Project: Can Composite Kirchhoff Plates Play Jazz?

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Spring 2023

1 Introduction

Plates are simplified models of the governing equations of three-dimensional elasticity which facilitate semi-analytical solutions through the use global shape functions. Typically, a plate is defined as a structure whose thickness is much smaller than the other two dimensions. Usually this thickness is constant, and for very thin plates, Kirchoff-Love plate theory can be used which assumes that shearing effects in bending are negligible, analogous to Euler beam theory, With simple geometries and boundary conditions, the Rayleigh-Ritz method with trigonometric shape functions is especially attractive. Composite plates are made by stacking plies made of parallel fibers embedded in a matrix material. These plies are anisotropic because the fibers are much stiffer than the matrix, thus each ply has a "preferred" direction for carrying load. Because each ply can be oriented in a different direction, the stacking sequence can be designed to tune the static or dynamic response of the structure. Composite plates are interesting because they form the basis of classic laminate theory, which is important for designing/analyzing thin composite structures in the aerospace industry. Furthermore, they provide a simple example of designing the material itself in order to optimize the response of a structure. Having the ability to choose ply angles in the stacking sequence allows a great deal of design freedom despite the relative simplicity of the problem.

2 Problem Statement

Inspired by the close connections between structural vibrations and musical instruments, we would like to choose the fiber angles in the stacking sequence of the composite plate such that it rings in a pleasing and specific way. For simplicity, we will restrict our design problem to thin square plates with pinned boundary conditions on all edges and fixed geometric dimensions. We will optimize the fiber angles to meet objectives on the structure's dynamic characteristic, and investigate the role of the number of plies through the thickness in meeting this objective. The eigenfrequencies of the structure determine the frequencies at which corresponding eigenmodes vibrate, and we will assume that these frequencies are the same frequencies which the ear picks up, obviously ignoring the added complexity of acoustics. We also know that lower order eigenfrequencies and modes typically dominate the dynamic response of a structure (independent of applied forcing). Thus, we also claim that tuning the lower order eigenfrequencies will control the sound the plate makes when vibrating. Call the eigenfrequencies ω_n and assume they are sorted from least to greatest. We want to design our plate so that the first four eigenfrequencies form a major seventh chord with an unspecified root (ie no explicit requirements on first eigenfrequency). This is a common but interesting chord which is built up from four notes: the root, a major third, perfect fifth, and major seventh. As is often noted, the frequencies of pleasing musical intervals are typically related by integer ratios. A major third has a 5/4 relation, a fifth 3/2 and major seventh 15/8. Thus we can write our objective for the composite plate design problem as

$$z(\underline{\theta}) = \frac{1}{2}\left[\Big(\frac{5}{4}\omega_1 - \omega_2\Big)^2 + \Big(\frac{3}{2}\omega_1 - \omega_3\Big)^2 + \Big(\frac{15}{8}\omega_1 - \omega_4\Big)^2\right]$$

The function z depends on the choice of ply angles in the stacking sequence and is non-negative. A value of zero corresponds to the first four eigenfrequencies exactly producing a major seventh chord. Thus we seek to minimize this objective by appropriately choosing the ply angles to tune the composite plate. Even if this objective is obtained exactly, there will be higher "harmonics" from the eigenfrequecies $\omega_5, \ldots$ which will influence the timbre of our composite "instrument." These may or not be pleasing, but we expect their contribution to the solution (volume) will decay.

3 Mechanics of Composite Plates

3.1 Equations of Motion

For free vibrations of the plate, the equations of motion are defined by the internal potential energy density from bending Π and the kinetic energy density T. The Lagrangian for the system is

$$\int_t \int_0^L \int_0^L \Big(T - \Pi\Big) dx_1 dx_2 dt$$

and the governing equation is

$$\delta \int_t \int_0^L \int_0^L \Big(T - \Pi\Big) dx_1 dx_2 dt = 0$$

We will write the energies in terms of the transverse displacement field $u_3(x_1, x_2)$ for the square composite Kirchoff plate, which we then discretize in terms of global shape functions, compute the spatial part of the integrals, and minimize in time. This will lead to definitions of the mass and stiffness matrices which are implicit functions of the fiber angles. Start with comparatively simple kinetic energy term:

$$T = \int_0^L \int_0^L \frac{1}{2}\rho h \left(\frac{\partial w}{\partial t}\right)^2 dx_1 dx_2$$

For thin plates where the thickness h is small, the kinetic energy of rotation can be neglected. The assumed form of the transverse displacement field $w(x_1, x_2)$ depends on the boundary conditions of the plate. We assume the simplest case where the plate is simply supported on all sides, which allows us to write

$$w(x_1, x_2) = \sum_i \sum_j \bar{w}_{ij}(t) \sin\left(\frac{i\pi x_1}{L}\right) \sin\left(\frac{j\pi x_2}{L}\right)$$

While the double series is more intuitive, it will be simpler to write the displacement as a single sum

$$w(x_1, x_2) = \sum_n w_n \Phi_n(x_1, x_2)$$

where Φ_n is a product of two sine functions whose frequencies are functions of n. Plugging this into the kinetic energy, we get

$$T = \frac{1}{2}\rho h \sum_n \sum_m \frac{\partial w_n}{\partial t}\frac{\partial w_m}{\partial t} \int\int \Phi_n \Phi_m dx_1 dx_2$$

Orthogonality properties will make many of these integrals vanish. For now, we write

$$\boxed{T = \frac{1}{2}\dot{w}_n \left(\rho h \int \Phi_n \Phi_m dA\right) \dot{w}_m}$$

Moving onto the bending strain energy, classic laminate theory says that for layups that are symmetric about the plates midplane, the in-plane and bending energies decouple so that the energy functional for the bending problem depends only on the traverse displacement:

$$\Pi = \frac{1}{2}\int\int \Bigg(D_{11} w_{,11}^2 + 2D_{12} w_{,11} w_{,22} + D_{22} w_{,22}^2 + 4D_{66} w_{,12}^2 \tag{1}$$

$$+4D_{16} w_{,11} w_{,12} + 4D_{26} w_{,22} w_{,12} \Bigg) dx_1 dx_2 \tag{2}$$

Note that the stiffness parameters $D = D(\theta)$ depend on the through-thickness structure of the composite layup and are the parameters that are adjusted in the optimization process. The composite is assumed to have constant layup in the domain so that these parameters do not depend on space. We will pull them out of the integrals and carry out term by term calculations:

$$\frac{D_{11}}{2}\int w_{,11}^2 dA = D_{11}\int\left(\sum_n w_n\Phi_{n,11}\right)\left(\sum_m w_m\Phi_{m,11}\right)dA$$

$$\boxed{=\frac{1}{2}w_n\left(D_{11}(\theta)\int\Phi_{n,11}\Phi_{m,11}dA\right)w_m}$$

$$\frac{1}{2}\int 2D_{12}w_{,11}w_{,22}dA = \frac{1}{2}2D_{12}\int\left(\sum_n w_n\Phi_{n,11}\right)\left(\sum_m w_m\Phi_{m,22}\right)dA$$

$$\boxed{=\frac{1}{2}w_n\left(2D_{12}(\theta)\int\Phi_{n,11}\Phi_{m,22}dA\right)w_m}$$

$$\frac{1}{2}D_{22}\int w_{,22}^2 dA = \frac{1}{2}D_{22}\int\left(\sum_n w_n\Phi_{n,22}\right)\left(\sum_m w_m\Phi_{m,22}\right)dA$$

$$\boxed{=\frac{1}{2}w_n\left(D_{22}(\theta)\int\Phi_{n,22}\Phi_{m,22}dA\right)w_m}$$

$$\frac{1}{2}4D_{66}\int w_{,12}^2 dA = \frac{1}{2}4D_{66}\int\left(\sum_n w_n\Phi_{n,12}\right)\left(\sum_m w_m\Phi_{m,12}\right)dA$$

$$\boxed{=\frac{1}{2}w_n\left(4D_{66}(\theta)\int\Phi_{n,12}\Phi_{m,12}dA\right)w_m}$$

$$\frac{1}{2}4D_{16}\int w_{,11}w_{,12}dA = \frac{1}{2}4D_{16}\int\left(\sum_n w_n\Phi_{n,11}\right)\left(\sum_m w_m\Phi_{m,12}\right)dA$$

$$\boxed{=\frac{1}{2}w_n\left(4D_{16}(\theta)\int\Phi_{n,11}\Phi_{m,12}dA\right)w_m}$$

$$\frac{1}{2}4D_{26}\int w_{,22}w_{,12}dA = \frac{1}{2}4D_{26}\int\left(\sum_n w_n\Phi_{n,22}\right)\left(\sum_m w_m\Phi_{m,12}\right)dA$$

$$\boxed{= \frac{1}{2} w_n \left(4D_{26}(\theta) \int \Phi_{n,22} \Phi_{m,12} dA \right) w_m}$$

Define the following matrices:

$$\boxed{M_{nm} := \rho h \int \Phi_n \Phi_m dA}$$

$$\boxed{\begin{aligned} K_{nm}(\theta) := D_{11}(\theta) \int \Phi_{n,11} \Phi_{m,11} dA + 2D_{12}(\theta) \int \Phi_{n,11} \Phi_{m,22} dA + D_{22}(\theta) \int \Phi_{n,22} \Phi_{m,22} dA \\ +4D_{66}(\theta) \int \Phi_{n,12} \Phi_{m,12} dA + 4D_{16}(\theta) \int \Phi_{n,11} \Phi_{m,12} dA + 4D_{26}(\theta) \int \Phi_{n,22} \Phi_{m,12} dA \end{aligned}}$$

These are the mass and stiffness matrices for the composite plate problem.

3.2 Eigenvalue Problem

The discretized dynamical system will be defined by the stiffness and mass matrices, and the generalized eigenvalue problem is

$$\left(K - \omega_n^2 M\right) \phi_n = 0$$

The solution to this problem defines the eigenfrequencies of interest $\omega_1, \ldots, \omega_4$. We now need to determine how the bending stiffness parameters used to construct K and M depend on the fiber angles before proceeding with the analysis.

3.3 Constitutive Laws

A composite plate is made up of a stacked sequence of lamina with given fiber directions. In order to understand the macroscopic stiffness properties of the plate, we first look at the constitutive law for an individual laminate (ply). Each ply is considered to be in a state of plane stress, and when the coordinate system is aligned with the fibers, we have

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} 1/E_1 & -v_{21}/E_1 & 0 \\ -v_{12}/E_2 & 1/E_2 & 0 \\ 0 & 0 & 1/G_{12} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

This is a material with no coupling between normal and shear strain and different stiffnesses in two perpendicular directions. However, when the fibers are oriented at an arbitrary angle w.r.t. the coordinate system, it can be seen that

$$\begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix}$$

The exact form of this constitutive matrix comes from applying a rotation to the fiber-aligned material tensor. There are six independent components. These six independent components can be stored in a vector and written as

$$\begin{bmatrix} C_{11} \\ C_{22} \\ C_{12} \\ C_{66} \\ C_{16} \\ C_{26} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cos 2\theta & \cos 4\theta \\ 1 & 1 & -\cos 2\theta & \cos 4\theta \\ 1 & -1 & 0 & -\cos 4\theta \\ 0 & 1 & 0 & -\cos 4\theta \\ 0 & 0 & \frac{1}{2}\sin 2\theta & \sin 4\theta \\ 0 & 0 & \frac{1}{2}\sin 2\theta & -\sin 4\theta \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$

$$\underline{C}(\theta) = \underline{\underline{\chi}}(\theta)\underline{\alpha}$$

The entries of the vector α are functions of the aligned Young's Moduli and Poisson Ratio's. As we will see, it will be useful to compute how these entries change with the fiber angle θ, so we also compute

$$\frac{\partial \underline{C}}{\partial \theta} = \frac{\partial \underline{\underline{\chi}}}{\partial \theta}\underline{\alpha}$$

$$\frac{\partial}{\partial \theta}\begin{bmatrix} C_{11} \\ C_{22} \\ C_{12} \\ C_{66} \\ C_{16} \\ C_{26} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -2\sin 2\theta & -4\sin 4\theta \\ 0 & 0 & 2\sin 2\theta & -4\sin 4\theta \\ 0 & 0 & 0 & 4\sin 4\theta \\ 0 & 0 & 0 & 4\sin 4\theta \\ 0 & 0 & \cos 2\theta & 4\cos 4\theta \\ 0 & 0 & \cos 2\theta & -4\cos 4\theta \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$

Now that we have a constitutive law for the individual lamina, we can turn to the composite plate. If we restrict ourselves to plates for which the "coupling stiffness" vanishes (as was assumed in the above energy calculations), we only need to compute the bending stiffness for the composite plate. This requirement forces the composite plate to have a symmetric stacking sequence about its midplane. Suppose there are an even number of equal thickness plies through the thickness h of the plate and that $x_3 = 0$ corresponds to the midplane of the plate. We can write the components of the bending stiffness as

$$\begin{aligned} \underline{D} &= \sum_{n=1}^{N} \int_{(n-1)(h/N)-h/2}^{n(h/N)-h/2} \underline{C}(\theta_n) x_3^2 dx_3 \\ &= \sum_{n=1}^{N} g_n \underline{C}(\theta_n) = \left(\sum_{n=1}^{N} g_n \underline{\underline{\chi}}(\theta_n) \right) \underline{\alpha} \\ g_n &:= \frac{1}{3}\left[\left(\frac{nh}{N} - \frac{h}{2} \right)^3 - \left(\frac{(n-1)h}{N} - \frac{h}{2} \right)^3 \right] \end{aligned}$$

From the symmetry of the stack, we require that there are an even number of equal thickness plies. Thus, the bending stiffness takes a slightly simpler form

$$\underline{D} = 2\left(\sum_{n=1}^{N/2} g_n \underline{\underline{\chi}}(\theta_n)\right)\underline{\alpha}$$

3.4 Sensitivities

When using a gradient-based optimization method, it is beneficial to have explicit means of computing gradients of the objective w.r.t. to design variables. If we we differentiate the objective $z(\underline{\theta})$, it is easy to see that the sensitivity $\frac{\partial \omega_n}{\partial \theta_i}$ is required. To obtain an expression for this sensitivity, differentiate the eigenvalue problem:

$$\left(\frac{\partial K}{\partial \theta_m} - \frac{\partial w_n^2}{\partial \theta_m} M - w_n^2 \frac{\partial M}{\partial \theta_m}\right)\phi_n + \left(K - w_n^2 M\right)\frac{\partial \phi_n}{\partial \theta_m} = 0$$

Pre-multiplying by ϕ_n^T, enforcing the governing equations, and using orthogonality and symmetry of the stiffness and mass matrices, this simplifies to

$$\frac{\partial w_n^2}{\partial \theta_m} = \phi_n^T \left(\frac{\partial K}{\partial \theta_m} - w_n^2 \frac{\partial M}{\partial \theta_m}\right)\phi_n$$

This derivative is evaluated using the current design to compute the stiffness/mass matrices along with the eigenvectors. The mass matrix does not depend on the fiber angles, thus this derivative is zero. In order to compute the sensitivity of the eigenvalues we require the gradient of the stiffness matrix. Given the above definition, the stiffness matrix can be written as

$$K_{nm}(\underline{\theta}) = \begin{bmatrix} D_{11}(\underline{\theta}) \\ D_{22}(\underline{\theta}) \\ D_{12}(\underline{\theta}) \\ D_{66}(\underline{\theta}) \\ D_{16}(\underline{\theta}) \\ D_{26}(\underline{\theta}) \end{bmatrix} \cdot \begin{bmatrix} \int \Phi_{n,11}\Phi_{m,11} dA \\ \int \Phi_{n,22}\Phi_{m,22} dA \\ 2\int \Phi_{n,11}\Phi_{m,22} dA \\ 4\int \Phi_{n,12}\Phi_{m,12} dA \\ 4\int \Phi_{n,11}\Phi_{m,12} dA \\ 4\int \Phi_{n,22}\Phi_{m,12} dA \end{bmatrix}$$

The integrals of the shape functions do not depend on the fiber angles. Thus, we can write

$$\frac{\partial K_{nm}}{\partial \theta_k} = \frac{\partial \underline{D}}{\partial \theta_k} \cdot \underline{\Phi}_{nm}$$

Given the definition of the bending stiffness, we can write its derivative as

$$\frac{\partial \underline{D}}{\partial \theta_k} = 2\left(\sum_{n=1}^{N/2} g_n \frac{\partial \underline{\underline{\chi}}(\theta_n)}{\partial \theta_k}\right)\underline{\alpha} = 2 g_k \frac{\partial \underline{\underline{\chi}}(\theta_k)}{\partial \theta_k}\underline{\alpha}$$

Returning to the sensitivity analysis, the eigenfrequency sensitivity is

$$\frac{\partial w_n^2}{\partial \theta_m} = \phi_i^n \frac{\partial K_{ij}}{\partial \theta_m}\phi_j^n$$

because the mass matrix doesn't depend on fiber angles and where ϕ_i^n is the i-th component of the n-th eigenvector. The gradient of the stiffness matrix can be written analytically as

$$\frac{\partial K_{nm}}{\partial \theta_k} = 2g_k \frac{\partial \underline{\underline{\chi}}(\theta_k)}{\partial \theta} \underline{\alpha} \cdot \begin{bmatrix} \int \Phi_{n,11}\Phi_{m,11} dA \\ \int \Phi_{n,22}\Phi_{m,22} dA \\ 2\int \Phi_{n,11}\Phi_{m,22} dA \\ 4\int \Phi_{n,12}\Phi_{m,12} dA \\ 4\int \Phi_{n,11}\Phi_{m,12} dA \\ 4\int \Phi_{n,22}\Phi_{m,12} dA \end{bmatrix}$$

4 Implementation

Though it is interesting to note that sensitivities have very explicit expressions owing to simple parameterization of the constitute laws in terms of the lay-up parameters, for the sake of easy implementation, we will not pursue these any further. Of course, it is more efficient to use the explicit sensitivities than rely on an optimization algorithm which must compute gradients on its own, but after all this whole problem is optional, so we opt for the lazy/inefficient route. We have developed techniques to compute the objective in terms of eigenfrequencies, eigenfrequencies in terms of the mass/stiffness matrices, mass/stiffnesss matrices in terms of constitutive relations, and constitutive relations in terms of the fiber angles. Thus, we create a function in matlab which computes the objective in terms of the layup angles and drop this into fmincon for minimization. We constrain the fiber angles to remove redundant rotations:

$$\frac{-\pi}{2} < \theta_i < \frac{\pi}{2}$$

which can be included in the upper and lower bound arguments of fmincon. Each optimization is carried out for a specified number of plies in the stack. We also need to choose the order/size of the Rayleigh-Ritz approximation of the transverse displacement field. See Table 1 for a summary of how the problem is implemented.

5 Results

We pre-compute the mass and stiffness matrices using symbolic differentiation and integration for $N = 8$, meaning the "highest order" shape function in the approximation is $\sin(8\pi x_1/L)\sin(8\pi x_2/L)$. Providing fmincon a matlab function which computes the eigenfrequencies and then the objective value as a function of the fiber angles, we first turn to a simple test problem. Analogous to our past homework, we seek fiber angles such that the second eigenfrequency is 10% greater than the first. For a range of ply numbers, fmincon can find a set of fiber angles which meet this criteria with little error. Basic music theory indicates that the interval created by 11/10 frequency ratios is quite dissonant,

Fiber-aligned Young's Modulus (E_1)	180E9
Matrix Young's Modulus (E_2)	10E9
Shear Modulus (G_{12})	7E9
Poisson Ratio (v_{12})	0.28
Density (ρ)	1500
Side Length Dimension (L)	1
Thickness (t)	0.05
Order of approximation (N)	8

Table 1: Parameters governing the implementation of the composite plate optimization problem similar to Graphite/Epoxy composite material.

thus our plate aspires to play something beautiful. Our original objective was to have the plates dominant natural frequencies play a major seventh chord:

$$z(\underline{\theta}) = \frac{1}{2}\left[\left(\frac{5}{4}\omega_1 - \omega_2\right)^2 + \left(\frac{3}{2}\omega_1 - \omega_3\right)^2 + \left(\frac{15}{8}\omega_1 - \omega_4\right)^2\right]$$

We will call the design *jazzy* if and only if the design of plate is such that

$$z(\underline{\theta}_j) < \epsilon$$

where ϵ is some small parameter for a given number of plies p. As Figure 1 suggests, it is difficult for aerospace structures to ever obtain this jazzy state. It seems that requiring these ratios on the first four eigenfrequencies in particular is more restrictive than the fiber angles can accommodate. To relax the design problem, we could require that *some eigenfrequencies in the spectrum were such that a major seventh chord is played*, but this would lose the guarantee that these frequencies would be salient, as the higher order modes contribute less to the solution.

6 Future Work

Focusing on more important problems.

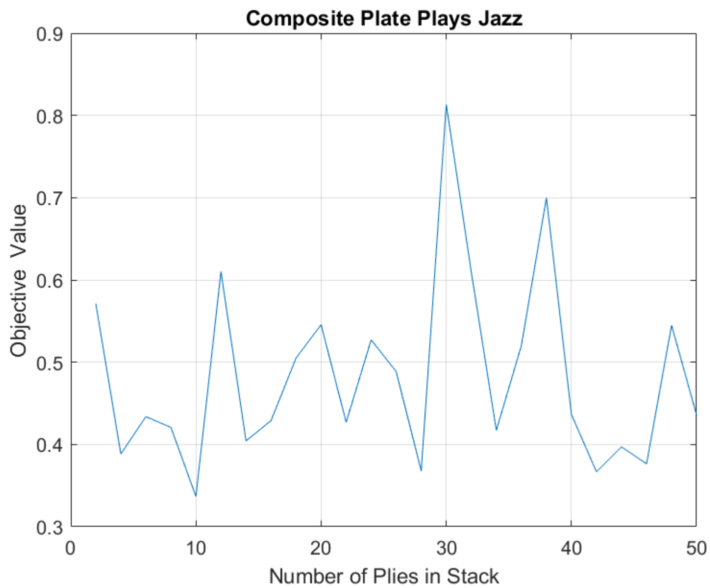

Figure 1: Objective value as a fraction of initial objective for randomly initialized fiber angles through stack plotted over range of ply numbers. Getting to within 10% of objective does not indicate good tuning, and even this is not reliably achieved. Lower values seem to indicate particularly poor guesses for initial fiber angles as opposed to good convergence. This composite plate fails to play jazz, or at least favors a kind of avant-garde jazz unknown to the authors.