Buckling

Conor Rowan

March 2024

1 Introduction

Buckling is a question of the geometric stability of a structure independent of material failure. The general idea of buckling is that the displacement of a structure acts to exaggerate the effect of the applied forces, leading to a kind of feedback and eventual instability. This is the case with the most familiar example of beam/column buckling: if the axially loaded beam/column somehow develops small bending displacements, these bending displacements increase the moment arm of the applied compressive force. This leads to more bending, which leads to larger offsets from the compressive load, and so on. The structure eventually becomes unstable (reaches a point where no further force can be equilibrated) as a result of this feedback. A structure can be unstable and "fail" through buckling well before material failure occurs. In fact, structural instability can occur even within the material's elastic range, when strains are still small. It is interesting to think of structural instability as an additional source of failure, outside of plasticity or fracture, which requires its own analysis. We survey a few common topics in buckling with this report. The hope is to put a together a nice general introduction to the subject for someone familiar with solid mechanics but with little exposure to buckling.

2 Simplest Example

Consider a vertical bar of length L loaded by a vertical force P with a torsional spring of stiffness k at its root. Assuming that angle θ measures the clockwise rotation from vertical, a torque balance yields

$$PL\sin\theta - k\theta = 0$$

For small angles θ, a Taylor series expansion of sine yields

$$(PL - k)\theta = 0$$

This is a strange problem. We apply an axial force which we would expect under the idealized conditions of this problem to produce no rotation. This is reflected in the solution to the linearized equation $\theta = 0$. Another possibility

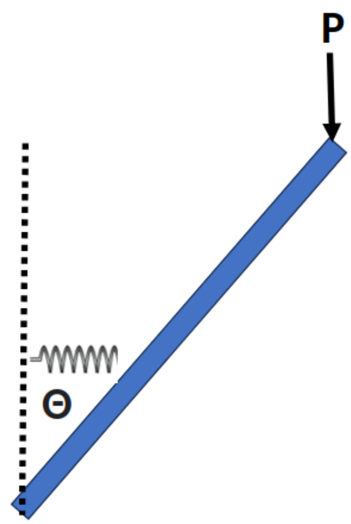

Figure 1: Some basic concepts from buckling can be introduced with this simple example of a pinned bar with a torsional spring.

is that $P = k/L$ for which there are non-zero θ that satisfy this equation. We do not, however, know the values of these angles. This is a common theme for buckling analysis in the linear regime–computing load values such that a linearized problem has non-zero equilibrium solutions with unknown displacement magnitudes. If we do not linearize the problem, the governing equation is

$$PL \sin\theta = k\theta$$

where for small values of P, the only solution is $\theta = 0$ but as the load becomes sufficiently large, solution(s) exist for non-zero rotation angles θ. In this case, the value of θ can be computed. Note that only two solutions will be physical, corresponding to the situation where the moment arm of the force begins to decrease again after θ passes through $\pi/2$. Any further solutions beyond this correspond to complete rotations of the bar.

3 Buckling of Euler Beams

Consider an Euler beam pinned at both ends with an applied compressive axial load P and length L. Though it is not clear that this scenario should produce transverse displacements $w(x)$, we ask the question: if there were to be transverse displacements, what displacements would satisfy equilibrium? The governing equation for this problem is

$$EIw_{,xx} + Pw = 0$$

The moment from axial bending stresses must balance the moment from the applied force (which obtains an offset from deflection w). The general solution is

$$w(x) = A\cos\left(\sqrt{\frac{P}{EI}}x\right) + B\sin\left(\sqrt{\frac{P}{EI}}x\right)$$

There is no displacement on either end of the beam, thus $w(0) = w(L) = 0$. This immediately implies $A = 0$, so we are left with

$$B\sin\left(\sqrt{\frac{P}{EI}}L\right) = 0$$

Similar to the rotation of the rigid rod, this requirement is trivially satisfied for zero displacement ($B = 0$). However, there is another solution which comes from treating the applied load as an unknown, namely $\sqrt{\frac{P}{EI}}L = n\pi$. Thus, for fixed material parameters and beam geometry, we have a solution which satisfies equilibrium when

$$P = \frac{n^2\pi^2 EI}{L^2}$$

Like the rigid rod, we cannot know the magnitude of the displacement B for this "critical" load. Buckling is said to occur at the first mode shape

$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

In summary, for a linear buckling theory we assume a non-zero transverse displacement and investigate the conditions of equilibrium. Satisfying equilibrium for the non-zero displacement gives a condition on the force which leaves the magnitude of the displacement unspecified. The force for which equilibrium is satisfied at non-zero transverse displacement is said to produce buckling. This is because arbitrarily large displacements satisfy force equilibrium, so that the displacement can grow without needing additional loading. The loss of uniqueness between force and displacement is a notion of instability that we encounter often in buckling.

Another approach to buckling is to assume an initial "imperfection" such that a displacement arises naturally out of the axial load. In the case of Euler beams, we can model this imperfection as a slight offset of the axial load so that a moment is produced even in the absence of the transverse deflection. The general solution to the problem of an offset compressive load is

$$w(x) = e\Big[\tan\left(\frac{L}{2}\sqrt{\frac{P}{EI}}\right)\sin\left(\sqrt{\frac{P}{EI}}x\right) + \cos\left(\sqrt{\frac{P}{EI}}x\right) - 1\Big]$$

where e is the offset from the centroid of the beam. As the applied load P approaches P_{cr}, the tangent function will become unbounded indicating collapse of the beam. The primary difference is that non-zero displacements occur even before P_{cr} unlike the idealized beam. This furnishes a second definition of linear buckling: the load at which the response of an imperfect system becomes unbounded.

4 Eigenvalue Buckling

For the case of linear buckling of a beam, we can modify the energy so that the axial force contributes a transverse displacement

$$\Pi = \int \frac{1}{2} EI w_{xx}^2 - \frac{1}{2} P w_x^2 dx$$

There is work both from the moments associated with stresses and from the axial force when the beam deflects. The first term in the energy is the usual energy as a result of bending displacements for an Euler beam. The second term is a work term which comes from computing the axial displacement as a result of small rotations of the beam. This term does not show up in typical beam analysis because we neglect the coupling between the axial and bending problems. If we expand the solution for the displacement $w(x)$ with global shape functions, it can be seen that the energy can be written in terms of coefficients $\underline{q}$ as

$$\Pi = \frac{1}{2} q_i K_{ij} q_j - \frac{1}{2} q_i P K_{ij}^G q_j$$

such that when we take the gradient to minimize energy we have

$$\left(K_{ij} - P K_{ij}^G \right) q_j = 0$$

The matrices arise from plugging in spatial shape functions to the energy, factoring out coefficients, and computing integrals. This expression tells us that either $q_j = 0$ and there is no transverse displacement or the matrix $K_{ij} - PK_{ij}^G$ is singular and the displacement can be of arbitrary magnitude. This is a generalized eigenvalue problem where eigenvalue/eigenvector pairs will correspond to buckling forces and buckling shapes. The lowest eigenvalue will be the load at which buckling initiates. A similar analysis can be carried out for buckling of plates, in which planar forces generate out of plane displacements by assumption, and thus contribute to the overall potential energy. The system is discretized with global shape functions, and the corresponding stiffness matrix is made singular by increasing the planar loading. This provides an estimate for the buckling mode shapes and applied load at buckling. The buckling behavior of plates is more complex and depends on their aspect ratio, and whether the force is shear or normal. Note that this is called an eigenvalue problem because we solve for the applied force P such that

$$\det\Big(K_{ij} - PK_{ij}^G\Big) = 0$$

A problem of this sort arises from an eigenvalue problem of the form

$$\underline{\underline{K}}\underline{q} = P\underline{\underline{K}}^G\underline{q}$$

This is a generalized eigenvalue problem with buckling modes (eigenvectors) $\underline{q}$ and corresponding eigenvalues P. Because solving this eigenvalue problem allows us to determine forces for which the displacement becomes unbounded, this is called eigenvalue buckling. It requires us to postulate a priori the way in which buckled displacement responses will generate energy.

5 Simple Geometric Nonlinearity

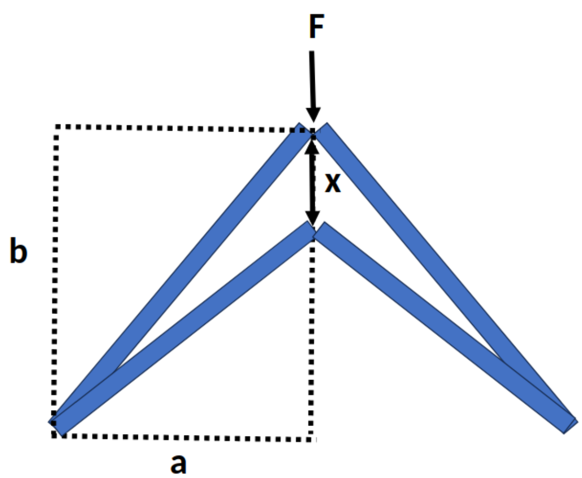

Figure 2: A nice large deformation instability problem with an analytical solution is the snapthrough of a two bar truss.

Consider the two bar truss structure shown in Figure 2. All joints are pinned, and we will assume that the displacement response is symmetric, meaning that a single coordinate describing the vertical displacement under the applied force characterizes the response of the structure. A useful way to formulate this problem is with the total potential energy. This reads

$$\Pi(x) = \frac{1}{2}A\int_0^L 2\sigma\epsilon dx - Fx$$

The stress/strain response of the two bars is equivalent, thus the factor of the two in the integrand. We assume a 1D stress state, thus the cross-sectional area appears outside the integral. We will make the additional assumption that the bars are linear elastic, and that the stress is constant over their length. The energy is then

$$\Pi(x) = EAL\epsilon(x)^2 - Fx$$

We need to write the strain in terms of the displacement coordinate x. Let's define the initial length of each bar as

$$L := \sqrt{a^2 + b^2}$$

The strain is then

$$\epsilon = \frac{\Delta L}{L} = \frac{L(x) - L}{L} = \frac{1}{L}\left(\sqrt{(a-x)^2 + b^2} - L\right)$$

Plugging this into the energy, we obtain

$$\Pi(x) = \frac{EA}{L}\left((a-x)^2 + b^2 + L^2 - 2L\sqrt{(a-x)^2 + b^2}\right) - Fx$$

Note that the strain is "geometrically nonlinear" even though we used the small strain definition $\epsilon = \Delta L/L$ because it is a nonlinear function of the displacement, which accounts for the fact that small variations in the displacement have different effects on the strain depending on the current state of the displacement. In other words, if the two bars are deformed to the point of being horizontal, changes in the displacement in either direction act to decrease the strain, whereas in the undeformed configuration, displacements obviously act to increase strain. This is a nonlinear effect which arises purely from the geometry of the deformation. Solutions to the elastic problem are governed by extrema of the energy. These can be computed with

$$\frac{\partial \Pi}{\partial x} = 0$$

The simplest way to study this problem is to look at an interactive plot. We can see that in general, the gradient of the energy has three zero crossings. This means that there are three solutions to this problem, or three configurations of the truss that satisfy force equilibrium. The first is the obvious one: a small displacement in the direction of the force. This is what we would obtain from a purely linear analysis. The second is quite counterintuitive, as the bars are pretty much horizontal. The last solution corresponds to the truss completely inverting. One can play with the force and material/geometric parameters to see how the displacement solutions, identified as the zero crossings of the gradient of the energy, vary.

The first and third solutions seem physically reasonable, though we can reason that to obtain the third solution, the force would need to be large enough to invert the structure, and then decreased to its current value. The necessity of this "loading path" can be observed by increasing the force in the plot until there is only one solution, corresponding to the inversion of the structure. The third solution would be obtained if a force of this magnitude or larger was applied then decreased. The second solution does not make much physical sense. In fact, we can show that this solution is unstable by looking at the second derivative of the energy. We can think of the concavity of the energy at an extremum as determining whether there are restoring forces to a displacement that satisfies force equilibrium. A positive second derivative indicates a concave up energy function, meaning that nudges to the displacement around the extremum increase the energy. There will be restoring forces in this situation, making the equilibrium stable. Alternatively, a negative second derivative means concave down, and nudges to the displacement will decrease the energy, moving the system away from this equilibrium. There are no restoring forces, and this configuration is unstable. It can be seen that the second solution corresponds to an unstable equilibrium. Thus, even though the balance of forces is satisfied, any disturbance to this displacement will cause the truss system to snap into another stable configuration. This is why we don't expect to observe this solution in reality. We can then say that

$$\frac{\partial \Pi}{\partial x} = 0, \quad \frac{\partial^2 \Pi}{\partial x^2} > 0 \implies \text{stable equilibrium}$$

$$\frac{\partial \Pi}{\partial x} = 0, \quad \frac{\partial^2 \Pi}{\partial x^2} < 0 \implies \text{unstable equilibrium}$$

The two bar truss can be considered an example of buckling because the system has the potential to "snap" from one stable equilibrium to another. Naturally, it will pass through the unstable configuration if there is any momentum to this process. Like buckling of beams, this is a situation where the geometry of the displacement response leads to drastic loss of stiffness, independent of failure of the material. Note that for linear problems, the energy functional has only one extremum and is always stable.

6 Buckling in General Nonlinear Analysis

When working in the finite strain setting, the interaction between the displacement and the loading is automatically captured. For example, an axially loaded beam described with finite strain models will be able to account for the fact that transverse deflections increase the moment arm of the force and thus reduce the effective stiffness. We were only able to accomplish this in the linear setting with some a priori knowledge of the coupling between the axial and transverse responses. A beam-like structure modeled in a general linear 3D finite element setting will not be able to account for the fact that transverse deflections amplify the effects of the axial load. In other words, nonlinear elastic models have the ability to model the "geometric" coupling between the displacement and the applied loads. Figure 3 and 4 show examples of how a nonlinear elastic model

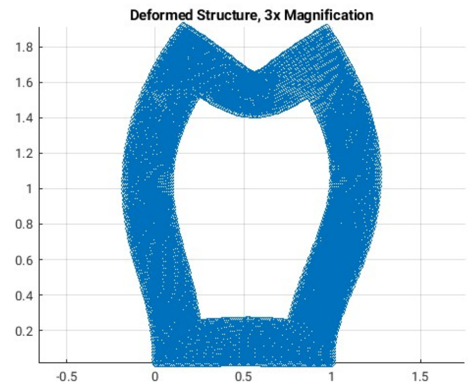

Figure 3: A hyperelastic frame structure loaded symmetrically by a downward traction on the upper horizontal surface. Because hyperleastic material models make use of finite strain measures, buckling effects are naturally incorporated into the analysis.

can naturally capture the displacement field's destabilizing interaction with the applied force. Note that some nonlinear finite element solvers will not predict buckling for a column loaded perfectly at its center, because there is nothing to introduce the initial transverse displacements. This likely depends on how the solution is initialized in the nonlinear solve. To account for this, small perturbations to the load can be introduced to break symmetry, modeling imperfections in any real material system.

7 Linearized Buckling Analysis

The above method requires numerically solving a nonlinear elastic problem, almost certainly with load stepping to carefully resolve the instability. In other words, applying a force large enough to cause structural instability all at once (without incrementally approaching it) would likely cause convergence issues in the nonlinear solve. This can be an expensive operation. Additionally, it may be somewhat vague to define the exact onset of instability. One remedy to these problems is "linearized buckling analysis." This method relies on using the St. Venant-Kirchhoff material model of a solid undergoing finite strains. In a sense, this is the simplest material model in nonlinear elasticity. It says that the second Piola-Kirchhoff stress tensor (PK2) is linearly proportional to the Green-Lagrange strain:

$$S_{IJ} = C_{IJKL} E_{KL}$$

where C_{IJKL} is a tensor of constants, and the Green-Lagrange strain is

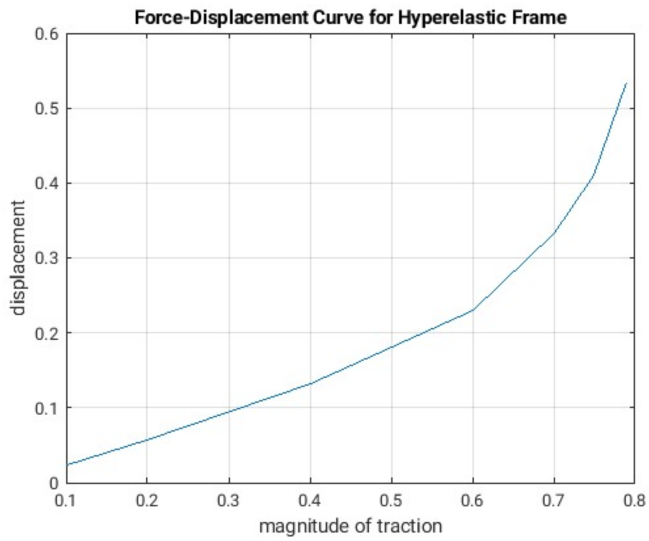

Figure 4: The symmetric compressive traction loading is applied to the frame structure at various magnitudes and the displacement at the center of the top surface of the structure is recorded. We see that the response is nonlinear, and that the structure moves towards instability as the force is increased. Note that the frame structure can also "buckle" to the side, and the response would be stiffer if it was made symmetric (no x_1 displacement along the vertical center line). The nonlinear elastic analysis models instability with no additional theory of buckling.

$$E_{IJ} = \frac{1}{2}\left(\frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I} + \frac{\partial u_K}{\partial X_I}\frac{\partial u_K}{\partial X_J}\right)$$

Both the PK2 stress and Green-Lagrange strain are defined fully in the reference configuration, thus uppercase letters are used by convention. It is outside the scope of these notes to clearly motivate these stress and strain measures, but we can briefly note that the Green-Lagrange strain measures changes in the squared length of differential line elements in the reference and deformed configurations. Imagine etching a small line on a body, deforming it, and taking the difference in the square of the length of this line before and after deformation. This is what E_{IJ} is getting at. Given a normal vector in the reference configuration, the PK2 outputs a force vector which is mapped back from the deformed configuration. It can be shown that the strain energy density for the St. Venant-Kirchhoff material model is

$$\Psi = \frac{1}{2}C_{IJKL}E_{IJ}E_{KL}$$

which is analogous to linear elasticity up to the definition of the strain. With these preliminaries established, our first task is to show that the discretized governing equations can be written as

$$\left(K^L_{QS} + K^{NL}_{QS}(\underline{U})\right)U_S = F_Q$$

This says that there is a linear part of the stiffness matrix, and a non-linear part which depends on the displacement, which both multiply the displacement degrees of freedom to obtain the force vector. It is natural that if there were to be an additive decomposition of this sort, the nonlinear part would depend on the displacement (otherwise the problem would be linear), but it is not clear that such an additive decomposition exists. We can motivate this in the following way. We start with the total potential energy of the problem, and compute its variation to find a minimum:

$$\Pi = \frac{1}{2}\int C_{IJKL}E_{IJ}E_{KL}d\Omega - \int T_I u_I dS$$

$$\implies \delta\Pi = \int C_{IJKL}E_{IJ}\frac{\partial E_{KL}}{\partial\left(\frac{\partial u_M}{\partial X_N}\right)}\frac{\partial \delta u_M}{\partial X_N}d\Omega - \int t_I \delta u_I dS = 0$$

This is how we derive governing equations for the variational nonlinear elasticity problem. With no loss of generality, discretize the test function with $\delta u_I = H_{IJ}W_J$ where $\underline{\underline{H}}$ is a matrix of spatial shape functions and W_J are arbitrary degrees of freedom. The displacement is discretized in the same way as $u_I = H_{IJ}U_K$. Using that the weights on the test function are arbitrary, we have at once that

$$\int t_I \delta u_I dS = \int t_I H_{IQ} dS = F_Q$$

Admittedly this is somewhat out of sequence but it agrees with the eventual result. This is the simple term. The strain energy term, which will give us the stiffness matrices, starts with working out the following derivative:

$$\frac{\partial E_{KL}}{\partial\left(\frac{\partial u_M}{\partial X_N}\right)} = \frac{1}{2}\left(\delta_{KM}\delta_{LN} + \delta_{LM}\delta_{NK} + \delta_{PM}\delta_{NL}\frac{\partial u_P}{\partial X_K} + \delta_{PM}\delta_{NK}\frac{\partial u_P}{\partial X_L}\right)$$

This comes from the definition of the Green-Lagrange strain in terms of the displacement gradients. Using that $\frac{\partial \delta u_M}{\partial X_N} = \frac{\partial H_{MQ}}{\partial X_N}W_Q$, we can evaluate more terms in the strain energy density

$$\frac{\partial E_{KL}}{\partial\left(\frac{\partial u_M}{\partial X_N}\right)}\frac{\partial \delta u_M}{\partial X_N} = \frac{1}{2}\left(\delta_{KM}\delta_{LN} + \delta_{LM}\delta_{NK} + \delta_{PM}\delta_{NL}\frac{\partial u_P}{\partial X_K} + \delta_{PM}\delta_{NK}\frac{\partial u_P}{\partial X_L}\right)\frac{\partial H_{MQ}}{\partial X_N}W_Q$$

$$= \frac{1}{2}\left(\frac{\partial H_{KQ}}{\partial X_L} + \frac{\partial H_{LQ}}{\partial X_K} + \frac{\partial u_M}{\partial X_K}\frac{\partial H_{MQ}}{\partial X_L} + \frac{\partial u_M}{\partial X_L}\frac{\partial H_{MQ}}{\partial X_K}\right)$$

where we have factored out the arbitrary W_Q as with the force vector. Plugging in the discretization of the displacement to this expression, we finally obtain

$$= \frac{1}{2}\left(\frac{\partial H_{KQ}}{\partial X_L} + \frac{\partial H_{LQ}}{\partial X_K} + U_T\left(\frac{\partial H_{MT}}{\partial X_K}\frac{\partial H_{MQ}}{\partial X_L} + \frac{\partial H_{MT}}{\partial X_L}\frac{\partial H_{MQ}}{\partial X_K}\right)\right)$$

Now, the discretized Green-Lagrange strain can be written as

$$E_{IJ} = \frac{1}{2}\left(\left(\frac{\partial H_{IS}}{\partial X_J} + \frac{\partial H_{JS}}{\partial X_I}\right)U_S + \frac{\partial H_{KS}}{\partial X_I}\frac{\partial H_{KR}}{\partial X_J}U_S U_R\right)$$

At long last, we can write the strain energy term of the total potential as

$$\int C_{IJKL}E_{IJ}\frac{\partial E_{KL}}{\partial\left(\frac{\partial u_M}{\partial X_N}\right)}\frac{\partial \delta u_M}{\partial X_N}d\Omega = \frac{1}{4}\int C_{IJKL}\left(\left[\frac{\partial H_{IS}}{\partial X_J} + \frac{\partial H_{JS}}{\partial X_I}\right]U_S + \frac{\partial H_{KS}}{\partial X_I}\frac{\partial H_{KR}}{\partial X_J}U_S U_R\right) *$$
$$\left(\left[\frac{\partial H_{KQ}}{\partial X_L} + \frac{\partial H_{LQ}}{\partial X_K}\right] + U_T\left(\frac{\partial H_{MT}}{\partial X_K}\frac{\partial H_{MQ}}{\partial X_L} + \frac{\partial H_{MT}}{\partial X_L}\frac{\partial H_{MQ}}{\partial X_K}\right)\right)d\Omega$$

This is quite a nasty expression. What we can see, however, is that we obtain the additive decomposition we were looking for. The terms in the square brackets, when multiplied together, form the linear part of the stiffness matrix. These only involve one power of the displacement degrees of freedom U_S, so they can be factored out to form a matrix-vector product. All of the other terms which arise from expanding the multiplication between the parentheses involve higher orders of the displacement. But the dependence is polynomial, so it is clear that one power can be factored out to form a matrix-vector product where the matrix depends on $\underline{U}$. Ultimately, this expression can be simplified to

$$= K^L_{QS}U_S + K^{NL}_{QS}(\underline{U})U_S$$

The governing equation for the finite element problem is then

$$\left(K^L_{QS} + K^{NL}_{QS}(\underline{U})\right)U_S = F_Q$$

We now have the tools in hand to formulate the linearized buckling problem which avoids the full nonlinear solve with incremental solution to determine the point of instability. The first step is to compute a "reference" solution. This involves choosing a configuration of the loads whose buckling effects we want to assess. We will call this force $\underline{F}_0$. Its magnitude is sufficiently small that no significant buckling or instability occurs. We then solve either a linear or non-linear numerical problem for the displacements corresponding to this reference solution. This would take one of the two forms:

$$\underline{U}_0 = \underline{\underline{K}}^{L,-1}\underline{F}_0, \quad \left(\underline{\underline{K}}^L + \underline{\underline{K}}^{NL}(\underline{U}_0)\right)\underline{U}_0 = \underline{F}_0$$

The linear problem can be solved explicitly whereas the nonlinear problem cannot. From what I understand, the goal of this step is to compute displacements which just model the onset of buckling. We then argue that the nonlinear part of the stiffness matrix, capable of modeling instability from geometric interactions between the force and displacement, is linearly proportional to the force vector. Mathematically, this reads

$$\Big(\underline{\underline{K}}^L + \lambda \underline{\underline{K}}^{NL}(\underline{U}_0)\Big)\underline{U} = \lambda \underline{F}_0$$

In other words, we take the reference displacements, use them to compute the nonlinear part of the stiffness matrix, and then assume that this contribution to the "total" stiffness matrix varies linearly with the magnitude of the force. This is accomplished with a load factor λ. Intuitively, this makes some sense: there is some initial loss of stiffness due to buckling at the reference solution, and increasing the force only acts to increase this effect. Mathematically, I do not see where this expression comes from. If this were a true linearization, there should be a Taylor series behind the scenes, but I have not been able to decipher where it is. Either way, the next step is to say that the onset of stability is when there is a loss of uniqueness between the force and displacement. This is to say that the effective stiffness matrix becomes singular. We thus solve the following problem for the load factor:

$$\det\big(\underline{\underline{K}}^L + \lambda \underline{\underline{K}}^{NL}(\underline{U}_0)\big) = 0$$

As in the case of linear eigenvalue buckling presented before, this problem is equivalent to a generalized eigenvalue problem:

$$\underline{\underline{K}}^L \underline{U} = -\lambda \underline{\underline{K}}^{NL}(\underline{U}_0)\underline{U}$$

Computing the value of the load factor then allows us to estimate the magnitude of the load for which the structure becomes unstable. After having seen this material, one can see that we what we are doing here can be summarized with: find the load level for which the displacement can be increased arbitrarily without changing the force. This indicates a point at which the structure stops being able to carry additional load. This is an instability.

8 Extended System Approach

There is an even better approach to buckling for nonlinear problems than linearization. Say that the solution to the discretized system is governed by a set of nonlinear equations

$$\underline{R}(\underline{a}, \lambda) = 0$$

where $\underline{a}$ are the coefficients on shape functions discretizing the displacement and λ is a load parameter that scales a given force vector. As in the example above, this system could be expressing the condition for the total potential energy being at a minimum. We want to determine the critical load at which the structure buckles, which is given by λ, and the nature of the buckled shape. As before, let's assume that the stiffness matrix has a linear and nonlinear part. The residual equation can be written as

$$\underline{\underline{K}}^L \underline{a} + \underline{\underline{K}}^{NL}(\underline{a})\underline{a} = \lambda \underline{F}$$

Say we want the system to accommodate a small increment of increased force. We can linearize the system around the current solution $\underline{a}_0$, where we define $\Delta\underline{a} = \underline{a} - \underline{a}_0$. The linearized system is

$$\underline{\underline{K}}^L \underline{a}_0 + \underline{\underline{K}}^{NL}(\underline{a}_0)\underline{a}_0 + \underline{\underline{K}}^L \Delta\underline{a} + \frac{\partial(\underline{\underline{K}}^{NL}\underline{a})}{\partial \underline{a}}(\underline{a}_0)\Delta\underline{a} = (\lambda + \Delta\lambda)\underline{F}$$

Because, by definition, the reference displacement already satisfies the system with the forcing scaled by λ, this can be written as

$$\left(\underline{\underline{K}}^L + \frac{\partial(\underline{\underline{K}}^{NL}\underline{a})}{\partial \underline{a}}(\underline{a}_0)\right)\Delta\underline{a} = \Delta\lambda \underline{F}$$

It is not possible to accommodate an increment in the force when this increment is not in the column space of this "tangent stiffness." This occurs when the tangent stiffness ceases to be invertible. Thus, we are looking for a reference displacement $\underline{a}_0$ which causes the tangent stiffness to lose full rank. This is taken to be a definition of buckling.

The extended system approach takes this insight and forms an expanded system of equations which enforces that the residual equations are satisfied, and that the force is such that reference solution causes the tangent stiffness to become singular. In addition to this, we can determine the buckling shape, or buckling mode, by finding the nullspace element $\Delta\underline{a} := \underline{\phi}$. The system of equations is then

$$\underline{\underline{K}}^L \underline{a} + \underline{\underline{K}}^{NL}(\underline{a})\underline{a} - \lambda \underline{F} = 0$$

$$\left(\underline{\underline{K}}^L + \frac{\partial(\underline{\underline{K}}^{NL}\underline{a})}{\partial \underline{a}}(\underline{a})\right)\underline{\phi} = 0$$

$$|\underline{\phi}| - 1 = 0$$

Given that the displacement $\underline{a}$, the buckling mode $\underline{\phi}$, and the force magnitude λ are all unknown, we have to introduce an additional equation which specifies the magnitude of the buckling mode. This nonlinear system can be solve, and the solution used to determine the buckled shape of the structure and the critical load.

9 Examples

The two methods used on geometrically nonlinear structures can be clarified by applying them to some examples. We need nonlinear physics that gives rise to instabilities. We can imagine a 1D material which loses stiffness with the displacement. The governing equation for a 1D elastic material is

$$\frac{\partial}{\partial X}\left(E(x)\frac{\partial u}{\partial X}\right) + b(X) = 0$$

We hypothesize a modulus of the form $E = 1 - u^2$, which means the material becomes unstable as the magnitude of the displacement approaches 1. The weak form of this governing equation is

$$\int (1-u^2)\frac{\partial u}{\partial X}\frac{\partial f_i}{\partial X} - b f_i dX = 0$$

We can discretize the displacement with $u(X) = \sum_j a_j f_j(X)$. Plugging this in to the above system, we weak form can be written as

$$K_{ij}a_j - N_{ijk\ell}a_j a_k a_\ell - F_i = 0$$

The definition of the matrices follows from the definition of the weak form. The linearized buckling approach proceeds by choosing a reference force $\underline{F}_0$, and computing the reference displacement. The nonlinear system needs to be solved in order to do this. With the reference force and displacement in hand, we can write

$$\left(\underline{\underline{K}} - \lambda \underline{\underline{\underline{\underline{N}}}}\,\underline{a}_0\underline{a}_0\right)\underline{a} = \lambda \underline{F}_0$$

The change in the geometric stiffness matrix is assumed to be linear with the increase in force. This is a weird, unobvious and often incorrect assumption. We then solve for the generalized eigenvalue problem

$$\underline{K}\,\underline{a} = \lambda \underline{\underline{\underline{\underline{N}}}}\,\underline{a}_0\underline{a}_0\underline{a}$$

The smallest eigenvalue determines the magnitude of the force that causes the stiffness to become singular, and thus creates an instability. The eigenvector corresponding to the smallest eigenvalue is the predicted buckling response. We can discretize the displacement with sines of integer frequencies, ensuring that the displacement boundaries are zero, and carry out this analysis. The shape functions are used to form the various matrices required for the problem. We will take $b(x) = \zeta \sin(\pi x)$, and load step the problem by gradually increasing ζ. See Figure 5.

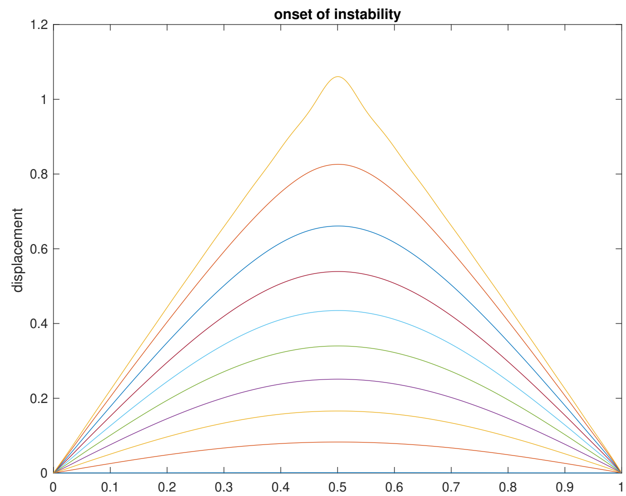

Figure 5: The force driving the problem is gradually increased over the load stepping process. The converged displacements are shown here. Instability is achieved around $\zeta = 8$, where the displacement reaches $u = 1$, and the material loses the ability to carry additional loads.

We can now compute a reference solution $\underline{a}_0$ around different load values and carry out the linearized buckling analysis. Take $\zeta = 5$ to be the reference force. We use the weak form to a compute the reference solution $\underline{a}_0$. We then solve the eigenvalue problem to determine the critical load and the buckled shape. See Figure 6 for results.

The linearized buckling analysis predicts the correct buckling mode but does not do well in recovering the critical force governing the onset of instability. At least we have demonstrated how the analysis works. We can now try out the extended system approach, and explore another nonlinear problem that exhibits instability. Returning to the St. Venant Kirchoff problem, we note that the strain energy is

$$\Psi = \frac{1}{2}E^2 = \frac{1}{2}\left(\frac{\partial u}{\partial X} + \frac{1}{2}\left(\frac{\partial u}{\partial X}\right)^2\right)^2$$

We note that the strain energy gives the second Piola Kirchhoff stress through

$$S = \frac{\partial \Psi}{\partial E} = \frac{\partial u}{\partial X} + \frac{1}{2}\left(\frac{\partial u}{\partial X}\right)^2$$

Though not obvious at first, this is a very weird expression for the stress. The tensile stress can become arbitrarily because it grows with positive dis-

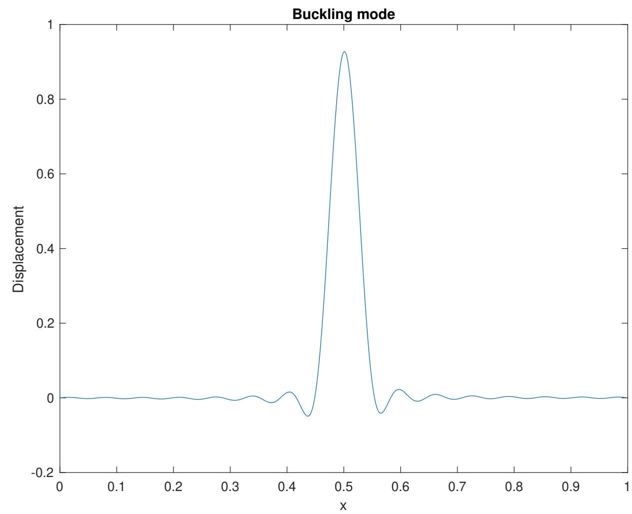

Figure 6: The buckling mode predicted when the problem is linearized around $\underline{a}_0$ corresponding to $\zeta = 5$. The buckling mode shows instability at the center of the bar, which is what we expect. However, the critical buckling it predicts is $\zeta_{cr} = 22$, which is much larger than the unstable value determined from load stepping the problem. It can be seen from numerical experimentation that the reference load needs to be very close to the critical load in order for this linearized analysis to predict the right critical value. This is likely a result of the erroneous assumption that the geometric stiffness scales linearly with the applied force.

placement gradients. As a result the quadratic term, the compressive stress is bounded from below. This means that as the displacement gradients becomes more negative, the stress eventually transitions to tension! This means that the bar reaches a limit of what it can handle in compression, so the problem is unstable in compression.

The easiest unstable problem we can analyze is the the compression driven bar with a clamped left end and a free right end. The displacement is discretized as $u = ax$, which means that when the domain has unit length, the energy is

$$\Pi = \frac{1}{2}\left(a + \frac{1}{2}a^2\right)^2 - Fa$$

When $F < 0$, the bar is put into compression. Solutions correspond to minima of the energy, which can be visualized here. Move the slider around to explore the set of solutions. When the bar is in tension, only one solution exists. When the bar is in compression and the force is small, two solutions exist: one corresponding to the displacement we would expect, and another corresponding to the bar totally inverting and being pulled in tension. As the force is made larger, the compressive solution doesn't exist because the bar can no longer equilibrate the force, and only the inversion solution exists. This is a nice model to begin thinking about nonlinear solid mechanics.

To illustrate the extended system approach, we can return to the Fourier discretization for the bar with zero displacement boundary conditions. The total potential energy for the bar is

$$\Pi = \int \frac{1}{2}\left(\frac{\partial u}{\partial X}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial X}\right)^3 + \frac{1}{8}\left(\frac{\partial u}{\partial X}\right)^4 - b(X)u(X)dX$$

This means that when the solution is discretized, the energy can be written as

$$\Pi = \frac{1}{2}K_{ij}a_i a_j + \frac{1}{2}N_{ijk}a_i a_j a_k + \frac{1}{8}L_{ijk\ell}a_i a_j a_k a_\ell - F_i a_i$$

If you are concerned about this, just plug in the discretization and moves the solution parameters out of the integral. These matrices have all the symmetries you could possibly want. This means that the condition for a minimum is

$$\frac{\partial \Pi}{\partial \underline{a}} = \underline{R}(\underline{a}) = \underline{\underline{K}}\,\underline{a} + \frac{3}{2}\underline{\underline{\underline{N}}}\,\underline{a}\,\underline{a} + \frac{1}{2}\underline{\underline{\underline{\underline{L}}}}\,\underline{a}\,\underline{a}\,\underline{a} - \underline{F} = 0$$

The tangent stiffness is the gradient of the residual system, and is

$$\underline{\underline{\kappa}} = \frac{\partial \underline{R}}{\partial \underline{a}} = \underline{\underline{K}} + 3\underline{\underline{\underline{N}}}\,\underline{a} + \frac{3}{2}\underline{\underline{\underline{\underline{L}}}}\,\underline{a}\,\underline{a}$$

Remember that the extended system approach finds the displacement such that instability occurs, the shape of the buckled structure, and the critical load value. This is accomplished by solving the following nonlinear system of equations:

$$\underline{\underline{K}}\,\underline{a} + \frac{3}{2}\underline{\underline{\underline{N}}}\,\underline{a}\,\underline{a} + \frac{1}{2}\underline{\underline{\underline{\underline{L}}}}\,\underline{a}\,\underline{a}\,\underline{a} - \lambda\underline{F} = 0$$

$$\left(\underline{\underline{K}} + 3\underline{\underline{\underline{N}}}\,\underline{a} + \frac{3}{2}\underline{\underline{\underline{\underline{L}}}}\,\underline{a}\,\underline{a}\right)\underline{\phi} = 0$$

$$|\underline{\phi}| - 1 = 0$$

A nonlinear solver is used so we need not compute gradients of the system. A body force of $b(X) = \sin(\pi X)$ is used again. See figure 7 for the results.

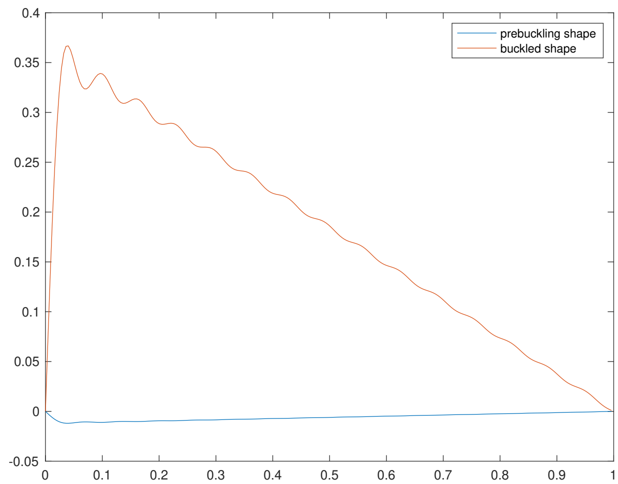

Figure 7: Not sure if this makes sense? Maybe this is the buckled shape because the slope of the compression part is such no load is carried anymore. The point was to learn the method, not to get good results :)