CHAPTER 13

Probability and Measurement

Then

You learned about experiments, outcomes, and events. You also found probabilities of simple events.

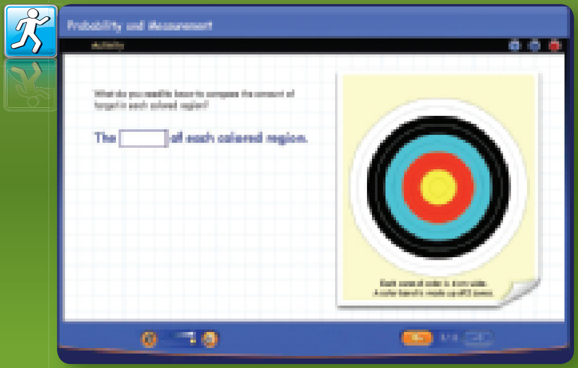
Now

In this chapter, you will:

- Represent sample spaces.
- Use permutations and combinations with probability.
- Find probabilities by using length and area.
- Find probabilities of compound events.

Why? ▲

GAMES Probability can be used to predict the likelihood of different outcomes of the games that we play.



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Vocabulary

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Foldables

Self-Check Practice

Worksheets

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Get Ready for the Chapter

Diagnose Readiness | You have two options for checking prerequisite skills.

1 **Textbook Option** Take the Quick Check below. Refer to the Quick Review for help.

QuickCheck

Simplify.

1. $\frac{1}{2} + \frac{3}{8}$

2. $\frac{7}{9} + \frac{2}{6}$

3. $\frac{2}{5} + \frac{7}{8}$

4. $\frac{2}{9} \cdot \frac{4}{8}$

5. $\frac{3}{7} \cdot \frac{21}{24}$

6. $\frac{3}{10} \cdot \frac{2}{9}$

7. **SOCCER** A soccer team brings a 4.5-gallon cooler of water to their games. How many 4-ounce cups can the team drink per game?

A die is rolled. Find the probability of each outcome.

8. *P*(greater than 1)

9. *P*(odd)

10. *P*(less than 2)

11. *P*(1 or 6)

12. **GAMES** Two friends are playing a game with a 20-sided die that has all of the letters of the alphabet except for Q, U, V, X, Y, and Z. What is the probability that the die will land on a vowel?

The table shows the results of an experiment in which a spinner numbered 1–4 was spun.

Outcome	Tally	Frequency
1	|||	3
2	𝍸 ||	7
3	𝍸 |	6
4	||||	4

13. What is the experimental probability that the spinner will land on a 4?

14. What is the experimental probability that the spinner will land on an odd number?

15. What is the experimental probability that the spinner will land on an even number?

QuickReview

Example 1

Simplify $\frac{6}{9} \cdot \frac{1}{2}$.

$\frac{6}{9} \cdot \frac{1}{2} = \frac{6 \cdot 1}{9 \cdot 2}$ Multiply the numerators and denominators.

$= \frac{6}{18}$ or $\frac{1}{3}$ Simplify.

Example 2

Suppose a die is rolled. What is the probability of rolling less than a five?

$P(\text{less than } 5) = \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}}$

$= \frac{4}{6}$ or $\frac{2}{3}$

The probability of rolling less than a five is $\frac{2}{3}$ or 67%.

Example 3

A spinner numbered 1–6 was spun. Find the experimental probability of landing on a 5.

Outcome	Tally	Frequency
1	||||	4
2	𝍸 ||	7
3	𝍸 |||	8
4	||||	4
5	||	2
6	𝍸	5

$P(5) = \frac{\text{number of times a 5 is spun}}{\text{total number of outcomes}}$ or $\frac{2}{30}$

The experimental probability of landing on a 5 is $\frac{2}{30}$ or 7%.

2 **Online Option** Take an online self-check Chapter Readiness Quiz at connectED.mcgraw-hill.com.

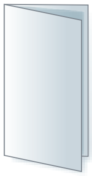
Get Started on the Chapter

You will learn several new concepts, skills, and vocabulary terms as you study Chapter 13. To get ready, identify important terms and organize your resources. You may wish to refer to Chapter 0 to review prerequisite skills.

FOLDABLES StudyOrganizer

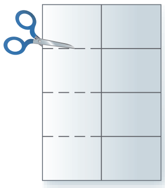
Probability and Measurement Make this Foldable to help you organize your Chapter 13 notes about probability. Begin with one sheet of paper.

1 **Fold** a sheet of paper lengthwise.

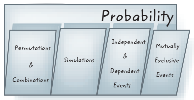


2 **Fold** in half two more times.

3 **Cut** along each fold on the left column.



4 **Label** as shown.



NewVocabulary

English		Español
sample space	p. 915	espacio muestral
tree diagram	p. 915	diagrama de árbol
permutation	p. 922	permutación
factorial	p. 922	factorial
circular permutation	p. 925	permutación circular
combination	p. 926	combinacion
geometric probability	p. 931	probabilidad geométrica
probability model	p. 939	modelo de la probabilidad
simulation	p. 939	simulacro
random variable	p. 941	variable aleatoria
expected value	p. 941	valor espenado
compound events	p. 947	eventos compuestos
independent events	p. 947	eventos independientes
dependent events	p. 947	eventos dependientes
conditional probability	p. 949	probabilidad condicional
probability tree	p. 949	árbol de la probabilidad
mutually exclusive	p. 956	mutuamente exclusivos
complement	p. 959	complemento

ReviewVocabulary

event evento one or more outcomes of an experiment

experiment experimento a situation involving chance such as flipping a coin or rolling a die

LESSON 13-1 Representing Sample Spaces

Then

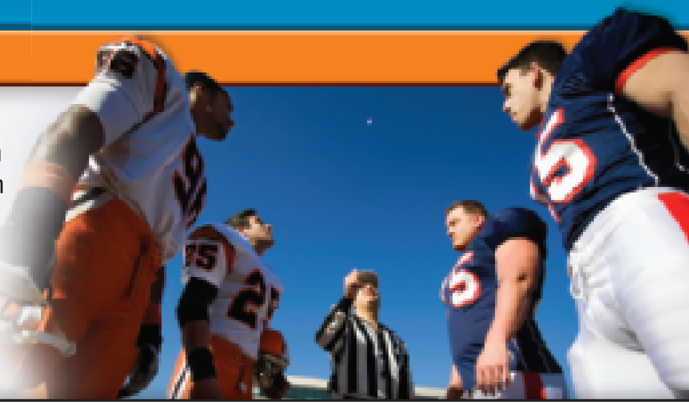
- You calculated experimental probability.

Now

- 1 Use lists, tables, and tree diagrams to represent sample spaces.
- 2 Use the Fundamental Counting Principle to count outcomes.

Why?

- In a football game, a referee tosses a fair coin to determine which team will take possession of the football first. The coin can land on heads or tails.



NewVocabulary

sample space
tree diagram
two-stage experiment
multi-stage experiment
Fundamental Counting Principle

Common Core State Standards

Content Standards

Preparation for S.CP.9 (+) Use permutations and combinations to compute probabilities of compound events and solve problems.

Mathematical Practices

1 Make sense of problems and persevere in solving them.
2 Reason abstractly and quantitatively.

1 Represent a Sample Space

You have learned the following about experiments, outcomes, and events.

Definition	Example
An *experiment* is a situation involving chance that leads to results called *outcomes*.	In the situation above, the experiment is tossing the coin.
An *outcome* is the result of a single performance or *trial* of an experiment.	The possible outcomes are landing on heads or tails.
An *event* is one or more outcomes of an experiment.	One event of this experiment is the coin landing on tails.

The **sample space** of an experiment is the set of all possible outcomes. You can represent a sample space by using an organized list, a table, or a **tree diagram**.

Example 1 Represent a Sample Space

A coin is tossed twice. Represent the sample space for this experiment by making an organized list, a table, and a tree diagram.

For each coin toss, there are two possible outcomes, heads H or tails T.

Organized List

Pair each possible outcome from the first toss with the possible outcomes from the second toss.

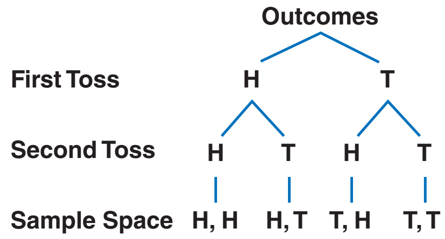
H, H
H, T
T, T
T, H

Table

List the outcomes of the first toss in the left column and those of the second toss in the top row.

Outcomes	Heads	Tails
Heads	H, H	H, T
Tails	T, H	T, T

Tree Diagram



GuidedPractice

1. A coin is tossed and then a number cube is rolled. Represent the sample space for this experiment by making an organized list, a table, and a tree diagram.

Robert Michael/CORBIS

The experiment in Example 1 is an example of a **two-stage experiment**, which is an experiment with two stages or events. Experiments with more than two stages are called **multi-stage experiments**.

Real-World Example 2 Multi-Stage Tree Diagrams

HAMBURGERS **To take a hamburger order, Keandra asks each customer the questions from the script shown. Draw a tree diagram to represent the sample space for hamburger orders.**

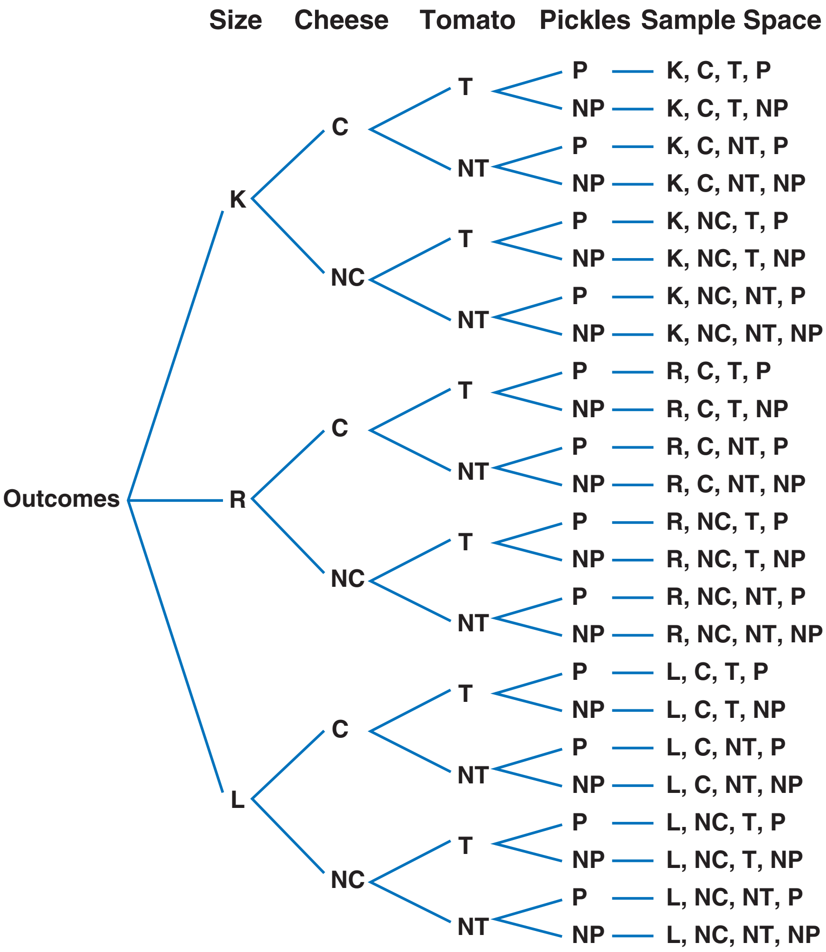
WatchOut!

Sense-Making The words *and/or* in the third question for Example 2 suggest an additional stage in the ordering process. By making separate stages for choosing with or without tomato and with or without pickles, you allow for the possibility of choosing *both* tomato and pickles.

The sample space is the result of four stages.

- Burger size (K, R, or L)
- Cheese (C or NC)
- Tomato (T or NT)
- Pickles (P or NP)

Draw a tree diagram with four stages.



ReadingMath

Tree Diagram Notation Choose notation for outcomes in your tree diagrams that will eliminate confusion. In Example 2, *C* stands for *cheese*, while *NC* stands for *no cheese*. Likewise, *NT* and *NP* stand for *no tomato* and *no pickles*, respectively.

GuidedPractice

2. **MUSIC** Yoki can choose a small MP3 player with a 4- or 8-gigabyte hard drive in black, teal, sage, or red. She can also get a clip and/or a dock to go with it. Make a tree diagram to represent the sample space for this situation.

2 Fundamental Counting Principle

For some two-stage or multi-stage experiments, listing the entire sample space may not be practical or necessary. To find the *number* of possible outcomes, you can use the **Fundamental Counting Principle**.

StudyTip

Multiplication Rule The Fundamental Counting Principle is sometimes called the *Multiplication Rule for Counting* or the *Counting Principle.*

KeyConcept Fundamental Counting Principle

Words The number of possible outcomes in a sample space can be found by multiplying the number of possible outcomes from each stage or event.

Symbols In a k-stage experiment, let

n_1 = the number of possible outcomes for the first stage.

n_2 = the number of possible outcomes for the second stage after the first stage has occurred.

$\vdots$

n_k = the number of possible outcomes for the kth stage after the first $k - 1$ stages have occurred.

Then the total possible outcomes of this k-stage experiment is

$$n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k.$$

Real-WorldLink

More than 95 percent of high school students order a traditional ring style, which includes the name of the school, a stone, and the graduation year.

Source: *Fort Worth Star-Telegram*

Real-World Example 3 Use the Fundamental Counting System

CLASS RINGS **Haley has selected a size and overall style for her class ring. Now she must choose from the ring options shown. How many different rings could Haley create in her chosen style and size?**

Ring Options	Number of Choices
metals	10
finishes	2
stone colors	12
stone cuts	5
side 1 activity logos	20
side 2 activity logos	20
band styles	2

Use the Fundamental Counting Principle.

metals		finishes		stone colors		stone cuts		side 1 logos		side 2 logos		band styles		possible outcomes
10	×	2	×	12	×	5	×	20	×	20	×	2	=	960,000

So, Haley could create 960,000 different rings.

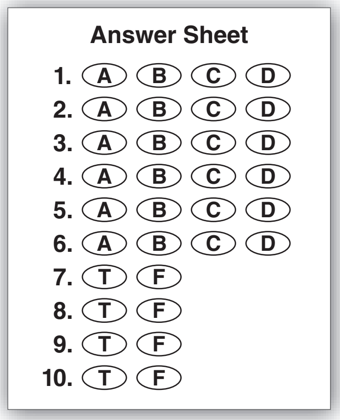
GuidedPractice

3. Find the number of possible outcomes for each situation.

A. The answer sheet shown is completed.

B. A die is rolled four times.

C. **SHOES** A pair of women's shoes comes in whole sizes 5 through 11 in red, navy, brown, or black. They can be leather or suede and are available in three different widths.



Answer Sheet

1. Ⓐ Ⓑ Ⓒ Ⓓ
2. Ⓐ Ⓑ Ⓒ Ⓓ
3. Ⓐ Ⓑ Ⓒ Ⓓ
4. Ⓐ Ⓑ Ⓒ Ⓓ
5. Ⓐ Ⓑ Ⓒ Ⓓ
6. Ⓐ Ⓑ Ⓒ Ⓓ
7. Ⓣ Ⓕ
8. Ⓣ Ⓕ
9. Ⓣ Ⓕ
10. Ⓣ Ⓕ

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Check Your Understanding

● = Step-by-Step Solutions begin on page R14.

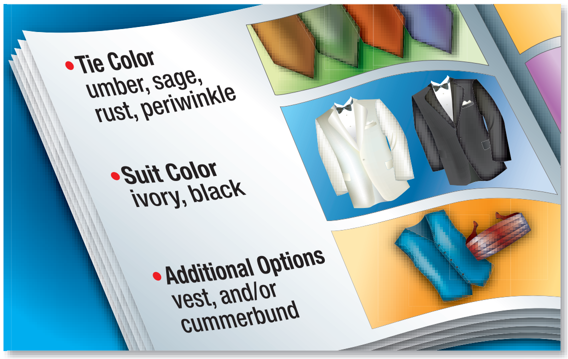
Example 1 **Represent the sample space for each experiment by making an organized list, a table, and a tree diagram.**

1. For each at bat, a player can either get on base or make an out. Suppose a player bats twice.

2. Quinton sold the most tickets in his school for the annual Autumn Festival. As a reward, he gets to choose twice from a grab bag with tickets that say "free juice" or "free notebook."

Example 2

3. **TUXEDOS** Patrick is renting a prom tuxedo from the catalog shown. Draw a tree diagram to represent the sample space for this situation.



Example 3 **Find the number of possible outcomes for each situation.**

4. Marcos is buying a cell phone and must choose a plan. Assume one of each is chosen.

Cell Phone Options	Number of Choices
phone style	15
minutes package	5
Internet access	3
text messaging	4
insurance	2

5. Desirée is creating a new menu for her restaurant. Assume one of each item is ordered.

Menu Titles	Number of Choices
appetizer	8
soup	4
salad	6
entree	12
dessert	9

Practice and Problem Solving

Extra Practice is on page R13.

Example 1 CCSS **REASONING** **Represent the sample space for each experiment by making an organized list, a table, and a tree diagram.**

6. Gina is a junior and has a choice for the next two years of either playing volleyball or basketball during the winter quarter.

7. Two different history classes in New York City are taking a trip to either the Smithsonian or the Museum of Natural History.

8. Simeon has an opportunity to travel abroad as a foreign exchange student during each of his last two years of college. He can choose between Ecuador or Italy.

9. A new club is formed, and a meeting time must be chosen. The possible meeting times are Monday or Thursday at 5:00 or 6:00 P.M.

10. An exam with multiple versions has exercises with triangles. In the first exercise, there is an obtuse triangle or an acute triangle. In the second exercise, there is an isosceles triangle or a scalene triangle.

11. **PAINTING** In an art class, students are working on two projects where they can use one of two different types of paints for each project. Represent the sample space for this experiment by making an organized list, a table, and a tree diagram.

Example 2 **Draw a tree diagram to represent the sample space for each situation.**

12. **BURRITOS** At a burrito stand, customers have the choice of beans, pork, or chicken with rice or no rice, and cheese and/or salsa.

13. **TRANSPORTATION** Blake is buying a vehicle and has a choice of sedan, truck, or van with leather or fabric interior, and a CD player and/or sunroof.

14. **TREATS** Ping and her friends go to a frozen yogurt parlor which has a sign like the one at the right. Draw a tree diagram for all possible combinations of cones with peanuts and/or sprinkles.


FROZEN YOGURT

Cones	Flavors
Cake Sugar Waffle	Strawberry Lime

Toppings: Peanuts and Sprinkles

Example 3 **CCSS PERSEVERANCE** **In Exercises 15–18, find the number of possible outcomes for each situation.**

15 In the Junior Student Council elections, there are 3 people running for secretary, 4 people running for treasurer, 5 people running for vice president, and 2 people running for class president.

16. When signing up for classes during his first semester of college, Frederico has 4 class spots to fill with a choice of 4 literature classes, 2 math classes, 6 history classes, and 3 film classes.

17. Niecy is choosing one each of 6 colleges, 5 majors, 2 minors, and 4 clubs.

18. Evita works at a restaurant where she has to wear a white blouse, black pants or skirt, and black shoes. She has 5 blouses, 4 pants, 3 skirts, and 6 pairs of black shoes.

19. **ART** For an art class assignment, Mr. Green gives students their choice of two quadrilaterals to use as a base. One must have sides of equal length, and the other must have at least one set of parallel sides. Represent the sample space by making an organized list, a table, and a tree diagram.

20. **BREAKFAST** A hotel restaurant serves omelets with a choice of vegetables, ham, or sausage that come with a side of hash browns, grits, or toast.

a. How many different outcomes of omelet and one side are there if a vegetable omelet comes with just one vegetable?

b. Find the number of possible outcomes for a vegetable omelet if you can get any or all vegetables on any omelet.


Omelets

All omelets served with your choice of hash browns, grits, or toast.

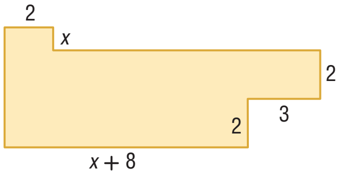
Vegetable Omelet

Ham Omelet

Sausage Omelet

Vegetable choices: green peppers, tomatoes, onions, mushrooms

21. **COMPOSITE FIGURES** Carlito is calculating the area of the composite figure at the right. List six different ways he can do this.



22. **TRANSPORTATION** Miranda got a new bicycle lock that has a four-number combination. Each number in the combination is from 0 to 9.

a. How many combinations are possible if there are no restrictions on the number of times Miranda can use each number?

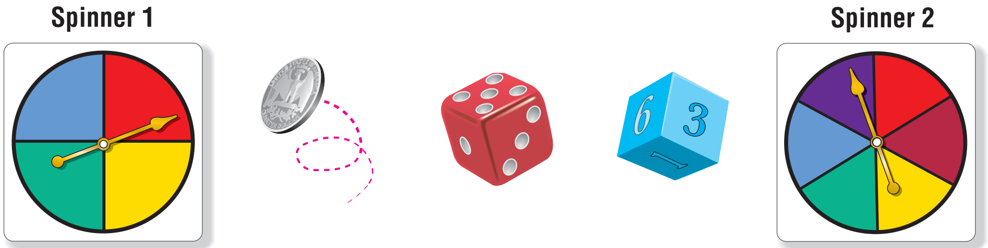
b. How many combinations are possible if Miranda can use each number only once? Explain.

㉓ **GAMES** Cody and Monette are playing a board game in which you roll two dice per turn.

a. In one turn, how many outcomes result in a sum of 8?

b. How many outcomes in one turn result in an odd sum?

24. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate a sequence of events. In the first stage of a two-stage experiment, you spin Spinner 1 below. If the result is red, you flip a coin. If the result is yellow, you roll a die. If the result is green, you roll a number cube. If the result is blue, you spin Spinner 2.



a. **Geometric** Draw a tree diagram to represent the sample space for the experiment.

b. **Logical** Draw a Venn diagram to represent the possible outcomes of the experiment.

c. **Analytical** How many possible outcomes are there?

d. **Verbal** Could you use the Fundamental Counting Principle to determine the number of outcomes? Explain.

H.O.T. Problems Use Higher-Order Thinking Skills

25. **CHALLENGE** A box contains n different objects. If you remove three objects from the box, one at a time, without putting the previous object back, how many possible outcomes exist? Explain your reasoning.

26. **OPEN ENDED** Sometimes a tree diagram for an experiment is not symmetrical. Describe a two-stage experiment where the tree diagram is asymmetrical. Include a sketch of the tree diagram. Explain.

27. **WRITING IN MATH** Explain why it is not possible to represent the sample space for a multi-stage experiment by using a table.

Example 3

28. **CCSS ARGUMENTS** Determine if the following statement is *sometimes*, *always*, or *never* true. Explain your reasoning.

When an outcome falls outside the sample space, it is a failure.

29. **REASONING** A multistage experiment has n possible outcomes at each stage. If the experiment is performed with k stages, write an equation for the total number of possible outcomes P. Explain.

30. **WRITING IN MATH** Explain when it is necessary to show all of the possible outcomes of an experiment by using a tree diagram and when using the Fundamental Counting Principle is sufficient.

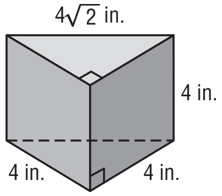
Standardized Test Practice

31. PROBABILITY Alejandra can invite two friends to go out to dinner with her for her birthday. If she is choosing among four of her friends, how many possible outcomes are there?

A 4 **C** 8

B 6 **D** 9

32. SHORT RESPONSE What is the volume of the triangular prism shown below?



33. Brad's password must be five digits long, use the numbers 0–9, and the digits must not repeat. What is the maximum number of different passwords that Brad can have?

F 15,120 **H** 59,049

G 30,240 **J** 100,000

34. SAT/ACT A pizza shop offers 3 types of crust, 5 vegetable toppings, and 4 meat toppings. How many different pizzas could be ordered by choosing 1 crust, 1 vegetable topping, and 1 meat topping?

A 12 **D** 60

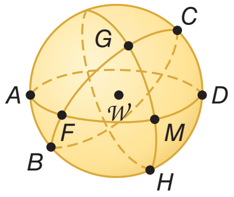
B 23 **E** infinite

C 35

Spiral Review

35. ARCHITECTURE To encourage recycling, the people of Rome, Italy, built a model of Basilica di San Pietro from empty beverage cans. The model was built to a 1:5 scale and was a rectangular prism that measured 26 meters high, 49 meters wide, and 93 meters long. Find the dimensions of the actual Basilica di San Pietro. (Lesson 12-8)

Using spherical geometry, name each of the following on sphere *W*. (Lesson 12-7)



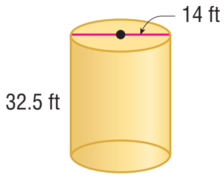
36. two lines containing point *F*

37. a segment containing point *G*

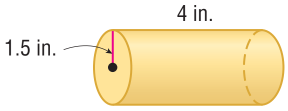
38. a triangle

Find the lateral area and surface area of each cylinder. Round to the nearest tenth. (Lesson 12-2)

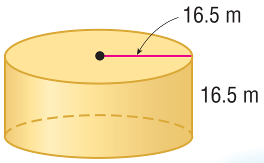
39.



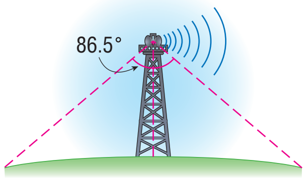
40.



41.



42. TELECOMMUNICATIONS The signal from a tower follows a ray that has its endpoint on the tower and is tangent to Earth. Suppose a tower is located at sea level as shown. Determine the measure of the arc intercepted by the two tangents. (Lesson 10-6)



Note: Art not drawn to scale

COORDINATE GEOMETRY Determine whether the figure with the given vertices has *line* symmetry and/or *rotational* symmetry. (Lesson 9-5)

43. $Q(2, 2), R(7, 2), S(6, 6), T(3, 6)$

44. $J(-2, 2), K(-5, -1), L(-2, -4), M(1, -1)$

Skills Review

Find each quotient.

45. $\frac{5^2}{2}$

46. $\frac{3^3}{3 \cdot 2}$

47. $\frac{2^4 \cdot 6}{8}$

48. $\frac{2^3 \cdot 12}{6}$

49. $\frac{4^4 \cdot 3}{24}$

LESSON 13-2 Probability with Permutations and Combinations

Then

- You used the Fundamental Counting Principle.

Now

1. Use permutations with probability.
2. Use combinations with probability.

Why?

- Lina, Troy, Davian, and Mary are being positioned for a photograph. There are 4 choices for who can stand on the far left, leaving 3 choices for who can stand in the second position. For the third position, just 2 choices remain, and for the last position just 1 is possible.

Digital Vision/Alamy

NewVocabulary

permutation
factorial
circular permutation
combination

Common Core State Standards

Content Standards

S.CP.9 (+) Use permutations and combinations to compute probabilities of compound events and solve problems.

Mathematical Practices

1 Make sense of problems and persevere in solving them.
4 Model with mathematics.

1 Probability Using Permutations

A **permutation** is an arrangement of objects in which order is important. One permutation of the four friends above is Troy, Davian, Mary, and then Lina. Using the Fundamental Counting Principle, there are 4 • 3 • 2 • 1 or 24 possible ordered arrangements of the friends.

The expression 4 • 3 • 2 • 1 used to calculate the number of permutations of these four friends can be written as 4!, which is read *4 factorial.*

KeyConcept Factorial

Words	The **factorial** of a positive integer n, written $n!$, is the product of the positive integers less than or equal to n.
Symbols	$n! = n \cdot (n - 1) \cdot (n - 2) \cdot \ldots \cdot 2 \cdot 1$, where $0! = 1$

PT

Example 1 Probability and Permutations of *n* Objects

SPORTS Chanise and Renee are members of the lacrosse team. If the 20 girls on the team are each assigned a jersey number from 1 to 20 at random, what is the probability that Chanise's jersey number will be 1 and Renee's will be 2?

Step 1 Find the number of possible outcomes in the sample space. This is the number of permutations of the 20 girls' names, or 20!.

Step 2 Find the number of favorable outcomes. This is the number of permutations of the other girls' names given that Chanise's jersey number is 1 and Renee's is 2: (20 − 2)! or 18!.

Step 3 Calculate the probability.

$$P(\text{Chanise 1, Renee 2}) = \frac{18!}{20!}$$

← number of favorable outcomes
← number of possible outcomes

$$= \frac{\overset{1}{\cancel{18!}}}{20 \cdot 19 \cdot \underset{1}{\cancel{18!}}}$$

Expand 20! and divide out common factors.

$$= \frac{1}{380}$$

Simplify.

GuidedPractice

1. **PHOTOGRAPHY** In the opening paragraph, what is the probability that Troy is chosen to stand on the far left and Davian on the far right for the photograph?

In the opening paragraph, suppose 6 friends were available, but the photographer wanted only 4 people in the picture. Using the Fundamental Counting Principle, the number of permutations of 4 friends taken from a group of 6 friends is $6 \cdot 5 \cdot 4 \cdot 3$ or 360.

Another way of describing this situation is the number of permutations of 6 friends taken 4 at a time, denoted ${}_6P_4$. This number can also be computed using factorials.

$${}_6P_4 = 6 \cdot 5 \cdot 4 \cdot 3 = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot \mathbf{2} \cdot \mathbf{1}}{\mathbf{2} \cdot \mathbf{1}} = \frac{6!}{2!} = \frac{6!}{(6-4)!}$$

This suggests the following formula.

ReadingMath

CCSS Precision The phrase *distinct objects* means that the objects are distinguishable as being different in some way.

KeyConcept Permutations

Symbols The number of permutations of n distinct objects taken r at a time is denoted by ${}_nP_r$ and given by ${}_nP_r = \frac{n!}{(n-r)!}$.

Example The number of permutations of 5 objects taken 2 at a time is

$${}_5P_2 = \frac{5!}{(5-2)!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!}} \text{ or } 20.$$

StudyTip

Randomness When outcomes are decided at random, they are equally likely to occur and their probabilities can be calculated using permutations and combinations.

Example 2 Probability and ${}_nP_r$

A class is divided into teams each made up of 15 students. Each team is directed to select team members to be officers. If Sam, Valencia, and Deshane are on a team, and the positions are decided at random, what is the probability that they are selected as president, vice president, and secretary, respectively?

Step 1 Since choosing officers is a way of ranking team members, order in this situation is important. The number of possible outcomes in the sample space is the number of permutations of 15 people taken 3 at a time, ${}_{15}P_3$.

$${}_{15}P_3 = \frac{15!}{(15-3)!} = \frac{15 \cdot 14 \cdot 13 \cdot \cancel{12!}}{\cancel{12!}} \text{ or } 2730$$

Step 2 The number of favorable outcomes is the number of permutations of the 3 students in their specific positions. This is 1!, or 1.

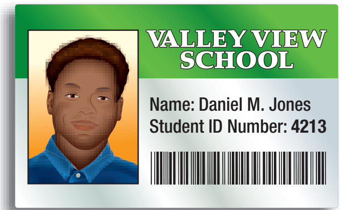
Step 3 So the probability of Sam, Valencia, and Deshane being selected as the three officers is $\frac{1}{2730}$.

GuidedPractice

2. A student identification card consists of 4 digits selected from 10 possible digits from 0 to 9. Digits cannot be repeated.

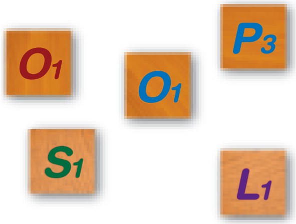
A. How many possible identification numbers are there?

B. Find the probability that a randomly generated card has the exact number 4213.



Name: Daniel M. Jones
Student ID Number: **4213**

In a game, you must try to create a word using randomly selected letter tiles. Suppose you select the tiles shown. If you consider the letters O and O to be distinct, then there are 5! or 120 permutations of these letters.



Four of these possible arrangements are listed below.

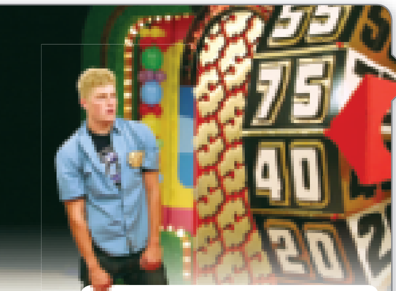
POOLS   POOLS   SPOOL   SPOOL

Notice that unless the Os are colored, several of these arrangements would look the same. Since there are 2 Os that can be arranged in 2! or 2 ways, the number of permutations of the letters O, P, O, L, and S can be written as $\frac{5!}{2!}$.

KeyConcept Permutations with Repetition

The number of distinguishable permutations of n objects in which one object is repeated r_1 times, another is repeated r_2 times, and so on, is

$$\frac{n!}{r_1! \cdot r_2! \cdot \ldots \cdot r_k!}.$$



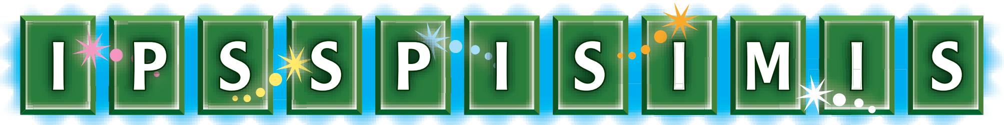
Real-WorldLink

Created in 1956, *The Price is Right* is the longest-running game show in the United States.

Source: IMDB

Example 3 Probability and Permutations with Repetition

GAME SHOW On a game show, you are given the following letters and asked to unscramble them to name a U.S. river. If you selected a permutation of these letters at random, what is the probability that they would spell the correct answer of MISSISSIPPI?



Step 1 There is a total of 11 letters. Of these letters, I occurs 4 times, S occurs 4 times, and P occurs 2 times. So, the number of distinguishable permutations of these letters is

$$\frac{11!}{4! \cdot 4! \cdot 2!} = \frac{39{,}916{,}800}{1152} \text{ or } 34{,}650.$$ Use a calculator.

Step 2 There is only 1 favorable arrangement—MISSISSIPPI.

Step 3 The probability that a permutation of these letters selected at random spells Mississippi is $\frac{1}{34{,}650}$.

GuidedPractice

3. TELEPHONE NUMBERS What is the probability that a 7-digit telephone number with the digits 5, 1, 6, 5, 2, 1, and 5 is the number 550-5211?

So far, you have been studying objects that are arranged in *linear* order. Notice that when the spices below are arranged in a line, shifting each spice one position to the right produces a different permutation—curry is now first instead of salt. There are 5! distinct permutations of these spices.

Tuukka Jantti/Stringer/Getty Images Entertainment/Getty Images

StudyTip

Turning the Circle Over If the circular object looks the same when it is turned over, such as a plain key ring, then the number of permutations must be divided by 2.

In a **circular permutation**, objects are arranged in a circle or loop. Consider the arrangements of these spices when placed on a turntable. Notice that rotating the turntable clockwise one position does *not* produce a different permutation—the order of the spices relative to each other remains unchanged.

Since 5 rotations of the turntable will produce the same permutation, the number of distinct permutations on the turntable is $\frac{1}{5}$ of the total number of arrangements when the spices are placed in a line.

$$\frac{1}{5} \cdot 5! = \frac{5 \cdot 4!}{5} \text{ or } 4!, \text{ which is } (5 - 1)!$$

KeyConcept Circular Permutations

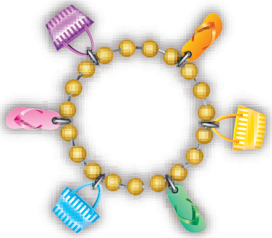
The number of distinguishable permutations of n objects arranged in a circle with no fixed reference point is

$$\frac{n!}{n} \text{ or } (n - 1)!.$$

If the n objects are arranged relative to a fixed reference point, then the arrangements are treated as linear, making the number of permutations $n!$.

Example 4 Probability and Circular Permutations

Find the indicated probability. Explain your reasoning.



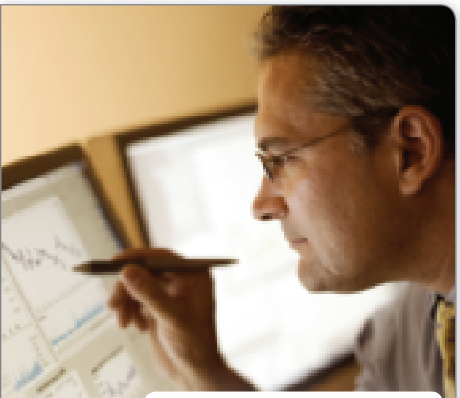
a. JEWELRY If the 6 charms on the bracelet shown are arranged at random, what is the probability that the arrangement shown is produced?

Since there is no fixed reference point, this is a circular permutation. So, there are (6 − 1)! or 5! distinguishable permutations of the charms. Thus, the probability that the exact arrangement shown is produced is $\frac{1}{5!}$ or $\frac{1}{120}$.

b. DINING You are seating a party of 4 people at a round table. One of the chairs around this table is next to a window. If the diners are seated at random, what is the probability that the person paying the bill is seated next to the window?

Since the people are seated around a table with a fixed reference point, this is a linear permutation. So there are 4! or 24 ways in which the people can be seated around the table. The number of favorable outcomes is the number of permutations of the other 3 diners given that the person paying the bill sits next to the window, 3! or 6.

So, the probability that the person paying the bill is seated next to the window is $\frac{6}{24}$ or $\frac{1}{4}$.



Real-WorldCareer

Statisticians Statisticians collect statistical data for various subject areas, including sports and games. They use computer software to analyze, interpret, and summarize the data. Most statisticians have a master's degree.

Steve Cole/Digital Vision/Getty Images

GuidedPractice

4. **FOOTBALL** A team's 11 football players huddle together before a play.

A. What is the probability that the fullback stands to the right of the quarterback if the team huddles together at random? Explain your reasoning.

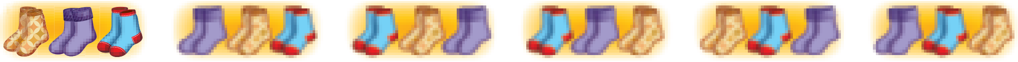
B. If a referee stands directly behind the huddle, what is the probability that the referee stands directly behind the halfback? Explain your reasoning.

StudyTip

Permutations and Combinations Use permutations when the order of an arrangement of objects is important and combinations when order is not important.

2 Probability Using Combinations

A **combination** is an arrangement of objects in which order is *not* important. Suppose you need to pack 3 of your 8 different pairs of socks for a trip. The order in which the socks are chosen does not matter, so the 3! or 6 groups of socks shown below would *not* be considered different. So, you would use combinations to determine the number of possible different sock choices.



A combination of n objects taken r at a time, or ${}_nC_r$, is calculated by dividing the number of permutations ${}_nP_r$ by the number of arrangements containing the same elements, $r!$.

KeyConcept Combinations

Symbols The number of combinations of n distinct objects taken r at a time is denoted by ${}_nC_r$ and is given by ${}_nC_r = \frac{n!}{(n-r)!\,r!}$.

Example The number of combinations of 8 objects taken 3 at a time is

$${}_8C_3 = \frac{8!}{(8-3)!\,3!} = \frac{8!}{5!3!} = \frac{8 \cdot 7 \cdot \cancel{6} \cdot \cancel{5!}}{\cancel{5!} \cdot \cancel{6}} \text{ or } 56.$$

Example 5 Probability and ${}_nC_r$

INVITATIONS **For her birthday, Monica can invite 6 of her 20 friends to join her at a theme park. If she chooses to invite friends at random, what is the probability that friends Tessa, Guido, Brendan, Faith, Charlotte, and Rhianna are chosen?**

Step 1 Since the order in which the friends are chosen does not matter, the number of possible outcomes in the sample space is the number of combinations of 20 people taken 6 at a time, ${}_{20}C_6$.

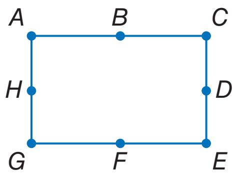
$${}_{20}C_6 = \frac{20!}{(20-6)!\,6!} = \frac{\cancel{20} \cdot 19 \cdot \cancel{18} \cdot 17 \cdot \overset{8}{\cancel{16}} \cdot 15 \cdot \cancel{14!}}{\cancel{14!} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}} \text{ or } 38{,}760$$

Step 2 There is only 1 favorable outcome—that the six students listed above are chosen. The order in which they are chosen is not important.

Step 3 So the probability of these six friends being chosen is $\frac{1}{38{,}760}$.

GuidedPractice

5. **GEOMETRY** If three points are randomly chosen from those named on the rectangle shown, what is the probability that they all lie on the same line segment?

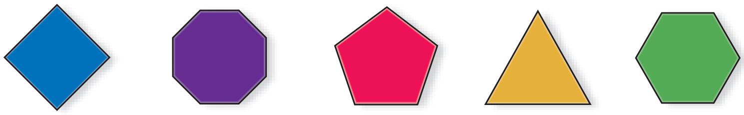


Check Your Understanding

● = Step-by-Step Solutions begin on page R14.

Example 1

1. **GEOMETRY** Five students are asked to randomly select and name a polygon from the group shown below. What is the probability that the first two students choose the triangle and quadrilateral, in that order?



Example 2

2. **PLAYS** A high school performs a production of *A Raisin in the Sun* with each freshman English class of 18 students. If the three members of the crew are decided at random, what is the probability that Chase is selected for lighting, Jaden is selected for props, and Emelina for spotlighting?

Example 3

3. **DRIVING** What is the probability that a license plate using the letters C, F, and F and numbers 3, 3, 3, and 1 will be CFF3133?

Example 4

4. **CHEMISTRY** In chemistry lab, you need to test six samples that are randomly arranged on a circular tray.

   a. What is the probability that the arrangement shown at the right is produced?

   b. What is the probability that test tube 2 will be in the top middle position?

Example 5

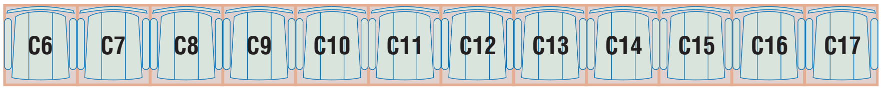
5. Five hundred boys, including Josh and Sokka, entered a drawing for two football game tickets. What is the probability that the tickets were won by Josh and Sokka?

Practice and Problem Solving

Extra Practice is on page R13.

Example 1

6. **CONCERTS** Nia and Chad are going to a concert with their high school's key club. If they choose a seat on the row below at random, what is the probability that Chad will be in seat C11 and Nia will be in C12?



⑦ **FAIRS** Alfonso and Colin each bought one raffle ticket at the state fair. If 50 tickets were randomly sold, what is the probability that Alfonso got ticket 14 and Colin got ticket 23?

Example 2

8. CCSS **MODELING** The table shows the finalists for a floor exercises competition. The order in which they will perform will be chosen randomly.

   a. What is the probability that Cecilia, Annie, and Kimi are the first 3 gymnasts to perform, in any order?

   b. What is the probability that Cecilia is first, Annie is second, and Kimi is third?

Floor Exercises Finalists
Eliza Hernandez
Kimi Kanazawa
Cecilia Long
Annie Montgomery
Shenice Malone
Caroline Smith
Jessica Watson

9. **JOBS** A store randomly assigns their employees work identification numbers to track productivity. Each number consists of 5 digits ranging from 1–9. If the digits cannot repeat, find the probability that a randomly generated number is 25938.

10. **GROUPS** Two people are chosen randomly from a group of ten. What is the probability that Jimmy was selected first and George second?

Example 3 ⑪ **MAGNETS** Santiago bought some letter magnets that he can arrange to form words on his fridge. If he randomly selected a permutation of the letters shown below, what is the probability that they would form the word BASKETBALL?

12. **ZIP CODES** What is the probability that a zip code randomly generated from among the digits 3, 7, 3, 9, 5, 7, 2, and 3 is the number 39372?

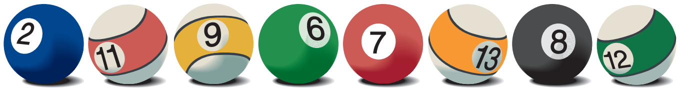
Example 4 **13.** **GROUPS** Keith is randomly arranging desks into circles for group activities. If there are 7 desks in his circle, what is the probability that Keith will be in the desk closest to the door?

14. **AMUSEMENT PARKS** Sylvie is at an amusement park with her friends. They go on a ride that has bucket seats in a circle. If there are 8 seats, what is the probability that Sylvie will be in the seat farthest from the entrance to the ride?

Example 5 **15.** **PHOTOGRAPHY** If you are randomly placing 24 photos in a photo album and you can place four photos on the first page, what is the probability that you choose the photos at the right?

16. **ROAD TRIPS** Rita is going on a road trip across the U.S. She needs to choose from 15 cities where she will stay for one night. If she randomly pulls 3 city brochures from a pile of 15, what is the probability that she chooses Austin, Cheyenne, and Savannah?

17. CCSS **SENSE-MAKING** Use the figure below. Assume that the balls are aligned at random.



a. What is the probability that in a row of 8 pool balls, the solid 2 and striped 11 would be first and second from the left?

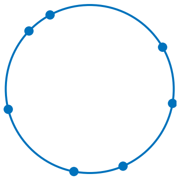
b. What is the probability that if the 8 pool balls were mixed up at random, they would end up in the order shown?

c. What is the probability that in a row of seven balls, with three 8 balls, three 9 balls, and one 6 ball, the three 8 balls would be to the left of the 6 ball and the three 9 balls would be on the right?

d. If the balls were randomly rearranged and formed a circle, what is the probability that the 6 ball is next to the 7 ball?

18. How many lines are determined by 10 randomly selected points, no 3 of which are collinear? Explain your calculation.

19. Suppose 7 points on a circle are chosen at random, as shown at the right.



a. Using the letters A through E, how many ways can the points on the circle be named?

b. If one point on the circle is fixed, how many arrangements are possible?

20. **RIDES** A carousel has 7 horses and one bench seat that will hold two people. One of the horses does not move up or down.

a. How many ways can the seats on the carousel be randomly filled by 9 people?

b. If the carousel is filled randomly, what is the probability that you and your friend will end up in the bench seat?

c. If 6 of the 9 people randomly filling the carousel are under the age of 8, what is the probability that a person under the age of 8 will end up on the horse that does not move up or down?

21 **LICENSES** A camera positioned above a traffic light photographs cars that fail to stop at a red light. In one unclear photograph, an officer could see that the first letter of the license plate was a Q, the second letter was an M or an N and the third letter was a B, P, or D. The first number was a 0, but the last two numbers were illegible. How many possible license plates fit this description?

22. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate permutations.

a. **Numerical** Randomly select three digits from 0 to 9. Find the possible permutations of the three integers.

b. **Tabular** Repeat part **a** for four additional sets of three integers. You will use some digits more than once. Copy and complete the table below.

Integers	Permutations	Average of Permutations	$\frac{\text{Average of Permutations}}{37}$
1, 4, 7	147, 174, 417, 471, 714, 741	444	12

c. **Verbal** Make a conjecture about the value of the average of the permutations of three digits between 0 and 9.

d. **Symbolic** If the three digits are x, y, and z, is it possible to write an equation for the average A of the permutations of the digits? If so, write the equation. If not, explain why not.

H.O.T. Problems Use Higher-Order Thinking Skills

23. **CHALLENGE** Fifteen boys and fifteen girls entered a drawing for four free movie tickets. What is the probability that all four tickets were won by girls?

24. **CHALLENGE** A student claimed that permutations and combinations were related by $r! \cdot {}_nC_r = {}_nP_r$. Use algebra to show that this is true. Then explain why ${}_nC_r$ and ${}_nP_r$ differ by the factor $r!$.

25. **OPEN ENDED** Describe a situation in which the probability is given by $\frac{1}{{}_7C_3}$.

26. **CCSS** **ARGUMENTS** Is the following statement *sometimes*, *always*, or *never* true? Explain.

$${}_nP_r = {}_nC_r$$

27. **PROOF** Prove that ${}_nC_n - r = {}_nC_r$.

28. **WRITING IN MATH** Compare and contrast permutations and combinations.

Standardized Test Practice

29. PROBABILITY Four members of the pep band, two girls and two boys, always stand in a row when they play. What is the probability that a girl will be at each end of the row if they line up in random order?

A $\frac{1}{24}$ **C** $\frac{1}{6}$

B $\frac{1}{12}$ **D** $\frac{1}{2}$

30. SHORT RESPONSE If you randomly select a permutation of the letters shown below, what is the probability that they would spell GEOMETRY?

31. ALGEBRA Student Council sells soft drinks at basketball games and makes $1.50 from each. If they pay $75 to rent the concession stand, how many soft drinks would they have to sell to make $250 profit?

F 116   **H** 167

G 117   **J** 217

32. SAT/ACT The ratio of 12:9 is equal to the ratio of $\frac{1}{3}$ to

A $\frac{1}{4}$ **D** 2

B 1   **E** 4

C $\frac{5}{4}$

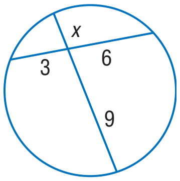
Spiral Review

33. SHOPPING A women's coat comes in sizes 4, 6, 8, or 10 in black, brown, ivory, and cinnamon. How many different coats could be selected? (Lesson 13-1)

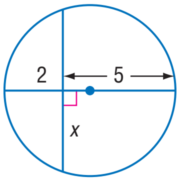
34. Two similar prisms have surface areas of 256 square inches and 324 square inches. What is the ratio of the height of the small prism to the height of the large prism? (Lesson 12-8)

Find *x*. Round to the nearest tenth, if necessary. (Lesson 10-7)

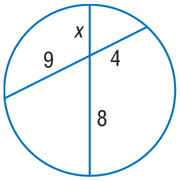
35.



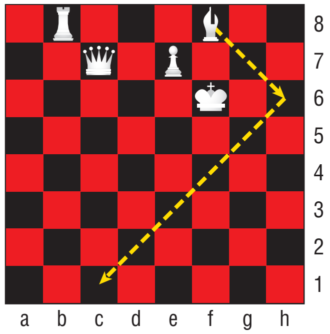
36.



37.



38. CHESS The bishop shown in square f8 can only move diagonally along dark squares. If the bishop is in c1 after two moves, describe the translation. (Lesson 9-2)



Skills Review

Use the number line to find each measure.

39. *DF*

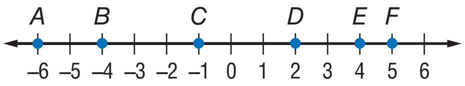
40. *AE*

41. *EF*

42. *BD*

43. *AC*

44. *CF*



LESSON 13-3 Geometric Probability

Then

- You found probabilities of simple events.

Now

1 Find probabilities by using length.

2 Find probabilities by using area.

Why?

- The object of the popular carnival game shown is to collect points by rolling a ball up an incline and into one of several circular target areas. The point value of each area is assigned based on the probability of a person landing a ball in that area.

NewVocabulary
geometric probability

Common Core State Standards

Content Standards
S.MD.7 (+) Analyze decisions and strategies using probability concepts (e.g., product testing, medical testing, pulling a hockey goalie at the end of a game).

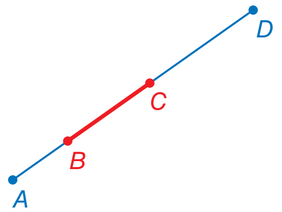
Mathematical Practices
1 Make sense of problems and persevere in solving them.
2 Reason abstractly and quantitatively.

1 **Probability with Length** The probability of winning the carnival game depends on the area of the target. Probability that involves a geometric measure such as length or area is called **geometric probability**.

KeyConcept Length Probability Ratio

Words If a line segment (1) contains another segment (2) and a point on segment (1) is chosen at random, then the probability that the point is on segment (2) is

$$\frac{\text{length of segment (2)}}{\text{length of segment (1)}}.$$



Example If a point E on $\overline{AD}$ is chosen at random, then $P(E \text{ is on } \overline{BC}) = \frac{BC}{AD}$.

Example 1 Use Lengths to Find Geometric Probability

Point X is chosen at random on $\overline{JM}$. Find the probability that X is on $\overline{KL}$.

$P(X \text{ is on } \overline{KL}) = \frac{KL}{JM}$ Length probability ratio

$= \frac{7}{14}$ $KL = 7$ and $JM = 3 + 7 + 4$ or 14

$= \frac{1}{2}$, 0.5, or 50% Simplify.

GuidedPractice

Point X is chosen at random on $\overline{JM}$. Find the probability of each event.

1A. $P(X \text{ is on } \overline{LM})$

1B. $P(X \text{ is on } \overline{KM})$

Geometric probability can be used in many real-world situations that involve an infinite number of outcomes.

fotog/Getty Images

Real-WorldLink

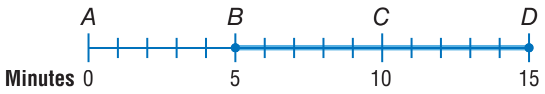
A Chicago Transit Authority train arrives or departs a station like Addison on the Red Line every 15 minutes.

Source: Chicago Transit Authority

Real-World Example 2 Model Real-World Probabilities

TRANSPORTATION Use the information at the left. Assuming that you arrive at Addison on the Red Line at a random time, what is the probability that you will have to wait 5 or more minutes for a train?

We can use a number line to model this situation. Since the trains arrive every 15 minutes, the next train will arrive in 15 minutes or less. On the number line below, the event of waiting 5 or more minutes is modeled by $\overline{BD}$.



Find the probability of this event.

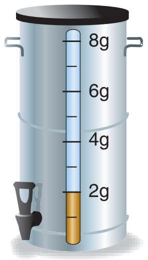
$$P(\text{waiting 5 or more minutes}) = \frac{BD}{AD} \qquad \text{Length probability ratio}$$

$$= \frac{10}{15} \text{ or } \frac{2}{3} \qquad BD = 10 \text{ and } AD = 15$$

So, the probability of waiting 5 or more minutes for the next train is $\frac{2}{3}$ or about 67%.

GuidedPractice

2. **TEA** Iced tea at a cafeteria-style restaurant is made in 8-gallon containers. Once the level gets below 2 gallons, the flavor of the tea becomes weak.



A. What is the probability that when someone tries to pour a glass of tea from the container, it is below 2 gallons?

B. What is the probability that the amount of tea in the container at any time is between 2 and 3 gallons?

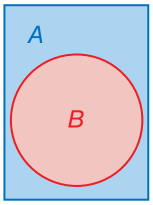
2 Probability with Area

Geometric probability can also involve area. The ratio for calculating geometric probability involving area is shown below.

KeyConcept Area Probability Ratio

Words If a region A contains a region B and a point E in region A is chosen at random, then the probability that point E is in region B is $\frac{\text{area of region } B}{\text{area of region } A}$.

Example If a point E is chosen at random in rectangle A, then $P(\text{point } E \text{ is in circle } B) = \frac{\text{area of region } B}{\text{area of region } A}$.



When determining geometric probabilities with targets, we assume

- that the object lands within the target area, and
- it is equally likely that the object will land anywhere in the region.

Ingram Publishing/SuperStock

Real-WorldLink

Champion accuracy skydivers routinely land less than two inches away from the center of a target.

Source: *SkyDiving News*

Real-World Example 3 Use Area to Find Geometric Probability

SKYDIVING Suppose a skydiver must land on a target of three concentric circles. If the diameter of the center circle is 2 yards and the circles are spaced 1 yard apart, what is the probability that the skydiver will land in the red circle?

You need to find the ratio of the area of the red circle to the area of the entire target. The radius of the red circle is 1 yard, while the radius of the entire target is 1 + 1 + 1 or 3 yards.

$$P(\text{skydiver lands in red circle}) = \frac{\text{area of red circle}}{\text{area of target}} \quad \text{Area probability ratio}$$

$$= \frac{\pi(1)^2}{\pi(3)^2} \quad A = \pi r^2$$

$$= \frac{\pi}{9\pi} \text{ or } \frac{1}{9} \quad \text{Simplify.}$$

The probability that the skydiver will land in the red circle is $\frac{1}{9}$ or about 11%.

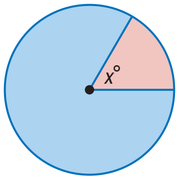
GuidedPractice

3. SKYDIVING Find each probability using the example above.

A. P(skydiver lands in the blue region)

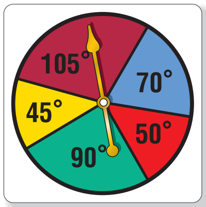
B. P(skydiver lands in white region)

You can also use an angle measure to find geometric probability. The ratio of the area of a sector of a circle to the area of the entire circle is the same as the ratio of the sector's central angle to 360. You will prove this in Exercise 27.



Example 4 Use Angle Measures to Find Geometric Probability

Use the spinner to find each probability.



a. ***P*(pointer landing on yellow)**

The angle measure of the yellow region is 45.

$P(\text{pointer landing on yellow}) = \frac{45}{360}$ or 12.5%

b. ***P*(pointer landing on purple)**

The angle measure of the purple region is 105.

$P(\text{pointer landing on purple}) = \frac{105}{360}$ or about 29%

c. ***P*(pointer landing on neither red nor blue)**

The combined angle measures of the red and blue region are 50 + 70 or 120.

$P(\text{pointer landing on neither red nor blue}) = \frac{360 - 120}{360}$ or about 67%

StudyTip

Use Estimation In Example 4b, the area of the purple sector is a little less than $\frac{1}{3}$ or 33% of the spinner. Therefore, an answer of 29% is reasonable.

GuidedPractice

4A. P(pointer landing on blue)

4B. P(pointer not landing on green)

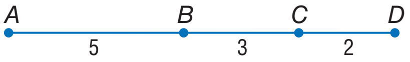
0 Furrer/F1online/age footstock

Check Your Understanding

● = Step-by-Step Solutions begin on page R14.

Example 1

Point X is chosen at random on $\overline{AD}$. Find the probability of each event.



1. $P(X \text{ is on } \overline{BD})$

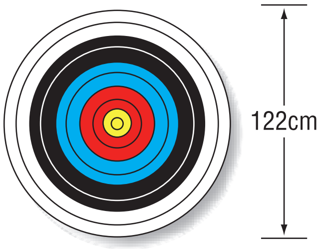
2. $P(X \text{ is on } \overline{BC})$

Example 2

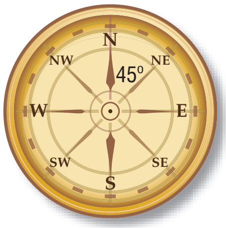
3. **CARDS** In a game of cards, 43 cards are used, including one joker. Four players are each dealt 10 cards and the rest are put in a pile. If Greg doesn't have the joker, what is the probability that either his partner or the pile have the joker?

Examples 3–4

4. **ARCHERY** An archer aims at a target that is 122 centimeters in diameter with 10 concentric circles whose diameters decrease by 12.2 centimeters as they get closer to the center. Find the probability that the archer will hit the center.



5. **NAVIGATION** A camper lost in the woods points his compass in a random direction. Find the probability that the camper is heading in the N to NE direction.



Practice and Problem Solving

Extra Practice is on page R13.

Example 1

CCSS REASONING **Point X is chosen at random on $\overline{FK}$. Find the probability of each event.**

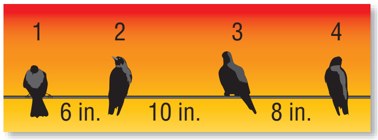
6. $P(X \text{ is on } \overline{FH})$

7. $P(X \text{ is on } \overline{GJ})$

8. $P(X \text{ is on } \overline{HK})$

9. $P(X \text{ is on } \overline{FG})$

10. **BIRDS** Four birds are sitting on a telephone wire. What is the probability that a fifth bird landing at a randomly selected point between birds 1 and 4 will sit at some point between birds 3 and 4?



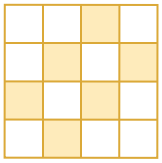
Example 2

11. **TELEVISION** Julio is watching television and sees an ad for a CD that he knows his friend wants for her birthday. If the ad replays at a random time in each 3-hour interval, what is the probability that he will see the ad again during his favorite 30-minute sitcom the next day?

Example 3

Find the probability that a point chosen at random lies in the shaded region. Assume that figures that seem to be regular and congruent are regular and congruent.

12.



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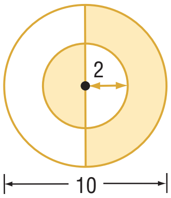

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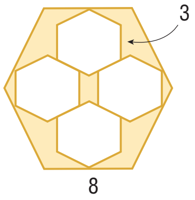
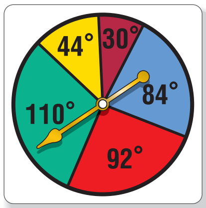


Example 4 **Use the spinner to find each probability. If the spinner lands on a line it is spun again.**



15. P(pointer landing on yellow)

16. P(pointer landing on blue)

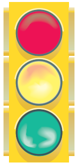
17. P(pointer not landing on green)

18. P(pointer landing on red)

19. P(pointer landing on neither red nor yellow)

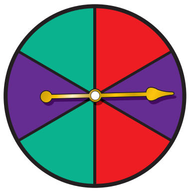
Describe an event with a 33% probability for each model.

20.



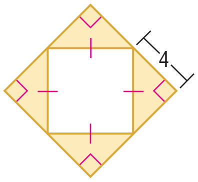
21.

22.



Find the probability that a point chosen at random lies in the shaded region.

23.



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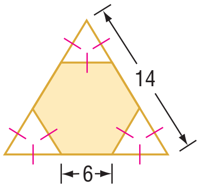

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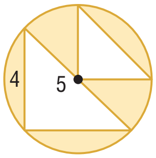
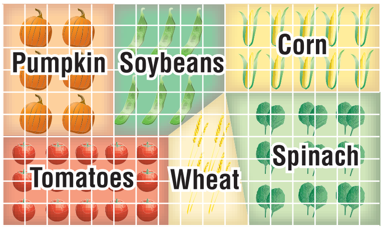


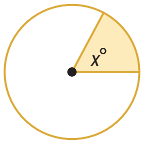
26. **FARMING** The layout for a farm is shown with each square representing a plot. Estimate the area of each field to answer each question.



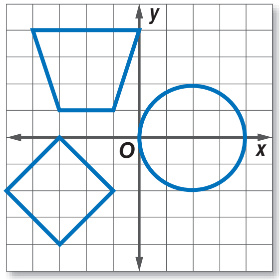
a. What is the approximate combined area of the spinach and corn fields?

b. Find the probability that a randomly chosen plot is used to grow soybeans.

27. **ALGEBRA** Prove that the probability that a randomly chosen point in the circle will lie in the shaded region is equal to $\frac{x}{360}$.



28. **COORDINATE GEOMETRY** If a point is chosen at random in the coordinate grid shown at the right, find each probability. Round to the nearest hundredth.



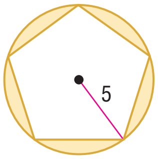
a. P(point inside the circle)

b. P(point inside the trapezoid)

c. P(point inside the trapezoid, square, or circle)

CCSS SENSE-MAKING **Find the probability that a point chosen at random lies in a shaded region.**

29.



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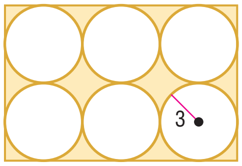

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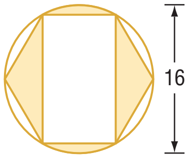
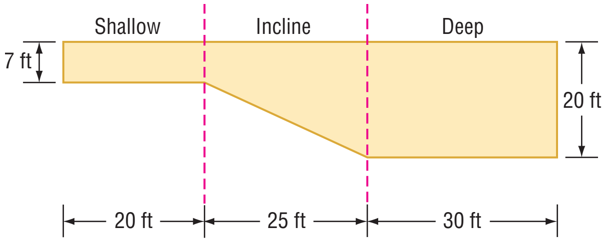


32. **COORDINATE GEOMETRY** Consider a system of inequalities, $1 \leq x \leq 6$, $y \leq x$, and $y \geq 1$. If a point (x, y) in the system is chosen at random, what is the probability that $(x - 1)^2 + (y - 1)^2 \geq 16$?

33 **VOLUME** The polar bear exhibit at a local zoo has a pool with the side profile shown. If the pool is 20 feet wide, what is the probability that a bear that is equally likely to swim anywhere in the pool will be in the incline region?



34. **DECISION MAKING** Meleah's flight was delayed and she is running late to make it to a national science competition. She is planning on renting a car at the airport and prefers car rental company A over car rental company B. The courtesy van for car rental company A arrives every 7 minutes, while the courtesy van for car rental company B arrives every 12 minutes.

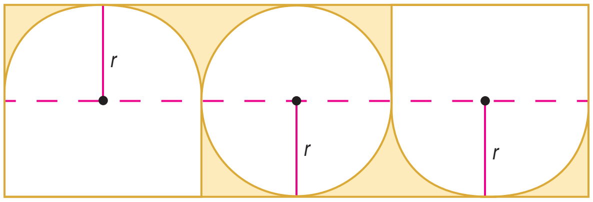
a. What is the probability that Meleah will have to wait 5 minutes or less to see each van? Explain your reasoning. (*Hint:* Use an area model.)

b. What is the probability that Meleah will have to wait 5 minutes or less to see one of the vans? Explain your reasoning.

c. Meleah can wait no more than 5 minutes without risking being late for the competition. If the van from company B should arrive first, should she wait for the van from company A or take the van from company B? Explain your reasoning.

H.O.T. Problems Use Higher-Order Thinking Skills

35. **CHALLENGE** Find the probability that a point chosen at random would lie in the shaded area of the figure. Round to the nearest tenth of a percent.

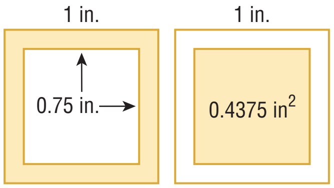


36. **CCSS** **REASONING** An isosceles triangle has a perimeter of 32 centimeters. If the lengths of the sides of the triangle are integers, what is the probability that the area of the triangle is exactly 48 square centimeters? Explain.

37. **WRITING IN MATH** Can athletic events be considered random events? Explain.

38. **OPEN ENDED** Represent a probability of 20% using three different geometric figures.

39. **WRITING IN MATH** Explain why the probability of a randomly chosen point falling in the shaded region of either of the squares shown is the same.



Standardized Test Practice

40. PROBABILITY A circle with radius 3 is contained in a square with side length 9. What is the probability that a randomly chosen point in the interior of the square will also lie in the interior of the circle?

A $\frac{1}{9}$

B $\frac{1}{3}$

C $\frac{\pi}{9}$

D $\frac{9}{\pi}$

41. ALGEBRA The area of Miki's room is $x^2 + 8x + 12$ square feet. A gallon of paint will cover an area of $x^2 + 6x + 8$ square feet. Which expression gives the number of gallons of paint that Miki will need to buy to paint her room?

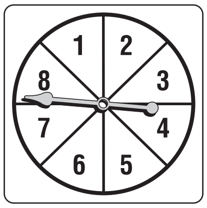
F $\frac{x+6}{x+4}$

G $\frac{x-4}{x-6}$

H $\frac{x+4}{x+6}$

J $\frac{x-4}{x+6}$

42. EXTENDED RESPONSE The spinner is divided into 8 equal sections.



a. If the arrow lands on a number, what is the probability that it will land on 3?

b. If the arrow lands on a number, what is the probability that it will land on an odd number?

43. SAT/ACT A box contains 7 blue marbles, 6 red marbles, 2 white marbles, and 3 black marbles. If one marble is chosen at random, what is the probability that it will be red?

A 0.11

B 0.17

C 0.33

D 0.39

E 0.67

Spiral Review

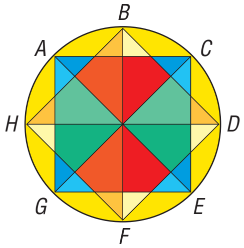
44. PROM Four friends are sitting at a table together at the prom. What is the probability that a particular one of them will sit in the chair closest to the dance floor? (Lesson 13-2)

Represent the sample space for each experiment by making an organized list, a table, and a tree diagram. (Lesson 13-1)

45. Tito has a choice of taking music lessons for the next two years and playing drums or guitar.

46. Denise can buy a pair of shoes in either flats or heels in black or navy blue.

STAINED GLASS In the stained glass window design, all of the small arcs around the circle are congruent. Suppose the center of the circle is point *O*. (Lesson 10-4)



47. What is the measure of each of the small arcs?

48. What kind of figure is $\triangle AOC$? Explain.

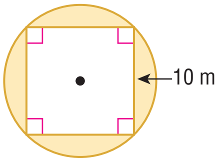
49. What kind of figure is quadrilateral *BDFH*? Explain.

50. What kind of figure is quadrilateral *ACEG*? Explain.

Skills Review

Find the area of the shaded region. Round to the nearest tenth.

51.



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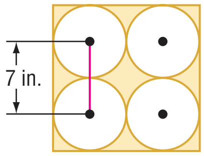

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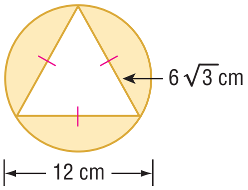


CHAPTER 13

Mid-Chapter Quiz

Lessons 13-1 through 13-3

1. **LUNCH** A deli has a lunch special, which consists of a sandwich, soup, dessert, and a drink for $4.99. The choices are in the table below. (Lesson 13-1)

Sandwich	Soup	Dessert	Drink
chicken salad	tomato	cookie	tea
ham	chicken noodle	pie	coffee
tuna	vegetable		cola
roast beef			diet cola
			milk

   a. How many different lunches can be created from the items shown in the table?

   b. If a soup and two desserts were added, how many different lunches could be created?

2. **FLAGS** How many different signals can be made with 5 flags from 8 flags of different colors? (Lesson 13-1)

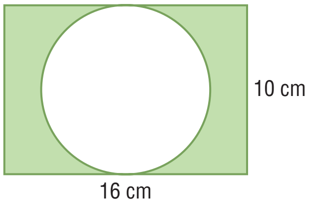
3. **CLOTHING** Marcy has six colors of shirts: red, blue, yellow, green, pink, and orange. She has each color in short-sleeved and long-sleeved styles. Represent the sample space for Marcy's shirt choices by making an organized list, a table, and a tree diagram. (Lesson 13-1)

4. **SPELLING** A bag contains one tile for each letter of the word TRAINS. If you selected a permutation of these letters at random, what is the probability that they would spell TRAINS? (Lesson 13-2)

5. **CHANGE** Augusto has 3 pockets and 4 different coins. In how many ways can he put one coin in each pocket? (Lesson 13-2)

6. **COINS** Ten coins are tossed simultaneously. In how many of the outcomes will the third coin turn up a head? (Lesson 13-2)

7. Find the probability that a point chosen at random lies in the shaded region. (Lesson 13-3)



8. **EXTENDED RESPONSE** A 320 meter long tightrope is suspended between two poles. Assume that the line has an equal chance of breaking anywhere along its length. (Lesson 13-3)

   a. Determine the probability that a break will occur in the first 50 meters of the tightrope.

   b. Determine the probability that the break will occur within 20 meters of a pole.

Point A is chosen at random on $\overline{BE}$. Find the probability of each event. (Lesson 13-3)

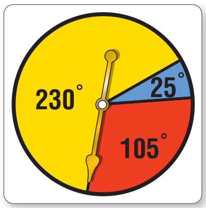
9. $P(A \text{ is on } \overline{CD})$

10. $P(A \text{ is on } \overline{BD})$

11. $P(A \text{ is on } \overline{CE})$

12. $P(A \text{ is on } \overline{DE})$

Use the spinner to find each probability. If the spinner lands on a line, it is spun again. (Lesson 13-3)

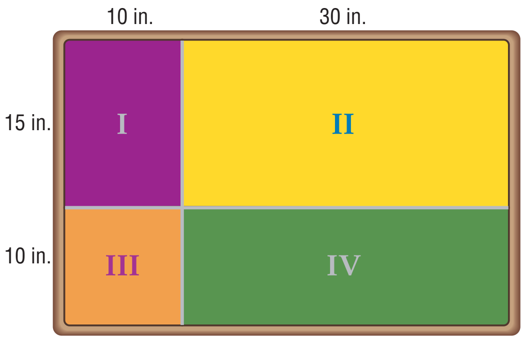


13. P(pointer landing on yellow)

14. P(pointer landing on blue)

15. P(pointer landing on red)

16. **GAMES** At a carnival, the object of a game is to throw a dart at the board and hit region III. (Lesson 13-3)



   a. What is the probability that it hits region I?

   b. What is the probability that it hits region II?

   c. What is the probability that it hits region III?

   d. What is the probability that it hits region IV?

LESSON 13-4 Simulations

Then

- You found probabilities by using geometric measures.

Now

1. Design simulations to estimate probabilities.
2. Summarize data from simulations.

Why?

- Based on practice, Allen knows that he makes 70% of his free throws. He wants to use this information to predict the number of free throws he is likely to make in games.

NewVocabulary

probability model
simulation
random variable
expected value
Law of Large Numbers

Common Core State Standards

Content Standards

G.MG.3 Apply geometric methods to solve problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios). ★

S.MD.6 (+) Use probabilities to make fair decisions (e.g., drawing by lots, using a random number generator).

Mathematical Practices

1 Make sense of problems and persevere in solving them.
4 Model with mathematics.

Hill Street Studios/Blend Images/Getty Images

1 Design a Simulation

A **probability model** is a mathematical model used to match a random phenomenon. A **simulation** is the use of a probability model to recreate a situation again and again so that the likelihood of various outcomes can be estimated. To design a simulation, use the following steps.

KeyConcept Designing a Simulation

Step 1 Determine each possible outcome and its theoretical probability.

Step 2 State any assumptions.

Step 3 Describe an appropriate probability model for the situation.

Step 4 Define what a trial is for the situation and state the number of trials to be conducted.

An appropriate probability model has the same probabilities as the situation you are trying to predict. Geometric models are common probability models.

Example 1 Design a Simulation by Using a Geometric Model

BASKETBALL Allen made 70% of his free throws last season. Design a simulation that can be used to estimate the probability that he will make his next free throw this season.

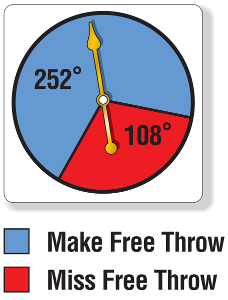
Step 1

Possible Outcomes	Theoretical Probability
• Allen makes a free throw. →	70%
• Allen misses a free throw. →	(100 − 70)% or 30%

Step 2 Our simulation will consist of 40 trials.

Step 3 One device that could be used is a spinner divided into two sectors, one containing 70% of the spinner's area and the other 30%. To create such a spinner, find the measure of the central angle of each sector.

Make Free Throw	Miss Free Throw
70% of 360° = 252°	30% of 360° = 108°



Step 4 A trial, one spin of the spinner, will represent shooting one free throw. A successful trial will be a made free throw and a failed trial will be a missed free throw. The simulation will consist of 40 trials.

GuidedPractice

1. **RESTAURANTS** A restaurant attaches game pieces to its large drink cups, awarding a prize to anyone who collects all 6 game pieces. Design a simulation using a geometric model that can be used to estimate how many large drinks a person needs to buy to collect all 6 game pieces.

Problem-Solving Tip

Use a Simulation
Simulations often provide a safe and efficient problem-solving strategy in situations that otherwise may be costly, dangerous, or impossible to solve using theoretical techniques. Simulations should involve data that are easier to obtain than the actual data you are modeling.

In addition to geometric models, simulations can also be conducted using dice, coin tosses, random number tables, and random number generators, such as those available on graphing calculators.

Example 2 Design a Simulation by Using Random Numbers

EYE COLOR **A survey of East High School students found that 40% had brown eyes, 30% had hazel eyes, 20% had blue eyes, and 10% had green eyes. Design a simulation that can be used to estimate the probability that a randomly chosen East High student will have one of these eye colors.**

Step 1

Possible Outcomes		Theoretical Probability
Brown eyes	→	40%
Hazel eyes	→	30%
Blue eyes	→	20%
Green eyes	→	10%

Step 2 We assume that a student's eye color will fall into one of these four categories.

Step 3 Use the random number generator on your calculator. Assign the ten integers 0–9 to accurately represent the probability data. The actual numbers chosen to represent the outcomes do not matter.

Outcome	Represented by
Brown eyes	0, 1, 2, 3
Hazel eyes	4, 5, 6
Blue eyes	7, 8
Green eyes	9

Step 4 A trial will represent selecting a student at random and recording his or her eye color. The simulation will consist of 20 trials.

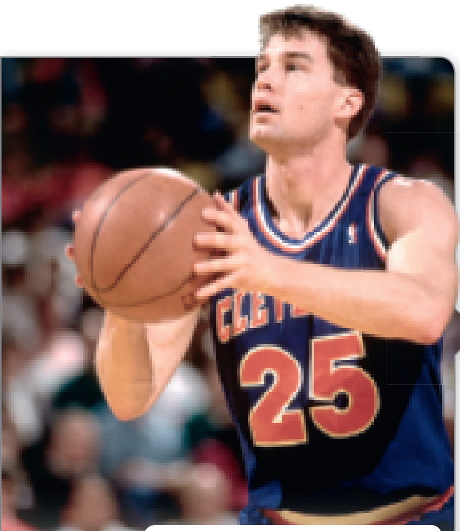
StudyTip

Random Number Generator
To generate a set of random integers on a graphing calculator, press MATH and select randInt(under the PRB menu. Then enter the beginning and ending integer values for your range and the number of integers you want in each trial.

GuidedPractice

2. **SOCCER** Last season, Yao made 18% of his free kicks. Design a simulation using a random number generator that can be used to estimate the probability that he will make his next free kick.

2 Summarize Data from a Simulation After designing a simulation, you will need to conduct the simulation and report the results. Include both numerical and graphical summaries of the simulation data, as well as an estimate of the probability of the desired outcome.



Real-WorldLink

Mark Price holds the record for the highest career free-throw percentage in the NBA at 90.4%.

Source: National Basketball Association

Andrew D. Bernstein/NBAE/Getty Images

Example 3 Conduct and Summarize Data from a Simulation

BASKETBALL **Refer to the simulation in Example 1. Conduct the simulation and report the results using appropriate numerical and graphical summaries.**

Make a frequency table and record the results after spinning the spinner 40 times.

Outcome	Tally	Frequency
Make Free Throw	𝍸 𝍸 𝍸 𝍸 𝍸 |	26
Miss Free Throw	𝍸 𝍸 ||||	14
Total		40

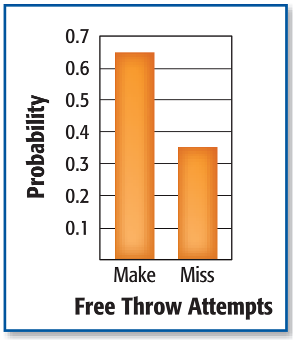
Based on the simulation data, calculate the probability that Allen will make his next free throw.

$$\frac{\text{number of made free throws}}{\text{number of free throws attempted}} = \frac{26}{40} \text{ or } 0.65$$

This is an *experimental probability.*

The probability that Allen makes his next free throw is 0.65 or 65%. Notice that this is close to the theoretical probability, 70%. So, the experimental probability of his missing the next free throw is 1 − 0.65 or 35%.

Make a bar graph of these results.



GuidedPractice

3. **EYE COLOR** Use a graphing calculator to conduct the simulation in Example 2. Then report the results using appropriate numerical and graphical summaries.

A **random variable** is a variable that can assume a set of values, each with fixed probabilities. For example, in the experiment of rolling two dice, the random variable X can represent the sums of the potential outcomes on the dice. The table shows some of the X-values assigned to outcomes from this experiment.

Sum of Outcomes of Rolling Two Dice

Outcome	X-Value
(1, 1)	2
(1, 2)	3
(2, 1)	3
(4, 5)	9
(6, 6)	12

Expected value, also known as mathematical expectation, is the average value of a random variable that one *expects* after repeating an experiment or simulation a theoretically infinite number of times. To find the expected value $E(X)$ of a random variable X, follow these steps.

KeyConcept Calculating Expected Value

Step 1 Multiply the value of X by its probability of occurring.

Step 2 Repeat Step 1 for all possible values of X.

Step 3 Find the sum of the results.

Since it is an average, an expected value does not have to be equal to a possible value of the random variable.

Example 4 Calculate Expected Value

DARTS Suppose a dart is thrown at the dartboard. The radius of the center circle is 1 centimeter and each successive circle has a radius 4 centimeters greater than the previous circle. The point value for each region is shown.

a. Let the random variable Y represent the point value assigned to a region on the dartboard. Calculate the expected value $E(Y)$ from each throw.

First calculate the geometric probability of landing in each region.

$$\text{Region 5} = \frac{\pi(1)^2}{\pi(1+4+4+4+4)^2} = \frac{1}{289} \qquad \text{Region 4} = \frac{\pi(4+1)^2 - \pi(1)^2}{\pi(17)^2} = \frac{24}{289}$$

$$\text{Region 3} = \frac{\pi(4+5)^2 - \pi(5)^2}{\pi(17)^2} = \frac{56}{289} \qquad \text{Region 2} = \frac{\pi(4+9)^2 - \pi(9)^2}{\pi(17)^2} = \frac{88}{289}$$

$$\text{Region 1} = \frac{\pi(4+13)^2 - \pi(13)^2}{\pi(17)^2} = \frac{120}{289}$$

$$E(Y) = 1 \cdot \frac{120}{289} + 2 \cdot \frac{88}{289} + 3 \cdot \frac{56}{289} + 4 \cdot \frac{24}{289} + 5 \cdot \frac{1}{289} \text{ or about } 1.96$$

The expected value of each throw is about 1.96.

b. Design a simulation to estimate the average value, or the average of the results of your simulation, of this game. How does this value compare with the expected value you found in part a?

Assign the integers 0–289 to accurately represent the probability data.

Region 1 = integers 1–120

Region 2 = integers 121–208

Region 3 = integers 209–264

Region 4 = integers 265–288

Region 5 = integer 289

Use a graphing calculator to generate 50 trials of random integers from 1 to 289. Record the results in a frequency table. Then calculate the average value of the outcomes.

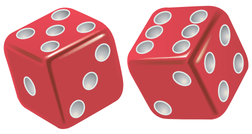
Outcome	Frequency
Region 1	16
Region 2	13
Region 3	13
Region 4	8
Region 5	0

$$\text{average value} = 1 \cdot \frac{16}{50} + 2 \cdot \frac{13}{50} + 3 \cdot \frac{13}{50} + 4 \cdot \frac{8}{50} + 5 \cdot \frac{0}{50} = 2.26$$

The average value 2.26 is greater than the expected value 1.96.

GuidedPractice

4. DICE If two dice are rolled, let the random variable X represent the sum of the potential outcomes.



A. Find the expected value $E(X)$.

B. Design and run a simulation to estimate the average value of this experiment. How does this value compare with the expected value you found in part A?

StudyTip

Geometric Probability Remember that when determining geometric probabilities with targets, we assume that the object lands within the target area, and that it is equally likely that the object will land anywhere in the region.

Math HistoryLink

Jakob Bernoulli

(1654–1705) Bernoulli was a Swiss mathematician. It seemed obvious to him that the more observations made of a given situation, the better one would be able to predict future outcomes. He provided scientific proof of his Law of Large Numbers in his work *Ars Conjectandi* (Art of Conjecturing), published in 1713.

North Wind/North Wind Picture Archives

The difference in the average value from the simulation and the expected value in Example 4 illustrates the **Law of Large Numbers**: as the number of trials of a random process increases, the average value will approach the expected value.

Check Your Understanding

● = Step-by-Step Solutions begin on page R14.

Examples 1, 3 **1. GRADES** Clara got an A on 80% of her first semester Biology quizzes. Design and conduct a simulation using a geometric model to estimate the probability that she will get an A on a second semester Biology quiz. Report the results using appropriate numerical and graphical summaries.

Examples 2–3 **2. FITNESS** The table shows the percent of members participating in four classes offered at a gym. Design and conduct a simulation to estimate the probability that a new gym member will take each class. Report the results using appropriate numerical and graphical summaries.

Class	Sign-Up %
tae kwon do	45%
yoga	30%
swimming	15%
kick-boxing	10%

Example 4 **3. CARNIVAL GAMES** The object of the game shown is to accumulate points by using a dart to pop the balloons. Assume that each dart will hit a balloon.

a. Calculate the expected value from each throw.

b. Design a simulation and estimate the average value of this game.

c. How do the expected value and average value compare?

Practice and Problem Solving

Extra Practice is on page R13.

Examples 1, 3 **Design and conduct a simulation using a geometric probability model. Then report the results using appropriate numerical and graphical summaries.**

4. BOWLING Bridget is a member of the bowling club at her school. Last season she bowled a strike 60% of the time.

⑤ **VIDEO GAMES** Ian works at a video game store. Last year he sold 95% of the new-release video games.

6. MUSIC Kadisha is listening to a CD with her CD player set on the random mode. There are 10 songs on the CD.

7. BOARD GAMES Pilar is playing a board game with eight different categories, each with questions that must be answered correctly in order to win.

Examples 2–3 **CCSS MODELING** **Design and conduct a simulation using a random number generator. Then report the results using appropriate numerical and graphical summaries.**

8. MOVIES A movie theater reviewed sales from the previous year to determine which genre of movie sold the most tickets. The results are shown at the right.

Genre	Ticket %
drama	40%
mystery	30%
comedy	25%
action	5%

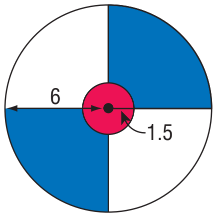
9. BASEBALL According to a baseball player's on-base percentages, he gets a single 60% of the time, a double 25% of the time, a triple 10% of the time, and a home run 5% of the time.

10. VACATION According to a survey done by a travel agency, 45% of their clients went on vacation to Europe, 25% went to Asia, 15% went to South America, 10% went to Africa, and 5% went to Australia.

11. TRANSPORTATION A car dealership's analysis indicated that 35% of the customers purchased a blue car, 30% purchased a red car, 15% purchased a white car, 15% purchased a black car, and 5% purchased any other color.

Example 4 **DARTBOARDS** **The dimensions of each dartboard below are given in inches. There is only one shot per game. Calculate the expected value of each dart game. Then design a simulation to estimate each game's average value. Compare the average and expected values. In each figure, 🟦 = 25, ☐ = 50, and 🟥 = 100 points.**

12.



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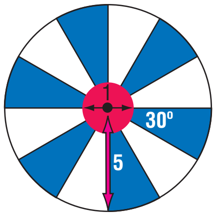

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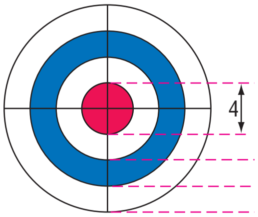


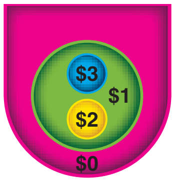
15. **CARDS** You are playing a team card game where a team can get 0 points, 1 point, or 3 points for a hand. The probability of your team getting 1 point for a hand is 60% and of getting 3 points for a hand is 5%.

a. Calculate your team's expected value for a hand.

b. Design a simulation and estimate your team's average value per hand.

c. Compare the values for parts **a** and **b**.

16. **DECISION MAKING** The object of the game shown is to win money by rolling a ball up an incline into regions with different payoff values. The probability that Susana will get \$0 in a roll is 55%, \$1 is 20%, \$2 is 20%, and \$3 is 5%.

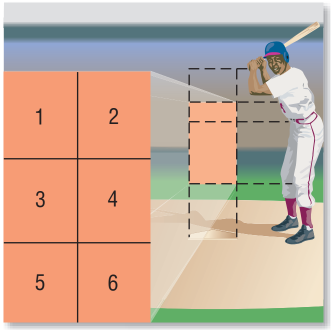


a. Suppose Susana pays \$1 to play. Calculate the expected payoff, which is the expected value minus the cost to play, for each roll.

b. Design a simulation to estimate Susana's average payoff for this game after she plays 10 times.

c. Should Susana play this game? Explain your reasoning.

17. **BASEBALL** Of his pitches thrown for strikes, a baseball pitcher wants to track which areas of the strike zone have a higher probability. He divides the strike zone into six congruent boxes as shown.



a. If a strike is equally likely to hit each box, what is the probability that he will throw a strike in each box?

b. Design a simulation to estimate the probability of a strike being thrown in each box.

c. Compare the values for parts **a** and **b**.

18. CCSS **MODELING** Cynthia used her statistics from last season to design a simulation using a random number generator to predict what she would score each time she got possession of the ball.

Integer Values	Points Scored	Frequency
1–14	0	31
15	1	0
16–28	2	17
29–30	3	2

a. Based on the frequency table, what did she assume was the theoretical probability that she would score two points in a possession?

b. What is Cynthia's average value for a possession? her expected value?

c. Would you expect the simulated data to be different? If so, explain how. If not, explain why.

19. **MULTIPLE REPRESENTATIONS** In this problem, you will investigate expected value.

a. Concrete Roll two dice 20 times and record the sum of each roll.

b. Numerical Use the random number generator on a calculator to generate 20 pairs of integers between 1 and 6. Record the sum of each pair.

c. Tabular Copy and complete the table below using your results from parts **a** and **b**.

Trial	Sum of Die Roll	Sum of Output from Random Number Generator
1		
2		
...		
20		

```
randInt(1,6,2)
                 {1 3}
                 {6 4}
                 {3 2}
                 {6 4}
                 {5 1}
```

d. Graphical Use a bar graph to graph the number of times each possible sum occurred in the first 5 rolls. Repeat the process for the first 10 rolls and then all 20 outcomes.

e. Verbal How does the shape of the bar graph change with each additional trial?

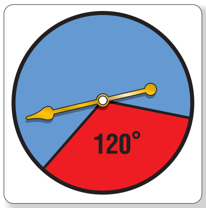
f. Graphical Graph the number of times each possible sum occurred with the random number generator as a bar graph.

g. Verbal How do the graphs of the die trial and the random number trial compare?

h. Analytical Based on the graphs, what do you think the expected value of each experiment would be? Explain your reasoning.

H.O.T. Problems Use Higher-Order Thinking Skills

20. **CCSS ARGUMENTS** An experiment has three equally likely outcomes *A*, *B*, and *C*. Is it possible to use the spinner shown in a simulation to predict the probability of outcome *C*? Explain your reasoning.



21 **REASONING** Can tossing a coin *sometimes*, *always*, or *never* be used to simulate an experiment with two possible outcomes? Explain.

22. **DECISION MAKING** A lottery consists of choosing 5 winning numbers from 31 possible numbers (0–30). The person who matches all 5 numbers, in any sequence, wins $1 million.

a. If a lottery ticket costs $1, should you play? Explain your reasoning by computing the expected payoff value, which is the expected value minus the ticket cost.

b. Would your decision to play change if the winnings increased to $5 million? if the winnings were only $0.5 million, but you chose from 21 numbers instead of 31 numbers? Explain.

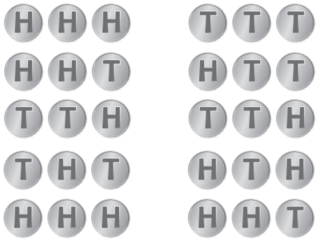
23. **REASONING** When designing a simulation where darts are thrown at targets, what assumptions need to be made and why are they needed?

24. **OPEN ENDED** Describe an experiment in which the expected value is not a possible outcome. Explain.

25. **WRITING IN MATH** How is expected value different from probability?

Standardized Test Practice

26. PROBABILITY Kaya tosses three coins at the same time and repeats the process 9 more times. Her results are shown below where H represents heads and T represents tails. Based on Kaya's data, what is the probability that at least one of the group of 3 coins will land with heads up?



A 0.1 **B** 0.2 **C** 0.3 **D** 0.9

27. ALGEBRA Paul collects comic books. He has 20 books in his collection, and he adds 3 per month. In how many months will he have a total of 44 books in his collection?

F 5 **G** 6 **H** 8 **J** 15

28. SHORT RESPONSE Alberto designed a simulation to determine how many times a player would roll a number higher than 4 on a die in a board game with 5 rolls. The table below shows his results for 50 trials. What is the probability that a player will roll a number higher than 4 two or more times in 5 rolls?

Number of Rolls Greater Than 4	Frequency
0	8
1	15
2	18
3	9
4	0
5	0

29. SAT/ACT If a jar contains 150 peanuts and 60 cashews, what is the approximate probability that a nut selected from the jar at random will be a cashew?

A 0.25 **C** 0.33 **E** 0.71

B 0.29 **D** 0.4

Spiral Review

Point X is chosen at random on $\overline{QT}$. Find the probability of each event. (Lesson 13-3)

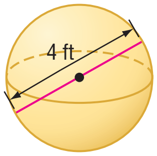
30. $P(X \text{ is on } \overline{QS})$

31. $P(X \text{ is on } \overline{RT})$

32. BOOKS Paige is choosing between 10 books at the library. What is the probability that she chooses 3 particular books to check out from the 10 initial books? (Lesson 13-2)

Find the surface area of each figure. Round to the nearest tenth. (Lesson 12-6)

33.



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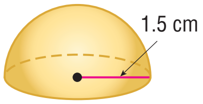

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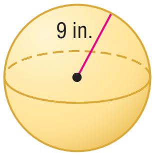


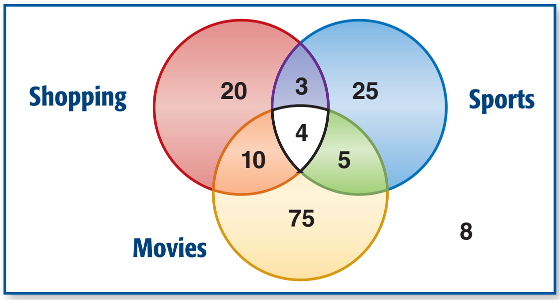
Skills Review

36. RECREATION A group of 150 students was asked what they like to do during their free time.

a. How many students like going to the movies or shopping?

b. Which activity was mentioned by 37 students?

c. How many students did *not* say they like movies?



LESSON 13-5

Probabilities of Independent and Dependent Events

Then

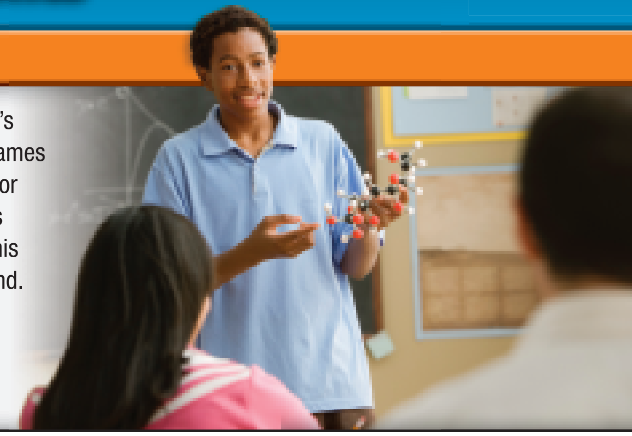
- You found simple probabilities

Now

1. Find probabilities of independent and dependent events.
2. Find probabilities of events given the occurrence of other events.

Why?

- The 18 students in Mrs. Turner's chemistry class are drawing names to determine who will give his or her presentation first. James is hoping to be chosen first and his friend Arturo wants to be second.



NewVocabulary

compound event
independent events
dependent events
conditional probability
probability tree

Common Core State Standards

Content Standards

S.CP.2 Understand that two events *A* and *B* are independent if the probability of *A* and *B* occurring together is the product of their probabilities, and use this characterization to determine if they are independent.

S.CP.3 Understand the conditional probability of *A* given *B* as $\frac{P(A \text{ and } B)}{P(B)}$, and interpret independence of *A* and *B* as saying that the conditional probability of *A* given *B* is the same as the probability of *A*, and the conditional probability of *B* given *A* is the same as the probability of *B*.

Mathematical Practices

2 Reason abstractly and quantitatively.
4 Model with mathematics.

1 Independent and Dependent Events

A **compound event** or *composite event* consists of two or more simple events. In the example above, James and Arturo being chosen to give their presentations first is a compound event. It consists of the event that James is chosen and the event that Arturo is chosen.

Compound events can be independent or dependent.

- Events *A* and *B* are **independent events** if the probability that *A* occurs does not affect the probability that *B* occurs.
- Events *A* and *B* are **dependent events** if the probability that *A* occurs in some way changes the probability that *B* occurs.

Consider choosing objects from a group of objects. If you replace the object each time, choosing additional objects are independent events. If you do not replace the object each time, choosing additional objects are dependent events.

Example 1 Identify Independent and Dependent Events

PT

Determine whether the events are *independent* or *dependent*. Explain your reasoning.

a. One coin is tossed, and then a second coin is tossed.

The outcome of the first coin toss in no way changes the probability of the outcome of the second coin toss. Therefore, these two events are *independent.*

b. In the class presentation example above, one student's name is chosen and not replaced, and then a second name is chosen.

After the first person is chosen, his or her name is removed and cannot be selected again. This affects the probability of the second person being chosen, since the sample space is reduced by one name. Therefore, these two events are *dependent.*

c. Wednesday's lottery numbers and Saturday's lottery numbers.

The numbers for one drawing have no bearing on the next drawing. Therefore, these two events are *independent*.

GuidedPractice

1A. A card is selected from a deck of cards and put back. Then a second card is selected.

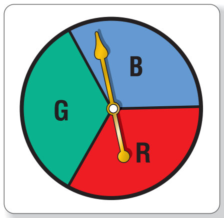
1B. Andrea selects a shirt from her closet to wear on Monday and then a different shirt to wear on Tuesday.

CORBIS

Suppose a coin is tossed and the spinner shown is spun. The sample space for this experiment is

{(H, B), (H, R), (H, G), (T, B), (T, R), (T, G)}.

Using the sample space, the probability of the compound event of the coin landing on heads and the spinner on green is $P(\text{H and G}) = \frac{1}{6}$.



Notice that this same probability can be found by multiplying the probabilities of each simple event.

$$P(\text{H}) = \frac{1}{2} \qquad P(\text{G}) = \frac{1}{3} \qquad P(\text{H and G}) = \frac{1}{2} \cdot \frac{1}{3} \text{ or } \frac{1}{6}$$

This example illustrates the first of two Multiplication Rules for Probability.

KeyConcept Probability of Two Independent Events

Words The probability that two independent events both occur is the product of the probabilities of each individual event.

Symbols If two events A and B are independent, then

$P(A \text{ and } B) = P(A) \cdot P(B)$.

This rule can be extended to any number of events.

ReadingMath

and The word *and* is a key word indicating to multiply probabilities.

Real-World Example 2 Probability of Independent Events

TRANSPORTATION Marisol and her friends are going to a concert. They put the slips of paper shown into a bag. If a person draws a yellow slip, he or she will ride in the van to the concert. A blue slip means he or she rides in the car.

Suppose Marisol draws a slip. Not liking the outcome, she puts it back and draws a second time. What is the probability that on each draw her slip is blue?

These events are independent since Marisol replaced the slip that she removed. Let B represent a blue slip and Y a yellow slip.

Draw 1 Draw 2

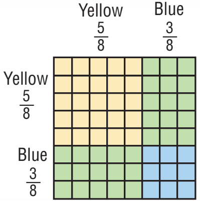
$$P(B \text{ and } B) = P(B) \cdot P(B) \qquad \text{Probability of independent events}$$

$$= \frac{3}{8} \cdot \frac{3}{8} \text{ or } \frac{9}{64} \qquad P(B) = \frac{3}{8}$$

So, the probability of Marisol drawing two blue slips is $\frac{9}{64}$ or about 14%.

StudyTip

Use an Area Model You can also use the area model shown below to calculate the probability that both slips are blue. The blue region represents the probability of drawing two successive blue slips. The area of this region is $\frac{9}{64}$ of the entire model.



GuidedPractice

Find each probability.

2A. A coin is tossed and a die is rolled. What is the probability that the coin lands heads up and the number rolled is a 6?

2B. Suppose you toss a coin four times. What is the probability of getting four tails?

The second of the Multiplication Rules of Probability addresses the probability of two dependent events.

KeyConcept Probability of Two Dependent Events

Words	The probability that two dependent events both occur is the product of the probability that the first event occurs and the probability that the second event occurs *after* the first event has already occurred.
Symbols	If two events A and B are dependent, then $P(A \text{ and } B) = P(A) \cdot P(B \mid A)$.

This rule can be extended to any number of events.

WatchOut!

Conditional Notation The "|" symbol in the notation $P(B|A)$ should not be interpreted as a division symbol.

The notation $P(B|A)$ is read *the probability that event B occurs given that event A has already occurred.* This is called **conditional probability**.

Real-WorldLink

A recent study found that with three or more teenage passengers, 85% of fatal crashes of passenger vehicles driven by teens involved driver error, almost 50% involved speeding, and almost 70% involved a single vehicle.

Source: National Safety Council

Example 3 Probability of Dependent Events

TRANSPORTATION Refer to Example 2. Suppose Marisol draws a slip and does not put it back. Then her friend Christian draws a slip. What is the probability that both friends draw a yellow slip?

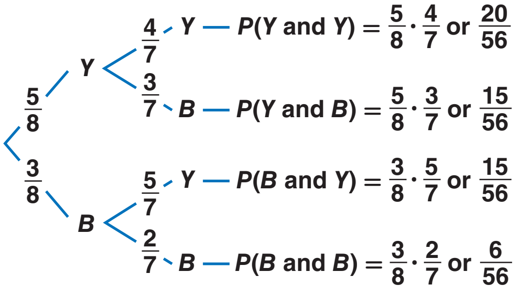
These events are dependent since Marisol does not replace the slip that she removed.

$P(Y \text{ and } Y) = P(Y) \cdot P(Y|Y)$ Probability of dependent events

$= \frac{5}{8} \cdot \frac{4}{7} \text{ or } \frac{5}{14}$ After the first yellow slip is chosen, 7 total slips remain, and 4 of those are yellow.

So, the probability that both friends draw yellow slips is $\frac{5}{14}$ or about 36%.

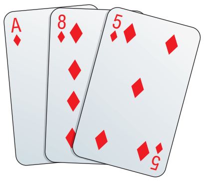
CHECK You can use a tree diagram with probabilities, called a **probability tree**, to verify this result. Calculate the probability of each simple event at the first stage and each conditional probability at the second stage. Then multiply along each branch to find the probability of each outcome.



The sum of the probabilities should be 1.

$\frac{20}{56} + \frac{15}{56} + \frac{15}{56} + \frac{6}{56} = \frac{56}{56}$ or 1 ✓

GuidedPractice



3. Three cards are selected from a standard deck of 52 cards. What is the probability that all three cards are diamonds if neither the first nor the second card is replaced?

Christoph Martin/Lifesize/Getty Images

2 Conditional Probabilities

Conditional Probabilities In addition to its use in finding the probability of two or more dependent events, conditional probability can be used when additional information is known about an event.

Suppose a die is rolled and it is known that the number rolled is odd. What is the probability that the number rolled is a 5?

There are only three odd numbers that can be rolled, so our sample space is reduced from {1, 2, 3, 4, 5, 6} to {1, 3, 5}. So, the probability that the number rolled is a 5 is $P(5|\text{odd}) = \frac{1}{3}$.

ReadingMath

Conditional Probability $P(5|\text{odd})$ is read *the probability that the number rolled is a 5 given that the number rolled is odd.*

Standardized Test Example 4 Conditional Probability

Ms. Fuentes' class is holding a debate. The 8 students participating randomly draw cards numbered with consecutive integers from 1 to 8.

- **Students who draw odd numbers will be on the Proposition Team.**
- **Students who draw even numbers will be on the Opposition Team.**

If Jonathan is on the Opposition Team, what is the probability that he drew the number 2?

A $\frac{1}{8}$ **B** $\frac{1}{4}$ **C** $\frac{3}{8}$ **D** $\frac{1}{2}$

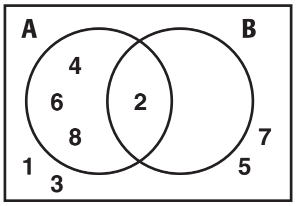
Read the Test Item

Since Jonathan is on the Opposition Team, he must have drawn an even number. So you need to find the probability that the number drawn was 2 given that the number drawn was even. This is a conditional probability problem.

Solve the Test Item

Let A be the event that an even number is drawn. Let B be the event that the number 2 is drawn.

Draw a Venn diagram to represent this situation. There are only four even numbers in the sample space, and only one out of these numbers is a 2. Therefore, the $P(B|A) = \frac{1}{4}$. The answer is B.



Test-TakingTip

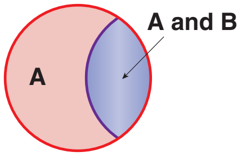
Use a Venn Diagram Use a Venn diagram to help you visualize the relationship between the outcomes of two events.

GuidedPractice

4. When two dice are rolled, what is the probability that one die is a 4, given that the sum of the two die is 9?

F $\frac{1}{6}$ **G** $\frac{1}{4}$ **H** $\frac{1}{3}$ **J** $\frac{1}{2}$

Since conditional probability reduces the sample space, the Venn diagram in Example 4 can be simplified as shown, with the intersection of the two events representing those outcomes in A and B. This suggests the following formula.



$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

KeyConcept Conditional Probability

The conditional probability of B given A is $P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$, where $P(A) \neq 0$.

Check Your Understanding

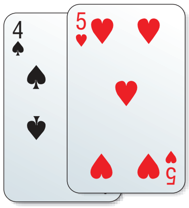
● = Step-by-Step Solutions begin on page R14.

Example 1 **Determine whether the events are *independent* or *dependent*. Explain.**

1. Jeremy took the SAT on Saturday and scored 1350. The following week he took the ACT and scored 23.

2. Alita's basketball team is in the final four. If they win, they will play in the championship game.

Example 2 **3.** **CARDS** A card is randomly chosen from a deck of 52 cards, replaced, and a second card is chosen. What is the probability of choosing both of the cards shown at the right?



Example 3 **4.** **TRANSPORTATION** Isaiah is getting on the bus after work. It costs $0.50 to ride the bus to his house. If he has 3 quarters, 5 dimes, and 2 nickels in his pocket, find the probability that he will randomly pull out two quarters in a row. Assume that the events are equally likely to occur.

Example 4 **5.** **GRIDDED RESPONSE** Every Saturday, 10 friends play dodgeball at a local park. To pick teams, they randomly draw cards with consecutive integers from 1 to 10. Odd numbers are on Team A, and even numbers are Team B. What is the probability that a player on Team B has drawn the number 10?

Practice and Problem Solving

Extra Practice is on page R13.

Examples 1–3 CCSS **REASONING** **Determine whether the events are *independent* or *dependent*. Then find the probability.**

6. In a game, you roll an even number on a die and then spin a spinner numbered 1 through 5 and get an odd number.

⑦ An ace is drawn, without replacement, from a deck of 52 cards. Then, a second ace is drawn.

8. In a bag of 3 green and 4 blue marbles, a blue marble is drawn and not replaced. Then, a second blue marble is drawn.

9. You roll two dice and get a 5 each time.

10. **GAMES** In a game, the spinner at the right is spun and a coin is tossed. What is the probability of getting an even number on the spinner and the coin landing on tails?

11. **GIFTS** Tisha's class is having a gift exchange. Tisha will draw first and her friend Brandi second. If there are 18 students participating, what is the probability that Brandi and Tisha draw each other's names?

12. **VACATION** A work survey found that 8 out of every 10 employees went on vacation last summer. If 3 employees' names are randomly chosen, with replacement, what is the probability that all 3 employees went on vacation last summer?

13. **CAMPAIGNS** The table shows the number of each color of Student Council campaign buttons Clemente has to give away. If given away at random, what is the probability that the first and second buttons given away are both red?

Button Color	Amount
blue	20
white	15
red	25
black	10

Example 4

14. A red marble is selected at random from a bag of 2 blue and 9 red marbles and not replaced. What is the probability that a second marble selected will be red?

15. A die is rolled. If the number rolled is greater than 2, find the probability that it is a 6.

16. A quadrilateral has a perimeter of 12 and all of the side lengths are odd integers. What is the probability that the quadrilateral is a rhombus?

17. A spinner numbered 1 through 12 is spun. Find the probability that the number spun is an 11 given that the number spun was an odd number.

18. **CLASSES** The probability that a student takes geometry and French at Satomi's school is 0.064. The probability that a student takes French is 0.45. What is the probability that a student takes geometry if the student takes French?

19 **TECHNOLOGY** At Bell High School, 43% of the students own a CD player and 28% own a CD player and an MP3 player. What is the probability that a student owns an MP3 player if he or she also owns a CD player?

20. **PROOF** Use the formula for the probability of two dependent events $P(A \text{ and } B)$ to derive the conditional probability formula for $P(B|A)$.

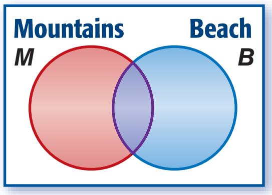
21. **TENNIS** A double fault in tennis is when the serving player fails to land their serve "in" without stepping on or over the service line in two chances. Kelly's first serve percentage is 40%, while her second serve percentage is 70%.

a. Draw a probability tree that shows each outcome.

b. What is the probability that Kelly will double fault?

c. Design a simulation using a random number generator that can be used to estimate the probability that Kelly double faults on her next serve.

22. **VACATION** A random survey was conducted to determine where families vacationed. The results indicated that $P(B) = 0.6$, $P(B \cap M) = 0.2$, and the probability that a family did not vacation at either destination is 0.1.



a. What is the probability that a family vacations in the mountains?

b. What is the probability that a family visiting the beach will also visit the mountains?

23. **DECISION MAKING** You are trying to decide whether you should expand a business. If you do not expand and the economy remains good, you expect \$2 million in revenue. If the economy is bad, you expect \$0.5 million. The cost to expand is \$1 million, but the expected revenue after the expansion is \$4 million in a good economy and \$1 million in a bad economy. You assume that the chances of a good and a bad economy are 30% and 70%, respectively. Use a probability tree to explain what you should do.

H.O.T. Problems Use Higher-Order Thinking Skills

24. CCSS **ARGUMENTS** There are n different objects in a bag. The probability of drawing object A and then object B without replacement is about 2.4%. What is the value of n? Explain.

25. **REASONING** If $P(A \mid B)$ is the same as $P(A)$, and $P(B \mid A)$ is the same as $P(B)$, what can be said about the relationship between events A and B?

26. **OPEN ENDED** Describe a pair of independent events and a pair of dependent events. Explain your reasoning.

27. **WRITING IN MATH** A medical journal reports the chance that a person smokes given that his or her parent smokes. Explain how you could determine the likelihood that a person's smoking and their parent's smoking are independent events.

Standardized Test Practice

28. PROBABILITY Shannon will be assigned at random to 1 of 6 P.E. classes throughout the day and 1 of 3 lunch times. What is the probability that she will be in the second P.E. class and the first lunch?

A $\frac{1}{18}$ **B** $\frac{1}{9}$ **C** $\frac{1}{6}$ **D** $\frac{1}{2}$

29. ALGEBRA Tameron downloaded 2 videos and 7 songs to his digital media player for $10.91. Jake downloaded 3 videos and 4 songs for $9.93. What is the cost of each video?

F $0.99 **H** $1.42

G $1.21 **J** $1.99

30. GRIDDED RESPONSE A bag of jelly beans contains 7 red, 11 yellow, and 13 green. Victoro picks two jelly beans from the bag without looking. What is the probability as a percent rounded to the nearest tenth that Victoro picks a green one and then a red one?

31. SAT/ACT If the probability that it will snow on Tuesday is $\frac{4}{13}$, then what is the probability that it will *not* snow?

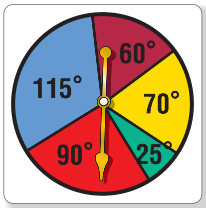
A $\frac{4}{9}$ **C** $\frac{13}{9}$ **E** $\frac{13}{4}$

B $\frac{9}{13}$ **D** $\frac{13}{5}$

Spiral Review

32. SOFTBALL Zoe struck out during 10% of her at bats last season. Design and conduct a simulation to estimate the probability that she will strike out at her next at bat this season. (Lesson 13-4)

Use the spinner to find each probability. The spinner is spun again if it stops on a line. (Lesson 13-3)



33. P(pointer landing on red)

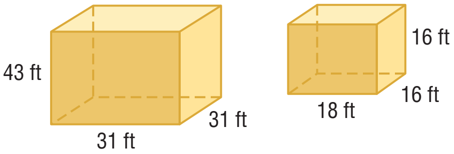
34. P(pointer landing on blue)

35. P(pointer landing on green)

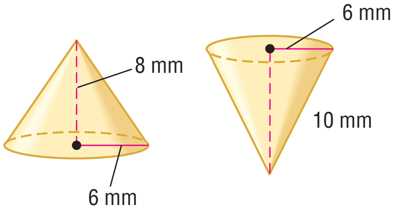
36. P(pointer landing on yellow)

Determine whether each pair of solids is *similar, congruent,* or *neither*. If the solids are similar, state the scale factor. (Lesson 12-8)

37.



38.

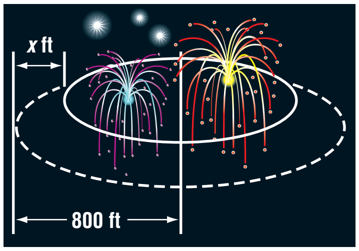


39. FIREWORKS Fireworks are shot from a barge on a river. There is an explosion circle inside which all of the fireworks will explode. Spectators sit outside a safety circle 800 feet from the center of the fireworks display. (Lesson 10-1)

a. Find the approximate circumference of the safety circle.

b. If the safety circle is 200 to 300 feet farther from the center than the explosion circle, find the range of values for the radius of the explosion circle.

c. Find the least and maximum circumferences of the explosion circle to the nearest foot.



Skills Review

Find the number of possible outcomes for each situation.

40. Blanca chooses from 5 different flavors of ice cream and 3 different toppings.

41. Perry chooses from 6 colors and 2 seat designs for his new mountain bike.

42. A rectangle has a perimeter of 12 and integer side lengths.

43. Three number cubes are rolled simultaneously.

EXTEND 13-5

Geometry Lab
Two-Way Frequency Tables

A **two-way frequency table** or *contingency table* is used to show the frequencies of data from a survey or experiment classified according to two variables, with the rows indicating one variable and the columns indicating the other.

CCSS Common Core State Standards
Content Standards
S.CP.4 Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities.
S.CP.6 Find the conditional probability of *A* given *B* as the fraction of *B*'s outcomes that also belong to *A*, and interpret the answer in terms of the model.
Mathematical Practices 5

Activity 1 Two-Way Frequency Table

PROM Michael asks a random sample of 160 upperclassmen at his high school whether or not they plan to attend the prom. He finds that 44 seniors and 32 juniors plan to attend the prom, while 25 seniors and 59 juniors do not plan to attend. Organize the responses into a two-way frequency table.

Step 1 Identify the variables. The students surveyed can be classified according *class* and *attendance.* Since the survey included only upperclassmen, the variable *class* has two categories: senior or junior. The variable *attendance* also has two categories: attending or not attending the prom.

Step 2 Create a two-way frequency table. Let the rows of the table represent *class* and the columns represent *attendance.* Then fill in the cells of the table with the information given.

Step 3 Add a *Totals* row and a *Totals* column to your table and fill in these cells with the correct sums.

Class	Attending the Prom	Not Attending the Prom	Totals
Senior	44	32	76
Junior	25	59	84
Totals	69	91	160

The frequencies reported in the *Totals* row and *Totals* column are called **marginal frequencies**, with the bottom rightmost cell reporting the total number of observations. The frequencies reported in the interior of the table are called **joint frequencies**. These show the frequencies of all possible combinations of the categories for the first variable with the categories for the second variable.

Analyze the Results

1. How many seniors were surveyed?

2. How many of the students that were surveyed plan to attend the prom?

A **relative frequency** is the ratio of the number of observations in a category to the total number of observations.

Activity 2 Two-Way Relative Frequency Table

PROM Convert the table from Activity 1 to a table of relative frequencies.

Step 1 Divide the frequency reported in each cell by the total number of respondents, 160.

Class	Attending the Prom	Not Attending the Prom	Totals
Senior	$\frac{44}{160}$	$\frac{32}{160}$	$\frac{76}{160}$
Junior	$\frac{25}{160}$	$\frac{59}{160}$	$\frac{84}{160}$
Totals	$\frac{69}{160}$	$\frac{91}{160}$	$\frac{160}{160}$

Step 2 Write each fraction as a percent rounded to the nearest tenth.

Class	Attending the Prom	Not Attending the Prom	Totals
Senior	27.5%	20%	47.5%
Junior	15.6%	36.9%	52.5%
Totals	43.1%	56.9%	100%

You can use joint and marginal relative frequencies to approximate conditional probabilities.

Activity 3 Conditional Probabilities

PROM Using the table from Activity 2, find the probability that a surveyed upperclassman plans to attend the prom given that he or she is a junior.

The probability that a surveyed upperclassman plans to attend the prom given that he or she is a junior is the conditional probability $P(\text{attending the prom} \mid \text{junior})$.

$$P(\text{attending the prom} \mid \text{junior}) = \frac{P(\text{attending the prom and junior})}{P(\text{junior})}$$

Conditional Probability

$$\approx \frac{0.156}{0.525} \text{ or } 29.7\%$$

P(attending the prom and junior) $= 15.6\%$ or 0.156, P(junior) $= 52.5\%$ or 0.525

Analyze and Apply

Refer to Activities 2 and 3.

3. If there are 285 upperclassmen, about how many would you predict plan to attend the prom?

4. Find the probability that a surveyed student is a junior and does not plan to attend the prom.

5. Find the probability that a surveyed student is a senior given that he or she plans to attend the prom.

6. What is a possible trend you notice in the data?

When survey results are classified according to variables, you may want to decide whether these variables are independent of each other. Variable A is considered independent of variable B if $P(A \text{ and } B) = P(A) \cdot P(B)$. In a two-way frequency table, you can test for the independence of two variables by comparing the joint relative frequencies with the products of the corresponding marginal relative frequencies.

Activity 4 Independence of Events

PROM Use the relative frequency table from Activity 2 to determine whether prom attendance is independent of class.

Calculate the expected joint relative frequencies if the two variables were independent. Then compare them to the actual relative frequencies.

For example, if 47.5% of respondents were seniors and 43.1% of respondents plan to attend the prom, then one would expect 47.5% • 43.1% or about 20.5% of respondents are seniors who plan to attend the prom.

Since the expected and actual joint relative frequencies are not the same, prom attendance for these respondents is not independent of class.

Class	Attending the Prom	Not Attending the Prom	Totals
Senior	27.5% (20.5%)	20% (27%)	47.5%
Junior	15.6% (22.6%)	36.9% (29.9%)	52.5%
Totals	43.1%	56.9%	100%

Note: The numbers in parentheses are the expected relative frequencies.

COLLECT DATA Design and conduct a survey of students at your school. Create a two-way relative frequency table for the data. Use your table to decide whether the data you collected indicate an independent relationship between the two variables. Explain your reasoning.

7. student gender and whether a student's car insurance is paid by the student or the student's parent(s)

8. student gender and whether a student buys or brings his or her lunch

LESSON 13-6 Probabilities of Mutually Exclusive Events

Then

- You found probabilities of independent and dependent events.

Now

- 1 Find probabilities of events that are mutually exclusive and events that are not mutually exclusive.
- 2 Find probabilities of complements.

Why?

- At Wayside High School, freshmen, sophomores, juniors, and seniors can all run for Student Council president. Dominic wants either a junior or a senior candidate to win the election. Trayvon wants either a sophomore or a female to win, but says, "If the winner is sophomore Katina Smith, I'll be thrilled!"

NewVocabulary

mutually exclusive events
complement

Common Core State Standards

Content Standards

S.CP.1 Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").

S.CP.7 Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, and interpret the answer in terms of the model.

Mathematical Practices

1 Make sense of problems and persevere in solving them.
4 Model with mathematics.

1 Mutually Exclusive Events

In Lesson 13-4, you examined probabilities involving the intersection of two or more events. In this lesson, you will examine probabilities involving the union of two or more events.

$$P(A \text{ and } B) \qquad\qquad P(A \text{ or } B)$$

Indicates an intersection of two sample spaces.

Indicates a union of two sample spaces.

To find the probability that one event occurs *or* another event occurs, you must know how the two events are related. If the two events cannot happen at the same time, they are said to be **mutually exclusive**. That is, the two events have no outcomes in common.

Real-World Example 1 Identify Mutually Exclusive Events

ELECTIONS **Refer to the application above. Determine whether the events are *mutually exclusive* or *not mutually exclusive*. Explain your reasoning.**

a. a junior winning the election or a senior winning the election

These events are mutually exclusive. There are no common outcomes—a student cannot be both a junior and a senior.

b. a sophomore winning the election or a female winning the election

These events are not mutually exclusive. A female student who is a sophomore is an outcome that both events have in common.

c. drawing an ace or a club from a standard deck of cards.

Since the ace of clubs represents both events, they are not mutually exclusive.

GuidedPractice

Determine whether the events are *mutually exclusive* or *not mutually exclusive*. Explain your reasoning.

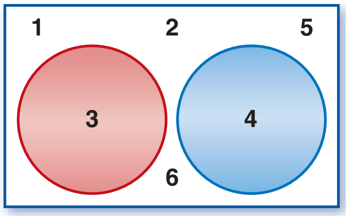
1A. selecting a number at random from the integers from 1 to 100 and getting a number divisible by 5 or a number divisible by 10

1B. drawing a card from a standard deck and getting a 5 or a heart

1C. getting a sum of 6 or 7 when two dice are rolled

Image Source Pink/Alamy

One way of finding the probability of two mutually exclusive events occurring is to examine their sample space.



When a die is rolled, what is the probability of getting a 3 or a 4? From the Venn diagram, you can see that there are two outcomes that satisfy this condition, 3 and 4. So,

$$P(3 \text{ and } 4) = \frac{2}{6} \text{ or } \frac{1}{3}.$$

Notice that this same probability can be found by adding the probabilities of each simple event.

$$P(3) = \frac{1}{6} \qquad P(4) = \frac{1}{6} \qquad P(3 \text{ and } 4) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} \text{ or } \frac{1}{3}$$

This example illustrates the first of two Addition Rules for Probability.

ReadingMath

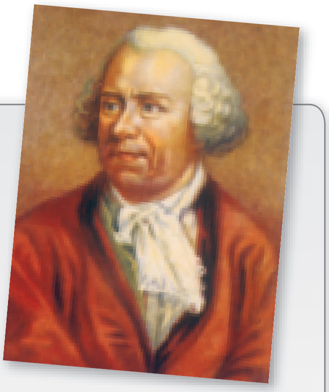
or The word *or* is a key word indicating that at least one of the events occurs. $P(A \text{ or } B)$ is read as *the probability that A occurs or that B occurs.*

KeyConcept Probability of Mutually Exclusive Events

Words If two events *A* and *B* are mutually exclusive, then the probability that *A* or *B* occurs is the sum of the probabilities of each individual event.

Example If two events *A* or *B* are mutually exclusive, then $P(A \text{ or } B) = P(A) + P(B)$.

This rule can be extended to any number of events.



Math HistoryLink

Leonhard Euler (1707–1783) Euler introduced graph theory in 1736 in a paper titled *Seven Bridges of Konigsburg*, a famous solved mathematics problem inspired by an actual place and situation. Also, Euler's formula relating the number of edges, vertices, and faces of a convex polyhedron is the origin of graph theory. Refer to Extend 13-6.

Real-World Example 2 Mutually Exclusive Events

MUSIC Ramiro makes a playlist that consists of songs from three different albums by his favorite artist. If he lets his MP3 player select the songs from this list at random, what is the probability that the first song played is from Album 1 or Album 2?

Ramiro's Playlist

Album	Number of Songs
1	10
2	12
3	13

These are mutually exclusive events, since the songs selected cannot be from both Album 1 and Album 2.

Let event $A1$ represent selecting a song from Album 1. Let event $A2$ represent selecting a song from Album 2. There are a total of $10 + 12 + 13$ or 35 songs.

$$P(A1 \text{ or } A2) = P(A1) + P(A2) \quad \text{Probability of mutually exclusive events}$$

$$= \frac{10}{35} + \frac{12}{35} \quad P(A1) = \frac{10}{35} \text{ and } P(A2) = \frac{12}{35}$$

$$= \frac{22}{35} \quad \text{Add.}$$

So, the probability that the first song played is from Album 1 or Album 2 is $\frac{22}{35}$ or about 63%.

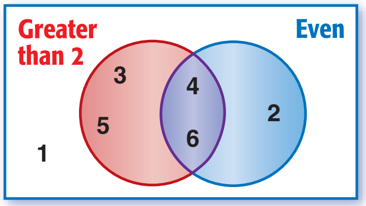
GuidedPractice

2A. Two dice are rolled. What is the probability that doubles are rolled or that the sum is 9?

2B. **CARNIVAL GAMES** If you win the ring toss game at a certain carnival, you receive a stuffed animal. If the stuffed animal is selected at random from among 15 puppies, 16 kittens, 14 frogs, 25 snakes, and 10 unicorns, what is the probability that a winner receives a puppy, a kitten, or a unicorn?

(l)akg-images, (r)Jill Braaten/The McGraw-Hill Companies

When a die is rolled, what is the probability of getting a number greater than 2 or an even number? From the Venn diagram, you can see that there are 5 numbers that are either greater than 2 or are an even number: 2, 3, 4, 5, and 6. So,



$$P(\text{greater than 2 or even}) = \frac{5}{6}.$$

Since it is possible to roll a number that is greater than 2 *and* an even number, these events are not mutually exclusive. Consider the probabilities of each individual event.

$$P(\text{greater than 2}) = \frac{4}{6} \qquad P(\text{even}) = \frac{3}{6}$$

If these probabilities were added, the probability of two outcomes, 4 and 6, would be counted twice—once for being numbers greater than 2 and once for being even numbers. You must subtract the probability of these common outcomes.

$$P(\textbf{greater than 2} \text{ or } \textbf{even}) = P(\textbf{greater than 2}) + P(\textbf{even}) - P(\textbf{greater than 2 and even})$$

$$= \frac{4}{6} + \frac{3}{6} - \frac{2}{6} \text{ or } \frac{5}{6}$$

This leads to the second of the Addition Rules for Probability.

KeyConcept Probability of Events That Are Not Mutually Exclusive

Words If two events A and B are not mutually exclusive, then the probability that A or B occurs is the sum of their individual probabilities minus the probability that both A and B occur.

Symbols If two events A and B are not mutually exclusive, then
$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

Real-World Example 3 Events That Are Not Mutually Exclusive

PT

Real-WorldLink

Juried art shows are shows in which artists are called to submit pieces and a panel of judges decides which art will be shown. They originated in the early 1800s to exhibit the work of current artists and educate the public.

Source: Humanities Web

ART The table shows the number and type of paintings Namiko has created. If she randomly selects a painting to submit to an art contest, what is the probability that she selects a portrait or an oil painting?

Namiko's Paintings

Media	Still Life	Portrait	Landscape
watercolor	4	5	3
oil	1	3	2
acrylic	3	2	1
pastel	1	0	5

Since some of Namiko's paintings are both portraits and oil paintings, these events are not mutually exclusive. Use the rule for two events that are not mutually exclusive. The total number of paintings from which to choose is 30.

$$P(\text{oil or portrait}) = P(\text{oil}) + P(\text{portrait}) - P(\text{oil and portrait})$$

$$= \frac{1+3+2}{30} + \frac{5+3+2+0}{30} - \frac{3}{30} \qquad \text{Substitution}$$

$$= \frac{6}{30} + \frac{10}{30} - \frac{3}{30} \text{ or } \frac{13}{30} \qquad \text{Simplify.}$$

The probability that Namiko selects a portrait or an oil painting is $\frac{13}{30}$ or about 43%.

GuidedPractice

3. What is the probability of drawing a king or a diamond from a standard deck of 52 cards?

Realistic Reflections/Getty Images

2 Probabilities of Complements

The **complement** of an event A consists of all the outcomes in the sample space that are not included as outcomes of event A.

When a die is rolled, the probability of getting a 4 is $\frac{1}{6}$. What is the probability of *not* getting a 4? There are 5 possible outcomes for this event: 1, 2, 3, 5, or 6. So, $P(\text{not } 4) = \frac{5}{6}$. Notice that this probability is also $1 - \frac{1}{6}$ or $1 - P(4)$.

KeyConcept Probability of the Complement of an Event

Words	The probability that an event will not occur is equal to 1 minus the probability that the event will occur.
Symbols	For an event A, $P(\text{not } A) = 1 - P(A)$.

Example 4 Complementary Events

RAFFLE Francisca bought 20 raffle tickets, hoping to win the $100 gift card to her favorite clothing store. If a total of 300 raffle tickets were sold, what is the probability that Francisca will not win the gift card?

Let event A represent selecting one of Francisca's tickets. Then find the probability of the complement of A.

$$P(\text{not } A) = 1 - P(A) \quad \text{Probability of a complement}$$

$$= 1 - \frac{20}{300} \quad \text{Substitution}$$

$$= \frac{280}{300} \text{ or } \frac{14}{15} \quad \text{Subtract and simplify.}$$

The probability that one of Francisca's tickets *will not* be selected is $\frac{14}{15}$ or about 93%.

ReadingMath

Complement The complement of event A can also be noted as A^C.

GuidedPractice

4. If the chance of rain is 70%, what is the probability that it will not rain?

ConceptSummary Probability Rules

Types of Events	Words	Probability Rule
Independent Events	The outcome of a first event *does not affect* the outcome of the second event.	If two events A and B are independent, then $P(A \text{ and } B) = P(A) \cdot P(B)$.
Dependent Events	The outcome of a first event *does affect* the outcome of the other event.	If two events A and B are dependent, then $P(A \text{ and } B) = P(A) \cdot P(B\|A)$.
Conditional	Additional information is known about the probability of an event.	The conditional probability of A given B is $P(A\|B) = \frac{P(A \text{ and } B)}{P(B)}$.
Mutually Exclusive Events	Events *do not share* common outcomes.	If two events A or B are mutually exclusive, then $P(A \text{ or } B) = P(A) + P(B)$.
Not Mutually Exclusive Events	Events *do share* common outcomes.	If two events A and B are not mutually exclusive, then $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.
Complementary Events	The outcomes of one event consist of all the outcomes in the sample space that are not outcomes of the other event.	For an event A, $P(\text{not } A) = 1 - P(A)$.

Real-WorldLink

About 81% of American motorists and their right-front passengers use a seat belt.

Source: National Highway Traffic Safety Administration

Real-World Example 5 Identify and Use Probability Rules

PT

SEAT BELTS Refer to the information at the left. Suppose two people are chosen at random from a group of 100 American motorists and passengers. If this group mirrors the population, what is the probability that at least one of them does not wear a seat belt?

Understand You know that 81% of Americans *do use* a seat belt. The phrase *at least one* means *one or more*. So, you need to find the probability that either

- the first person chosen does not use a seat belt *or*
- the second person chosen does not use a seat belt *or*
- both people chosen do not use a seat belt.

Plan The complement of the event described above is the event that both people chosen *do use* a seat belt. Find the probability of this event, and then find the probability of its complement.

Let event A represent choosing a person who does use a seat belt.

Let event B represent choosing a person who does use a seat belt after the first person has already been chosen.

These are two dependent events, since the outcome of the first event affects the probability of the outcome of the second event.

Solve

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$ Probability of dependent events

$$= \frac{81}{100} \cdot \frac{80}{99}$$ $P(A) = \frac{0.81(100)}{100}$ or $\frac{81}{100}$

$$= \frac{6480}{9900} \text{ or } \frac{36}{55}$$ Multiply.

$$P[\text{not } (A \text{ and } B)] = 1 - P(A \text{ and } B)$$ Probability of a complement

$$= 1 - \frac{36}{55}$$ Substitution

$$= \frac{19}{55}$$ Subtract.

So, the probability that at least one of the passengers does not use a seat belt is $\frac{19}{55}$ or about 35%.

Check Use logical reasoning to check the reasonableness of your answer.

The probability that one person chosen out of 100 *does not* wear his or her seat belt is (100 − 81)% or 19%. The probability that two people chosen out of 100 wear their seat belt should be greater than 19%. Since 35% > 19%, the answer is reasonable.

StudyTip

Key Probability Words When determining what type of probability you are dealing with in a situation, look for key words and correctly interpret their meaning.

and → independent or dependent events

or → mutually exclusive or not mutually exclusive

not → complementary events

and then → conditional

at least n → *n* or more

at most n → *n* or less

GuidedPractice

5. CELL PHONES According to an online poll, 35% of American motorists routinely use their cell phones while driving. Three people are chosen at random from a group of 100 motorists. What is the probability that

A. at least two of them use their cell phone while driving?

B. no more than one use their cell phone while driving?

Digital Vision/Alamy

Check Your Understanding

● = Step-by-Step Solutions begin on page R14.

Example 1 **Determine whether the events are *mutually exclusive* or *not mutually exclusive*. Explain your reasoning.**

1. drawing a card from a standard deck and getting a jack or a club

2. adopting a cat or a dog

Example 2 **3.** **JOBS** Adelaide is the employee of the month at her job. Her reward is to select at random from 4 gift cards, 6 coffee mugs, 7 DVDs, 10 CDs, and 3 gift baskets. What is the probability that an employee receives a gift card, coffee mug, or CD?

Example 3 **4.** **CLUBS** According to the table, what is the probability that a student in a club is a junior or on the debate team?

Club	Soph.	Junior	Senior
Key	12	14	8
Debate	2	6	3
Math	7	4	5
French	11	15	13

Example 4 **Determine the probability of each event.**

5. If you have a 2 in 10 chance of bowling a spare, what is the probability of missing the spare?

6. If the chance of living in a particular dorm is 75%, what is the probability of living in another dorm?

Example 5 **7.** **PROM** In Armando's senior class of 100 students, 91% went to the senior prom. If two people are chosen at random from the entire class, what is the probability that at least one of them did not go to prom?

Practice and Problem Solving

Extra Practice is on page R13.

Examples 1–3 **Determine whether the events are *mutually exclusive* or *not mutually exclusive*. Then find the probability. Round to the nearest tenth of a percent, if necessary.**

8. drawing a card from a standard deck and getting a jack or a six

⑨ rolling a pair of dice and getting doubles or a sum of 8

10. selecting a number at random from integers 1 to 20 and getting an even number or a number divisible by 3

11. tossing a coin and getting heads or tails

12. drawing an ace or a heart from a standard deck of 52 cards

13. rolling a pair of dice and getting a sum of either 6 or 10

14. **SPORTS** The table includes all of the programs offered at a sports complex and the number of participants aged 14–16. What is the probability that a player is 14 or plays basketball?

Graceland Sports Complex			
Age	Soccer	Baseball	Basketball
14	28	36	42
15	30	26	33
16	35	41	29

15. CCSS **MODELING** An exchange student is moving back to Italy, and her homeroom class wants to get her a going away present. The teacher takes a survey of the class of 32 students and finds that 10 people chose a card, 12 chose a T-shirt, 6 chose a video, and 4 chose a bracelet. If the teacher randomly selects the present, what is the probability that the exchange student will get a card or a bracelet?

Example 4 **Determine the probability of each event.**

16. rolling a pair of dice and not getting a 3

17. drawing a card from a standard deck and not getting a diamond

18. flipping a coin and not landing on heads

19. spinning a spinner numbered 1–8 and not landing on 5

20. **RAFFLE** Namid bought 20 raffle tickets. If a total of 500 raffle tickets were sold, what is the probability that Namid will not win the raffle?

21 **JOBS** Of young workers aged 18 to 25, 71% are paid by the hour. If two people are randomly chosen out of a group of 100 young workers, what is the probability that exactly one is paid by the hour?

Example 5 **22.** **RECYCLING** Suppose 31% of Americans recycle. If two Americans are chosen randomly from a group of 50, what is the probability that at most one of them recycles?

CARDS **Suppose you pull a card from a standard 52-card deck. Find the probability of each event.**

23. The card is a 4.

24. The card is red.

25. The card is a face card.

26. The card is not a face card.

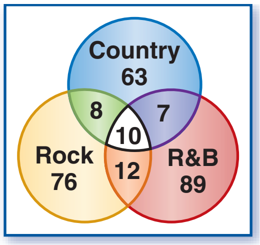
27. **MUSIC** A school carried out a survey of 265 students to see which types of music students would want played at a school dance. The results are shown in the Venn Diagram. Find each probability.

a. *P*(country or R&B)

b. *P*(rock and country or R&B and rock)

c. *P*(R&B but not rock)

d. *P*(all three)



H.O.T. Problems Use Higher-Order Thinking Skills

28. **CCSS** **CRITIQUE** Tetsuya and Mason want to determine the probability that a red marble will be chosen out of a bag of 4 red, 7 blue, 5 green, and 2 purple marbles. Is either of them correct? Explain your reasoning.

Tetsuya

$P(R) = \frac{4}{17}$

Mason

$P(R) = 1 - \frac{4}{18}$

29. **CHALLENGE** You roll 3 dice. What is the probability that the outcome of at least two of the dice will be less than or equal to 4? Explain your reasoning.

REASONING **Determine whether the following are mutually exclusive. Explain.**

30. choosing a quadrilateral that is a square and a quadrilateral that is a rectangle

31. choosing a triangle that is equilateral and a triangle that is equiangular

32. choosing a complex number and choosing a natural number

33. **OPEN ENDED** Describe a pair of events that are mutually exclusive and a pair of events that are not mutually exclusive.

34. **WRITING IN MATH** Explain why the sum of the probabilities of two mutually exclusive events is not always 1.

Standardized Test Practice

35. PROBABILITY Customers at a new salon can win prizes during opening day. The table shows the type and number of prizes. What is the probability that the first customer wins a manicure or a massage?

Prize	Number
manicure	10
pedicure	6
massage	3
facial	1

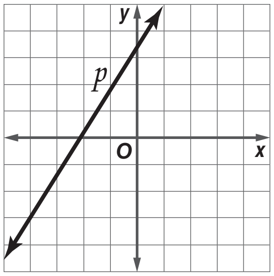
A 0.075 **C** 0.5

B 0.35 **D** 0.65

36. SHORT RESPONSE A cube numbered 1 through 6 is shown.

If the cube is rolled once, what is the probability that a number less than 3 or an odd number shows on the top face of the cube?

37. ALGEBRA What will happen to the slope of line *p* if it is shifted so that the *y*-intercept stays the same and the *x*-intercept approaches the origin?



F The slope will become negative.

G The slope will become zero.

H The slope will decrease.

J The slope will increase.

38. SAT/ACT The probability of choosing a peppermint from a certain bag of candy is 0.25, and the probability of choosing a chocolate is 0.3. The bag contains 60 pieces of candy, and the only types of candy in the bag are peppermint, chocolate, and butterscotch. How many butterscotch candies are in the bag?

A 25 **D** 33

B 27 **E** 45

C 30

Spiral Review

Determine whether the events are *independent* or *dependent*. Then find the probability. (Lesson 13-5)

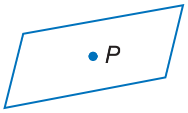
39. A king is drawn, without replacement, from a standard deck of 52 cards. Then, a second king is drawn.

40. You roll a die and get a 2. You roll another die and get a 3.

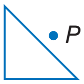
41. SPORTS A survey at a high school found that 15% of the athletes at the school play only volleyball, 20% play only soccer, 30% play only basketball, and 35% play only football. Design a simulation that can be used to estimate the probability that an athlete will play each of these sports. (Lesson 13-4)

Copy the figure and point *P*. Then use a ruler to draw the image of the figure under a dilation with center *P* and the scale factor *r* indicated. (Lesson 9-6)

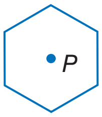
42. $r = \frac{1}{2}$



43. $r = 3$



44. $r = \frac{1}{5}$

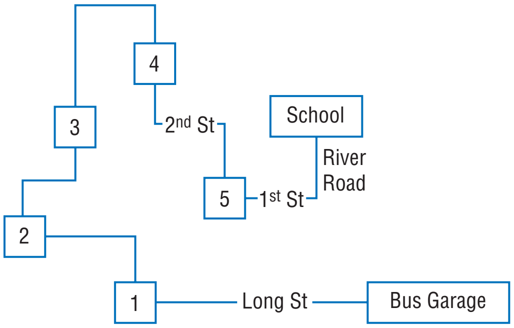


EXTEND 13-6 Geometry Lab
Graph Theory

Mathematical structures can be used to model relationships in a set. The study of these graphs is called *graph theory*. These **vertex-edge graphs** are not like graphs that can be seen on a coordinate plane. Each graph, also called a **network**, is a collection of vertices, called **nodes**, and segments, called **edges**, that connect the nodes.

The bus route in the figure is an example of a network. The school, each stop, and the garage are nodes in the network. The connecting streets, such as Long Street, are edges.

This is an example of a **traceable network** because all of the nodes are connected, and each edge is used once in the network.



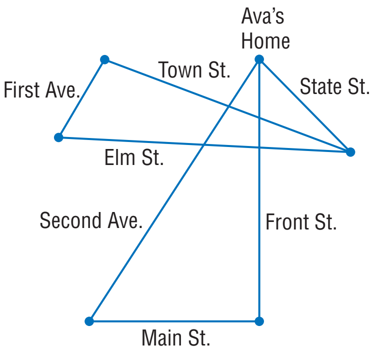
Activity 1

The graph represents the streets on Ava's newspaper route. To complete her route as quickly as possible, how can Ava ride her bike down each street only once?

Step 1 Copy the graph onto your paper.

Step 2 Beginning at Ava's home, trace over her route without lifting your pencil. Remember to trace each edge only once.

Step 3 Describe Ava's route.



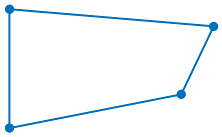
Analyze

1. Is there more than one traceable route that begins at Ava's house? If so, how many?

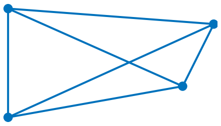
2. If it does not matter where Ava starts, how many traceable routes are possible?

Is each graph traceable? Write *yes* or *no*. Explain your reasoning.

3.



4.



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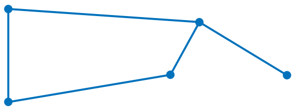


6. The campus for Centerburgh High School has five buildings built around the edge of a circular courtyard. There is a sidewalk between each pair of buildings.

a. Draw a graph of the campus. Is the graph traceable?

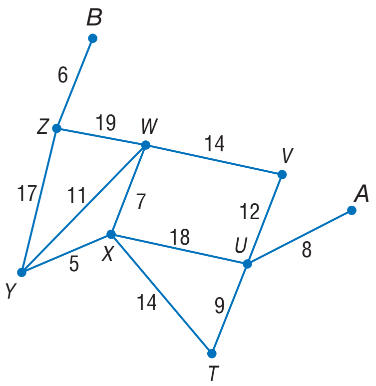
b. Suppose there are no sidewalks between pairs of adjacent buildings. Is it possible to reach all five buildings without walking down any sidewalk more than once?

7. REASONING Write a rule for determining whether a graph is traceable.

In a network, routes from one vertex to another are also called *paths*. **Weighted vertex-edge graphs** are graphs in which a value, or **weight**, is assigned to each edge. The **weight of a path** is the sum of the weights of the edges along the path. The **efficient route** is the path with the minimum weight.

Activity 2

The edges of the network have different weights. Find the efficient route from *A* to *B*.



Step 1 Find all of the possible paths from *A* to *B*. Label each path with the letters of the nodes along the path.

Step 2 Trace each path and add the weights of each edge. The path with the least weight is the efficient route: *A-U-X-Y-Z-B*. The weight is 54.

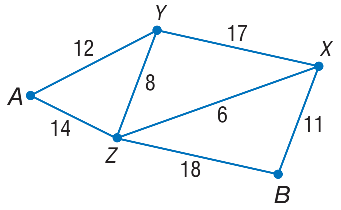
Pay attention to the weights when determining the efficient route. It may not be the path with the fewest edges.

Model and Analyze

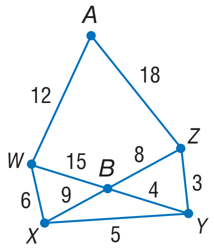
8. What is the longest path from *A* to *B* that does not cover any edges more than once?

Determine the efficient route from *A* to *B* for each network.

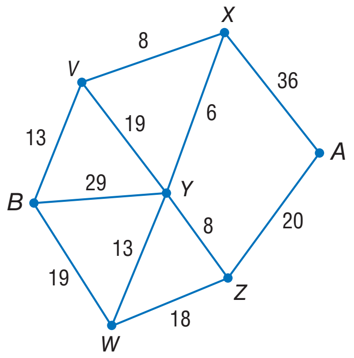
9.



10.



11.

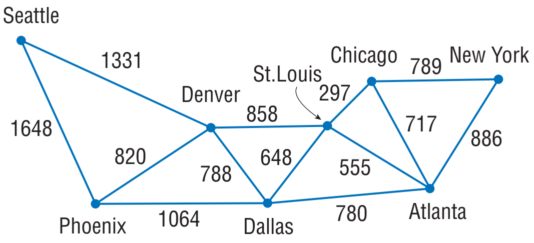


12. OPEN ENDED Create a network with 8 nodes and an efficient route with a value of 25.

13. WRITING IN MATH Explain your method for determining the efficient route of a network.

14. TRAVEL Use the graph at the right to find each efficient route.

- **a.** from Phoenix to New York
- **b.** from Seattle to Atlanta



15. *Six Degrees of Separation* is a well-known example of graph theory. In this case, each person is a node and people are linked by an edge when they know each other.

- **a.** Make a graph of the situation. Directly connect yourself to three other people that you know personally. This represents the first degree of separation.
- **b.** Expand the graph to show the first three degrees of separation. Name a person who is within 3 degrees of you, and list the path.

CHAPTER 13 Study Guide and Review

Study Guide

KeyConcepts

Representing Sample Spaces (Lesson 13-1)

- The sample space of an experiment is the set of all possible outcomes. It can be determined by using an organized list, a table, or a tree diagram.

Permutations and Combinations (Lesson 13-2)

- A permutation of n objects taken r at a time is given by $_nP_r = \frac{n!}{(n-r)!}$.
- A combination of n objects taken r at a time is given by $_nC_r = \frac{n!}{(n-r)!r!}$.
- Permutations should be used when order is important, and combinations should be used when order is not important.

Geometric Probability (Lesson 13-3)

- If a region A contains a region B and a point E in region A is chosen at random, then the probability that point E is in region B is $\frac{\text{area of region } B}{\text{area of region } A}$.

Simulations (Lesson 13-4)

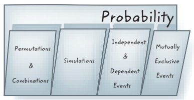
- A simulation uses a probability model to recreate a situation again and again so that the likelihood of various outcomes can be estimated.

Probabilities of Compound Events (Lessons 13-5 and 13-6)

- If event A does not affect the outcome of event B, then the events are independent and $P(A \text{ and } B) = P(A) \cdot P(B)$.
- If two events A and B are dependent, then $P(A \text{ and } B) = P(A) \cdot P(B|A)$.
- If two events A and B cannot happen at the same time, they are mutually exclusive and $P(A \text{ or } B) = P(A) + P(B)$.
- If two events A and B are not mutually exclusive, then $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

FOLDABLES® **Study**Organizer

Be sure the Key Concepts are noted in your Foldable.



KeyVocabulary

circular permutation	(p. 925)
combination	(p. 926)
complement	(p. 959)
compound events	(p. 947)
conditional probability	(p. 949)
dependent events	(p. 947)
expected value	(p. 941)
factorial	(p. 922)
Fundamental Counting Principle	(p. 917)
geometric probability	(p. 931)
independent events	(p. 947)
mutually exclusive events	(p. 956)
permutation	(p. 922)
probability model	(p. 939)
probability tree	(p. 949)
random variable	(p. 941)
sample space	(p. 915)
simulation	(p. 939)
tree diagram	(p. 915)

VocabularyCheck

State whether each sentence is *true* or *false*. If *false*, replace the underlined term to make a true sentence.

1. A <u>tree diagram</u> uses line segments to display possible outcomes.
2. A <u>permutation</u> is an arrangement of objects in which order is NOT important.
3. Determining the arrangement of people around a circular table would require <u>circular permutation</u>.
4. Tossing a coin and then tossing another coin is an example of <u>dependent events</u>.
5. <u>Geometric probability</u> involves a geometric measure such as length or area.
6. $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, is an example of a <u>factorial</u>.
7. The set of all possible outcomes is the <u>sample space</u>.
8. Combining a coin toss and a roll of a die makes a <u>simple</u> event.
9. Grant flipped a coin 200 times to create a <u>probability tree</u> of the experiment.
10. Drawing two socks out of a drawer without replacing them are examples of <u>mutually exclusive events</u>.

Lesson-by-Lesson Review

13-1 Representing Sample Spaces

11. **POPCORN** A movie theater sells small (S), medium (M), and large (L) size popcorn with the choice of no butter (NB), butter (B), and extra butter (EB). Represent the sample space for popcorn orders by making an organized list, a table, and a tree diagram.

12. **SHOES** A pair of men's shoes comes in whole sizes 5 through 13 in navy, brown, or black. How many different pairs could be selected?

Example 1

Three coins are tossed. Represent the sample space for this experiment by making an organized list.

Pair each possible outcome from the first toss with the possible outcomes from the second toss and third toss.

HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

13-2 Probability with Permutations and Combinations

13. **DINING** Three boys and three girls go out to eat together. The restaurant only has round tables. Fred does not want any girl next to him and Gena does not want any boy next to her. How many arrangements are possible?

14. **DANCE** The dance committee consisted of 10 students. The committee will select three officers at random. What is the probability that Alice, David, and Carlene are selected?

15. **COMPETITION** From 32 students, 4 are to be randomly chosen for an academic challenge team. In how many ways can this be done?

Example 2

For a party, Lucita needs to seat four people at a round table. How many combinations are possible?

Since there is no fixed reference point, this is a circular permutation.

$P_n = (n - 1)!$ Formula for circular permutation

$P_4 = (4 - 1)!$ $n = 4$

$= 3!$ or 6 Simplify.

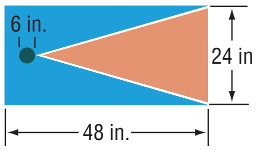
So, there are 6 ways for Casey to seat four people at a round table.

13-3 Geometric Probability

16. **GAMES** Measurements for a beanbag game are shown. What is the probability of each event?

   a. P(hole)

   b. P(no hole)



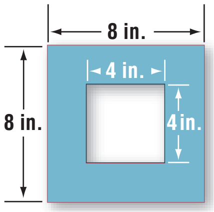
17. **POOL** Morgan, Phil, Callie, and Tyreese are sitting on the side of a pool in that order. Morgan is 2 feet from Phil. Phil is 4 feet from Callie. Callie is 3 feet from Tyreese. Oscar joins them.

   a. Find the probability that Oscar sits between Morgan and Phil.

   b. Find the probability that Oscar sits between Phil and Tyreese.

Example 3

A carnival game is shown.



a. If Khianna threw 10 beanbags at the board, what is the probability that the beanbag went in the hole?

Area of hole $= 4 \cdot 4 = 16$

Area of board $= (8 \cdot 8) - 16 = 64 - 16$ or 48

$P(\text{hole}) = \frac{16}{64}$ or about 25%

b. What is the probability that the beanbag did not go in the hole?

$P(\text{no hole}) = \frac{48}{64}$ or about 75%

Study Guide and Review *Continued*

13-4 Simulations

For each of the following, describe how you would use a geometric probability model to design a simulation.

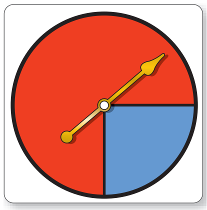
18. **POLO** Max scores 35% of the goals his team earns in each water polo match.

19. **BOOKS** According to a survey, people buy 30% of their books in October, November, and December, 22% during January, February, and March, 23% during April, May, and June, and 25% during July, August, and September.

20. **OIL** The United States consumes 17.3 million barrels of oil a day. 63% is used for transportation, 4.9% is used to generate electricity, 7.8% is used for heating and cooking, and 24.3% is used for industrial processes.

Example 4

Darius made 75% of his field goal kicks last season. Design a simulation that can be used to estimate the probability that he will make his next field goal kick this season.



Use a spinner that is divided into 2 sectors. Make one sector red containing 75% of the spinner's area and the other blue containing 25% of the spinner's area.

Spin the spinner 50 times. Each spin represents kicking a field goal. A successful trial will be a made field goal, and a failed trial will be a missed field goal.

13-5 Probabilities of Independent and Dependent Events

21. **MARBLES** A box contains 3 white marbles and 4 black marbles. What is the probability of drawing 2 black marbles and 1 white marble in a row without replacing any marbles?

22. **CARDS** Two cards are randomly chosen from a standard deck of cards with replacement. What is the probability of successfully drawing, in order, a three and then a queen?

23. **PIZZA** A nationwide survey found that 72% of people in the United States like pizza. If 3 people are randomly selected, what is the probability that all three like pizza?

Example 5

A bag contains 3 red, 2 white, and 6 blue marbles. What is the probability of drawing, in order, 2 red and 1 blue marble without replacement?

Since the marbles are not being replaced, the events are dependent events.

$$P(\text{red, red, blue}) = P(\text{red}) \cdot P(\text{red}) \cdot P(\text{blue})$$

$$= \frac{3}{11} \cdot \frac{2}{10} \cdot \frac{6}{9}$$

$$= \frac{2}{55} \text{ or about } 3.6\%$$

13-6 Probabilities of Mutually Exclusive Events

24. **ROLLING DICE** Two dice are rolled. What is the probability that the sum of the numbers is 7 or 11?

25. **CARDS** A card is drawn from a deck of cards. Find the probability of drawing a 10 or a diamond.

26. **RAFFLE** A bag contains 40 raffle tickets numbered 1 through 40.

- **a.** What is the probability that a ticket chosen is an even number or less than 5?
- **b.** What is the probability that a ticket chosen is greater than 30 or less than 10?

Example 6

Two dice are rolled. What is the probability that the sum is 5 or doubles are rolled?

These are mutually exclusive events because the sum of doubles can never equal 5.

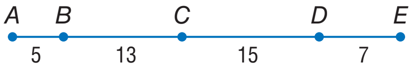
$$P(\text{sum is 5 or doubles}) = P(\text{sum is 5}) + P(\text{doubles})$$

$$= \frac{4}{36} + \frac{6}{36}$$

$$= \frac{5}{18} \text{ or about } 27.8\%$$

CHAPTER 13 Practice Test

Point X is chosen at random on $\overline{AE}$. Find the probability of each event.



1. $P(X \text{ is on } \overline{AC})$

2. $P(X \text{ is on } \overline{CD})$

3. **BASEBALL** A baseball team fields 9 players. How many possible batting orders are there for the 9 players?

4. **TRAVEL** A traveling salesperson needs to visit four cities in her territory. How many distinct itineraries are there for visiting each city once?

Represent the sample space for each experiment by making an organized list, a table, and a tree diagram.

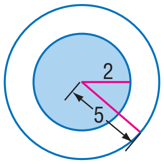
5. A box has 1 red ball, 1 green ball, and 1 blue ball. Two balls are drawn from the box one after the other, without replacement.

6. Shinsuke wants to adopt a pet and goes to his local humane society to find a dog or cat. While he is there, he decides to adopt two pets.

7. **ENGINEERING** An engineer is analyzing three factors that affect the quality of semiconductors: temperature, humidity, and material selection. There are 6 possible temperature settings, 4 possible humidity settings, and 6 choices of materials. How many combinations of settings are there?

8. **SPELLING** How many distinguishable ways are there to arrange the letters in the word "bubble"?

9. **PAINTBALL** Cordell is shooting a paintball gun at the target. What is the probability that he will shoot the shaded region?



10. **SHORT RESPONSE** What is the probability that a phone number using the numbers 7, 7, 7, 2, 2, 2, and 6 will be 622-2777?

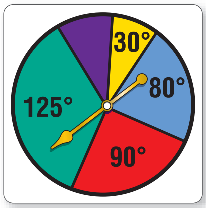
11. **TICKETS** Fifteen people entered the drawing at the right. What is the probability that Jodi, Dan, and Pilar all won the tickets?

Determine whether the events are *independent* or *dependent*. Then find the probability.

12. A deck of cards has 5 yellow, 5 pink, and 5 orange cards. Two cards are chosen from the deck with replacement. Find P(the first card is pink and the second card is pink).

13. There are 6 green, 2 red, 2 brown, 4 navy, and 2 purple marbles in a hat. Sadie picks 2 marbles from the hat without replacement. What is the probability that the first marble is brown and the second marble is not purple?

Use the spinner to find each probability. If the spinner lands on a line, it is spun again.



14. P(pointer landing on purple)

15. P(pointer landing on red)

16. P(pointer not landing on yellow)

17. **FOOTBALL** According to a football team's offensive success rate, the team punts 40% of the time, kicks a field goal 30% of the time, loses possession 5% of the time, and scores a touchdown 25% of the time. Design a simulation using a random number generator. Report the results using appropriate numerical and graphical summaries.

Determine whether the events are *mutually exclusive* or *not mutually exclusive*. Explain your reasoning.

18. a person owning a car and a truck

19. rolling a pair of dice and getting a sum of 7 and 6 on the face of one die

20. a playing card being both a spade and a club

21. **GRADES** This quarter, Todd earned As in his classes 45% of the time. Design and conduct a simulation using a geometric probability model. Then report the results using appropriate numerical and graphical summaries.

CHAPTER 13 Preparing for Standardized Tests

Organize Data

Sometimes you may be given a set of data that you need to analyze in order to solve items on a standardized test. Use this section to practice organizing data and to help you solve problems.

Mike Kemp/Rubberball Productions/Getty Images

Strategies for Organizing Data

Step 1

When you are given a problem statement containing data, consider:

- **making a list** of the data.
- **using a table** to organize the data.
- **using a data display** (such as a *bar graph, Venn diagram, circle graph, line graph, box-and-whisker plot,* etc.) to organize the data.

Step 2

Organize the data.

- Create your table, list, or data display.
- If possible, fill in any missing values that can be found by intermediate computations.

Step 3

Analyze the data to solve the problem.

- Reread the problem statement to determine what you are being asked to solve.
- Use the properties of geometry and algebra to work with the organized data and solve the problem.
- If time permits, go back and check your answer.

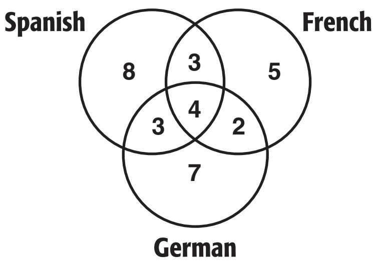
Standardized Test Example

Read the problem. Identify what you need to know. Then use the information in the problem to solve.

Of the students who speak a foreign language at Marie's school, 18 speak Spanish, 14 speak French, and 16 speak German. There are 8 students who only speak Spanish, 7 who speak only German, 3 who speak Spanish and French, 2 who speak French and German, and 4 who speak all three languages. If a student is selected at random, what is the probability that he or she speaks Spanish or German, but not French?

A $\frac{7}{12}$ B $\frac{9}{16}$ C $\frac{2}{5}$ D $\frac{5}{18}$

Read the problem carefully. The data is difficult to analyze as it is presented. Use a Venn diagram to organize the data and solve the problem.



Step 1 Draw three circles, each representing a language.

Step 2 Fill in the data given in the problem statement.

Step 3 Fill in the missing values. For example, you know that 18 students speak Spanish and 14 students speak French.

$18 - 8 - 3 - 4 = 3$ (Spanish and German)
$14 - 3 - 4 - 2 = 5$ (only French)

Step 4 Solve the problem. You are asked to find the probability that a randomly selected student speaks Spanish or German, but not French. From the Venn diagram, you can see that there are 32 total students. Of these, $8 + 3 + 7$, or 18 students speak Spanish or German, but not French. So, the probability is $\frac{18}{32}$ or $\frac{9}{16}$. So, the correct answer is B.

Exercises

Read the problem. Identify what you need to know. Then organize the data to solve the problem.

1. Alana has the letter tiles A, H, M, and T in a bag. If she selects a permutation of the tiles at random, what is the probability she will spell the word MATH?

A $\frac{1}{4}$
C $\frac{3}{50}$
B $\frac{1}{12}$
D $\frac{1}{24}$

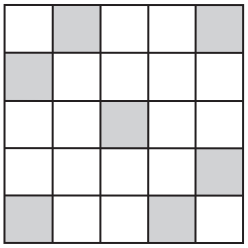
2. The table below shows the number of freshmen, sophomores, juniors, and seniors involved in basketball, soccer, and volleyball. What is the probability that a randomly selected student is a junior or plays volleyball?

Sport	Fr	So	Jr	Sr
Basketball	7	6	5	6
Soccer	6	4	8	7
Volleyball	9	2	4	6

F $\frac{4}{21}$
H $\frac{5}{17}$
G $\frac{5}{21}$
J $\frac{17}{35}$

3. Find the probability that a point chosen at random lies in the shaded region.

A 0.22
C 0.28
B 0.25
D 0.32



4. There are 10 sophomore, 8 junior, and 9 senior members in student council. Each member is assigned to help plan one school activity during the year. There are 4 sophomores working on the field day and 6 working on the pep rally. Of the juniors, 2 are working on the field day and 5 are working on the school dance. There are 2 seniors working on the pep rally. If each activity has a total of 9 students helping to plan it, what is the probability that a randomly selected student council member is a junior or is working on the field day?

F $\frac{1}{5}$
H $\frac{5}{9}$
G $\frac{4}{18}$
J $\frac{2}{3}$

CHAPTER 13

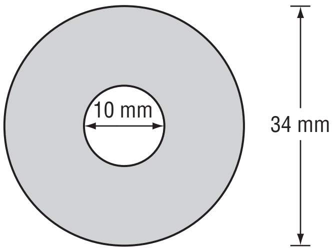
Standardized Test Practice

Cumulative, Chapters 1 through 13

Multiple Choice

Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. A machine is making steel washers by cutting out 10-millimeter circular disks from 34-millimeter circular disks as shown below. What is the area of each washer to the nearest tenth?



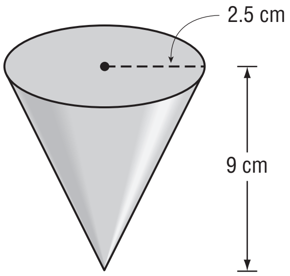
A 75.4 mm^2

B 829.4 mm^2

C 986.5 mm^2

D 3317.5 mm^2

2. How much paper is needed to make the drinking cup below? Round to the nearest tenth.



F 73.4 cm^2

G 70.7 cm^2

H 67.9 cm^2

J 58.8 cm^2

3. Which of the following properties of real numbers justifies the statement below?

If $3x - 2 = 7x + 12$, then
$3x - 2 + 2 = 7x + 12 + 2$.

A Addition Property of Equality

B Reflection Property of Equality

C Subtraction Property of Equality

D Symmetric Property of Equality

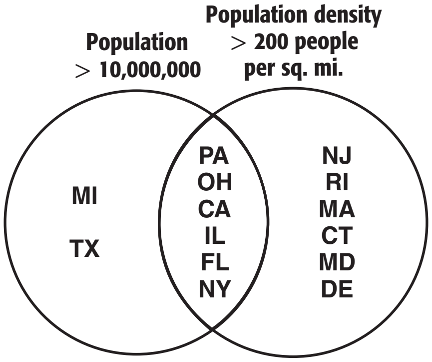
Test-Taking Tip

Question 4 What is the probability of rolling doubles with two number cubes? Multiply this by the number of trials.

4. What is the expected number of times Clarence will roll doubles with two number cubes in 90 trials? (Doubles occur when both number cubes show the same number in a trial.)

F 6
G 9
H 10
J 15

5. The Venn diagram shows the states in the U.S. in which the population is greater than 10,000,000 and the population density is greater than 200 people per square mile. Which statement is false?



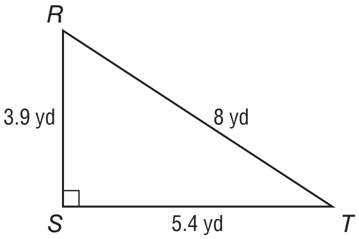
A In California (CA), the population is greater than 10,000,000, and the density is greater than 200.

B In Maryland (MD), the density is greater than 200.

C 14 states have a density greater than 200.

D 8 states have a population greater than 10,000,000.

6. Which of the following correctly shows the relationship between the angle measures of triangle *RST*?



F $m\angle S < m\angle R < m\angle T$

G $m\angle T < m\angle S < m\angle R$

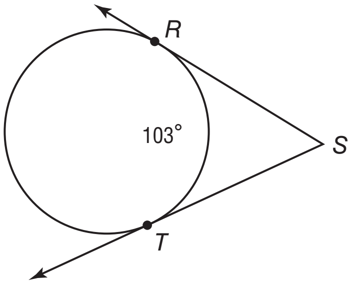
H $m\angle R < m\angle S < m\angle T$

J $m\angle T < m\angle R < m\angle S$

Short Response/Gridded Response

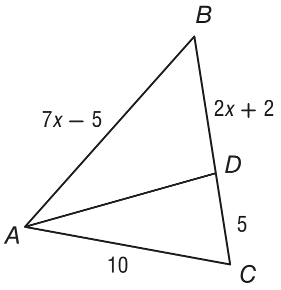
Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

7. GRIDDED RESPONSE What is $m\angle S$ in the figure below? Express your answer in degrees.



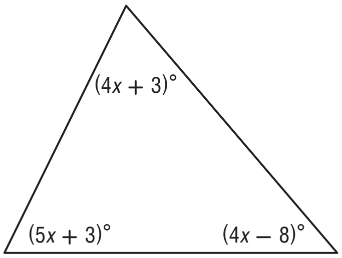
8. Does the figure have rotational symmetry? If so, give the order of symmetry.

9. GRIDDED RESPONSE Segment AD bisects $\angle CAB$ in the triangle below. What is the value of x?

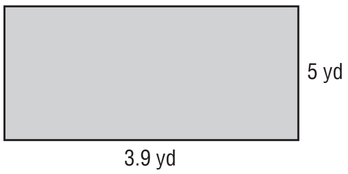


10. GRIDDED RESPONSE Armando leans an 18-foot ladder against the side of his house to clean out the gutters. The base of the ladder is 5 feet from the wall. How high up the side of the house does the ladder reach? Express your answer in feet, rounded to the nearest tenth.

11. Solve for x in the triangle below.



12. What effect does doubling the dimensions of the rectangle below have on its area and perimeter?



Extended Response

Record your answers on a sheet of paper.
Show your work.

13. A bag contains 3 red chips, 5 green chips, 2 yellow chips, 4 brown chips, and 6 purple chips. One chip is chosen at random, the color noted, and the chip returned to the bag.

- **a.** Suppose two trials of this experiment are conducted. Are the events independent or dependent? Explain.
- **b.** What is the probability that both chips are purple?
- **c.** What is the probability that the first chip is green and the second is brown?

Need ExtraHelp?

If you missed Question...	1	2	3	4	5	6	7	8	9	10	11	12	13
Go to Lesson...	11-3	12-3	2-6	13-4	2-2	5-5	10-6	9-5	5-1	8-2	4-2	1-6	13-5