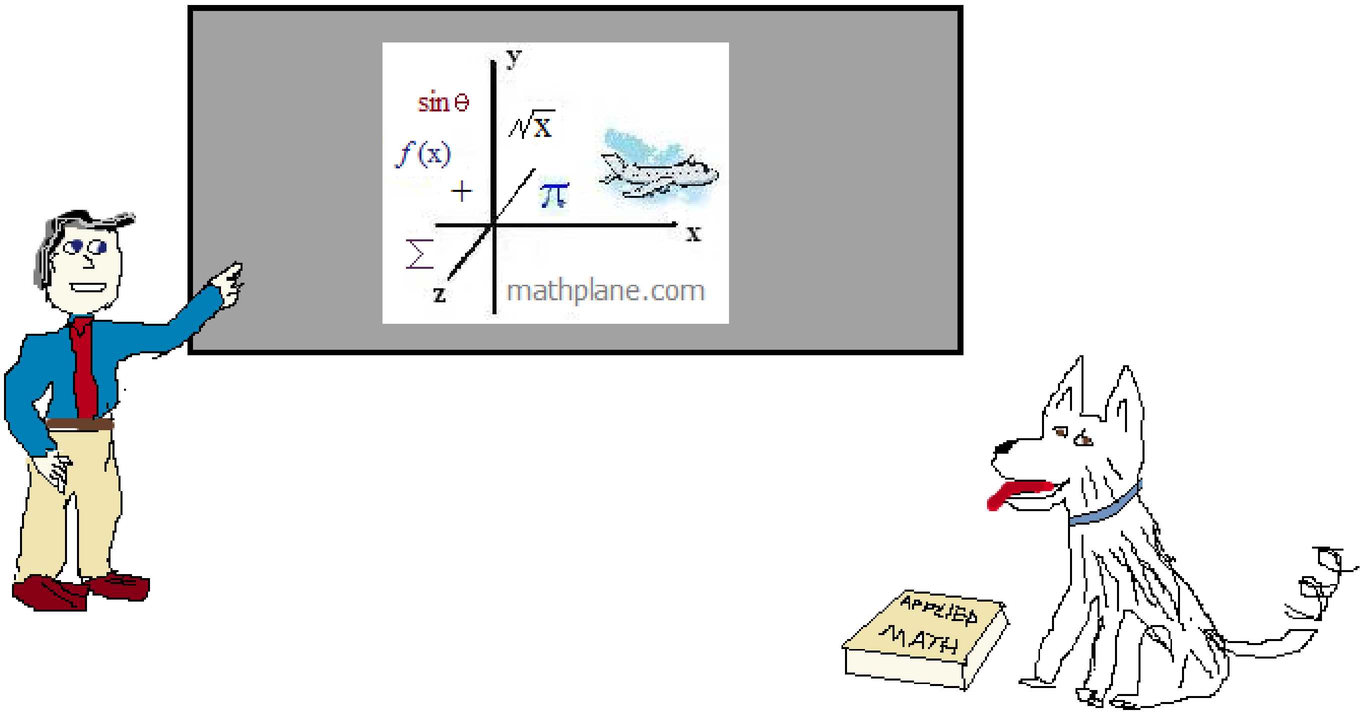
$$\int \frac{x^2 - 2x - 2}{x^3 - 1}\,dx = \int \frac{-1}{(x-1)}\,dx + \int \frac{2x+1}{(x^2+x+1)}\,dx$$

$$\boxed{-\ln|x-1| + \ln|x^2+x+1| + C}$$

Thanks for visiting. (Hope it helps!)

If you have questions, suggestions, or feedback, let us know.

Cheers.



Also, at TeacherPayTeachers and TES

And, Mathplane *Express* for mobile at mathplane.ORG

One more question:

$$\int \sin^5 x \bullet (\cos^{20} x) \; dx$$

Answer on next page-→

Trigonometry Integral Question

Steps and Answer

$$\int \sin^5 x \cdot (\cos^{20} x)\ dx$$

$$\int \sin x \cdot \sin^4 x \cdot (\cos^{20} x)\ dx$$

$$\int \sin x \cdot (\sin^2 x)^2 \cdot (\cos^{20} x)\ dx$$

$$\int \sin x \cdot (1 - \cos^2 x)^2 \cdot (\cos^{20} x)\ dx$$

$$\int \sin x \cdot (1 - 2\cos^2 x + \cos^4 x) \cdot (\cos^{20} x)\ dx$$

$$\int \sin x \cdot (\cos^{20} x - 2\cos^{22} x + \cos^{24} x)\ dx$$

$$\int \sin x(\cos^{20} x) - 2\sin x(\cos^{22} x) + \sin x(\cos^{24} x)\ dx$$

$$\int \sin x(\cos^{20} x)\ dx + \int -2\sin x(\cos^{22} x)\ dx + \int \sin x(\cos^{24} x)\ dx$$

$$-\int -\sin x(\cos^{20} x)\ dx + \int -2\sin x(\cos^{22} x)\ dx + -\int -\sin x(\cos^{24} x)\ dx$$

$$-\left[\frac{1}{21}\cos^{21} x\right] + \frac{2}{23}\cos^{23} x + -\left[\frac{1}{25}\cos^{25} x\right]$$

$$-\frac{1}{21}\cos^{21} x + \frac{2}{23}\cos^{23} x - \frac{1}{25}\cos^{25} x + C$$