Aeroelastic Plate

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1 Introduction

We wish to create a semi-analytical aeroelastic wing model for a deformable cantilevered Kirchoff plate of thickness h in which the plate deflects and rotates according to lifting forces from an inviscid flow. It is known that the aerodynamic forces will depend strongly on the local angle of rotation of the plate (angle of attack), so if we assume a transverse displacement field composed of an (unknown) midplane displacement and rotation, the loading term will depend on the configuration of the plate thus manifesting as a non-linearity. We will also account for three dimensional flow effects which modify the effective angle of the attack of each section of the plate. A number of models/methods will be integrated in tackling this problem: Kirchoff plate theory will model the deformation of the simplified wing, potential flow theory will be used to find the sectional lift coefficient of a thin plate, and Prandtl lifting line theory will be used to account for 3D flow effects. In essence, the latter two methods are needed to understand the applied forces in the plate bending model. Finally, we will formulate a method to solve the resultant governing equations.

2 Kirchoff Plate

We will use variational methods to derive governing equations for the aeroelastic plate, thus we must formulate the bending strain energy in terms of the transverse displacement $u_3(x_1, x_2)$. It can be shown that the strain energy is

$$U = \frac{D}{2}\int_A u_{3,11}^2 + u_{3,22}^2 + 2vu_{3,11}u_{3,22} + 2(1-v)u_{3,12}^2 dA$$

where $D = \frac{Eh^3}{12(1-v^2)}$ is the constant bending stiffness of the plate. To simplify the problem and enforce the sort of deformation compatible with fluid equations, we restrict the transverse displacement to a midplane deflection $u(x_1)$ and a change in angle of attack $\phi(x_1)$. This guarantees that the x_1 cross-sections remain straight but still permits realistic elastic responses. The transverse displacement field is then

$$u_3(x_1, x_2) = u(x_1) + x_2\phi(x_1)$$

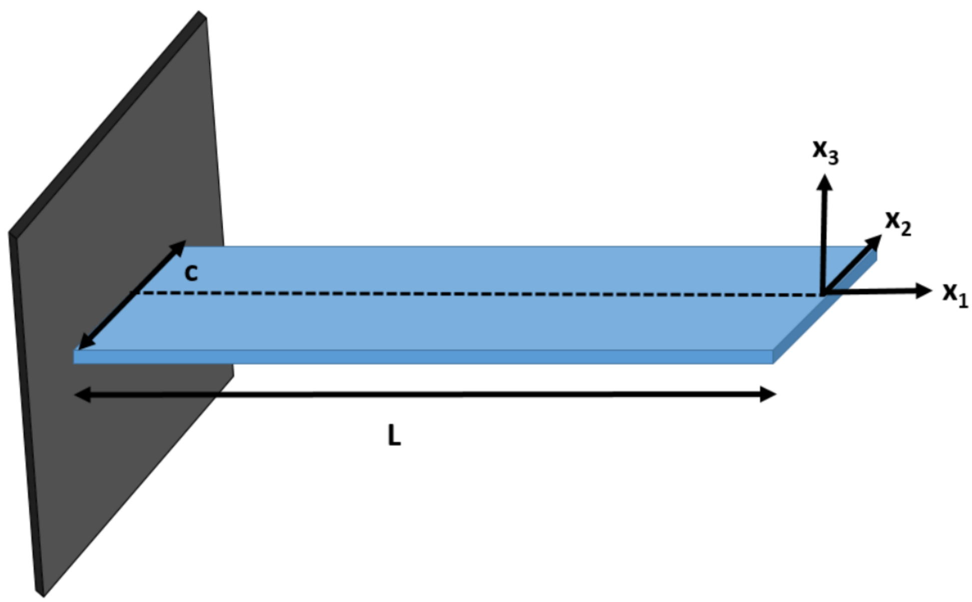

Figure 1: Thin cantilevered Kirchoff plate which will deform according to aerodynamic lifting forces.

where the angle of rotation ϕ is positive according to the right-hand rule. This form of displacement can be substituted into the strain energy and simplified. This reads

$$U = \frac{D}{2} \int_0^L \int_{-c/2}^{c/2} u_{11}^2 + 2x_2 u_{11} \phi_{11} + x_2^2 \phi_{11}^2 + 2(1-v)\phi_1^2 dx_2 dx_1$$

Due to the simplified form of displacement, the x_2 integration can be carried out explicitly, making the problem effectively one-dimensional. The second term, which indicates coupling between bending and torsion, vanishes due to the symmetry of the integration interval. The strain energy is then

$$U = \frac{1}{2} \int_0^L \left[\int_{-c/2}^{c/2} D dx_2 \right] u_{11}^2 + \left[\int_{-c/2}^{c/2} D x_2^2 dx_2 \right] \phi_{11}^2 + 2(1-v) \left[\int_{-c/2}^{c/2} D dx_2 \right] \phi_1^2 dx_1$$

The bracketed terms are geometric parameters and can be renamed to simplify this expression. Thus, the bending strain energy can be written as

$$U = \frac{1}{2} \int_0^L S u_{11}^2 + J \phi_{11}^2 + 2(1-v) S \phi_1^2 dx_1$$

where $S := \int D dx_2$ and $J := \int D x_2^2 dx_2$ are geometric stiffness parameters. The bending strain energy is a single integral of a simple form for the aeroelastic plate. Note that the plate is cantilevered at $x_1 = 0$ and free at $x_2 = L$, thus the boundary conditions on the two unknown functions are

$$u(0) = u_1(0) = \phi(0) = \phi_1(0) = 0$$
$$u_{11}(L) = u_{111}(L) = \phi_{11}(L) = -J\phi_{111}(L) + 2(1-v) S \phi_1(L) = 0$$

We will now turn to the work done by the aerodynamic forces to find the second term in the energy functional. Lift forces are positive in the $+x_3$ direction and depend on the angle of rotation ϕ, acting through the quarter chord point $x_2 = -\frac{c}{4}$. The center of pressure of symmetric airfoils has been observed to be nearly stationary at this point for small angles of attack. This assumption simplifies the problem in that only the magnitude of the lift depends on the rotation of the plate, not the point of application. We will assume that the lifting force distribution $L(\phi, x_1)$ can be represented as a cosine series:

$$L(\phi, x_1) = \mathcal{L} \sum_{n=0}^{N-1} A_n(\phi) \cos\Big(\frac{n\pi}{L} x_1\Big)$$

which will be motivated and specified by the potential flow and Lifting-line derivations of the aerodynamic characteristics of the flat plate airfoil. The cosine shape functions construct the x_1 dependence of the lift force with the coefficients A_n. These coefficients, in turn, depend on the unknown torsion response of the plate. The work done by the aerodynamic forces can be written as

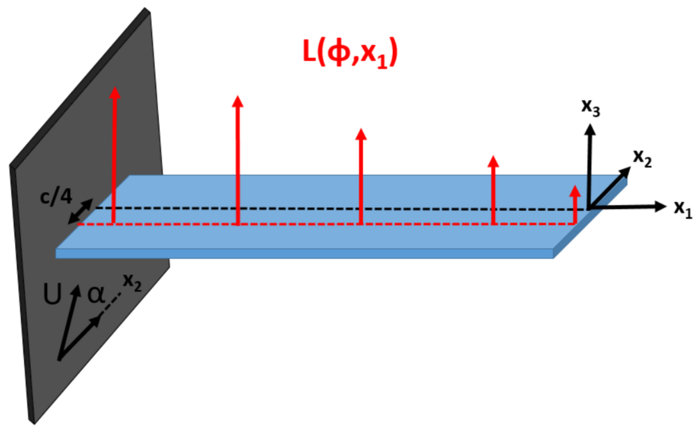

Figure 2: Lift forces are fixed at the quarter chord point for the flow of velocity U making an angle α up from the x_2 axis. The lift force varies with the span position x_1 and the angle of rotation of the elastic plate, making this problem non-linear in the applied loads.

$$V = \int_0^L \int_{-c/2}^{c/2} \delta\Big(\frac{-c}{4}\Big) L(\phi, x_1)\Big[u + x_2\phi\Big] dx_2 dx_1 = \int_0^L L(\phi, x_1)\Big[u - \frac{c}{4}\phi\Big] dx_1$$

$$= \int_0^L \mathcal{L}\Bigg[u \sum_{n=0}^{N-1} A_n(\phi)\cos\Big(\frac{n\pi}{L}x_1\Big) - \frac{c}{4}\phi \sum_{n=0}^{N-1} A_n(\phi)\cos\Big(\frac{n\pi}{L}x_1\Big)\Bigg] dx_1$$

The energy functional for statics problems in elasticity is $\Pi = U - V$ and will be used to derive the governing equations for the problem. The strain energy of the elastic plate can be combined with work done by aerodynamic loads:

$$\Pi = \int_0^L \frac{1}{2}\Big[S u_{11}^2 + J\phi_{11}^2 + 2(1-v)S\phi_1^2\Big] + \mathcal{L}\Big(\frac{c}{4}\phi - u\Big) \sum_{n=0}^{N-1} A_n(\phi)\cos\Big(\frac{n\pi}{L}x_1\Big) dx_1$$

3 Potential Flow

We use potential flow theory to find the circulation around a two-dimensional flate plate airfoil, which lets us calculate the sectional lift and lift coefficient. These parameters can then be incorporated into a 3D inviscid flow model.

The potential flow around a circle parameterized by $z = ce^{i\theta}$ with circulation Γ can be shown to be

$$w(z) = U\left(ze^{-i\alpha} + \frac{c^2}{z}e^{i\alpha}\right) - \frac{i\Gamma}{2\pi}\ln z$$

The free-stream flow velocity is U, traveling left to right, and the flow makes an angle α from horizontal. From the definition of potential flow, the horizontal and vertical velocity components are found from the potential by

$$\frac{dw}{dz} = U\left(e^{-i\alpha} - \frac{c^2}{z^2}e^{i\alpha}\right) - \frac{i\Gamma}{2\pi z} = u - iv$$

The Joukowksy transform can be used to model flow around more interesting geometries. Though it is known to generate airfoil-like shapes, a simple case of the Joukowksky transform takes a circle centered at the origin to a line. This transformation reads

$$Z = \frac{1}{4}\left(z + \frac{c^2}{z}\right)$$

This can be seen by finding the image of the parameterization $z = ce^{i\theta}$ under this mapping. Plugging in, we see that

$$Z = \frac{c}{4}\left(e^{i\theta} + e^{-i\theta}\right) = \frac{c}{2}\cos\theta$$

Thus, as the angle $\theta = 0$ we are at the trailing edge $Z = \frac{c}{2}$, and at $\theta = \pi$, we find the leading edge $Z = -\frac{c}{2}$. Of course, the total chord length of this airfoil is c as required. We can now calculate the velocity components around the plate using the theory of conformal maps. If we have an expression for flow around a circle, and map which takes that circle to a plate, we can calculcate the flow velocity around the plate with

$$\frac{dw}{dZ} = \frac{dw}{dz}\frac{dz}{dZ} = \frac{dw/dz}{dZ/dz}$$

which, when written this way, keeps our description of the flow in the (untransformed) z plane. The velocity components around the plate are then

$$u - iv = 4\left[U\left(e^{-i\alpha} - \frac{c^2}{z^2}e^{i\alpha}\right) - \frac{i\Gamma}{2\pi z}\right]\left(1 - \frac{c^2}{z^2}\right)^{-1}$$

Using the parameterization of the circle in the z plane, this reads

$$= \left[U\left(e^{-i\alpha} - e^{i\alpha}e^{-i2\theta}\right) - \frac{i\Gamma}{2\pi c}e^{-i\theta}\right]\left(1 - e^{-i2\theta}\right)^{-1}$$

$$= e^{-i\theta}\left[U\left(e^{i(\theta-\alpha)} - e^{-i(\theta-\alpha)}\right) - \frac{i\Gamma}{2\pi c}\right]\left(1 - e^{-i2\theta}\right)^{-1}$$

$$= e^{-i\theta}\left[2Ui\sin(\theta-\alpha) - \frac{i\Gamma}{2\pi c}\right]\left(1-e^{-i2\theta}\right)^{-1}$$

The Kutta condition requires that there are finite fluid velocity at the trailing edge of the airfoil, corresponding to $\theta = 0$. This can be accomplished by choosing the circulation Γ such that the numerator of the expression for velocity is zero at this point. This implies that

$$2U\sin(-\alpha) = \frac{\Gamma}{2\pi c} \implies \Gamma = -4\pi Uc\sin\alpha$$

The Kutta-Joukowsky theorem states that the lift force per unit span is

$$\ell = -\rho U\Gamma = 4\pi U^2\rho c\sin\alpha$$

The sectional lift coefficient follows from the expression for the lift force

$$c_\ell = \frac{\ell}{\frac{1}{2}\rho c U^2} = 8\pi\rho\ \sin\alpha$$

which allows us to show that slope of the lift coefficient is approximately constant

$$c_{\ell\alpha} := \frac{dc_\ell}{d\alpha} = 8\pi\rho\cos\alpha \approx 8\pi\rho$$

It is clear that the zero lift angle of attack is $\alpha = 0$. These results, obtained from potential flow around a flat plate, will be required in the lifting-line theory, which is used to account for 3D flow effects.

4 Lifting Line Theory

It can be observed that vertical flow is induced from changes in lift over the span of a wing, and this varying "downwash" velocity acts to change the effective angle of attack of each airfoil section. In other words, the effective angle of attack must incorporate the geometric orientation of the airfoil, the direction of the free-stream flow, and this induced velocity. To begin, note that from the definition of the sectional lift coefficient and the Kutta-Joukowksy theorem, we can write

$$c_\ell = \frac{-2\Gamma}{Uc}$$

Also, note that

$$c_\ell = c_{\ell\alpha}\alpha_{eff} = 8\pi\rho\Big(\alpha - \phi(x_1) - \beta(x_1)\Big)$$

where the effective angle of attack α_{eff} incorporates the freestream/rigid angle α, the elastic rotation angle ϕ, and the downwash angle β. By the definition of ϕ, a positive rotation reduces the angle of attack of the plate. At this point, we

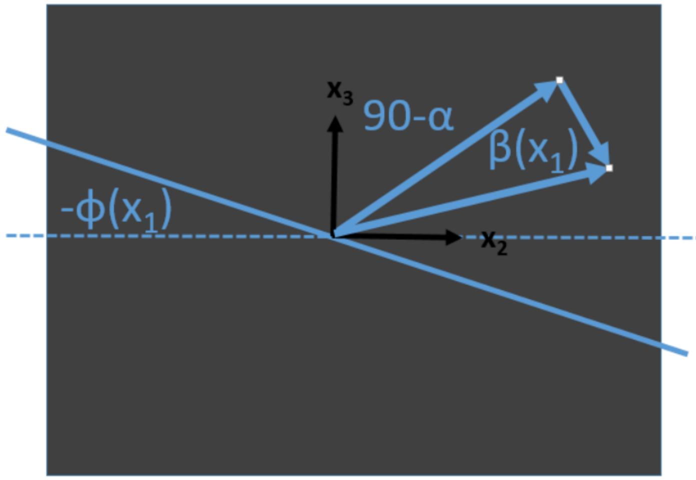

Figure 3: Cross-section of deformed plate with freestream velocity at angle α to x_2 axis, correction angle β from downwash velocity, and elastic rotation ϕ.

are relating the plate model to the airfoil geometry and aerodynamic characteristics with the deformation angle ϕ and the coordinate system of the plate. By assumption, the lifting forces are not affected by the transverse deflection u. The lifting line theory will allow us to calculate the downwash angle which governs the modified circulation distribution. Combining the two expressions for the lift coefficient, we can find the circulation in terms of the effective angle of attack as

$$\Gamma(x_1) = -4\pi\rho U c\Big(\alpha - \phi(x_1) - \beta(x_1)\Big)$$

This reproduces the expression for circulation from the Joukowksy airfoil with the effective angle of attack and where the sine function is linearized by the assumption of small angles. Previously, we imagined a plate cantilevered to a wall whereas now we imagine a full wing planform of span $2L$ such that the imagined "fuselage" at $x_1 = 0$ has no relevant aerodynamic properties but does act as a cantilever for the plate on either side. We represent the unknown circulation distribution as a cosine series, indicating its symmetry about the origin:

$$\Gamma(x_1) = 4LU\sum_{n=0}^{N} A_n \cos\Big(\frac{n\pi}{L}x_1\Big)$$

We want to express the induced angle of attack β in terms of the circulation, and we will do this by finding the downwash velocity w. When this velocity is small, which is perpendicular to the freestream flow of velocity U, then $\beta(x_1) \approx$

$\frac{w(x_1)}{U}$. We have chosen not to use the angular coordinate transformation which is typical in Lifting-lione derivations, as it complicates the aero-structural problem with the use of multiple coordinate systems. It will also be seen that staying in the Cartesian coordinates facilitates numerical solutions to the governing equations. We can now write

$$\Gamma(x_1) = 4LU \sum_{n=0}^{N} A_n \cos\Big(\frac{n\pi}{L} x_1\Big) = -4\pi\rho U c\Big(\alpha - \phi(x_1) - \frac{w(x_1)}{U}\Big)$$

The angle of rotation ϕ is governed by statics considerations and can be treated as known *in this context*. Thus, if we can solve for the distribution of downwash velocity in terms of the series representation of circulation, we can find an expression for the coefficients A_n. To do this, note that a theorem from Helmholtz tells us that span-wise changes in the lift distribution, through the circulation Γ, must be accompanied by a shed vortex of equivalent strength. These shed vortices induce downward velocity on the underside of the airfoil, and act to reduce the effective angle of attack. A differential element of circulation at location x' induces a small velocity at the span position x_1. The shed vortices act like semi-infinite vortex lines, and the Biot-Savart law takes a simpler form for this geometry:

$$dw = \frac{d\Gamma}{4\pi(x' - x_1)}$$

We can incorporate all such shed vortices by integrating over the span of the wing to find the downwash velocity at span position x_1. We use the series definition of the circulation to say that

$$\begin{aligned} w(x_1) &= \frac{1}{4\pi} \int \frac{d\Gamma}{x' - x_1} = \frac{1}{4\pi} \int_{-L}^{L} \frac{(d\Gamma/dx')}{x' - x_1} dx' \\ &= -U \int_{-L}^{L} \frac{\sum n A_n \sin\big(\frac{n\pi}{L} x'\big)}{x' - x_1} dx' \\ &= -U \sum_{n=0}^{N} n A_n \int_{-L}^{L} \frac{\sin\big(\frac{n\pi}{L} x'\big)}{x' - x_1} dx' \end{aligned}$$

This integral is non-trivial to evaluate, and despite its singularities, should be convergent based on physical considerations. For the time being, we can replace it with an undefined function $f(n, x_1)$, as the dependence on the field variable x' is integrated away. The induced velocity will be negative, thus it will naturally produce the negative sign used in the effective angle of attack. Returning to the two expressions for circulation, we can now write

$$4LU \sum_{n=0}^{N} A_n \cos\Big(\frac{n\pi}{L} x_1\Big) = -4\pi\rho U c\Big(\alpha - \phi(x_1) + \sum_{n=0}^{N} n A_n f(n, x_1)\Big)$$

This expression can be rearranged to isolate the coefficients on circulation

$$\sum_{n=0}^{N} A_n \left[\frac{L}{\pi \rho c} \cos\Big(\frac{n\pi}{L} x_1\Big) + n f(n, x_1) \right] = \phi(x_1) - \alpha$$

This expression can be turned into a linear system by discretizing the span coordinate and forcing the relation to be satisfied at N points x_j:

$$\sum_{n=0}^{N} A_n \left[\frac{L}{\pi \rho c} \cos\Big(\frac{n\pi}{L} x_j\Big) + n f(n, x_j) \right] = \phi(x_j) - \alpha$$

Discretizing in this way produces an approximate solution to the problem where coefficients is such that the governing equation is satisfied at a finite number of points, which we will take to be on a grid of uniform spacing. Note that in order to solve this system with matrix inversion, the number of grid points must be equivalent to the number of terms in the series for circulation, thus $j = 0, 1, \ldots, N-1$. The linear system can be written as

$$H_{jn} A_n = \phi_j - \alpha_j$$

where α_j is the constant α multuipying a column vector of 1's. The matrix of coefficients comes out of the discretized lifting-line equation, and can be inverted to solve for the coefficients.

$$H_{jn} := \frac{L}{\pi \rho c} \cos\Big(\frac{n\pi}{L} x_j\Big) + n f(n, x_j)$$

$$A_n = H_{nj}^{-1} \phi(x_j) - v_n$$

The circulation distribution can finally be written as

$$\Gamma(x_1, \phi) = 4LU \sum_{n=0}^{N} \left[H_{nj}^{-1} \phi(x_j) - v_n \right] \cos\Big(\frac{n\pi}{L} x_1\Big)$$

which means that the lift force is

$$L(x_1, \phi) = 4\rho L U^2 \sum_{n=0}^{N} \left[H_{nj}^{-1} \phi(x_j) - v_n \right] \cos\Big(\frac{n\pi}{L} x_1\Big)$$

5 Integral

To proceed with the analysis, We need to evaluate the integral

$$f(n, x_1) = \int_{-L}^{L} \frac{\sin\big(\frac{n\pi}{L} x'\big)}{x' - x_1} dx'$$

Even numerical integration struggles to handle this expression due to complex oscillatory behavior and singularities, thus an analytic expression is sought. From [**?**], we have the tabulated indefinite integral

$$\int \frac{\sin kx}{a+bx}dx = \frac{1}{b}\left[\cos\left(\frac{ka}{b}\right)\mathrm{si}(u) - \sin\left(\frac{ka}{b}\right)\mathrm{ci}(u)\right]$$

which makes use of the following definitions

$$u := \frac{k}{b}(a+bx)$$

$$\mathrm{ci}(x) := -\int_x^\infty \frac{\cos t}{t}dt$$

$$\mathrm{si}(x) := -\int_x^\infty \frac{\sin t}{t}dt$$

Evidently this is a complicated integral. The sine and cosine integrals are stored in Python as special functions. In our case, the constants are $k = \frac{n\pi}{L}$, $a = -x_1$, and $b = 1$. The definite integral is computed from the indefinite integral as

$$\begin{aligned} f(n, x_1) = &\left[\cos\Big(\frac{n\pi x_1}{L}\Big)\mathrm{si}\Big(\frac{n\pi}{L}(L-x_1)\Big) + \sin\Big(\frac{n\pi x_1}{L}\Big)\mathrm{ci}\Big(\frac{n\pi}{L}(L-x_1)\Big)\right] \\ &- \left[\cos\Big(\frac{n\pi x_1}{L}\Big)\mathrm{si}\Big(-\frac{n\pi}{L}(L+x_1)\Big) + \sin\Big(\frac{n\pi x_1}{L}\Big)\mathrm{ci}\Big(-\frac{n\pi}{L}(L+x_1)\Big)\right] \end{aligned} \tag{1}$$

6 Solution Methods

The distribution of applied lift is a function of the span position x_1 and the state of elastic wing twist ϕ. The coefficients on the cosine series for circulation are a linear function of the rotation angle at the N positions x_i. Our approach to solving the structural problem must respect this representation of the lift distribution, which is difficult to accomplish if we used Euler-Lagrange equations to derive governing differential equations from the energy functional Π. Namely, we must calculate $\frac{\partial A_n}{\partial \phi}$ without a continuous representation of their relationship. In lieu of the variational approach, we can compute a stationary point of the discretized energy functional by evaluating the bending displacement u and rotation angle ϕ at N points, and computing derivatives with finite differencing. The energy is then a function of $2N$ parameters u_i and ϕ_i, and we use the gradient to accomplish the minimization. This allows us to unambiguously use the lifting-line results. Combining the discretization, lifting-line theory, and finite differencing the energy is

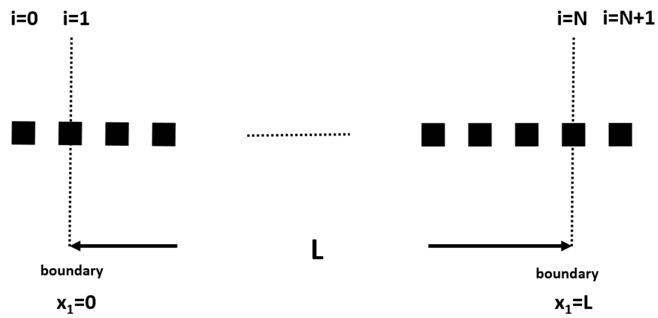

Figure 4: Values of state variables ϕ and u are stored on a discretized interval. Boundary conditions are enforced with algebraic equations by adding a fictitious point on both sides of the interval.

$$\Pi = \sum \Bigg[\frac{S}{2}\Big(\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}\Big)^2 + \frac{J}{2}\Big(\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2}\Big)^2$$

$$+(1-v)S\Big(\frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x}\Big)^2 + \mathcal{L}\Big(\frac{c}{4}\phi_i - u_i\Big)\sum_{n=0}^{N-1}\Big(H_{nj}^{-1}\phi_j + v_n\Big)\cos\Big(\frac{n\pi}{L}x_i\Big)\Bigg]\Delta x$$

A solution to this problem is a point in the finite space $[\vec{u}, \vec{\phi}]$ such that the energy function is at a stationary point. We can locate such a solution with a gradient descent algorithm of the sort

$$\begin{bmatrix} \vec{u}_{i+1} \\ \vec{\phi}_{i+1} \end{bmatrix} = \begin{bmatrix} \vec{u}_i \\ \vec{\phi}_i \end{bmatrix} - \gamma \nabla \Pi\Big(\vec{u}_i, \vec{\phi}_i\Big)$$

Only the interior points are updated with the gradient of the energy, as the boundary conditions determine the value of the endpoints in any given configuration of the system. The differential boundary conditions for both state variables can be replaced with the appropriate differencing schemes and transformed into algebraic equations. The continuous boundary conditions are

$$u(0) = u_1(0) = \phi(0) = \phi_1(0) = 0$$
$$u_{11}(L) = u_{111}(L) = \phi_{11}(L) = -J\phi_{111}(L) + 2(1-v)S\phi_1(L) = 0$$

We incorporate a single fictitious point outside the physical boundary of the problem which helps enforce the differenced boundary conditions. The fictitious point on the left is at index $i = 0$ and $i = N + 1$ on the right. The boundary conditions on the left end are

$$u_1 = 0$$

$$\left.\frac{\partial u}{\partial x_1}\right|_{i=1} = 0 = \frac{u_2 - u_0}{\Delta x} \implies u_0 = u_2$$

$$\phi_1 = 0$$

$$\left.\frac{\partial \phi}{\partial x_1}\right|_{i=1} = 0 = \frac{\phi_2 - \phi_0}{\Delta x} \implies \phi_0 = \phi_2$$

Because of the clamped wing root, these conditions are straightforward to enforce. The free end on the right side has more complicated behavior:

$$\left.\frac{\partial^2 u}{\partial x_1^2}\right|_{i=N} = 0 = \frac{1}{\Delta x^2}\Big(u_{N+1} - 2u_N + u_{N-1}\Big)$$

$$\left.\frac{\partial^3 u}{\partial x_1^3}\right|_{i=N} = 0 = \frac{1}{\Delta x^3}\Big(u_{N+1} - 3u_N + 3u_{N-1} - u_{N-2}\Big)$$

$$\left.\frac{\partial^2 \phi}{\partial x_1^2}\right|_{i=N} = 0 = \frac{1}{\Delta x^2}\Big(\phi_{N+1} - 2\phi_N + \phi_{N-1}\Big)$$

$$-J\left.\frac{\partial^3 \phi}{\partial x_1^3}\right|_{i=N} + 2(1-v)S\left.\frac{\partial \phi}{\partial x_1}\right|_{i=N} = 0 = \frac{-J}{\Delta x^3}\Big(\phi_{N+1} - 3\phi_N + 3\phi_{N-1} - \phi_{N-2}\Big) + 2(1-v)\frac{S}{\Delta x}\Big(\phi_{N+1} - \phi_{N-1}\Big)$$

The free end boundary conditions result in a system of two equations for the two unknowns at $i = N$ and $i = N + 1$ for each state variable. We initialize a guess for the system state $[\vec{u}_0, \vec{\phi}_0]$, compute the gradient of the energy, update the interior points, then use the boundary conditions to find the values at the fictitious points/boundaries. The values of the functions at the boundary are required to calculate the energy even though they are not updated by it. At this point, we can either repeat the process or halt if the magnitude of the gradient is below a certain threshold, indicating a stationary point. We now turn to computing the gradient of the energy Π. More precisely, the energy is

$$\Pi = \sum_{i=1}^{N} w_i \Bigg[\frac{S}{2}\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2}\right)^2 + \frac{J}{2}\left(\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2}\right)^2$$
$$+ (1-v)S\left(\frac{\phi_{i+1} - \phi_{i-1}}{2\Delta x}\right)^2 + \mathcal{L}\Big(\frac{c}{4}\phi_i - u_i\Big)\sum_{n=0}^{N-1}\Big(H_{nj}^{-1}\phi_j + v_n\Big)\cos\Big(\frac{n\pi}{L}x_i\Big)\Bigg]\Delta x$$

We start at the left-hand boundary point at $i = 1$, split the interval into N subdivisions, and stop at the right-hand boundary at $i = N$. To numerically integrate in this manner, we use a centered approximation method, but this requires the introduction of a weight function which is 1/2 on either boundary and 1 otherwise. We take derivatives of the energy with respect to the parameters u_j, ϕ_j where $j = 2, 3, \ldots, N-1$ indicating that the boundary points are not updated by the gradient, but follow from the interior points and boundary conditions. We see from the expressions for derivatives that the fictitious points are use in computing the gradient even for points inside the boundary, and are thus not only for enforcing boundary conditions, but actually show up in the expression for energy.

$$\frac{\partial \Pi}{\partial u_j} = \Bigg[\frac{S}{\Delta x^4} \Bigg(w_{j-1}(u_j - 2u_{j-1} + u_{j-2}) - 2w_j(u_{j+1} - 2u_j + u_{j-1}) + w_{j+1}(u_{j+2} - 2u_{j+1} + u_j) \Bigg) - \mathcal{L} w_j \sum_{n=0}^{N-1} \left(H_{nj}^{-1} \phi_j + v_n \right) \cos\!\left(\frac{n\pi}{L} x_j \right) \Bigg] \Delta x$$

$$\frac{\partial \Pi}{\partial \phi_k} = \Bigg[\frac{J}{\Delta x^4} \Bigg(w_{k-1}(\phi_k - 2\phi_{k-1} + \phi_{k-2}) - 2w_k(\phi_{k+1} - 2\phi_k + \phi_{k-1}) + w_{k+1}(\phi_{k+2} - 2\phi_{k+1} + \phi_k) \Bigg) + \frac{S(1-v)}{2\Delta x^2} \Bigg(w_{k-1}(\phi_k - \phi_{k-2}) - w_{k+1}(\phi_{k+2} - \phi_k) \Bigg) + \mathcal{L} \frac{c}{4} w_k \sum_n (H_{nj}^{-1} \phi_j + v_n) \cos\!\left(\frac{n\pi x_k}{L} \right) + \sum_i \mathcal{L} w_i \left(\frac{c}{4} \phi_i - u_i \right) \sum_n (H_{nk}^{-1} + v_n) \cos\!\left(\frac{n\pi x_i}{L} \right) \Bigg]$$

7 I never finished this!