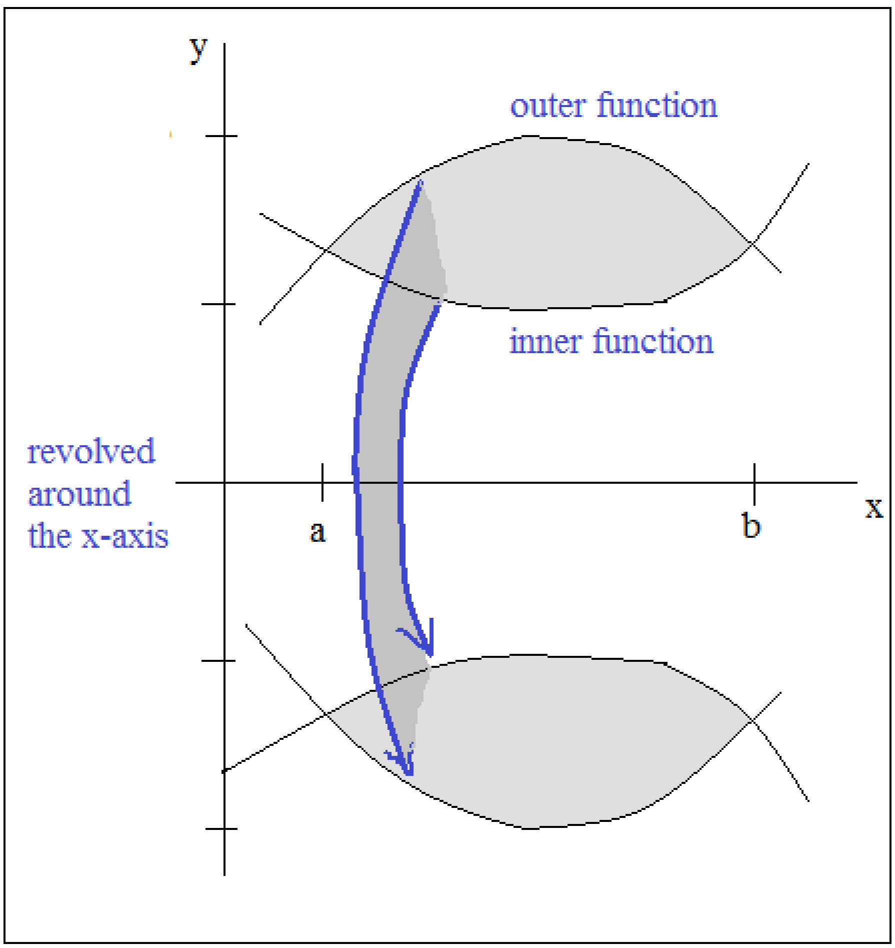
Calculus: Volume of Solids

Definite Integral Notes, Examples, and Formulas related to Disc/Washer and Shell/Cylinder Methods

Includes Practice Test (with solutions)



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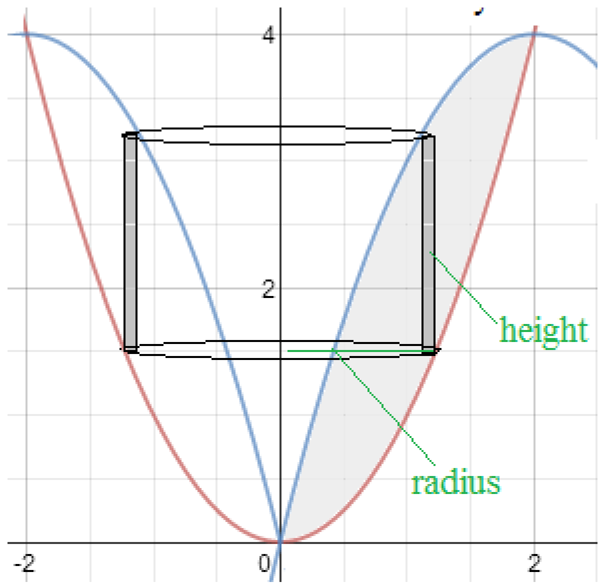
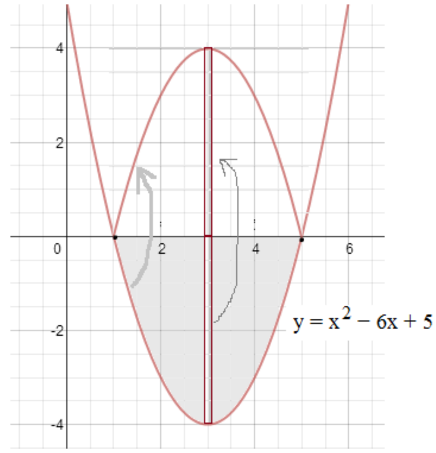


Shell (or, Cylinder) Method

Utilizing the Shell (Cylinder) Method

Example: Find the volume of a solid formed by revolving the area bounded by $y = x^2 - 6x + 5$ and $y = 0$ around the x-axis.



Disc Method

$$\text{Volume} = \int_a^b \pi\,(\text{function})^2\,dx$$

$$\text{Volume} = \int_1^5 \pi\,(x^2 - 6x + 5)^2\,dx$$

$$(x^2 - 6x + 5)(x^2 - 6x + 5)$$

$$\begin{array}{r} x^4 - 6x^3 + 5x^2 \qquad\qquad \\ -6x^3 + 36x^2 - 30x \qquad \\ 5x^2 - 30x + 25 \\ \hline x^4 - 12x^3 + 46x^2 - 60x + 25 \end{array}$$

$$\text{Volume} = \pi \int_1^5 x^4 - 12x^3 + 46x^2 - 60x + 25\,dx$$

$$\pi\left(\frac{x^5}{5} - 3x^4 + \frac{46x^3}{3} - 30x^2 + 25x\,\Bigg|_1^5\right)$$

$$\boxed{\frac{512}{15}\pi}$$

Now, suppose we want to find the volume of a solid formed from the same area revolved around the y-axis.

If we use the disc method, we need to express the equations in terms of y.

What is the inverse of $y = x^2 - 6x + 5$?

This is difficult to figure out..

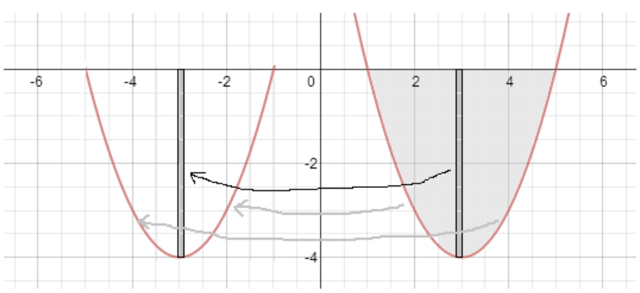
However, there is another approach to finding the volume of a solid. the "shell method".

It takes partitions that are parallel to the axis of rotation.

> When a partition is revolved around the axis, it creates a cylinder ('shell').

Shell (Cylinder) Method

$$\text{Volume} = \int_a^b 2\pi\,(\text{radius})(\text{height})\,dx$$

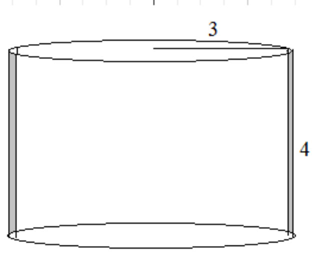


$$\text{Volume} = 2\pi \int_1^5 (x)(x^2 - 6x + 5)\,dx$$

$$\text{Volume} = 2\pi \int_1^5 x^3 - 6x^2 + 5x\,dx$$

$$2\pi \cdot \left(\frac{x^4}{4} - 2x^3 + \frac{5x^2}{2}\,\Bigg|_1^5\right)$$

$$2\pi \cdot \left(\frac{624}{4} - 248 + \frac{120}{2}\right) = -64\pi$$



The middle partition revolved around the y-axis forms a cylinder

This lateral area is 24π

(the sum of lateral areas from all the cylinders is the volume of the solid)

since it's area under the x-axis, the value was negative...

But, area cannot be negative...

$$\boxed{64\pi}$$

Volume of Solids: Shell vs. Disc Method

Find the volume of a solid formed by revolving the area bounded by

$y = \sqrt{x}$ $\quad y = 0$ and $x = 9$

around the x-axis

Disc Method

$$\text{Volume} = \int_a^b \pi(\text{function})^2 \, dx$$

$$\text{Volume} = \int_0^9 \pi(\sqrt{x})^2 \, dx$$

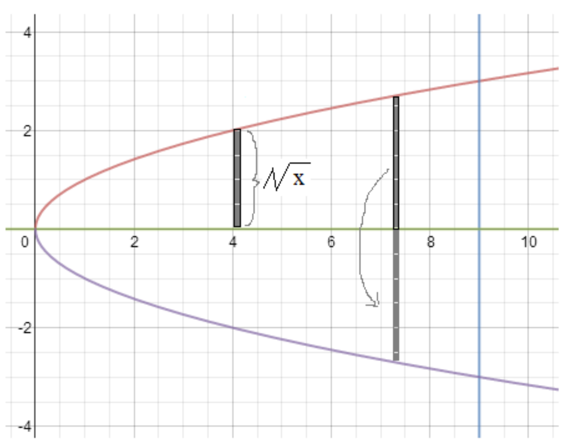
$$\text{Volume} = \pi \int_0^9 x \, dx$$

$$\pi \cdot \frac{x^2}{2}\Bigg|_0^9 = \frac{81}{2}\pi$$

Partitions are perpendicular to the x-axis

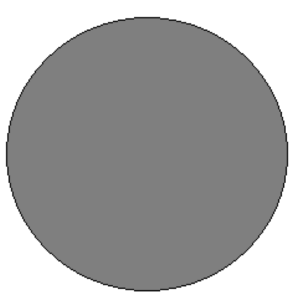
Each partition is a disc!

(the radius of each disc is the function)



front view of partition

side view of partition



Lateral Area of cylinder:

LA = 2π(radius)(height)

Shell Method

$$\text{Volume} = \int_a^b 2\pi(\text{radius})(\text{height}) \, dy$$

$$y = \sqrt{x} \longrightarrow x = y^2$$

$$\text{Volume} = \int_0^3 2\pi(y)(y^2) \, dy$$

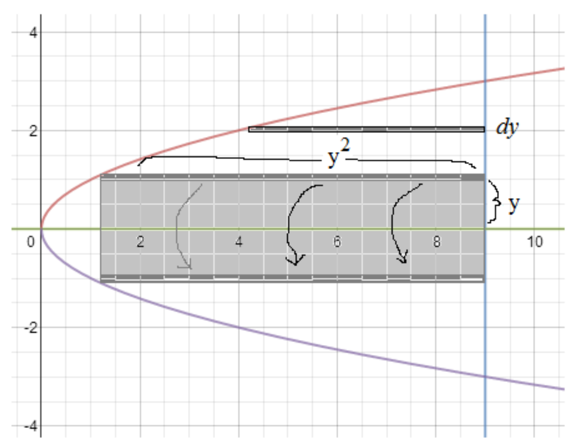
$$\text{Volume} = 2\pi \int_0^3 (y^3) \, dy$$

$$2\pi \cdot \frac{y^4}{4}\Bigg|_0^3 = \frac{81}{2}\pi$$

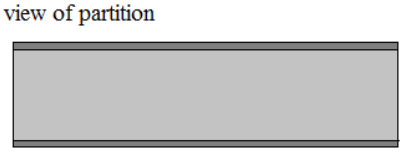
Partitions are parallel to the x-axis

Each partition is a cylinder!

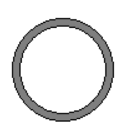
(the height of each cylinder is the function, and the radius of each cylinder is y)



front view of partition



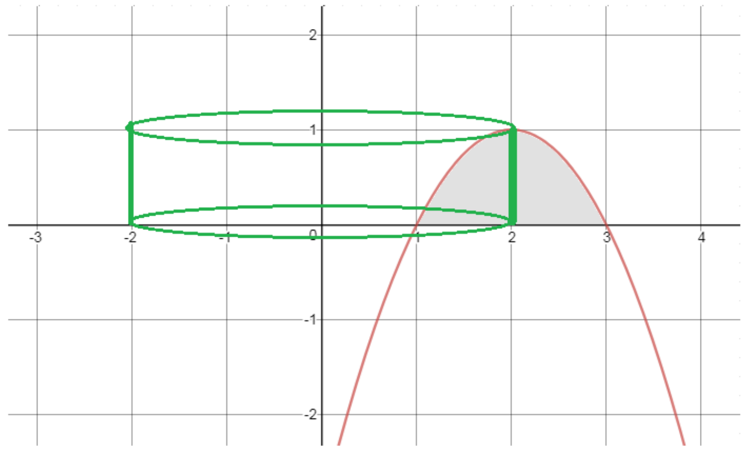
side view of partition
(hollow cylinder)



Each cylinder is a 'shell'.
When all the different sized shells are added together, they form the solid.

(Depending on the function, orientation, and/or rotation, one method may be easier than the other....)

Example: What is the volume of the solid from the region between the x-axis and $y = -x^2 + 4x - 3$ revolved around the y-axis?



Method 1: Apply the "cylinder method" (or "shell method")

Note: each partition is a cylinder with

radius: x

height: $-x^2 + 4x - 3$

formula for surface area of cylinder:

$SA = 2\pi\,(\text{radius})\,(\text{height})$

We'll construct an definite integral that represents cylinder partitions from x = 1 to 3

$$\int_1^3 2\pi x\,(-x^2 + 4x - 3)\,dx$$

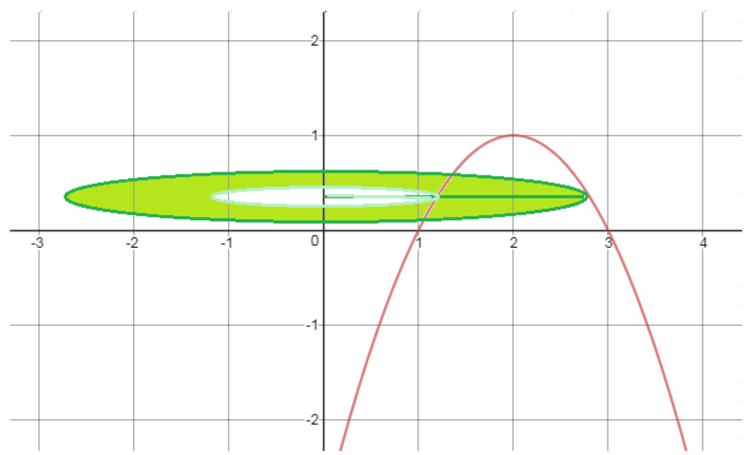
(radius) (height)

$$2\pi \int_1^3 -x^3 + 4x^2 - 3x\;dx \Longrightarrow 2\pi \left[\frac{-x^4}{4} + \frac{4x^3}{3} - \frac{3x^2}{2}\right]_1^3 =$$

$$2\pi\left(\frac{-81}{4} + 36 - \frac{27}{2} + \frac{1}{4} - \frac{4}{3} + \frac{3}{2}\right) = \boxed{\frac{16}{3}\pi}$$

Method 2: Using "Disk Method" (and horizontal partitions)

Since we're using horizontal partitions, we need to solve for x....



$y = -x^2 + 4x - 3$ solve for x:

$y + 3 = -x^2 + 4x$

$y + 3 = -1(x^2 - 4x \quad)$

(complete the square)

$-4 + y + 3 = -1(x^2 - 4x + 4)$

$y - 1 = -1(x - 2)^2$

$1 - y = (x - 2)^2$

$\pm\sqrt{1 - y} = (x - 2)$

$x = 2 \pm \sqrt{1 - y}$

right half: $x = 2 + \sqrt{1 - y}$

left half: $x = 2 - \sqrt{1 - y}$

Note: each partition is a "washer"

Outer disk: radius extends from y-axis to right half of curve

Inner disk (that is cut out): radius extends from y-axis to left half of curve

We'll construct definite integrals that represent the horizontal disk partitions extending from y = 0 to y = 1

$$\int_0^1 \pi\left(2 + \sqrt{1 - y}\right)^2 dy - \int_0^1 \pi\left(2 - \sqrt{1 - y}\right)^2 dy$$

(radius of 'outer' disk) (radius of 'inner' disk that is being hollowed out)

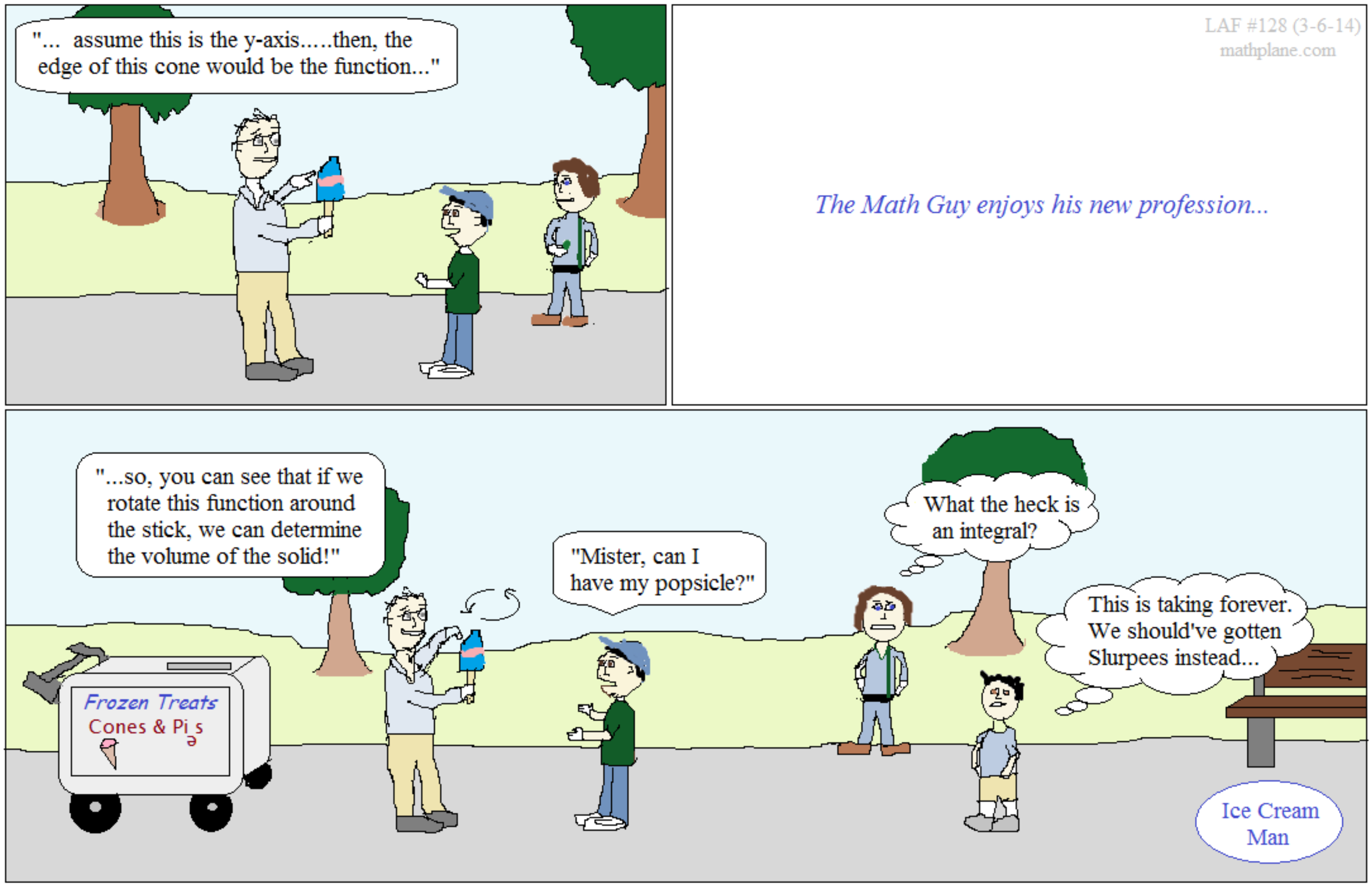
$$\pi\int_0^1 4 + 4\sqrt{1 - y} + (1 - y)\,dy - \pi\int_0^1 4 - 4\sqrt{1 - y} + (1 - y)\,dy$$

$$\pi\int_0^1 5 + 4\sqrt{1 - y} - y\,dy - \pi\int_0^1 5 - 4\sqrt{1 - y} - y\,dy$$

$$\frac{43}{6}\pi - \frac{11}{6}\pi = \boxed{\frac{16}{3}\pi}$$

$$\pi\int_0^1 8\sqrt{1 - y}\,dy$$

$$8\pi \left[\frac{-(1 - y)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^1 = \boxed{\frac{16}{3}\pi}$$



Practice Test→

Volume of Solids Quiz: Disc and Shell Methods

1) Find the volume of a solid with area bounded by $y = x^2$ and $y = 4x - x^2$, and

a) revolved around the y-axis

b) revolved around $x = 4$

2) Find the volume of a solid with area bounded by

$y = x^3$ $y = 0$ and $x = 2$

and revolved around:

a) x-axis

b) y-axis

c) the line $x = 4$

d) the line $y = 8$

3) The region R is bounded by $y = \sqrt{x}$

$x = 0$

$y = 2$

Find the volume of the region R rotated around x = 4,

a) using the disk method

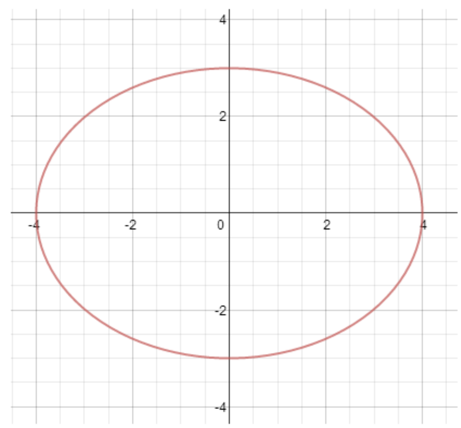
b) using the shell method

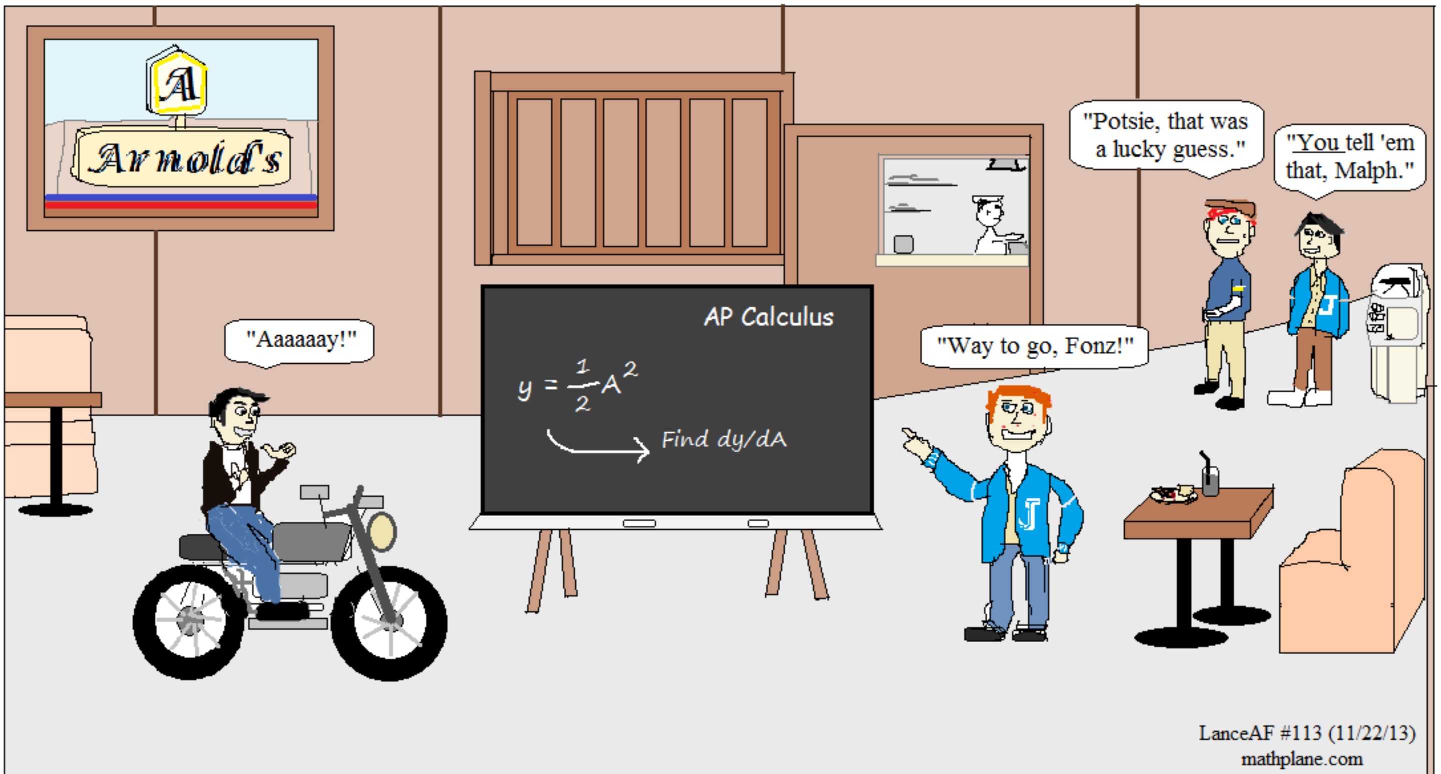
4) Find the volume of the solid generated by revolving the ellipse $9x^2 + 16y^2 = 144$

a) around its major axis

b) around its minor axis

Use disk and shell methods...





Happy Days

Despite Richie's help, Fonzie dropped out of Calculus.
(... although he did have some success with velocity and acceleration!)

Solutions-→

Volume of Solids Quiz: Disc and Shell Methods

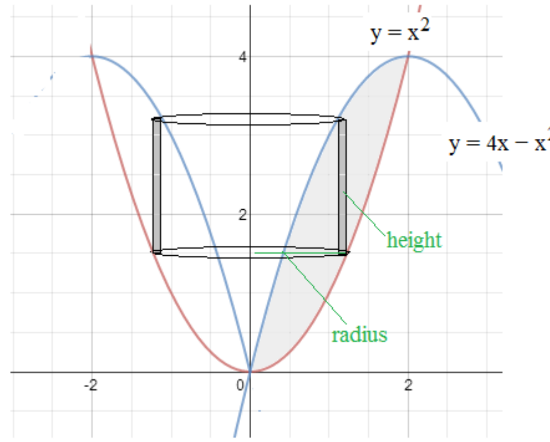
SOLUTIONS

1) Find the volume of a solid with area bounded by $y = x^2$ and $y = 4x - x^2$, and

a) revolved around the y-axis

b) revolved around $x = 4$

Since it is difficult to put the second equation in terms of y, we'll utilize the 'shell method' (and use parallel partitions)



a) $$\text{Volume} = \int_a^b 2\pi\,(\text{radius})(\text{height})\,dx$$

$$\text{Volume} = \int_0^2 2\pi\,(x)\,((4x - x^2) - x^2)\,dx$$

$$\text{Volume} = 2\pi \int_0^2 4x^2 - 2x^3\,dx$$

$$= 2\pi\left(\frac{4x^3}{3} - \frac{x^4}{2}\right)\Bigg|_0^2 = 2\pi\left(\frac{32}{3} - 8\right) = \boxed{\frac{16\pi}{3}}$$

Observe: Each cylinder (shell) has radius x and height $(4x - x^2) - x^2$

and, 2π(radius)(height) is the surface area of each cylinder!

b) the 'radius': $(4 - x)$

the 'height': $(4x - x^2) - x^2$

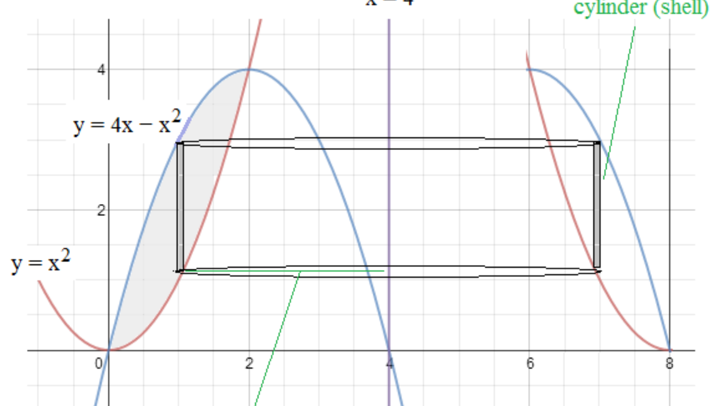
$$\text{Volume} = 2\pi \int_0^2 (4 - x)(4x - 2x^2)\,dx$$

radius & height of each shell/cylinder

$$\text{Volume} = 2\pi \int_0^2 16x - 8x^2 - 4x^2 + 2x^3\,dx$$

$$= 2\pi\left(8x^2 - 4x^3 + \frac{x^4}{2}\right)\Bigg|_0^2$$

$$2\pi\,(32 + 32 + 8 - 0 + 0 - 0) = \boxed{16\pi}$$



Lateral Area of Cylinder is 2π(radius)(height)

Volume of Solids Quiz: Disc and Shell Methods

2) Find the volume of a solid with area bounded by

$y = x^3 \qquad y = 0 \quad \text{and} \quad x = 2$

and revolved around:

a) x-axis

b) y-axis

c) the line x = 4

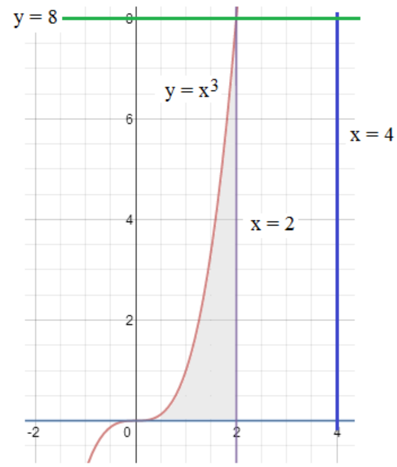
d) the line y = 8

a) Utilizing the disc method:
x-axis

$$\text{Volume} = \int_a^b \pi (\text{radius})^2 \, dx$$

$$\text{Volume} = \int_0^2 \pi \, (x^3)^2 \, dx$$

$$\pi \frac{x^7}{7} \Big|_0^2 = \boxed{\frac{128\pi}{7}}$$



b) Utilizing the shell method:
y-axis

$$\text{Volume} = \int_a^b 2\pi (\text{radius})(\text{height}) \, dx$$

$$\text{Volume} = 2\pi \int_0^2 (x)(x^3) \, dx$$

$$2\pi \frac{x^5}{5} \Big|_0^2 = \boxed{\frac{64\pi}{5}}$$

c) Using the shell (cylinder) method:
the line x = 4

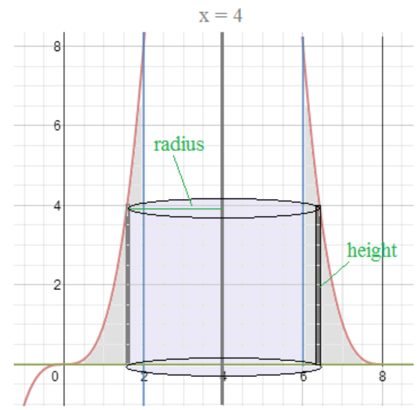
$$\text{Volume} = \int_0^2 2\pi (4 - x)(x^3) \, dx$$

radius height

$$\text{Volume} = 2\pi \int_0^2 4x^3 - x^4 \, dx$$

$$2\pi \left(x^4 - \frac{x^5}{5} \right) \Big|_0^2$$

$$= 32\pi - \frac{64\pi}{5} \qquad \boxed{\frac{96\pi}{5}}$$



d) Using the shell method:
the line y = 8

$$\text{Volume} = 2\pi \int_0^8 (8 - y)(2 - \sqrt[3]{y}) \, dy$$

$$\text{Volume} = 2\pi \int_0^8 16 - 8y^{\frac{1}{3}} - 2y + y^{\frac{4}{3}} \, dy$$

$$2\pi \left(16y - 6y^{\frac{4}{3}} - y^2 + \frac{3y^{\frac{7}{3}}}{7} \right) \Big|_0^8$$

$$2\pi \left(128 - 96 - 64 + \frac{384}{7}\right) = \boxed{\frac{320\pi}{7}}$$

d) Using the disk / washer method:

$$\text{Volume} = \int_0^2 \pi (8)^2 - \pi (8 - x^3)^2 \, dx$$

$$\text{Volume} = \pi \int_0^2 64 - 64 - 16x^3 + x^6 \, dx$$

$$\pi \left(-4x^4 + \frac{x^7}{7} \right) \Big|_0^2 = -64 + \frac{128}{7}$$

$$\boxed{-\frac{320}{7}\pi}$$

It's negative because
the region is below y = 8
Of course, area must be
positive...

SOLUTIONS

3) The region R is bounded by $y = \sqrt{x}$

$x = 0$

$y = 2$

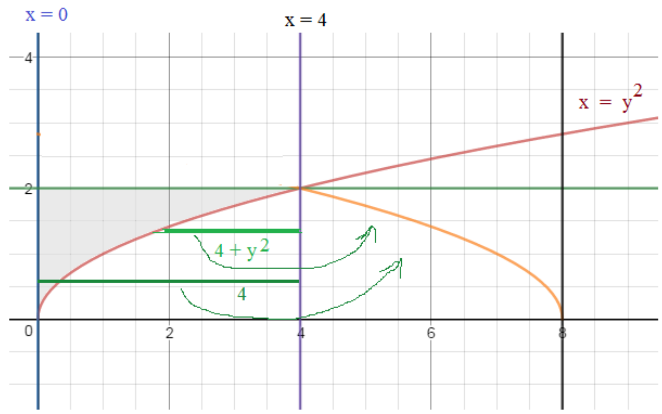
Find the volume of the region R rotated around x = 4,

a) using the disk method

b) using the shell method

DISK

$\pi(\text{radius})^2$ (area of a disc/circle)

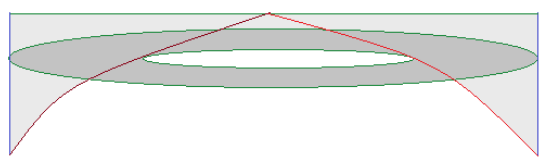


$$\pi\int_0^2 (4)^2\,dy \;-\; \pi\int_0^2 (4 - y^2)^2\,dy$$

radius of outside — radius of inside

$$\pi\, 16y\Big|_0^2 \;-\; \pi\left(16y - \frac{8y^3}{3} + \frac{y^5}{5}\right)\Big|_0^2$$

$$32\pi \;-\; \left(32 - \frac{64}{3} + \frac{32}{5}\right)\pi \;=\; \boxed{\frac{224\pi}{15}}$$



each partition is a flat disc with the center hallowed out...

SHELL

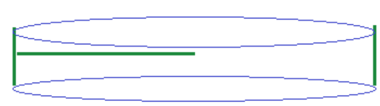
$2\pi(\text{radius})(\text{height})$ (lateral area of a cylinder)

$$\int_0^4 2\pi\,(4 - x)\,(2 - \sqrt{x})\,dx$$

radius height

$$2\pi\int_0^4 8 - 2x - 4\sqrt{x} + x^{\frac{3}{2}}\,dx$$

$$2\pi\left(8x - x^2 - \frac{8x^{\frac{3}{2}}}{3} + \frac{2x^{\frac{5}{2}}}{5}\right)\Big|_0^4$$



each partition is a cylinder shell

$$2\pi\left(32 - 16 - \frac{64}{3} + \frac{64}{5}\right) = 2\pi\left(16 - \frac{128}{15}\right) = \boxed{\frac{224\pi}{15}}$$

4) Find the volume of the solid generated by revolving the ellipse $9x^2 + 16y^2 = 144$ SOLUTIONS

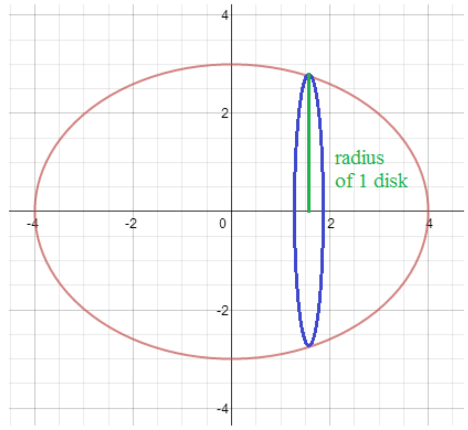
a) around its major axis

b) around its minor axis

Use disk and shell methods...

In standard form:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$



a). major axis is the x-axis

Since the ellipse (ellipsoid) is symmetric, we'll focus on the top 1/2 of the figure...

Then, using the disk method...

$$\frac{y^2}{9} = 1 - \frac{x^2}{16}$$

$$y^2 = 9 - \frac{9x^2}{16}$$

each radius is

$$y = \pm\sqrt{9 - \frac{9x^2}{16}}$$

The boundries are -4 to 4

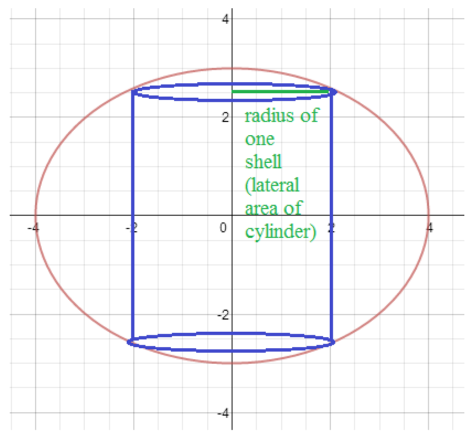
$$\int_{-4}^{4} \pi \left(\text{radius}\right)^2 dx$$

$$\int_{-4}^{4} \pi \left(9 - \frac{9x^2}{16}\right) dx = \pi\left(9x - \frac{3x^3}{16}\right)\Bigg|_{-4}^{4} = \pi\Big((36 - 12) - (-36 + 12)\Big) = \boxed{48\pi}$$

b) minor axis is the y-axis

Again, the ellipse (and ellipsoid) is symmetric over the axis, so we can work on half.

Then, using the shell method...



$$\frac{y^2}{9} = 1 - \frac{x^2}{16}$$

$$y^2 = 9 - \frac{9x^2}{16}$$

$$y = \pm\sqrt{9 - \frac{9x^2}{16}}$$

The cylinders/shells have vertical heights, and we'll focus on quadrant I...
So the boundaries will be 0 to 4..

$$\int_{0}^{4} 2\pi\,(\text{radius})(\text{height})$$

$$\int_{0}^{4} 2\pi\, x \sqrt{9 - \frac{9x^2}{16}}\, dx$$

Radii Height

Using a calculator ----> the integral is 32π

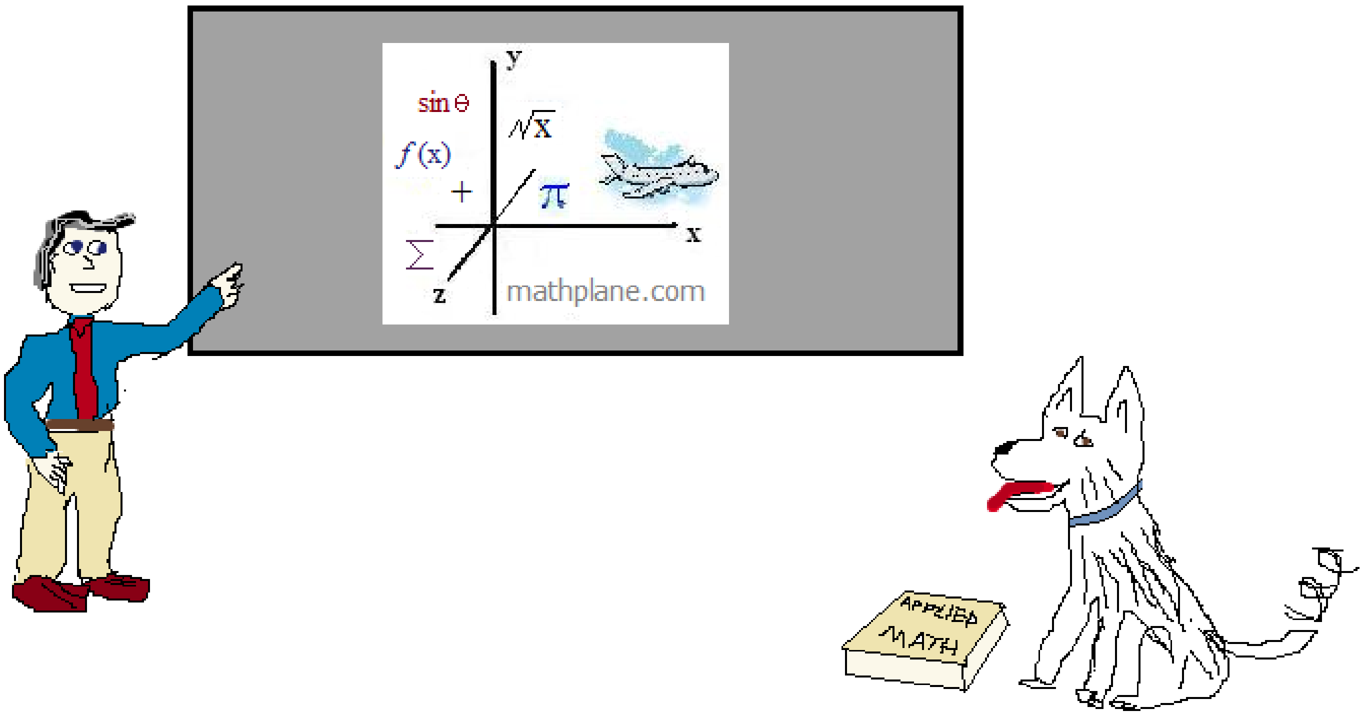
Finally, we double the answer to account for the bottom half of the ellipsoid!

$\boxed{64\pi}$

Thanks for visiting. (Hope it helped!)

If you have questions, suggestions, or requests, let us know.

Cheers



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