Trigonometry: Law of Sines and Cosines III

Notes, Examples, and practice questions (with solutions)

Example: $\overline{AC} = 18$

$\overline{BC} = 20$

$\angle B = 38^{\circ}$

Find the other angles and sides...

We are given Side-Side-Angle and an acute angle... So, this is an ambiguous case (law of sines)
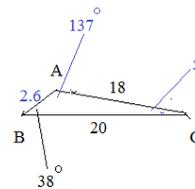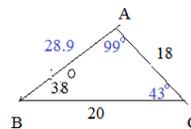
Two solutions (ambiguous case)

If $\overline{AB}$ is largest side $\implies$ 43° 38° 99° 20 - 18 - 28.9   obtuse triangle
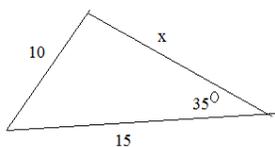
$$\frac{18}{\sin 38} = \frac{20}{\sin(A)}$$ angle A = 43° approximately

If $\overline{AB}$ is smallest side $\implies$ 137° 38° 5° 20 - 18 - 2.6   obtuse triangle

angle A = 43° $\implies$ supplement is 137°

Example: Find X:

Method 1: using law of cosines

$$10^2 = x^2 + 15^2 - 2(x)(15)\cos 35$$

$$0 = x^2 + 125 - (30x)\cos 35$$

(quadratic)

$\implies$ x = 7.19   or   17.38

Method 2: using law of sines (and ambiguous case)

$$\frac{10}{\sin 35} = \frac{15}{\sin A}$$ angle A = 59 degrees (approximately)

(and, supplement is 121 degrees)

Triangle 1: 59° 35° 86° sides are 15 - 10 - 17.4

Triangle 2: 121° 35° 24° sides are 15 - 10 - 7.2

Example: Find the measures of the 3 angles.
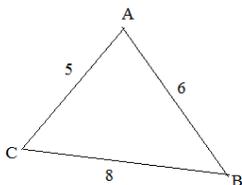
Using Law of Cosines

$$8^2 = 5^2 + 6^2 - 2(5)(6)\cos A$$
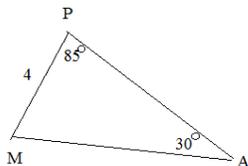
$$3 = -60\cos A$$

A = 92.87°

$$6^2 = 5^2 + 8^2 + 2(5)(8)\cos C$$

$$-53 = -80\cos C$$

C = 48.5°

180 - 92.87 - 48.5 = 38.63° is angle B

Example: Solve the triangle.

Angle M is 65 degrees...

(sum of angles is 180)

Use law of sines

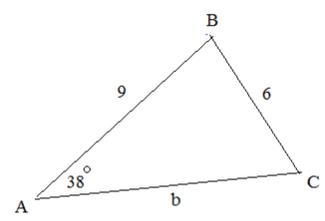$$\frac{\sin(30)}{4} = \frac{\sin(85)}{p}$$

$$\frac{1}{2} p = 4\sin(85)$$

p = 7.97

$$\frac{\sin(30)}{4} = \frac{\sin(65)}{m}$$

$$\frac{1}{2} m = 4\sin(65)$$

m = 7.25

Example: Solve the triangle (finding all angles and sides) <u>using law of cosines</u> (only)

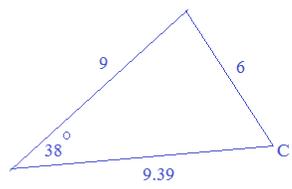$$6^2 = 9^2 + b^2 - 2(9)(b)(\cos 38)$$

$$36 = 81 + b^2 - 14.1842b$$

$$b^2 - 14.1842b + 45 = 0$$

quadratic formula; 2 solutions

$$b = 4.79 \text{ or } 9.39$$

Note: This is "angle-side-side", so it's the
'ambiguous case'...

If b = 9.39,
it's an acute triangle

If b = 4.79,
it's an obtuse triangle
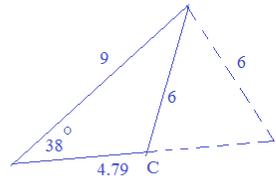

$$9^2 = 9.39^2 + 6^2 - 2(9.39)(6)(\cos C)$$

$$81 = 124.172 - 112.68(\cos C)$$

$$\cos C = .383138$$

Angle C = 67.47 degrees....

and, Angle B = 180 - 67.47 - 38 = 74.53°

supplements!

$$9^2 = 4.79^2 + 6^2 - 2(4.79)(6)(\cos C)$$

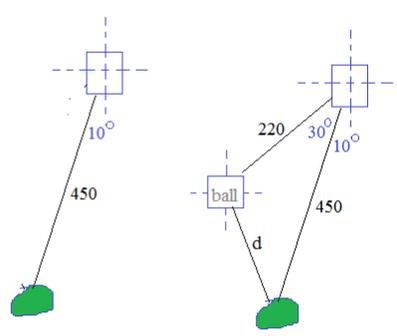$$81 = 58.944 - 57.48(\cos C)$$

$$\cos C = -.3837$$

Angle C = 112.53 degrees
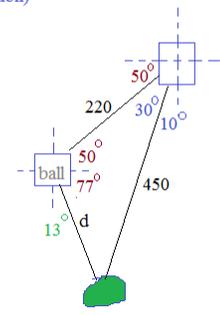
and, Angle B = 180 - 112.53 - 38 = 29.47°

Example: A golf green lies 450 yards S10W from the tee.
A golfer slices his shot, driving the ball 220 yards S40W.

How far and in what direction should the 2nd shot be?

Using law of cosines: $d^2 = 220^2 + 450^2 - 2(220)(450)\cos 30$
(to get distance)

$$d = 281.83$$

Using law of sines:
(to get direction)

$$\frac{\sin(\text{angle/ball})}{450} = \frac{\sin(30)}{281.83}$$
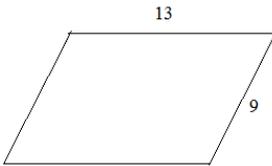
angle/ball = 53 degrees
or 127 degrees

"ambiguous case",
but using the diagram

the angle is 127 degrees!

2nd shot direction: S13E

1) The parallelogram has side lengths 9 and 13.
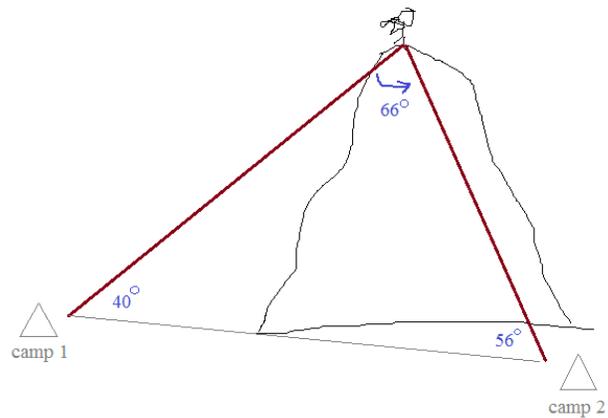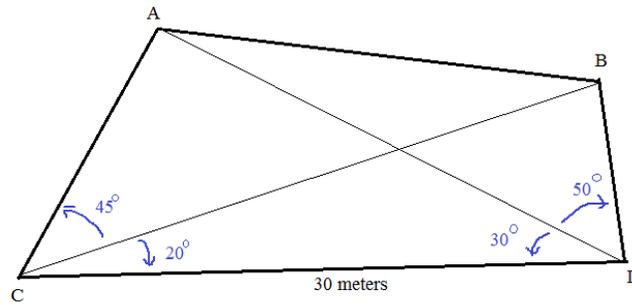
If one diagonal is 16, what is the other diagonal?

13

9

2) A surveyor stands atop a 2400 foot mountain...
He can look down at an angle of depression of 40 degrees and see Base Camp 1.
If the surveyor swivels 66 degrees to his left,
he can look down at an angle of depression of 56 degrees and see Base Camp 2.

What is the approximate distance between Base Camps 1 and 2?

66°

40°

56°

camp 1

camp 2

3) Can you find the length of $\overline{AB}$?

A

B

45°

50°

20°

30°

C

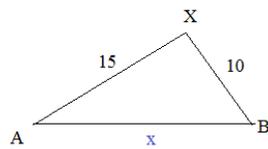30 meters

D

4) Given: Angle A = 40 degrees

      side a = 10
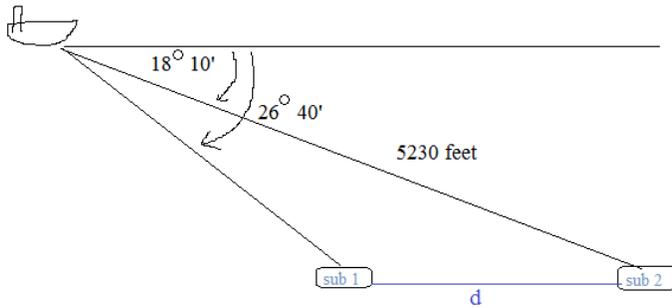
      side b = 15

X

15

10

In the following 'ambiguous case',
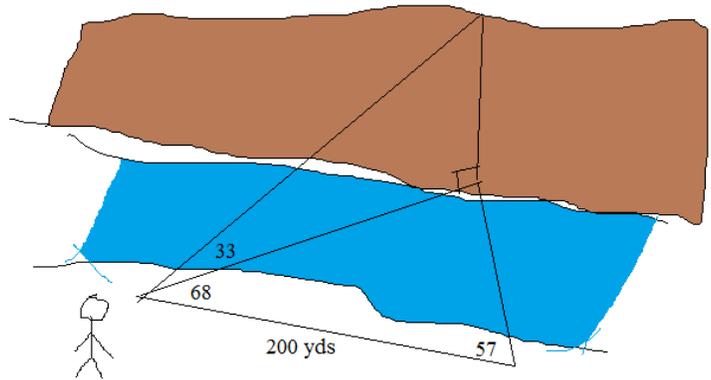find the length of x

A

x

B

   a) using law of sines

   b) using law of cosines

5) The diagram depicts a boat on the surface of the sea,
   and 2 submarines below the water level.
   What is the distance between the submarines?

18° 10'
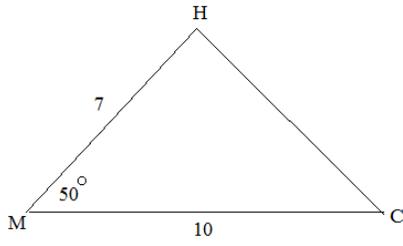
26° 40'

5230 feet

sub 1   sub 2

d

6) A surveyor is standing across a (blue) river, facing (brown) cliffs..
   Taking 2 measurements from spots 200 yards apart, he finds
   the measures are 68 degrees and 57 degrees to a specific point
   across the river.
   If the angle of elevation from the surveyor to the top of the
   cliffs is 33 degrees, then what is the approximate height of the
   cliffs?

33

68

200 yds   57

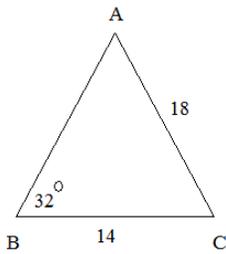7) Using Law of Sines ONLY, find ∠H, ∠C, and $\overline{HC}$

```
        H
       /\
      /  \
   7 /    \
    / 50°   \
   /_____\
  M     10     C
```

8) Find the missing angles and sides of △ABC using

a) Law of Cosines (only)

b) Law of Sines   (only)

```
        A
       /\
      /  \
     /    \ 18
    / 32°   \
   /_____\
  B    14      C
```

SOLUTIONS-→

1) The parallelogram has side lengths 9 and 13.

   If one diagonal is 16, what is the other diagonal?

13

A

9   16

9

Note: we know that consecutive angles in a
parallelogram are supplementary...

13

use law of sines/cosines to find
angles...

$16^2 = 9^2 + 13^2 - 2(9)(13)\cos(A)$

$256 = 81 + 169 - 234\cos(A)$

$A = 91.5°$

180 - A

88.5°

d

9

then, use law of cosines to find diagonal...

$d^2 = 9^2 + 13^2 - 2(9)(13)\cos(88.5)$
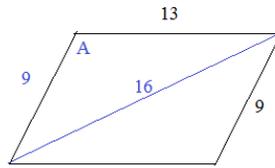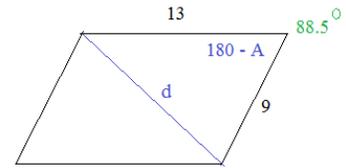
law of cosines

$c^2 = a^2 + b^2 - 2(a)(b)(\cos C)$

d = 15.6   (approx.)

2) A surveyor stands atop a 2400 foot mountain...
   He can look down at an angle of depression of 40 degrees and see Base Camp 1.
   If the surveyor swivels 66 degrees to his left,
   he can look down at an angle of depression of 56 degrees and see Base Camp 2.
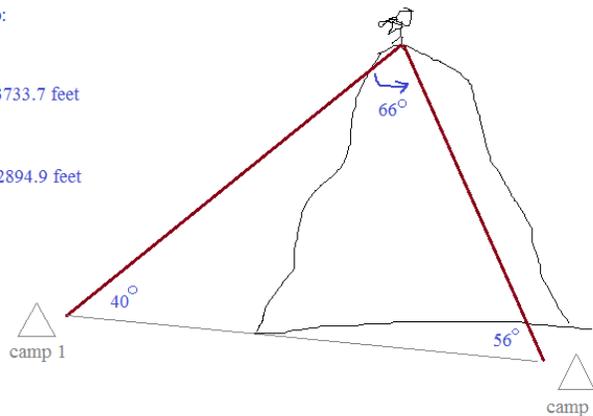
   What is the approximate distance between Base Camps 1 and 2?

   First, find the distances from the surveyor to each base camp:

   $\sin(40°) = \dfrac{2400 \text{ feet}}{\text{distance to camp 1}}$ distance is approx. 3733.7 feet

   $\sin(56°) = \dfrac{2400 \text{ feet}}{\text{distance to camp 1}}$ distance is approx. 2894.9 feet

   Then, using law of cosines, we can find
   the distance between the camps!

66°

40°
camp 1

56°
camp 2

$\left(\begin{array}{c}\text{distance} \\ \text{between camps}\end{array}\right)^2 = (3733.7)^2 + (2894.9)^2 - 2(3733.7)(2894.9)\cos(66°)$

distance between camps is approximately 3678 feet
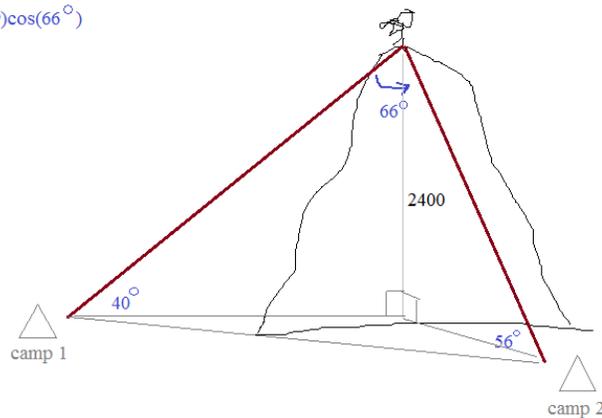
66°

2400

40°
camp 1

56°
camp 2

3) Can you find the length of $\overline{AB}$?

Step 1: Using Law of Sines and $\triangle ACD$

find $\overline{AD}$

$$\frac{\sin(85)}{30} = \frac{\sin(65)}{AD}$$

$\overline{AD} = 27.3$ approx.

Step 2: Using Law of Sines and $\triangle BDC$

find $\overline{BD}$

$$\frac{\sin(80)}{30} = \frac{\sin(20)}{BD}$$
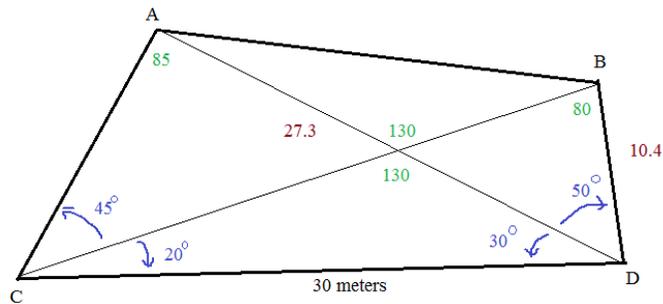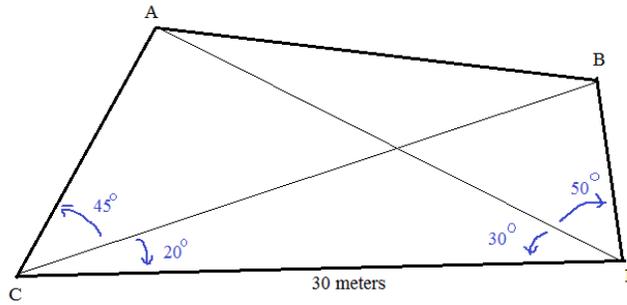
$\overline{BD} = 10.4$ approx.

Step 3: Using Law of Cosines and $\triangle ABD$

find $\overline{AB}$

$$\overline{AB}^2 = (10.4)^2 + (27.3)^2 - 2(10.4)(27.3)\cos(50^\circ)$$

$\overline{AB} = 22.1$ approximately

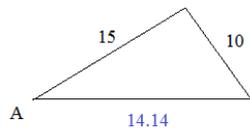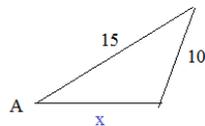4) Given: Angle A = 40 degrees

side a = 10

side b = 15

In the following 'ambiguous case', find the length of x

a) using law of sines

b) using law of cosines
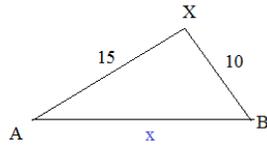
Method 1: Using law of sines...

$$\frac{\sin(40)}{10} = \frac{\sin(B)}{15}$$

$\sin(B) = .9641$

$B = 74.61^\circ$ or 105.39°

If $B = 74.61^\circ$,

$X = 65.39^\circ$

$$\frac{\sin(40)}{10} = \frac{\sin(65.39)}{x}$$

x = 14.14

If $B = 105.39^\circ$,

$X = 34.61^\circ$

$$\frac{\sin(40)}{10} = \frac{\sin(34.61)}{x}$$

x = 8.84

Method 2: Using law of cosines...

$$100 = 15^2 + x^2 - 2(15)(x)\cos(40)$$

$$-125 = x^2 - 30(x)(.766)$$

$$x^2 - 22.98x + 125 = 0$$

quadratic formula (calculator)

x = 8.84 or 14.14

5) The diagram depicts a boat on the surface of the sea, and 2 submarines below the water level. What is the distance between the submarines?
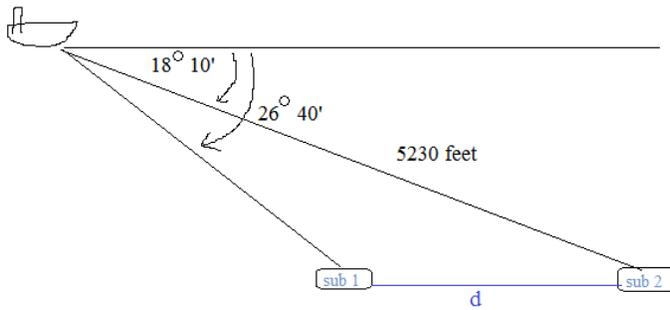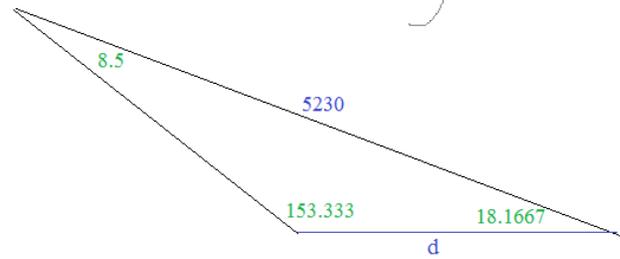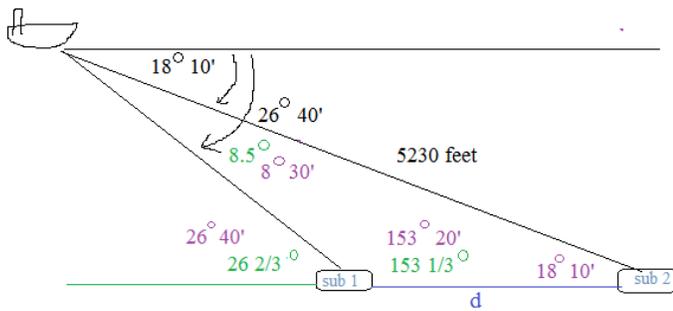
18° 10'

26° 40'

5230 feet

sub 1   d   sub 2

Using Law of Sines

$$\frac{\sin(8.5)}{d} = \frac{\sin(153.333)}{5230}$$

d = 1722 feet (approx.)

18° 10'

26° 40'

8.5°   8° 30'

26° 40'   153° 20'

26 2/3°   153 1/3°   18° 10'

sub 1   d   sub 2

8.5

5230

153.333   18.1667

d

6) A surveyor is standing across a (blue) river, facing (brown) cliffs..
Taking 2 measurements from spots 200 yards apart, he finds
the measures are 68 degrees and 57 degrees to a specific point
across the river.
If the angle of elevation from the surveyor to the top of the
cliffs is 33 degrees, then what is the approximate height of the cliffs?

Step 1: use law of sines to get the base of the triangle...
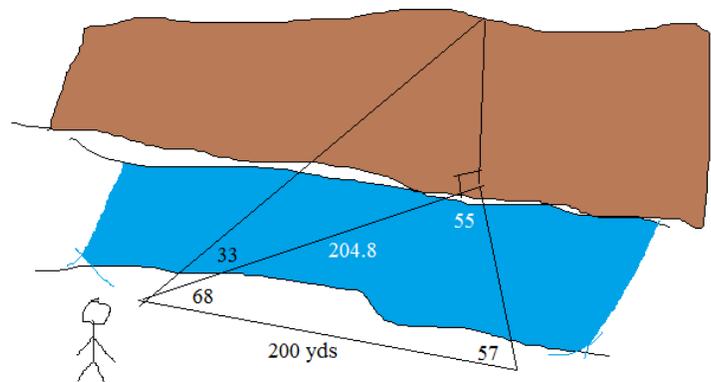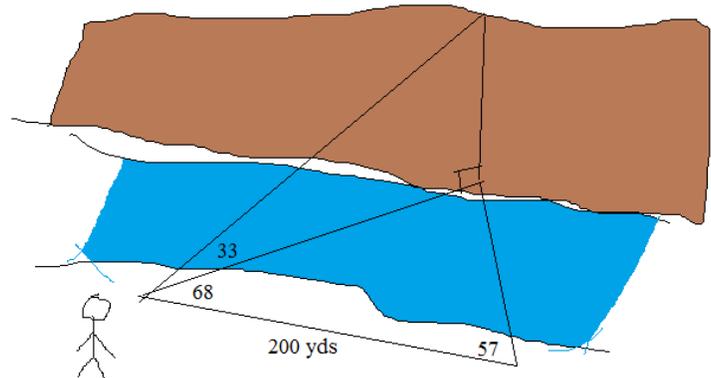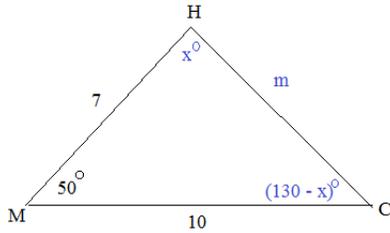
$$\frac{\sin(55)}{200} = \frac{\sin(57)}{d}$$

d = 204.8

Step 2: Using trig ratios, find the height of the cliffs..

$$\tan(33) = \frac{height}{204.8}$$

height = 133 yards (approx.)

33

68

200 yds   57

33   204.8   55

68

200 yds   57

7) Using Law of Sines ONLY, find $\angle$H, $\angle$C, and $\overline{HC}$

proportion 1 $\quad \dfrac{\sin(50)}{m} = \dfrac{\sin(x)}{10} \quad \Longrightarrow \quad m = \dfrac{10\sin(50)}{\sin(x)}$

proportion 2 $\quad \dfrac{\sin(50)}{m} = \dfrac{\sin(130 - x)}{7} \quad \Longrightarrow \quad m = \dfrac{7\sin(50)}{\sin(130 - x)}$

using substitution $\quad \dfrac{10\sin(50)}{\sin(x)} = \dfrac{7\sin(50)}{\sin(130 - x)}$

$\dfrac{7.66}{\sin(x)} = \dfrac{5.36}{\sin(130 - x)}$

$\dfrac{7.66}{\sin(x)} = \dfrac{5.36}{\sin130\cos x - \cos130\sin x} \qquad$ applying difference formula

$\dfrac{7.66}{\sin(x)} = \dfrac{5.36}{.766\cos x + .643\sin x}$

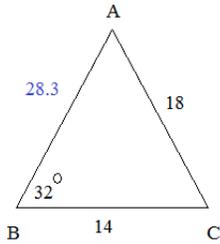cross multiply $\quad 5.36\sin(x) = 5.87\cos(x) + 4.93\sin(x)$

$.43\sin(x) = 5.87\cos(x)$

trig quotient identity $\quad \tan(x) = 13.65$

$x = 85.8^{\circ}$

If we had used law of cosines:

$m^2 = 7^2 + 10^2 - 2(7)(10)\cos(50^{\circ})$

$m^2 = 149 - 140(.6428) \qquad m = 7.68$ ✓

$\angle H = 85.8$

$\angle C = 44.2$

$m = 7.68$

8) Find the missing angles and sides of $\triangle$ABC using

    a) Law of Cosines (only)

    b) Law of Sines (only)

a) Using Law of Cosines

$18^2 = c^2 + 14^2 - 2(14)(c)\text{Cos}(32^{\circ})$

$128 = c^2 - 23.75c$

$c^2 - 23.75c - 128 = 0 \qquad$ quadratic equation

$c = -4.5 \text{ or } 28.28 \qquad$ since sides much be greater than zero, side c is 28.28

$28.28^2 = 14^2 + 18^2 - 2(14)(18)\text{Cos}(C)$

$279.76 = -504\text{Cos}(C)$

$C = 123.7^{\circ}$

$14^2 = 28.28^2 + 18^2 - 2(28.28)(18)\text{Cos}(A)$

$-927.758 = -1018.08\text{Cos}(A)$

$A = 24.3^{\circ}$

b) Using Law of Sines $\qquad$ Side - Side - Angle (ambiguous case of sines)

$\dfrac{\sin(32)}{18} = \dfrac{\sin(A)}{14}$

$\sin(A) = \dfrac{14\sin(32)}{18}$

$A = 24.34^{\circ}$

since this is an ambiguous case, we'll check the other possibility... Supplement of 24.34 is 155.66... However, since 155 + 32 > 180, there is only 1 triangle...

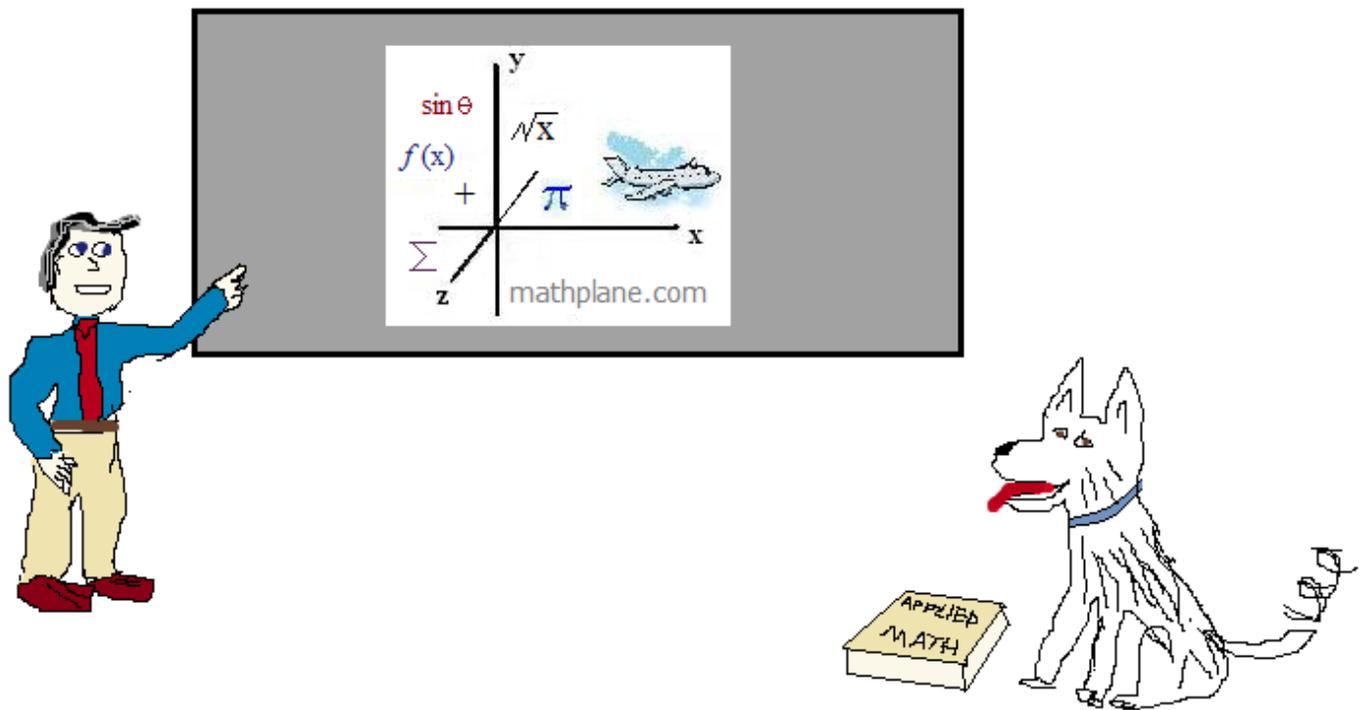then, we know angle C is 123.66°

$\dfrac{\sin(123.66)}{c} = \dfrac{\sin(32)}{18} \quad \Longrightarrow \quad$ and, the other side is 28.27

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers

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