Space: The Role of Dimensionality in Physics

by Conor Rowan

Concentrations: Mathematics and Mechanical Engineering

Thesis Advisor: Eric Kramer

Second Reader: Mike Bergman

Provost and Vice President: Ian Bickford

Completed Summer 2018

Bard College at Simon's Rock
Great Barrington, Massachusetts

Table of Contents

Contents

Abstract

The number of spatial dimensions clearly manifests in the laws of physics, and generalizing these laws to different dimensions gives hints as to why we live in 3 + 1 spacetime. Spacetime is a notion that emerged with relativity in the early 20th century and is a way of unifying space and time where a N + D spacetime is a world with N spatial dimensions and D temporal ones. For the sake of sanity, we assume throughout that D=1 and N is variable given that a world with multiple time dimensions is an even greater challenge to think and talk about. Dimensionality implicitly influences the form of force laws for gravity and electromagnetism and changing the rate at which these forces vary with distance has far-reaching consequences. In different dimensions, the stability of circular orbits is no longer a safe assumption, nor is our "traditional" model of 3d quantum mechanics, which describes the behavior of atoms and their electrons. Assuming that a prospective dimensionality must be a positive whole number, we would expect it to be very large if picked at random. After all, a number chosen randomly between 1 and 1,000,000 is less than 100 only 0.01 percent of the time, and this is with an artificial upper limit. Intuitively, a world with this many dimensions seems chaotic and improbable–the Anthropic Cosmological Principle tells us that the world we observe must be conducive to the development of intelligent life, and perhaps our low-dimensional universe is exactly that. Thus, in addition to discussing a few interesting anecdotes and purely mathematical results, my thesis explores the various ways in which the undergraduate physics curriculum would differ in higher/lower dimensional space, offering insight into the nature of different-dimensional universes. Topics such as generalized waves and geometries are touched on, but the majority of the thesis is concerned with orbits, rocketry and quantum mechanics. It is shown that the theories of both Newton and Einstein predict the impossibility of stable orbits (circular or otherwise) in higher dimensions. The equation of a self-powered two-dimensional rocket travelling in a variable gravitational field

is formulated, and the feasibility of 2d rocketry is investigated with a computer simulation. In essence, this chapter analyzes the likelihood of space travel/exploration in a world where gravity is much stronger. Lastly, the arbitrary-dimensional atom is studied via the behavior of its electrons' orbitals and through generalized solutions to the governing equation of quantum mechanics, the Schrodinger equation. Many of these findings point to the convenience, if not necessity of $3+1$ dimensional spacetime for the evolution of advanced life.

1 Introduction

And even as we, who are now in Space, look down on Flatland and see the insides of all things, certainly there is yet above us some higher, purer region, whither thou dost surely purpose to lead me.

-Edwin Abbott, Flatland, 1884

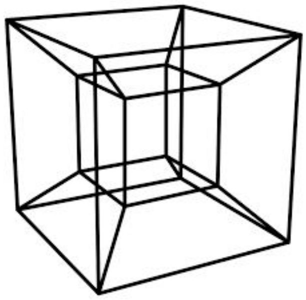
The dimension of a space, or its “dimensionality,” is the minimum number of coordinates needed to uniquely specify a point. The familiar x-y-z space of math and physics is an example relevant to our own universe–fixing one coordinate describes a plane (z=constant, for example) and two coordinates describe a line. Thus, three is the minimum number needed to identify a single point in space. If we want to describe an “event” in our $3 + 1$ spacetime, which requires both a time and place, three spatial coordinates in addition to a temporal one are necessary. It’s almost impossible to imagine a world having more than three indepedent directions; a world where our entire universe is but a tiny slice of some more complicated reality, a world where a chair with three legs could not stand up. These inscrutable questions and abstract wonderings were the exact reason I became interested in this topic–I hoped that the language of math (in the context of physics, of course) might help ease the burden of understanding the consequences of higher dimensional space. While mathematics and intuition can be two very separate things, it seems that math is the only meaningful point of access into questions about different universes. Generalizing the laws of physics to different dimensions necessitates the assumption that the structure of these worlds is qualitatively similar to ours; that the fundamental forces still exist, that the speed of light is still constant, etc. We make as few assumptions as possible, follow the math and hopefully come to appreciate some of the strangeness that tampering with the very fabric of our reality entails.

Preamble The following few examples are non-mathematical introductions into thinking about dimensionality. We look first at the omnipresence of a higher dimensional being. Then, just as a sphere in 3d space is built up of many planar rings, a four dimensional sphere comprises a vast array of spherical cross sections. This is impossible to envision in a Zen-riddle kind of way–it is so at odds with intuition that it seems like a ridiculous statement. But despite the inherent challenges in visualizing hyperspheres (more to come in the next chapter), the tesseract is an interesting tool for visualizing a hypercube. The Necker cube provides another enigmatic glimpse into higher dimensions, and the next section discusses the relevance of mirror images to higher dimensions. These few anecdotes illustrate the fundamental challenge of seeing beyond our three-dimensional bias, and the various ways that we can try. Things get more mathematical with the treatment of waves, which hints at the privileged nature of our specific universe. The Anthropic Principle is way of making sense of this claim, and is touched on afterwards.

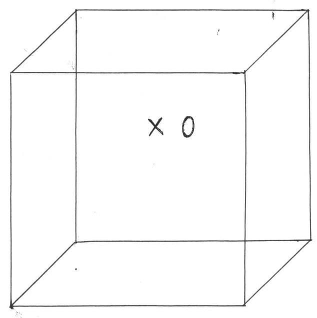
Higher Dimensions It seems that being in a higher dimension gives a privileged look into lower dimensional worlds. In Edwin Abbott's "Flatland," a square inhabiting a flat universe is contacted by a sphere and brought news of the third dimension. This sphere convinces our four-sided protagonist of his three-dimensionality by describing the contents of a locked cabinet, then proceeding to (gently) poke and prod the square's "inside." In A.K. Dewdney's "Planiverse", university students contact a different two-dimensional world (named Arde) through a compupter simulation, and marvel at how Ardean engineers managed to construct a steam engine without being able to access its inside directly. Perhaps a four-dimensional being would be impressed by our internal combustion engines, which prohibit direct examination of their sealed chambers. An important aspect of observing some subspace from a higher dimension, and perhaps a tool for visualization, is seeing the inside of all things in that subspace. There is nowhere for a Flatlander to hide an object from a three dimensional observer. A four dimensional surgeon could peform heart surgery without ever making an incision.

Cross Sections A sphere crossing through a plane would look like a point at first, would grow in size (to the sphere's radius), then shrink back down until disappearing. The Flatlander's only see a changing series of cross sections from which the sphere is built. Analogously, a hypersphere crossing through our universe would suddenly manifest as a point, grow to become a sphere with the same radius as the hypersphere, then contract back down to a point and vanish. The rate at which this took place would depend on how quickly the hypersphere passed through our space. Cross sections of a hypersphere are, well, spherical. Picture a person, or any other sufficiently complex object, falling through Flatland. The cross sections would be confusingly variable and irregular. A complicated 4d object crossing through our space would be totally inexplicable.

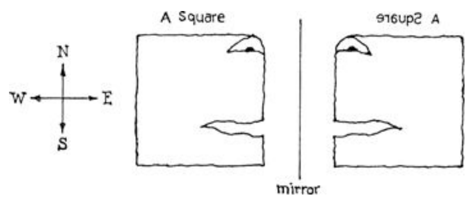
Tesseract A tesseract is conventionally understood as a representation of a hypercube. To see why this is the case, consider looking at a regular cube head on. What does it look like? Simply a large square with a smaller square inside of it, with vertices connected by lines. A tesseract is the exact anaolgue–assuming we are looking at a hypercube head on, it should appear as two nested cubes with connected vertices.



Necker Cube In 1832, the Swiss crystallographer Louis Albert Necker published the now famous optical illusion called a Necker Cube (he is also responsible for another illusion called the "impossible cube"). It is a line drawing of a cube with no cues as to its orientation, so the front side of the cube, or the side perceived to be closest to the viewer, could be the lower-left or upper-right face of the line drawing. With some effort, the viewer can make the cube flip, where it appears to change its orientation from one state to the other. This alone is interesting; however, if we mark the cube with an X and an O on one of the two potential front faces, we gain information about the cube's orientation, but are still able to make it flip. Seeing the lower-left face as the front, the markings are interpreted as being on the outside of the back face (we are, in effect, seeing through the cube). If we were to rotate the cube 180 degrees around a vertical axis in order to bring the marked side closest to the viewer, the markings would face us and the O would be on the left, the X on the right. Return to the original marked Necker Cube and flip it so the upper-right face is outwards, as are the markings. Now the X is on the left and the O on the right. There is no three-dimensional rotation that can take one state of the marked Necker cube to the other. What we have observed is a rotation through four-dimensional space. This anecdote is a useful way of imagining a higher spatial dimension, a rotation which essentially turns the cube inside out.

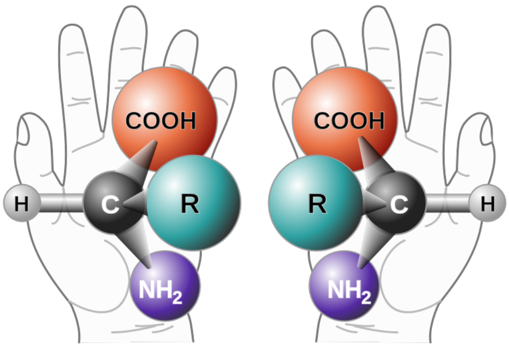


An object's mirror image is equivalent to being flipped in a fourth spatial dimension. The states of the marked Necker cube are mirrored. Let's use the Flatland analogy to better illustrate this point. Say there were two Flatlander twins that were mirror images of each other. No rotation in their plane could make them the same; however, lifting one twin into the third dimension and flipping them "upside down" would make the two twins indistinguishable. The philosopher Immanuel Kant proposed an analogous problem concerning left and right hands. In a universe empty but for a single human hand, does it make sense to label it right or left? To me, it seems that the answer is: it depends. If we confine the hand to three dimensions, it has an inherent orientation, independent of what we happen to name it. We do not need an oriented object to recognize a right hand versus a left one. However, the possibility of rotating it four-dimensionally clearly destroys the utility of left- and right-handedness.



Envision the world of Flatland as the x-y plane (z=0) and yourself as a three-dimensional observer able to freely move up and down the z-axis, taking on either positive or negative values. You observe Flatland from above or below, but neither orientation is preferential. Everytime you cross through the plane, Flatland becomes its own mirror image. In other words, it appears one way from above, and another from below. It's odd to imagine that "handedness," (meaning three-dimensional objects that are mirror images of each other) is arbitrary to a higher dimensional observer.

Chirality An interesting application of handedness appears in chemistry. Chirality is a property of certain molecules that is best understood with an image. Interestingly, the word chirality comes the Greek word for hand. Chiral molecules are made of the same constituent elements in the same relative configurations but cannot be interchanged by rotation. A chiral molecule is not superimposable on its mirror image. An enantiomer is a molecule that exhibits chirality. Enantiomers have many of the same chemical properties such as boiling point and solubility, but behave differently in some situations. In fact, enantiomers only exhibit divergent chemical behaviors when interacting with other chiral molecules, and will act the same in any other reaction. There is a striking parallel with our everyday experience of chirality (handedness). The only time we notice our handedness is when interacting with other oriented objects. Gloves, instruments and cars are chiral objects and this property influences how we interact with them. Left handed guitarists could not play a right handed guitar, for example. Although handedness is arbitrary in the context of higher dimensions, it influences the ongoings (human, chemical, etc) of our world in important ways.



Waves The n-dimensional wave equation can be written in the form

$$\frac{\partial^2\psi}{\partial x_1^2} + \cdots + \frac{\partial^2\psi}{\partial x_n^2} = \frac{\partial^2\psi}{\partial t^2}$$

where we assume for simplicity that the speed at which the wave propagates is unity. Given that the wave is spherically symmetric, ψ can be written exclusively as a function of radius (r) and time (t) where $r^2 = x_1^2 + ... + x_n^2$. A few useful facts follow from this:

$$\sum(\frac{\partial r}{\partial x_i})^2 = 1 \qquad \sum\frac{\partial^2 r}{\partial x_i^2} = \frac{n-1}{r}$$

Using the multivariate chain rule and product rule, the derivatives of $\psi(r,t)$ can be expressed in terms of r's and x_i's. We then manipulate and employ the previous identities to cast the wave equation only in terms of radius.

$$\frac{\partial\psi}{\partial x_i} = \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x_i} \quad \frac{\partial^2\psi}{\partial x_i^2} = \frac{\partial r}{\partial x_i}\frac{\partial^2\psi}{\partial r\partial x_i} + \frac{\partial\psi}{\partial r}\frac{\partial^2 r}{\partial x_i^2}$$

The mixed partial can be simplified

$$\frac{\partial^2\psi}{\partial r\partial x_i} = \frac{\partial}{\partial r}\frac{\partial\psi}{\partial x_i} = \frac{\partial}{\partial r}(\frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x_i}) = \frac{\partial r}{\partial x_i}\frac{\partial^2\psi}{\partial r^2}$$

Therefore,

$$\frac{\partial^2\psi}{\partial x_i^2} = \frac{\partial\psi}{\partial r}\frac{\partial^2 r}{\partial x_i^2} + (\frac{\partial r}{\partial x_i})^2\frac{\partial^2\psi}{\partial r^2}$$

Summing over all i and using the identities introduced earlier, we arrive at

$$\sum\frac{\partial^2\psi}{\partial x_i^2} = \frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial r^2} + (\frac{n-1}{r})\frac{\partial\psi}{\partial r}$$

Now define a new variable $\phi(r,t) = r^k\psi(r,t)$ so that

$$\frac{\partial^2\phi}{\partial r^2} = r^k\frac{\partial^2\psi}{\partial r^2} + 2kr^{k-1}\frac{\partial\psi}{\partial r} + k(k-1)r^{k-2}\psi$$

If we divide through by r^k then set $k = \frac{n-1}{2}$, we end up with

$$\frac{1}{r^{\frac{n-1}{2}}}\frac{\partial^2\phi}{\partial r^2} = \frac{\partial^2\psi}{\partial r^2} + \frac{n-1}{r}\frac{\partial\psi}{\partial r} + \frac{(n-1)(n-3)}{4r^2}\psi$$

At this point, its easy to recognize the first two terms on the right side of the expession as $\frac{\partial^2\psi}{\partial t^2}$, so we write

$$\frac{\partial^2\psi}{\partial t^2} = \frac{1}{r^{\frac{n-1}{2}}}\frac{\partial^2\phi}{\partial r^2} - \frac{(n-1)(n-3)}{4r^2}\psi$$

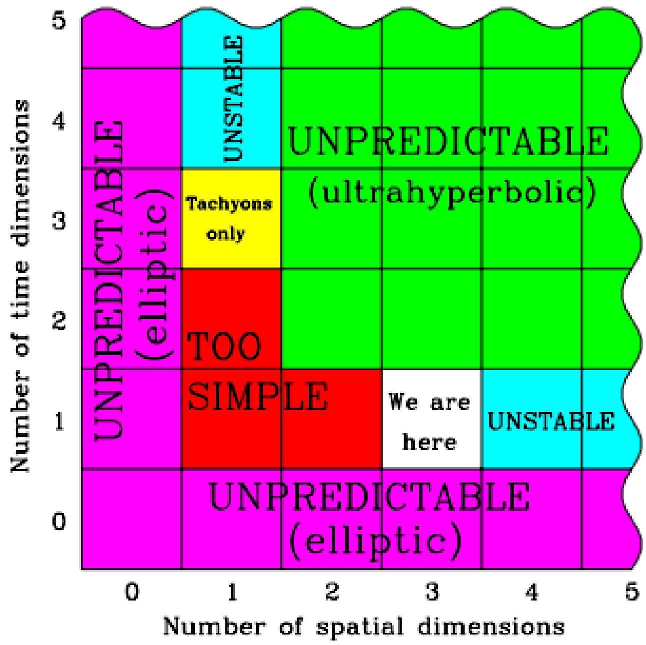
Multiplying through by $r^{\frac{n-1}{2}}$, we get

$$\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial r^2} - \frac{(n-1)(n-3)}{4r^2}\phi$$

The wave equation has been transformed in such a way to show how dimensionality manifests in its solution. For n=1, the second term on the right side vanishes and the solution is simple, as would be expected in the one-dimensional case. However, n=3 makes the second term vanish and is effectively one-dimensional, the difference being that $\phi = r\psi$ which corresponds to the amplitude of the wave decreasing as it travels outward. It is now clear that the wave equation exhibits special and simple behavior in the case of n=3.

Huygen's principle states that every point on a wavefront can be thought of as a source of future waves and the resulting wave is the superposition of all these tiny contributions. This is the same as saying that every disturbance in the medium supporting the wave travels at single, definite speed. It can be shown that Huygen's principle is only true in odd-dimensional spaces. Odd-dimensional waves have a sharp front and rear, as wide as the wave speed multiplied by the time it took to create the disturbance. They do not get wider or more diffuse as they propagate. This is in contrast to even-dimensional waves, such as those on the surface of a pond. Dropping a stone into water doesn't create a single plane wave, rather a series of waves whose amplitudes quickly diminish with time. The ability to process clear, undistorted signals is key in the development of an advanced civilization, and an even number of dimensions does not allow for sharply defined signals in the form of waves. A close analysis of solutions of the traditional wave equation in higher odd dimensions shows that waves traversing large distances become distorted, thus three dimensions is the only case of high-fidelity wave transmission.

Anthropic Cosmological Principle The Anthropic Cosmological Principle states that we must observe the universe to be conducive to the evolution of conscious, intelligent life. Dimensionality is just one of many facets of the principle–scientists turn to the measured numerical values of fundamental constants as proof. Changing the strength of the electromagnetic force by fractions of a percent drastically upsets the production of oxygen and carbon in stars, two crucial elements for life. A balance between a variety of these constants also leads to the atomic stability of matter. The fact that the dimensionality of our universe corresponds to the most convenient behavior of waves, and thus the ability to transmit information, also evinces the anthropic principle. Throughout the coming explorations of the role of dimensionality in physics, it is interesting to consider how unique the case of n=3 is for the development and evolution of conscious life as we know it.

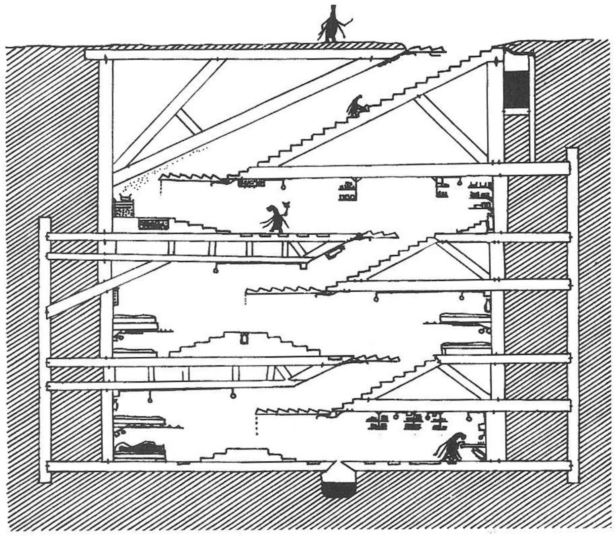


UNSTABLE
UNPREDICTABLE (elliptic)
UNPREDICTABLE (ultrahyperbolic)
Tachyons only
TOO SIMPLE
We are here
UNSTABLE
UNPREDICTABLE (elliptic)
Number of time dimensions
Number of spatial dimensions
0 1 2 3 4 5

2 Hypersphere

Instead of moving, you merely exercise some magic art of vanishing and returning to sight; and instead of any lucid description of your new World, you simply tell me the number and sizes of some forty of my retinue, facts known to any child in my capital.

-Edwin Abbott, Flatland, 1884



Preamble A hypersphere is a generalization of a sphere to higher dimensions (think hyperspace). A unit circle is expressed mathematically as $x^2 + y^2 = 1$, and a unit sphere is $x^2 + y^2 + z^2 = 1$. Adding a dimension translates to adding another squared coordinate to the sum on the left side of the equation. Thus, an arbitrary dimensional unit hypersphere is simply $\sum x_n^2 = 1$. This short chapter investigates mathematical consequences of geometry in very high dimensions, and is not particularly relevant to physics. That being said, the results of looking at these limiting cases are fascinatingly counterintuitive, thus illustrating the utility of math as tool for peering into these divergent worlds.

We begin with the formula for an n-dimensional sphere expressed in Cartesian coordinates $x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2$ and note that the n-dimensional volume is the integral of the (n-1)-dimensional surface area integrated from 0 to R. This can be easily checked for the familiar two and three dimensional formulas. We also write the volume as the integral of the differential volume element.

$$V_n(R) = \int_0^R S_{n-1}(R)dr$$

$$V_n(R) = \int_V dV = \int \cdots \int_{x_1^2+x_2^2+\cdots+x_n^2 \leq R^2} dx_1 \ldots dx_n$$

Additionally, we know that the volume formula should be of the form $C_n R^n$ ie. a constant dependent on the dimensionality times the n-th power of the radius, which is easily seen from dimensional analysis. Equivalently, $S_{n-1}(R) = \frac{dV_n n-1}{dR} = nC_n R^{n-1}$. Therefore

$$\int \cdots \int_{x_1^2+x_2^2+\cdots+x_n^2 \leq R^2} dx_1 \ldots dx_n = \int_0^R S_{n-1}(R)dR = nC_N \int_0^R r^{n-1}dr$$

The n-dimensional Cartesian differential volume element can be written in spherical coordinates as $dx_1 \ldots dx_n = r^{n-1}drd\Omega_{n-1}$ where the Ω term represents the n-1 angular coordinates. Looking back to the previous expression, its clear that $\int \cdots \int d\Omega_{n-1} = nC_n$. We now consider the following expression in order to determine the constant C_n:

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-(x_1+\cdots+x_2^2)} dx_1 \ldots dx_n = \int e^{-x_1^2}dx_1 \cdots \int e^{-x_n^2}dx_n = \pi^{\frac{n}{2}}$$

Writing the same integral in spherical coordinates and using earlier results yields

$$\int_0^{\infty} r^{n-1}e^{-r^2}dr \int d\Omega_{n-1} = nC_n \int_0^{\infty} r^{n-1}e^{-r^2}dr = \frac{1}{2}nC_n\Gamma(\frac{n}{2}) = C_n\Gamma(1+\frac{n}{2})$$

This means that $C_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}$and using the formula for volume as a function of this constant and the relation between surface area and volume we can write

$$V_n(R) = \frac{\pi^{\frac{n}{2}}R^n}{\Gamma(1+\frac{n}{2})} \quad S_{n-1}(R) = \frac{n\pi^{\frac{n}{2}}R^{n-1}}{\Gamma(1+\frac{n}{2})}$$

As Figure 1a shows, the surface area and volume for a unit hypersphere reach a maximum and then eventually decay away to 0. Unfortunately, the meaning of a comparison among volumes with different units is unclear, so in Figure 1b, the n-th root is taken of the volume and surface area formulas so as to keep the units as distance[1].

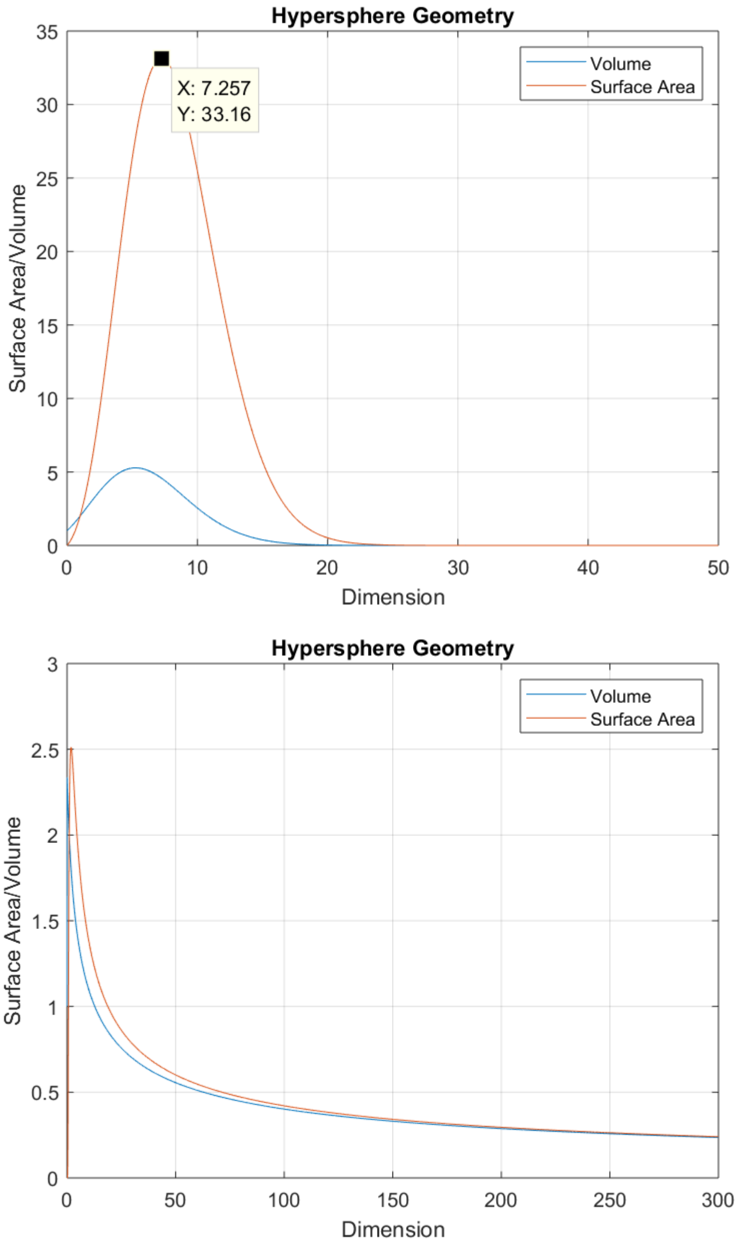


Figure 1: a) Surface area and volume peak at low n values, then decay away and b) the n-th root of volume and surface still decay away, but at a much lower rate

Using the data cursor in matlab, we can find the dimensionality that corresponds to maximum unit surface area and volume. Figure 1a shows the coordinates for the point of maximal area, revealing that n=7 is the integer dimension of largest surface area. The same approach shows that n=5 maximizes the unit volume.

Other strange things happen in these high dimensional geometries. The hypercube that contains the unit hypersphere (side length of 2) has a volume of 2^n and the distance from its center to a vertex is $||(1, \ldots, 1)|| = \sqrt{(1 + \cdots + 1)} = \sqrt{n}$, which both diverge as n gets large, whereas the hypersphere's volume goes to 0.

We now look at another interesting geometric result involving hyperspheres. First, we notice that $V_n(R) = V_n(1)R^n$ which follows from the formulas derived in the past section. Consider a shell at the surface of an arbitrary dimensional hypersphere with thickness a. The percent of the volume concentrated in this shell is

$$\frac{V_n(R) - V_n(R-a)}{V_n(R)}$$

which can be rewritten as

$$\frac{V_n(1)(R^n - (R-a)^n)}{V_n(1)R^n} = 1 - (1 - \frac{a}{R})^n$$

This expression can be graphed against the number of dimensions n for some fixed value of a. Choose $a = 0.01R$ so that we are plotting the function $1 - .99^n$. As Figure 2 shows, for extremely high dimensionality, nearly all the volume is concentrated in this thin shell at the hypersphere's surface.

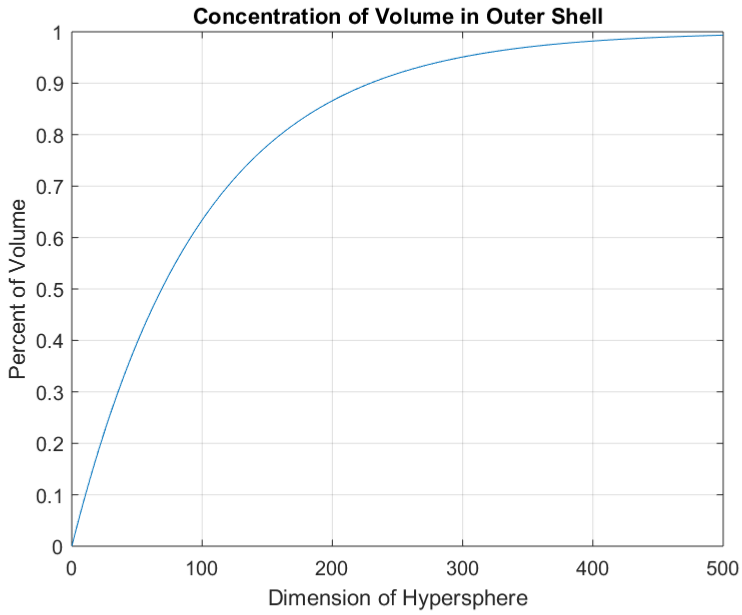
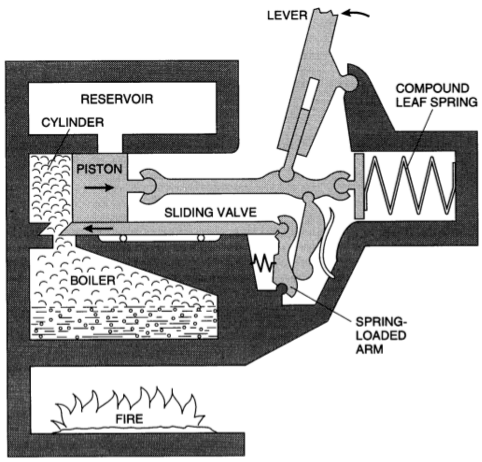


Figure 2: When the dimension is large, nearly all of the hypersphere's volume is found in the shell comprising the outermost 1 percent of the radius

3 Orbits

For the light comes to us alike in our homes and out of them, by day and night, equally at all times and in all places, whence we know not.

-Edwin Abbott, Flatland, 1884



Preamble Earlier, we showed that waves exhibited special behavior in three dimensions–special in the sense of being different from all other cases, but also particularly conducive to the sending and recieving of complex signals. As it turns out, the same can be said for orbits. In fact, many have cited the necessity of three dimensions for stable orbits as a reason why life would not develop in alternate universes–a reliable source of energy is central in the world of biology. A rogue planet, following some non-closed trajectory (a parabola or hyperbola probably) is unlikely to spend much time near a star, and very unlikely to be at the "right" distance from the star to foster a climate amenable to the formation of life. Fortunately, there is no disagreement between Newtonian and General Relativistic theories on this topic. This is a classic example of evoking the anthropic cosmological principle–we, as intelligent observers, find ourselves in a universe that seems to be perfectly tailored to our needs. Conversely, if the universe was not fit for the evovlution of conscious life, there would be no life to observe it. The convenience of 3d orbital mechanics is one of many fine-tuned elements of our laws of physics and is commonly cited as support for the anthropic principle.

History Isaac Newton was born on December 25, 1642 in Lincolnshire, England.[37] Preceeding Newton, the only thing known about planetary orbits were the three laws of Johannes Kepler (1571-1630) which stated that orbits follow elliptical paths, that at every point along their path an equal area is swept out within the plane of the orbit and that the period of the orbit squared is proportional to the cubed distance to the center of the orbit.[38] These laws explained the character of orbital mechanics without claims about the origin of these phenomenon. Newton knew that an object would follow a straight path unless acted on by an external force, and surmised that some invisible force must be causing the curved trajectories observed in the motion of the planets. At once connecting why (famously) apples fall to the ground and why the earth revolves around the sun, he conceived of gravity and formulated its accompanying force law to desribe it quantitatively. As the following analysis will illustrate, the case of n=3 is a sweet spot with respect to Newton's (and Einstein's) theories of gravity.

The most straightforward way the dimensionality of space manifests in classical mechanics is through the gravitational potential. The potential is a power law of the form $V(r) = \frac{-A}{r^{n-2}}$ where A is a constant and r represents the radial distance from a mass. In this power law, n represents the dimensionality of space. This can be seen employing a few basic results from mechanics. First, recall that $V(r) = -\int F(r)dr$ and that $F(r) = mg(r)$ where m is the mass of the smaller object, (the one whose motion we are interested in) and $g(r)$ is the gravitational field. Gauss' Law of Gravity, an example of the well-known divergence theorem, tells us that in three dimensions

$$\int\int g \cdot dA = -4\pi GM$$

where the integral is taken around the boundary of a Gaussian surface. The law can be easily generalized to n-dimensions

$$\int \cdots \int g \cdot dA = -nC_n GM$$

where C_n is the coefficient of n-dimensional volume derived in the last chapter and nC_n is the coefficient for surface area. Making this generalization, we solve for g(r) using the problem's inherent spherical symmetry

$$g \cdot S_{n-1}(r) = g \cdot nC_n r^{n-1} = -nC_n GM \rightarrow g(r) = -\frac{GM}{r^{n-1}}$$

Therefore, we recover the familiar form of the force law and potential but with different exponents

$$F(r) = mg(r) = -\frac{GMm}{r^{n-1}} \quad V(r) = -\frac{GMm}{r^{n-2}} =$$

We want to say something about the relationship between dimensionality and the behavior of orbits. A result from classical mechanics is that the effective potential, defined as $V_{eff}(r) = \frac{l^2}{2mr^2} - V(r)$, fully describes the sort of orbits a planet can take on. The constant l is angular momentum, a conserved quantity. A stable orbit is one for which small disturbances do not drastically alter its character (a small push will not send the planet to infinity, for example). An effective potential with a minimum is one that allows for stability. Elementary calculus tells us that a minimum must satisfy $\frac{dV_{eff}}{dr} = 0$ and $\frac{d^2V_{eff}}{dr^2} > 0$ so we investigate the relationship between these expressions and the dimensionality n.

$$\frac{d}{dr}V_{eff} = \frac{d}{dr}(\frac{l^2}{2mr^2} - \frac{A}{r^{n-2}}) = -\frac{l^2}{mr^3} + \frac{(n-2)A}{r^{n-1}} = 0 \to A = \frac{l^2r^{n-4}}{m(n-2)}$$

$$\frac{d^2V_{eff}}{dr^2} = \frac{3l^2}{mr^4} - (n-1)(n-2)Ar^{-n} > 0 \to \frac{3l^2}{mr^4} > (n-1)(n-2)Ar^{-n}$$

Plugging in for A and simplifying we get $n < 4$. This rules out all dimensions greater than our's for orbital stability. One might wonder whether the angular momentum l is in fact a conserved quantity in an arbitrary number of dimensions. The n-dimensional Lagrangian $L = T - V$ (the difference between kinetic and potential energy) expressed in plane polar coordinates reads $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{A}{r^{n-2}}$. Applying Lagrange's equations, we obtain

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) = \frac{\partial L}{\partial \theta} = 0 = \frac{d}{dt}(mr^2\dot{\theta}) \to mr^2\dot{\theta} = l$$

This is the standard argument to show the conservation of angular momentum, but it relies on the fact that all three-dimensional orbits occur in a plane. If we can show that orbits in n-dimensional space are confined to a 2d subspace, we are free to choose θ as the angular coordinate within this plane and invoke the preceeding result. We know from linear algebra that two linearly independent vectors can be transformed into an orthogonal basis via Graham-Schmidt decomposition and that a plane is, by definition, a space spanned by two basis elements. Regardless of the dimensionality, an orbiting planet will have a velocity vector tangent to its path and a force vector along the line connecting it to the mass it orbits. Therefore its motion is fully determined by two independent vectors, and the orbit must remain in a plane.

Stability and Dimensionality Continued Although we have shown that a necessary condition for orbital stability is $n < 4$, a more advanced argument, called Bertrand's Theorem, rules out any force law except Hooke's Law (n=0) and the inverse square law of our universe (n=3). Because a zero dimensional universe is not particulary interesting, we conclude that n=3 is the only dimensionality that allows for stable orbits in classical mechanics.

General Relativity In 1963, F.R. Tangherlini showed that general relativity agrees with the result that n<4 is required for orbital stability. Assuming Schwarzschild geometry outside of a large mass M, we generalize the metric to read (in G=c=1 units)

$$ds^2 = (1 - \frac{2M}{r^{n-2}})dt^2 - (1 - \frac{2M}{r^{n-2}})^{-1}dr^2 - r^2d\theta^2$$

where all other angular coordinates introduced to account for the higher dimensionality are set to 0 because the orbital motion is planar. Following Tangherlini's notation, we introduce two conserved quantities corresponding to relativistic energy and angular momentum

$$(1 - \frac{2M}{r^{n-2}})\frac{dt}{ds} = k_0 \quad r^2\frac{d\theta}{ds} = k_\theta$$

Next, we make the subsititution $u = \frac{1}{r}$ and obtain the expression

$$\frac{1}{2}(\frac{du}{d\theta})^2 + \frac{1}{2}u^2 - \frac{Mu^{n-2}}{k_\theta^2} - Mu^n = \frac{k_0^2 - 1}{2k_\theta^2}$$

Which can be verified via a lengthy calculation by substituting in for u, k_θ, k_0 and rearranging until the Schwarzschild metric is recovered. The first term on the left is analogous to kinetic energy whereas the constant on the right side can be interpreted as total energy, thus we define the effective potential

$$V_{eff}(u) = \frac{1}{2}u^2 - \frac{Mu^{n-2}}{k_\theta^2} - Mu^n$$

and perform the same analysis that we did on the Newtonian effective potential. Setting $\frac{dV_{eff}}{du} = 0$ we find that $k_\theta^{-2} = \frac{u - Mnu^{n-1}}{M(n-2)u^{n-3}}$. Remembering that a necessary condition for a minimum to exist is that $\frac{d^2V_{eff}}{du^2} > 0$, we compute the second derivative and plug in for k_θ^{-2} to find that

$$1 - (n-3) - 2Mnu^{n-2} > 0$$

The third term on the left side is always greater than zero, but can be made small for certain choices of M and u. Thus, this expression demonstrates that for $n > 3$, the inequality cannot be satisfied and no minimum in the effective potential exists. General Relativity agrees with the simpler Newtonian result.

The Evolution of Life The fact that orbital stability requires the three-dimensionality of space is a fascinating result. Although we implicitly assume the laws of physics take the same basic form in different-dimensional universes, this tells us that our universe, among all the possible ones, is very well-suited for the development of intelligent life. After all, if the dimensionality of the universe were totally random, we would expect it to be a very large number. Although there is a non-zero chance that natural nuclear reactions could occur and supply energy for some strange life form on a non-orbiting mass in space, orbital stability is instrumental in the evolution of complex organisms. Without it, we would not have reliable energy from a star, nor predictable seasons or a consistent climate. Perhaps the structure of the universe we observe is not entirely coincidental.

4 Rocketry

It seemed that this poor ignorant monarch–as he called himself–was persuaded that the straight line which he called his Kingdom, and in which he passed his existence, constituted the whole of the world, and indeed the whole of space.

-Edwin Abbott, Flatland, 1884

Preamble Looking into lower dimensions is quite a bit simpler than dealing with higher ones–not necessarily because the math is inherently easier, but because visualization presents no challenges. In addition, our ability to peer down on Flatland makes the fourth dimension feel closer than ever; where and what is this unseen fourth dimension that makes cubes turn inside out and planets fly off into space? As it turns out, Flatland (where n=2) is totally pathological with respect to planets, gravity and space travel. The reason originates in the drastic change of behavior of the gravitational potential energy when n=2. Escape velocities become meaningless and the distances at which signals can be transmitted gain a theoretical upper limit. This chapter focuses on mathematically simulating a 2d space mission and investigates its feasibility. Unfortunately, we are forced to recycle three dimensional constants such as the mass of the earth and the distance to the moon in lieu of any better information. However, to make this Flatland planet as earthlike as possible, the gravitational constant has been adjusted to fix surface gravity at the good old 9.81 m/s^2. The merciless 2d gravity that the Flatlanders are stuck with presents serious difficulties for any hopes of sending a rocket ship into space. Celebrated earthen intellect be damned, imagine the acuity (and size, presumably) of the two-dimensional brain that manages to power a Flatlander space craft through a logarithmic gravitational potential energy!

Escape velocity is the minimum speed at which an object moving radially away from a gravitational mass must travel in order to come to rest at a distance of $r = \infty$. It is obtained by setting the kinetic energy equal to the gravitational potential at ∞. Using the n-dimensional force law, this reads

$$\frac{1}{2}m\dot{r}^2 = \int_R^\infty \frac{GMm}{r^{n-1}} dr$$

This expression can usually be solved for the escape velocity $\dot{r}$ as a function of an arbitrary radial distance R. However, in the case of n=2, the integral diverges and we end up with a logarthmic potential function. This means there is no finite velocity which, on its own, will get an object to $r = \infty$. Given this lack of escape velocity, and the fact that the force of gravity is proportional to r^{-1}, it is natural to ask about the possibility of space travel in two dimensions.

Escape Velocity and Horizons In three dimensions, escape velocity ensures that the kinetic energy is zero at infinity, where the potential also goes to zero. This is not possible in two dimensions, as the previous paragraph outlines, because the logarithmic potential diverges. There must be a finite radius where an object sent radially outward with initial velocity comes to rest. This radius will be denoted r_{stop}. Conservation of energy tells us that $T_1 + V_1 = T_2 + V_2 \rightarrow \frac{1}{2}mv^2 + GMmln(R) = 0 + GMmln(r_{stop})$ where v is the initial velocity, R is the initial radius, m is the mass of the object and M is the mass of the planet. Consequently,

$$v = \sqrt{2GMln(\frac{r_{stop}}{R})}$$

This is the closest analogue to escape velocity that exists in two dimensions– we choose a certain radius we want the object to arrive at, and this sets the speed. Farther distances require higher velocities. Certain values of r_{stop} will require velocities greater than light (M and R are assumed to be given). We will call the radius at which the required velocity is the speed of light "c" the communication horizon, reminiscent of the term "event horizon" which comes from the study of black holes. The event horizon is the outer-most boundary of a blackhole, which marks where the escape velocity becomes greater than light. No concept of escape velocity exists in two-dimensions, so we instead talk about the communication horizon, which delineates the distance that information can be communicated given the details of a planet. It can be expressed as

$$r_{CH} = R * exp(\frac{c^2}{2GM})$$

This is a startling conclusion. Because of the nature of the potential energy, every mass has a communication horizon of varying radius. No information sent out from this object (electromagnetic or otherwise) can cross this barrier. The math tells us that the smallest possible communication horizon is at the surface of the mass, as the argument of the exponential scaling R can never go negative. In essence, every mass in two dimensional space behaves like a black hole, preventing information from travelling past a certain radius. While this seems problematic, we might imagine planets or galaxies existing within a communication horizon so large as to be unimportant. If, for example, the horizon of a galaxy is a large fraction of the observable universe, it may have no consequences in space travel and commmunication. Before we can insert values into this expression, we must come up with a 2d version of the gravitational constant G. Let's call it G_2– its necessity is obvious as the argument of an exponential must always be dimensionless and this condition is only met if the units of G_2 change as outlined by the 2d force law. Because the gravitationl constant is measured, (ie does not follow analytically from any physical laws), we must make some assumptions about this flat universe. For the sake of comparison, we'll say that acceleration at the surface of 2d earth is still $9.81\frac{m}{s^2}$. Then, using $F = \frac{G_2 Mm}{r}$, $G_2 = 1.046 * 10^{-17}$. The new gravitational constant should be smaller, as force decays with distance much more slowly.

Plugging in the three dimensional values for the mass/radius of earth and the new gravitational constant, we find that our planet has a communication horizon that is too large for a handheld calculator to display. The radius of our observable universe is on the order of 10^{26}, thus the earth poses no problems for any intelligent 2d life forms' galactic exploration. Similarly, the communication horizon of the sun can be found. It is also so large as to be essentially non-existent.

A little more rigor Normally, Newton's shell theorem allows us to replace spherically symmetric bodies with a point at their center of mass. This is why the force law and potential energy appear as simple expressions independent of the specific spherical geometry of the planet under consideration. The coordinate r is the distance from the center (not the surface, for example) and M is the total mass. In the preceeding analysis, we have assumed that the shell theorem holds in two dimensions. This is to say that circular 2d planets can be replaced with point masses at their center. This result follows from a specific interaction between the form of the force law and the dimensionality, and is not guaranteed. A brief proof ensures that we are justified in the assumptions previously made, and offers more evidence for the generalized form of the force law I have claimed. To start, imagine a ring of total mass M and radius R, with linear mass density $\sigma = \frac{M}{2\pi R}$. We calculate the potential at a point outside the planet by summing the contributions from each infinitessimal section around the ring. The set up is illustrated in Figure 3.

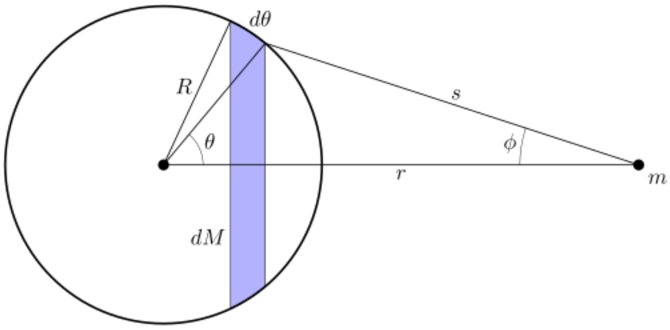


Figure 3: Visual representation of 2d shell theorem method

The potential is a function of r (lower case), where this is the distance from the ring's center to somewhere outside. The variable ϕ is irrelevant because the potential is a scalar. S is rewritten using the law of cosines. The integral then reads

$$V(r) = \int_0^{2\pi} G\frac{M}{2\pi R}Rd\theta ln(\sqrt{R^2 + r^2 - 2Rrcos\theta})$$

We know force is $-\frac{dV}{dr}$ so we take an r derivative and bring it into the integral. Some algebra shows that this yields

$$-\frac{GM}{2r} - \frac{GM(r^2 - R^2)}{4\pi}\int_0^{2\pi}\frac{d\theta}{R^2 + r^2 - 2Rrcos\theta}$$

This integral is difficult to evaluate using traditional calculus methods, but Maple can solve it to yield $\frac{2\pi}{r^2-R^2}$ so that the expression evaluates to $-\frac{GM}{2r}$, thus recovering the two dimensional force law $F(r) = -\frac{GM}{r}$. This treatment has been dealing with force and potential per unit mass, thus m does not appear.

Rocketry We are now tasked with finding the equation of motion for a rocket travelling in a variable gravitational field. For simplicity, drag forces will be ignored. That being said, one might speculate that the effect of drag in two dimensions would be much greater than in three, as there is less "room" for the air to be parted around the rocket. Take the surface of our 2d planet as an inertial reference frame with the radial distance r pointing upwards. Newton's second law states that $\frac{d}{dt}(m\dot{r}(t)) = \sum F_{ext}$ but we are dealing with a variable mass system, as the rocket's mass decreases with time as fuel is burned off.

$$\dot{m}(t)\dot{r}(t) + m(t)\ddot{r}(t) = \sum F_{ext}$$

This expression tells us that external forces are responsible for any net change in momentum of the rocket. The two forces acting on the rocket are thrust propelling the rocket upwards and gravity pulling the rocket back towards its starting point. We are considering a universe that is empty except for the rocket and planet.

$$\sum F_{ext} = T - G = \dot{m}(t)V_{muzzle} - \frac{GMm(t)}{r}$$

V_{muzzle} is the speed at which fuel is ejected from the rocket's engines and M is the mass of the planet. The units for the thrust term are easily verified as force. To make this problem more tractable, we assume that $\dot{m}(t) = -c$, a negative constant because mass is decreasing with time. Additionally, the total mass m(t) can be rewritten as $m_0 - ct$ where m_0 is the initial mass of the rocket + fuel. Lastly, we assume that the fraction of the rocket's mass that is fuel is denoted by p, so $m_{fuel} = 0$ at $t = \frac{p*m_0}{c}$. This will be the time where the thrust term goes to zero and the only external force becomes gravity. Nasa's Saturn V moon rocket was over 90 percent fuel by mass, and a 2d rocket would need to carry more fuel and would have less space to store it. The rocket equation is now

$$(m_0 - ct)\ddot{r}(t) - c\dot{r}(t) + \frac{GM(m_0 - ct)}{r} = cV_{muzzle}$$

which is subject to the initial conditions $r(0) = 0, \dot{r}(0) = 0, t_{stage1} = \frac{p*m_0}{c}$. At $t > t_{stage1}$, we solve

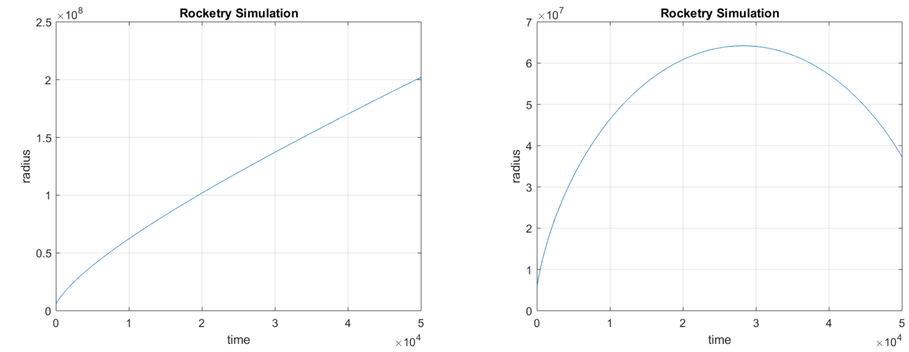
$$(1-p)m_0\ddot{r}(t) + \frac{GM(1-p)m_0}{r} = 0 \rightarrow \ddot{r} = \frac{GM}{r}$$

where the initial conditions of this equation are given by the final position and velocity of the rocket when it runs out of fuel ensuring continuity of the rocket's trajectory.

Although this is the rocket equation for a two dimensional universe, we can simply substitute r^2 in the denominator of the gravitational force term to obtain the 3d solution. In this way, we can compare the physics of space travel in the two universes. We have addressed the problem of communication horizons already–the laws of physics allow the rocket ship to travel any reasonable distance in space. Would going to the moon from an earth-like planet be possible for a flat civilization? We first must solve this second order non-linear differential equation for position. No elementary technique can solve it analytically, so we must turn to numerical methods instead.

Euler's step approximation method can be used to numerically evalulate the differential equations of interest. Plotting radius as a function of time will give insight into the rocket's behavior. Before moving to the two dimensional case, the code can be tested with 3d force laws and known values of constants. Every appearance of r can be replaced with r^2 and the dimensionality is effectively changed. We can use data from real rocket missions to assign values to muzzle velocity, rate of change of mass and mass of the rocket in addition to using the usual gravitational constant G/the radius of the earth as an initial condition. Furthermore, we should observe the rocket escaping earth's gravitational field for the appropriate choice of velocity, and the rocket should turn around for velocities below v_{escape}. We know the thrust of the rocket cV_{muzzle} must be greater than the rocket's initial weight $m_0 * g = 9.81m_0$ for the rocket to achieve lift off. This is another aspect of the code to check; we fix a mass rate of change c and test muzzle velocities above and below the threshold. The following subplots of Figure 4 show these situations.

Above and Below Escape Velocity



Above and Below Necessary Muzzle Velocity

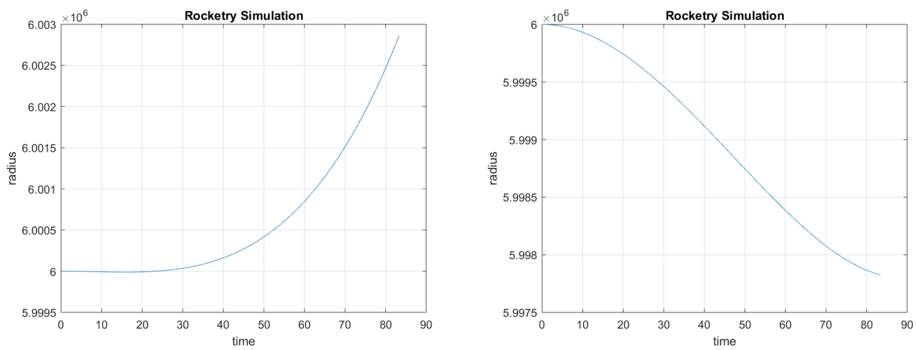


Figure 4: Testing rocket's trajectory against a variety of known outcomes

It appears the code is working! As a last check, we can change the step size in the approximation to see if the nature of the solution changes. Reducing and increasing this value does not alter the rocket's trajectory, so we conclude the code is functioning correctly and proceed to use it to generate results. At this point, its time to transition to the two dimensional rocket. We know that there should be no escape velocity–the rocket should always turn around regardless of how large the muzzle velocity or how far away it gets. Earlier, we chose G_2 so that gravity at the surface of the earth was still $9.81\frac{m}{s^2}$ which means that the necessary thrust is the same in our 2d world (assuming the mass rate of change is the same constant). It is reasonable to expect that the rocket needs a higher muzzle velocity to travel any significant distance due to the nature of the gravitational field. Another component of the simulation we have control over is the burn time, which is related to what percent of the rocket's mass is fuel. For two and three dimensions, an excessive burn time will put the rocket so far away from earth that gravity is negligible and it will appear to escape, even for muzzle velocities marginally larger than what is required for lift off. In the 2d case, keeping the burn time small simply makes the rocket's behavior easier to detect. We know it will always stop and turn back towards earth, but that might be an impractically large time if the engines fire until the rocket is millions of kilometers away.

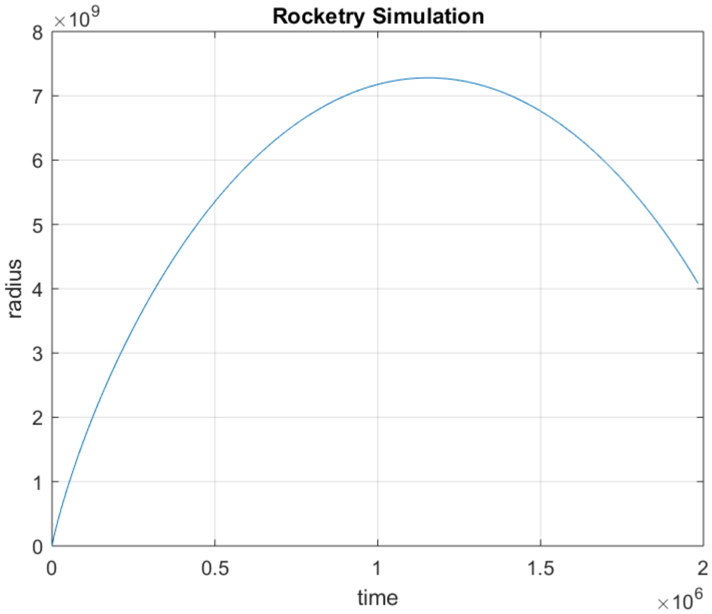


Figure 5: The rocket ship, despite travelling the distance to Pluto, still turns around in the two-dimensional gravitational field

Figure 5 shows a two-dimensional rocket ship travelling in the r^{-1} gravitational field. We have assumed the rocket is 35 percent fuel, it starts at rest on the surface of the flat earth in an otherwise empty universe and that it's muzzle velocity is $5.7 * 10^4 \frac{m}{s}$, approximately an order of magnitude larger than that of a real rocket. It travels billions of kilometers away, comes to rest and falls back to earth. This is a shocking result, and confirms the analytical predictions the physics made. There is no escape velocity, thus every unpowered rocket will eventually turn around and fall back towards the planet from which it came, or some other massive celestial object. Our rocket makes it all the way to Pluto (7.5 billion kilometers, at its farthest), but still has not escaped the r^{-1} gravitational field of earth.

Moon Mission Now let's simulate a rocket going to the moon. In the spirit of adopting real values from our galaxy, we'll say the moon is approximately 400 million meters away. Additionally, we will assume that the rocket is mostly fuel, as was the case for all real space missions. Setting $p = 0.8$ makes the approximate minimum muzzle velocity to achieve this distance $7.6 * 10^4 \frac{m}{s}$. Figure 6 shows the rocket's trajectory.

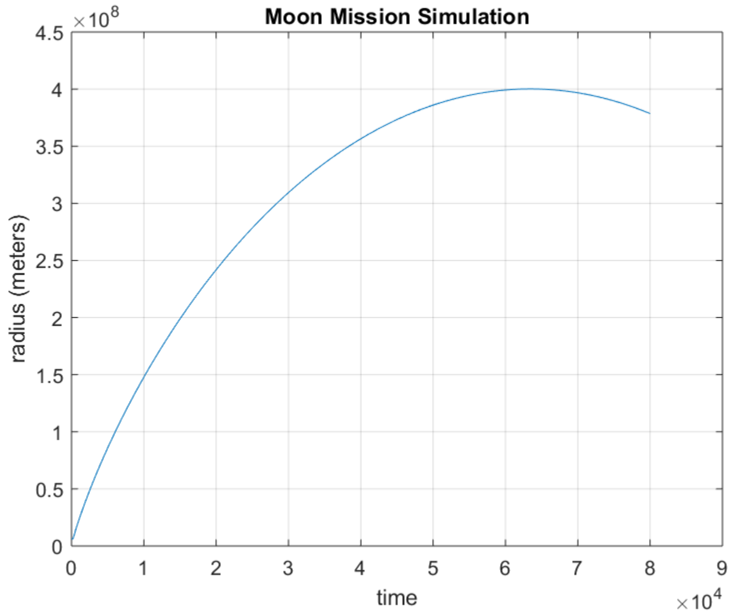


Figure 6: The 2d rocket travels to the moon before being turned around by the ever-present earth gravity

To discuss the topic of two dimensional rocketry, it is necessary to draw parallels between our universe and the hypothetical flat one. There is no way to know what scale rockets and planets would exist on with respect to our units of meters, kilograms and seconds, so we simply recycle known constant values for the sake of the simulation. True, we can calculate a new gravitational constant under the assumption that surface gravity is the same, but we are forced to use 3d masses and distances in the 2d rocket problem. These assumptions are not always viable. Consider the case of the 2d Moon Mission, where 80 percent of a 500,000kg rocket is fuel. How big would a flat rocket need to be to contain this? First, note that atomic bond lengths are on the order of $10^{-10}m$, so we can expect 10^{20} atoms to fit in a square meter at most. Fuel's are hydrocarbons, so we can use the mass of a carbon atom to find an estimate for surface density of rocket fuel.

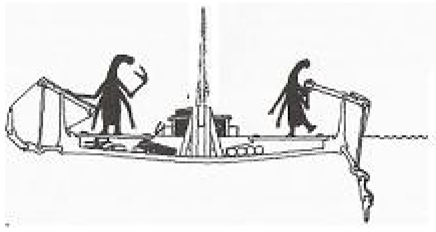
$$10^{20}\frac{atoms}{m^2} * 2 * 10^{-26}\frac{kg}{atom} = 2 * 10^{-6}\frac{kg}{m^2}$$

Given that the rocket requires 400,000kg of fuel, this equates to a storage area of $2*10^{11}m^2$. The width of the Saturn V rocket was about one-quarter of its height, which would make the 2d rocket $2.25*10^5$ meters wide, a preposterously large value, especially considering that it would be an insurmountable obstacle for 2d beings to try to get around. Although borrowing from three dimensions offers an easy way to gain qualitative insight into the physics of a flat universe, there are many necessary assumptions which do injustice to the details of such a world. Could a rocket ever balance fuel capacity and size? If it could, it seems that 2d chemistry would have to be different either in nature or scale. Perhaps two dimensional drag would have unexpected effects on the rocket's motion. What does it even mean to talk about the mass of an infinitely thin object? We appear to be unequipped to answer seemingly simple questions like these, whereas many of the physical results come easily.

5 The N-Dimensional Atom

All is confined to the narrowest band of vision imaginable; an infinitesimal line, encompassing me, contains my entire visual world.

-A.K. Dewdney, The Planiverse, 1984



Preamble This chapter is by far the longest and most technical. We begin by proving bizarre generalizations of standard quantum theory to different dimensions–first discussing the improbably large Bohr radius of higher dimensional atoms, then to the question of atomic stability, and finally treating energy eigenvalues/ionization energy. The consequences of this analysis suggest either that our quantum model does not apply in different dimensions (given the strangeness of its predictions) or that, once again, a three dimensional universe is custom-fit to the needs of biological life. This is especially true with regards to the size and stability of atoms. As before, we are forced to begrudgingly reuse values of 3d fundamental constants such as ϵ_0. The anthropic cosmological principle should be kept in mind throughout these sections–the instability of atoms and planetary orbits are sufficient grounds to disqualify alternate universes as potentially life-bearing. The bulk of this chapter is dedicated to the laborious undertaking of solving the radial Schrodinger equation in N dimensions. To make this goal tractable, it was necessary to generalize the Laplacian operator but not the form of the potential energy, as we've done thus far. Further work could certainly be done in generalizing both the Laplacian and the potential, as solutions to this radial Schrodinger equation should reflect the predictions of the generalized Bohr radius, energy eigenvalues, ionization energy, etc. of the first sections of the chapter.

Bohr Radius In traditional 3 + 1 dimensional quantum mechanics, the Bohr radius and the orbital radii of higher energy levels can be calculated by claiming that the centrepital force on the electron is balanced by the

Coulomb force from the nucleus. Formally, $\frac{mv^2}{r} = \frac{Ze^2}{r^2}$ where Gaussian units are used to avoid having to write out the constant $4\pi\epsilon_0$ in Coulomb's law. Equivalently, $r = \frac{Ze^2}{mv^2}$ (Z is the atomic number and e is the charge of the electron). Standard quantum mechanics tells us that the maximum allowed value of angular momentum is quantized by energy level n, meaning that $L = mvr = n\hbar = \frac{nh}{2\pi}$, so $v = \frac{nh}{2\pi mr}$. Plugging in for v in the first expression, we find that

$$r = \frac{n^2h^2}{4\pi^2 mZe^2}$$

Thus the familiar Bohr radius for the hydrogen atom is $r = \frac{h^2}{4\pi^2 me^2}$. In higher dimensions, the Couolmb force takes a different form–for dimensionality N, we say that $\frac{mv^2}{r} = \frac{Ze^2}{r^{N-1}}$ which is simply generalizing in the same way that we did for gravitational force. In this case, $r^{N-2} = \frac{Ze^2}{mv^2}$. Given that the expression for angular momentum is independent of dimension, plugging in as before yields

$$r = (\frac{4\pi^2 mZe^2}{n^2h^2})^{\frac{1}{N-4}}$$

With this result, there are a number of aspects of the different-dimensional atom to analyze. First, let's look at how the ground state radius of the hydrogen atom varies with dimension (Z=n=1). In Gaussian units, the charge of the electron is $4.803*10^{-10}cm^{3/2}g^{1/2}s^{-1} = 1.5188*10^{-14}m^{3/2}kg^{1/2}s^{-1}$ It's mass is $9.109 * 10^{-31}kg$. The table below shows the Bohr radii as a function of dimension.

Dimensionality	Bohr Radii (m)
1	3.7545e-04
2	7.2750e-06
3	5.2926e-11
4	Undefined
5	1.8894e+10
6	1.3746e+05
7	2.6634e+03
8	370.7509
9	113.5706
10	51.6085

In Gaussian units, the vacuum permittivity ϵ_0 is built into the definition of charge so it's been implicitly assumed that the value of ϵ_0 does not vary with the dimension, but its units do to keep the Bohr radius as a measure of meters. As expected, the three-dimensional Bohr radius is reproducued by this generalized expression. Interestingly, it is by the far the smallest value in the table. The expression breaks down for N=4 because the exponent diverges. The five-dimensional Bohr radius is enormous, as are all the other higher dimensional ones compared to the 10^{-10} m scale we are accustomed to.

When the dimension N is less than 5, the exponent in the atomic radius expression is negative and increasing principal quantum number n increases the orbital radius. Figure 7 depicts this behavior:

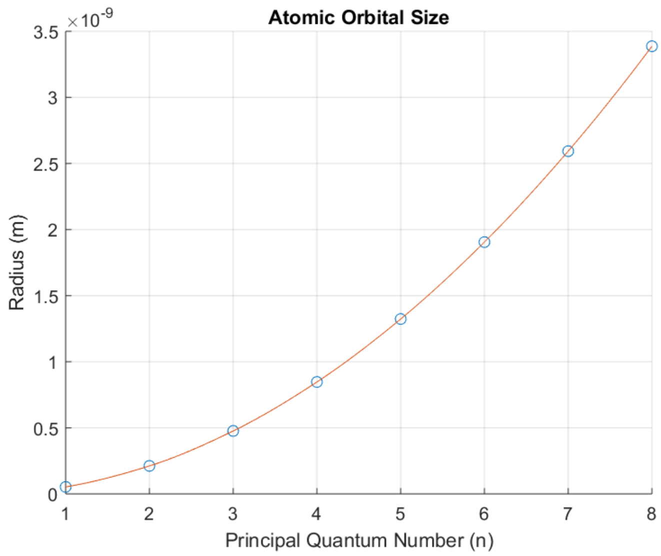


Figure 7: Orbital radius increases with principal quantum number in traditional 3d quantum mechanics

Once N≥5, the exponent is positive and the orbital radius decreases with increasing quantum number. Because n corresponds to the energy levels of the atom, higher energy states are held closer to the nucleus. This suggests that electrons will fall into the nucleus. The huge orbital radii of the five-dimensional atom in addition to their decreasing size are highlighted in Figure 8.

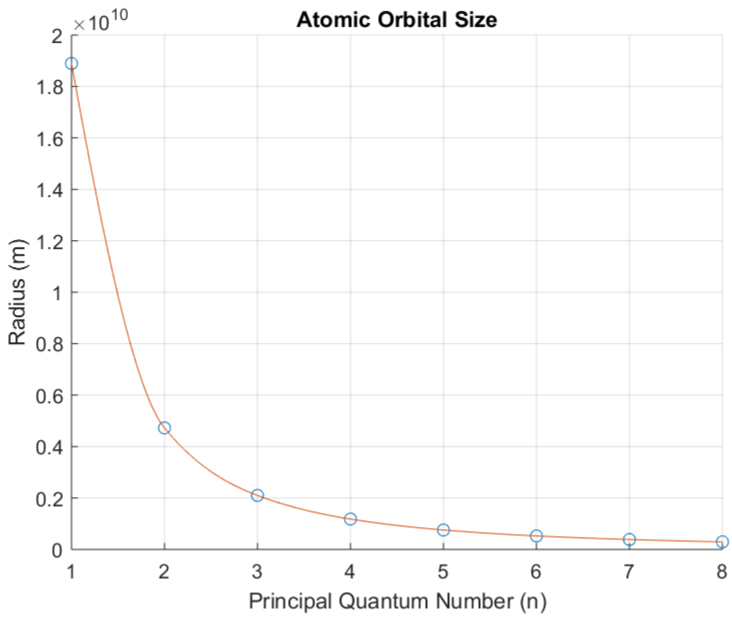


Figure 8: Higher dimensional atoms display behavior contrary to their 3d counterparts

Although this does not prove the instability of atomic orbitals in higher dimensions, it certainly does hint at it. If the orbitals were to remain stable, an atom's outer electrons would have the lowest energy. The consequences of this for higher dimensional chemistry are far-reaching and undoubtedly, very complicated. We would expect that it would be almost unrecognizable to us.

Atomic Instability To assess whether the electron is confined to an orbital or falls into the nucleus/flies away, we can look for a minimum in it's energy. It has kinetic energy and is subjected to the potential energy from the atomic nucleus. In general, $E = KE + PE = \frac{p^2}{2m} - \frac{Ze^2}{r^{N-2}}$ where we again employ Gaussian units for the sake of simplicity. The orbital angular momentum of the electron is $n\hbar = mvr = pr$, therefore the linear momentum can be written as $p = \frac{n\hbar}{r}$. Plugging in, (and deciding to look at the hydrogen atom where Z=1), we find that the total energy of an electron in N-dimensional space is

$$E = \frac{n^2\hbar^2}{2mr^2} - \frac{e^2}{r^{N-2}}$$

When N=3, the energy E(r) has a minimum at which radius the electron has a stable orbital. Again, there is intersting behavior around N=5. For

N≤3, there is always a minimum. The case of N=4 is ambiguous and will be addressed separately. Once N≥5, a minimum becomes impossible because the potential (in other words, the negative term) dominates as r→ 0 and the energy can become arbitrarily negative. Figure 9 shows qualitatively the distribution of the electron's energy as a function of radius for any dimension greater than 4. There is a maximum, but no minimum and the electron either flies away or falls into the nucleus depending on which side of the peak it is located. No stable orbit is possible, thus atoms themselves are unstable and would not exist in high dimensions.

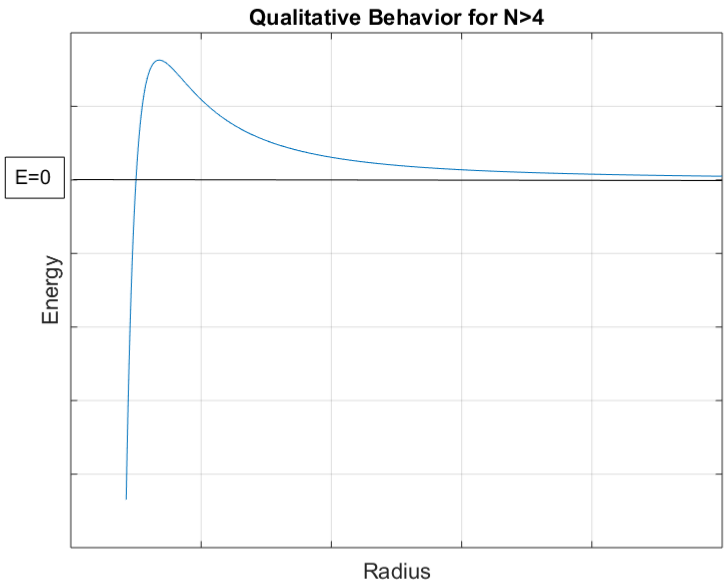


Figure 9: The lack of a minimum in the potential of higher dimensional atoms indicates instability of the electron's orbit

The case of four dimensions has been delayed because it does not follow the same line of logic as above. When N=4, both terms in the energy are inverse squares and can be combined so that the energy is now $E = \frac{n^2\hbar^2 - 2me^2}{2mr^2}$ where the sign depends on a balance among the constants $\hbar$, e, m and n. Using these constants' three-dimesional values, and for any reasonable energy level n, the numerator is negative so all electrons fall into the nucleus. However, we don't know how these constants could vary with dimensionality, so the 4d electrons could also fly off. Either way, they are not stable. To be more precise, we can appeal to the special relativistic formulation of kinetic energy where we replace $\frac{p^2}{2m}$ with $\sqrt{p^2c^2 + m^2c^4}$. Using the fact that

$p = \frac{n\hbar}{r}$, the energy reads

$$E = \sqrt{\frac{n^2\hbar^2c^2}{r^2} + m^2c^4} - \frac{e^2}{r^2}$$

Now it is clear that as r→0, the $-\frac{1}{r^2}$ potential term dominates and the energy can become arbitrarily negative. Thus the ambiguity is cleared up and the electrons fall into the nucleus in four dimensions.

Energy Eigenvalues In three dimensions, it is commonly known that the allowed energies of a hydrogen atom are quantized by the principal quantum number n and obey the relation $E = -\frac{13.6eV}{n^2}$. It is easy to generalize a relation between energy and principal quantum number to N-dimensions. First, note that the electron's energy is $-\frac{1}{2}mv^2$ and its velocity can be determined as a function of radius with $\frac{mv^2}{r} = \frac{Ze^2}{r^{N-1}} \rightarrow v = \sqrt{\frac{Ze^2}{mr^{N-2}}}$ which is a simple force balance. Plugging in for v, we get

$$E = -\frac{Ze^2}{2r^{N-2}}$$

Now, we use the previous result about atomic orbital radii, namely $r = (\frac{Zme^2}{\hbar^2n^2})^{1/(N-4)}$ and plug this in for r:

$$E = -\frac{Ze^2}{2}(\frac{\hbar^2n^2}{Zme^2})^{\frac{N-2}{N-4}}$$

For N=3, this expression tells us that $-(\frac{Z^2e^4m}{2\hbar^2})\frac{1}{n^2}$ which for the hydrogen atom (Z=1), correctly yields $-13.6eV\frac{1}{n^2}$. Interestingly, the case of N=4 brings about a dramatic divergence in behavior. When $N < 4$, the exponent is negative and the factor n^2 ends up in the denominator, meaning that increasing principal quantum number makes the energy less negative (bigger), so the electron's energy and energy level increase together as expected. However, when $N > 4$, n^2 is in the numerator so as the energy level increases, the energy becomes more negative (smaller). The highest energy state is the ground state n=1, totally opposite of 3d quantum mechanics. We have now found a potential reason why the orbital radii decrease with the PQN n–the largest orbital is the "ground state" n=1 (no longer an accurate term) which also corresponds with the highest energy orbital. As the energy becomes more negative with increasing n, the size of the orbital radius goes down. Perhaps the principal quantum number for $N > 4$ simply has a reciprocal relationship to its 3d analogue.

Ionization Energy In three dimensions, the virial theorem allows us to simply express the total energy of an electron at a given energy level. Because electron energies are negative, and E=0 corresponds to the electron being at $r = \infty$ where the potential has decayed away, the ionization energy is the absolute value of the total energy. Adding this amount of energy would ensure the electron "escapes" the potential well of the nucleus and is said to ionize the atom. As we have seen, when $N \geq 5$, increasing the principal quantum number decreases the radius of the orbital, a huge qualitative shift in behavior from the typical 3d atom. Interestingly, the virial theorem analysis parallels this shift, breaking down when N=4 and yielding unusable results for $N \geq 5$ (it predicts a positive energy of the electron, and thus a negative ionization energy which is non-sensical). Consequently, we are able to find an exact ionization energy for the two-dimensional atom and must resort to a different approach for higher dimensions.

Virial Theorem and the Two-Dimensioinal Atom For forces arising from a potential of the form $V(r) = \frac{a}{r^{N-2}}$, the virial theorem reads

$$2\langle T\rangle = -(N-2)\langle V\rangle$$

where T is the kinetic energy and V is the potential. The angular brackets represent time averages, but will be dropped moving forward. The total energy is then

$$E = V + T = V + \frac{-N+2}{2}V = (2 - \frac{N}{2})V$$

We are interested in the case of N=2, (and Z=1, assuming we're working with the hydrogen atom), so we plug this in for the virial theorem coefficient and write out the potential using Gaussian units $E = e^2 ln(r)$, but we know from previous work that $r = (\frac{me^2}{n^2\hbar^2})^{\frac{1}{N-4}}$, therefore

$$E = e^2 ln(\sqrt{\frac{n^2\hbar^2}{me^2}})$$

In writing $V = e^2 ln(r)$, we've assumed that the base unit definition of the electron's charge e has changed to keep the units of energy the same, namely that $[e] = m * g^{1/2} * s^{-1}$. As before, we'll fix the value of e in lieu of any better information. In that case, we can plug in to find the ground state ionization energy of the two-dimensional hydrogen atom. The ionization energy is $1.7036 * 10^{-8}$ eV, many orders of magnitude less than the 13.6 eV ground state ionization energy of the 3d hydrogen atom.

Higher Dimensional Ionization Energy The virial theorem fails to be able to meaningfully express the total energy of the electron for higher dimensional atoms. There is also the added quirk that the "ground state" of the electron is the farthest out, and increasing the PQN brings electrons closer to the nucleus. As a result, we must find the integer value of the principal quantum number n that keeps the electron outside of a nucleus of finite size. In other words, we fix a size of the nucleus for the higher dimensional atom (it could be constant, it could depend on dimension but it is an assumption either way), and use the expression for orbital radii as a function of PQN to find the largest n-value that does not locate the electron within the nucleus. Using this PQN, we can use the generalized energy eigenvalue expression to write the total energy of the closest held electron. The absolute value of this quantity is the ionization energy, and is the analogue to the ground state ionization energy. It is easy enough to test two different assumptions about the nucleus: one, that it does not vary with dimension and has a radius of $8.783 * 10^{-16}$ (hydrogen atom) or two, that the radius of the nucleus is always the same fraction of the Bohr radius as it is in the 3d case. By plotting the orbital radius as a function of PQN for a wide range of n-values, we use the data cursor on matlab to find the largest n-value that keeps the electron out of the nucleus for a given dimension. Plugging in this integer value and taking the absolute value gives ionization energy in Joules (J), which can be converted to electron volts (eV) knowing that $1eV = 1.602 * 10^{-19}J$. The following table lists higher-dimensional ionization energies for nucleii fixed at the 3d size.

Dimension N	Max PQN (n)	Ionization Energy (eV)
5	4.638e+12	1.0624e+36
6	1.565e+20	1.2097e+51
7	5.281e+27	1.3777e+66

These values are astronomical–the assumption that the nucleii have a constant size is likely illogical, especially considering the huge Bohr radii of the 5, 6 and 7 dimensional hydrogen atoms. We know that the Bohr radius of the 3d atom is $5.2926 * 10^{-11}$ m and that the hydrogen nucleus has a radius of $8.783 * 10^{-16}$ m. Dividing the latter by the former gives the fraction of the Bohr radius that the nucleus occupies. Assuming that this ratio is fixed gives the following ionization energies:

Dimension N	Nucleus Radius (m)	Max PQN (n)	Ionization Energy (eV)
5	3.1355e5	245	2.3084e-26
6	2.2811	6.025e4	2.6576e-11
7	0.0442	1.478e7	4.30e-3

These values, while closer by order of magnitude, are extremely small. A universe governed by either assumption would present many challenges for harnessing electricity and magnetism. Throughout this analysis, (as before), it has been implicitly assumed that the constant ϵ_0, which is built into Gaussian units of charge, is independent of dimension. Perhaps finely tuning it's value would lead to more reasonable ionization energies.

Schrodinger Equation To understand more thoroughly the structure of the N-dimensional atom, we can solve a generalized version of the Schrodinger equation. It reads

$$-\frac{\hbar^2}{2m}\nabla^2\psi - \frac{Ze^2}{r}\psi = E\psi$$

Although until now, the potential has been generalized to an inverse N-2 power rule, we will assume it keeps the same r^{-1} form and analyze how the radial solutions change with the N-dimensional Laplacian. This makes the already laborious algebra more tractable and still offers insight into the behavior of the atom with the introduction of new angular coordinates. To tackle this problem, it is first necessary to establish how the Laplacian varies with the dimensionality.

For arbitrary curvilinear coordinates in N-dimensions, the laplacian can be written as a function of the metric tensor

$$\nabla^2 = \frac{1}{\sqrt{detg}}\frac{\partial}{\partial x^i}(\sqrt{detg}\ g^{ij}\frac{\partial}{\partial x^j})$$

Where "det g" means the determinant of the metric g and g^{ij} represents the components of the inverse metric. The Einstein summation convention is used to avoid having to write a sum explicitly. All metrics of interest will be diagonal, thus the determinant is the product of the diagonal entries and the only non-zero terms are when i=j. To check the validity of this formula, let's use three-dimensional spherical coordinates and compare to the corresponding laplacian which is tabulated in many places. The metric is

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 sin\theta \end{bmatrix}$$

Conveniently, the inverse of a diagonal metric is computed by taking the reciprocal of it's elements. It's easy to see that $\sqrt{detg} = r^2 sin\theta$. The first component of the laplacian (i=j=r) is

$$\frac{1}{r^2 sin\theta}\frac{\partial}{\partial r}(r^2 sin\theta g^{rr}\frac{\partial}{\partial r}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r})$$

The second component is

$$\frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}(r^2 sin\theta\ g^{\theta\theta}\frac{\partial}{\partial\theta}) = \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial}{\partial\theta})$$

The third component is checked similarly. The 3d laplacian in spherical coordinates is the sum of all these. The formula gives the desired result, which in its entirety is $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2 sin\phi}\frac{\partial^2}{\partial\phi^2}$. The same process can be repeated for cylindrical coordinates.

Having confirmed the validity of this prescription, we are tasked with generalizing the spherical coordinate metric to an arbitrary number of dimensions. This coordinate system is chosen because solutions to the Schrodinger equation exhibit spherical symmetry. In plane polar coordinates (two dimensions), $x_1 = rcos\phi_1$ and $x_2 = rsin\phi_1$. In 3d spherical coordinates, $x_1 = rcos\phi_1$,$x_2 = rsin\phi_1 cos\phi_2$ and $x_3 = rsin\phi_1 sin\phi_2$. ϕ_1 is the angle measured down from the vertical axis (x_1) and ranges between 0 and π. The angle ϕ_2 goes across the equator of the sphere and ranges between 0 and 2π. In N-dimensions, there is one radial coordinate and N-1 angular coordinates. One of the angular coordinates will range between 0 and 2π and the rest between 0 and π.

$$\begin{aligned} x_1 &= rcos\phi_1 \\ x_2 &= rsin\phi_1 cos\phi_2 \\ x_3 &= rsin\phi_1 sin\phi_2 cos\phi_3 \\ &\vdots \\ x_{N-1} &= rsin\phi_1 \dots sin\phi_{N-2} cos\phi_{N-1} \\ x_N &= rsin\phi_1 \dots sin\phi_{N-2} sin\phi_{N-1} \end{aligned}$$

In order for the x_N's to be positive or negative, and being consistent with the naming scheme we have introduced, the highest indexed angular coordinate (ϕ_{N-1}) will be in the range $[0,2\pi)$. We are now in a position to formulate the N-dimensional spherical coordinate metric tensor. Define R=$(x_1, x_2, \dots, x_N)$ ie the vector of all the coordinate with their associated formulas. The metric is then

$$g_{ij} = \begin{bmatrix} |\frac{\partial R}{\partial r}|^2 & & & \\ & |\frac{\partial R}{\partial \phi_1}|^2 & & \\ & & \ddots & \\ & & & |\frac{\partial R}{\partial \phi_{N-1}}|^2 \end{bmatrix}$$

which is simply a concise way of writing a coordinate transformation for the line element $ds^2 = g_{ij}dx_i dx_j$. Computing the magnitude of these derivatives and generalizing the pattern, the metric works out to be

$$g_{ij} = \begin{bmatrix} 1 & & & & & \\ & r^2 & & & & \\ & & r^2 sin^2\phi_1 & & & \\ & & & r^2 sin^2\phi_1 sin^2\phi_2 & & \\ & & & & \ddots & \\ & & & & & r^2 sin^2\phi_1 sin^2\phi_2 \dots sin^2\phi_{N-2} \end{bmatrix}$$

Taking the product of the diagonal elements,

$$detg = r^{2(N-1)} sin^{2(N-2)}\phi_1 sin^{2(N-3)}\phi_2 \dots sin^2\phi_{N-2}$$

Using the formula for the laplacian, the first term (i=j=r) is $\frac{1}{r^{N-1}}\frac{\partial}{\partial r}(r^{N-1}\frac{\partial}{\partial r})$. Although we are mostly interested in the radial component of the laplacian, the second term (ϕ_1) can be computed as well. It is $\frac{1}{r^2 sin^{N-2}\phi_1}\frac{\partial}{\partial\phi_1}(sin^{N-2}\phi_1\frac{\partial}{\partial\phi_1})$.

Separation of Variables Finally, we have all the necessary pieces to solve the Schrodinger equation. We are interested in the radial solution, so we write the laplacian as

$$\nabla^2 = \frac{1}{r^{N-1}}\frac{\partial}{\partial r}(r^{N-1}\frac{\partial}{\partial r}) + \frac{1}{r^2}\Lambda^2$$

where Λ contains all information about angular coordinates and will eventually be factored out. Following the convention of separation of variables, (the method used to solve the 3d Schrodinger equation and many other partial differential equations), we assume a solution that is a product of two functions, each depending on different variables. In this case, let's say $\psi = R(r)f(\phi)$ so that the function R only depends on the radial coordinate r and f can depend on any of the angles $(\phi_1, \dots, \phi_{N-1})$. Once again, the Schrodinger equation reads

$$-\frac{\hbar^2}{2m}\nabla^2\psi - \frac{Ze^2}{r}\psi = E\psi$$

Plugging in the assumed solution, we get

$$-\frac{\hbar^2}{2m}(f(\phi)\frac{1}{r^{N-1}}\frac{\partial}{\partial r}(r^{N-1}\frac{\partial R}{\partial r})) + \frac{1}{r^2}R(r)\lambda^2 f(\phi) - (\frac{Ze^2}{r} - E)R(r)f(\phi) = 0$$

where E has been replaced with $-|E|$ becasue the energy for a bound state is always negative. Following the convention, we divide through by $R(r)f(\phi)$ to obtain

$$-\frac{\hbar^2}{2mR(r)}(\frac{1}{r^{N-1}}\frac{d}{dr}(r^{N-1}\frac{dR}{dr})) + \frac{1}{r^2}\frac{\lambda^2 f(\phi)}{f(\phi)} - (\frac{Ze^2}{r} - E) = 0$$

Partial derivative have been replaced by regular derivatives because the functions only depend on one variable. It is now necessary to rearrange this expression so that one side of the equality depends on r and the other on the angular coordinates ϕ. If two functions are the same but depend on different variables, they must equal a constant β.

$$\frac{\hbar^2 r^2}{2mR(r)}(\frac{1}{r^{N-1}}\frac{d}{dr}(r^{N-1}\frac{dR}{dr})) + r^2(\frac{Ze^2}{r} - E) = \frac{\lambda^2 f(\phi)}{f(\phi)} = \beta$$

We now have an ordinary differential equation in the coordinate r. After some manipulation, it reads

$$\frac{1}{r^{N-1}}\frac{d}{dr}(r^{N-1}\frac{dR}{dr}) - \frac{\beta}{r^2}R + \frac{2m}{\hbar^2}(\frac{Ze^2}{r} - E)R = 0$$

Knowing that β is constant, define it as $\beta = L(L+N-2)$ where L is a positive integer. To de-dimensionalize this equation, we can define

$$\rho = (8mE/\hbar^2)^{1/2}r \quad \lambda = (\frac{Z^2e^4m}{2\hbar^2 E})^{1/2}$$

which are both dimensionless. With these new quantities defined, a substitution and some algebra shows that the new radial Schrodinger equation is

$$\frac{d^2R}{d\rho^2} + \frac{N-1}{\rho}\frac{dR}{d\rho} - \frac{L(L+N-2)}{\rho^2}R + (\frac{\lambda}{\rho} - \frac{1}{4})R = 0$$

Assuming a solution of form $R(\rho) = \rho^L e^{\rho/2}S(\rho)$ (analogous to the three-dimensional approach) reduces this equation to a Laguerre ODE with known solutions. Taking derivatives and plugging in yields

$$e^{-\rho/2}(\rho^L\tfrac{d^2S}{d\rho^2} + (2L\rho^{L-1} - \rho^L)\tfrac{dS}{d\rho} + (L(L-1)\rho^{L-2} - L\rho^{L-1} + \tfrac{1}{4}\rho^L)S +$$
$$\frac{N-1}{\rho}(\frac{\rho^{L-1}}{2}((2L-\rho)S + 2\rho\frac{dS}{d\rho}) + (\frac{-L(L+N-2)}{\rho^2} + \frac{\lambda}{\rho} - \frac{1}{4})(\rho^L S))$$

Which after tedious algebra, reduces to

$$\frac{d^2S}{d\rho^2} + (\frac{2L+N-1}{\rho} - 1)\frac{dS}{d\rho} + (\frac{\lambda - L - (N-1)/2}{\rho})S = 0$$

The Laguerre equation is

$$[\frac{d^2}{d\rho^2}+(\frac{p+1}{\rho}-1)\frac{d}{d\rho}+\frac{(q-p)}{\rho}]L_q^p(\rho)=0$$

Therefore, $p=2L+N-2$ and $q=\lambda+L+\frac{N-3}{2}$.

The differential equation for S can be solved with a series solution of the form $S(\rho)=\sum_{j=0}^{\infty}a_j\rho^j$ which terminates at some $j=n_f$ and is therefore a polynomial. The radial wave function must be normalizable, (as it is a probability distribution), so $\psi\to 0$ as $r\to\infty$. Plugging in the solution, we can find a recursion relation for the coefficients a_j:

$$\sum_{j=0}^{\infty}j(j-1)a_j\rho^{j-2}+(\frac{2L+N-1}{\rho}-1)\sum_{j=0}^{\infty}ja_j\rho^{j-1}+(\frac{\lambda-L-(N-1)/2}{\rho}\sum_{j=0}^{\infty}a_j\rho^j=0$$

$$\sum_{j=1}^{\infty}((j-1)+(2L+N-1))ja_j\rho^{j-2}+\sum_{j=0}^{\infty}(\lambda-L-(N-1)/2-j)a_j\rho^{j-1}=0$$

$$\sum_{j=0}^{\infty}((j+2L+N-1)(j+1)a_{j+1}+(\lambda-L-(N-1)/2-j)a_j)\rho^{j-1}=0$$

The only way for this sum to be 0 is if every individual term vanishes, so

$$a_{j+1}=\frac{j+L+(N-1)/2-\lambda}{(j+1)(j+2L+N-1)}$$

It is now clear why the series solution must terminate: the sign on the coefficients a_j is always positive, so even if an infinite series converged to an analytic function, it would not meet the condition that $\psi\to 0$ with increasing r. Additionally, because $a_{n_f+1}=0$ by definition, the recursion relation tells us $\lambda=n_f+L+\frac{N-1}{2}$. Let's call this the generalize principal quantum number (traditionally named "n") in analogy with the three-dimensional solution. Therefore, $n=n_f+L+1$ in three dimensions.

Effective Potential Let's return to the radial Schrodinger equation with $m=\hbar=1$ units. It is

$$\frac{1}{r^{N-1}}\frac{d}{dr}(r^{N-1}\frac{dR}{dr})-\frac{L(L+N-2)}{r^2}R+2(-V-E)R=0$$

Which for N=3 becomes

$$-\frac{1}{2}\frac{1}{r^2}\frac{d}{dr}(r^2\frac{dR}{dr})+(\frac{L(L+1)}{2r^2}+V(r))R=E*R$$

Traditionally, the subsitution $u(r) = r * R(r)$ is made, which transforms the previous equation to

$$-\frac{1}{2}\frac{d^2u}{dr^2} + (V(r) + \frac{L(L+1)}{2r^2})u = E * u$$

Consequently, $V_{eff} = V(r) + \frac{L(L+1)}{2r^2}$ for the three-dimensional atom. For arbitrary dimension N, we can make the substitution $u(r) = r^{\frac{N-1}{2}}R(r)$:

$$Eu(r)r^{\frac{1-N}{2}} = -\frac{1}{2}(\frac{d^2u}{dr^2}r^{\frac{1-N}{2}} + \frac{du}{dr}r^{\frac{-1-N}{2}} + \frac{1-N}{2}\frac{du}{dr}r^{\frac{-1-N}{2}} + \frac{1-N}{2}\frac{-1-N}{2}u(r)r^{\frac{-3-N}{2}})$$
$$-\frac{N-1}{2}(\frac{du}{dr}r^{\frac{1-N}{2}} + \frac{1-N}{2}u(r)r^{\frac{-1-N}{2}}) + (\frac{L(L+N-2)}{2r^2} + V(r))u(r)r^{\frac{1-N}{2}}$$

After a great deal of algebra, this expression simplifies to

$$-\frac{1}{2}\frac{d^2u}{dr^2} + (V(r) + \frac{L(L+N-2)}{2r^2} + \frac{(N-3)(N-1)}{8r^2})u(r) = Eu(r)$$

meaning that the generalized effective potential is

$$V_{eff} = V(r) + \frac{L(L+N-2)}{2r^2} + \frac{(N-3)(N-1)}{8r^2}$$

This expression clearly reduces to the known three-dimensional effective potential when plugging in N=3. Equating the coefficients on $1/r^2$ for the 3d and N-dimensional effective potentials $L'(L'+1) = L(L+N-2) + \frac{(N-1)(N-3)}{4}$, it's easy to see that

$$L' = L + \frac{N-3}{2}$$

where the primed L is the three dimensional version. This equation is useful because it shows the relation between the constant L in 3 and N dimensions. Again, the 3d principal quantum number $n = n_r + L + 1$ so we can claim that the N-dimensional principal quantum number $n' = n + \frac{N-3}{2}$ where the prime indicates the generalized PQN. We now synthesize a handful of previous results: the radial Schrodinger equation was manipulated to look like a Laguerre equation for S(ρ), which also depended on the two functions $p = 2L+N-2$ and $q = \lambda + L + \frac{N-3}{2} = n+L+N-3$ because $\lambda = n' = n + \frac{N-3}{2}$. The radial solution was defined to be $R(\rho) = \rho^L e^{-\rho/2}S(\rho)$ so that multiplying the solutions to the Laguerre equation ($L_q^p(\rho)$, known functions) by $\rho^L e^{-\rho/2}$ gives the radial solution R as a function of the modified radius ρ. Therefore,

$$R(\rho) = A\rho^L e^{-\rho/2} L_{(n+L+N-3)}^{(2L+N-2)}(\rho)$$

where the normalization constant A has been introduced because the square of the wave equation is a probability distribution and must equal 1 when integrated over all space.

$$A^2 \int_0^\infty [\rho^L e^{\rho/2} L^{(2L+N-2)}_{(n+L+N-3)}(\rho)]^2 r^{N-1} dr = 1$$

The variable ρ can be rewritten using it's original definition and that $\lambda = n' = n + \frac{N-3}{2}$ as

$$\rho = \frac{2me^2}{\hbar^2(n + \frac{N-3}{2})} r$$

The potential has remained a $\frac{1}{r}$ power law, (unlike in the earlier treatment of generalized atomic orbital radii), so the Bohr radius a_0 ($=\frac{\hbar^2}{me^2}$)does not depend on the dimension N. After making these substitutions, the integral expressing normalization of the wave function reads

$$A^2 (\frac{a_0(n + \frac{N-3}{2})}{2})^N \int_0^\infty e^{-\rho} \rho^{2(L+(N-3)/2)} L^{(2L+N-2)}_{(n+L+N-3)}(\rho)^2 \rho^2 d\rho = 1$$

Recycling a result from the 3d atom, the N-dimensional normalization constant is

$$A = (\frac{2Z}{a(n+(N-3)/2})^{N/2} (\frac{(n-L-1)!}{2(n+(N-3)/2)((n+L+N-3)!)^3})^{1/2}$$

Separation of variables led us to assume a solution that was the product of radial and angular functions R(r) and f($\phi_1, \phi_2, \ldots, \phi_{N-1}$). Having determined the normalization constant, we now know the radial solution. The angular solutions, known as hyperspherical harmonics (denoted Y(ϕ)), are less interesting and much more complicated. Thus we conclude the analysis at this point, content to understand the radial solution for the N-dimensional atom. At long last, we can write

$$\psi(r, \phi_1, \ldots, \phi_{N-1}) = A * R(\rho) * Y(\phi_1, \ldots, \phi_{N-1}) =$$

$$(\frac{2Z}{a(n+(N-3)/2)})^{N/2} (\frac{(n-L-1)!}{2(n+(N-3)/2)((n+L+N-3)!)^3})^{1/2} e^{-\rho/2} \rho^L L^{(2L+N-2)}_{(n+L+N-3)}(\rho) Y(\phi)$$

This is the full solution but moving forward the hyperspherical harmonics will not be included because the goal is to plot radial solutions. The final step is to write out the relevant Laguerre polynomials. We will change the form of the Laguerre solutions to make this task easier. Until now, the Laguerre equation was

$$(\rho\frac{d^2}{d\rho^2} + (p+1-\rho)\frac{d}{d\rho} + (q-p))L_q^p = 0$$

The constants p and q were found when compared to the differential equation for S

$$(\rho\frac{d^2}{d\rho^2} + (2L+N-1-\rho)\frac{d}{d\rho} + (\lambda - L - (N-1)/2))S = 0$$

But it was determined that $\lambda = n' = n + \frac{N-3}{2}$ (λ was defined as the N-dimensional PQN, which is related to the 3d PQN). The equation can be rewritten as

$$(\rho\frac{d^2}{d\rho^2} + (2L+N-1-\rho)\frac{d}{d\rho} + (n-L-1))S = 0$$

and compared to the more commonly tabulated Laguerre differential equation

$$(\rho\frac{d^2}{d\rho^2} + (p+1-\rho)\frac{d}{d\rho} + k)L_k^p = 0$$

The new Laguerre polynomials that will appear in the solution to the radial wave equation are $L_{(n-L-1)}^{(2L+N-2)}(\rho)$ which are simpler not only because they are common, but also because the dimensionality N drops out of the lower index. As in the 3d atom, L is the angular momentum quantum number and ranges from 0 to n-1, where n is the principal quantum number. Let's plot the solution for a number of n, N and L values.

$L_0^0(x) = L_0(x)$	$L_0^2(x) = 2$
$L_1^0(x) = L_1(x)$	$L_3^0(x) = L_3(x)$
$L_1^1(x) = -2x + 4$	$L_3^1(x) = -4x^3 + 48x^2 - 144x + 96$
$L_0^1(x) = 1$	$L_2^3(x) = 60x^2 - 600x + 1200$
$L_2^0(x) = L_2(x)$	$L_3^3(x) = -120x^3 + 2160x^2 - 10800x + 14400$
$L_2^1(x) = 3x^2 - 18x + 18$	$L_3^2(x) = -20x^3 + 300x^2 - 1200x + 1200$
$L_2^2(x) = 12x^2 - 96x + 144$	$L_1^3(x) = -24x + 96$
$L_1^2(x) = -6x + 18$	$L_0^3(x) = 6$

Figure 10: Table of relevant Laguerre polynomials

These are the Laguerre polynomials required to generate solutions. Let's start with reproducing the three-dimensional results, namely the 1s (N=3, n=1, L=0), 2p (N=3, n=1, L=1) and 3s orbitals (N=3, n=3, L=0). Let's plot against the modified radius ρ (which is just a constant $*$ r) and set the normalization constant A=1 for simplicity. These solutions are displayed in red, and are obtained by plugging in to the indices we found on the Laguerre solutions and using the table. Higher and lower dimensional radial solutions are then overlaid for ease of comparison.

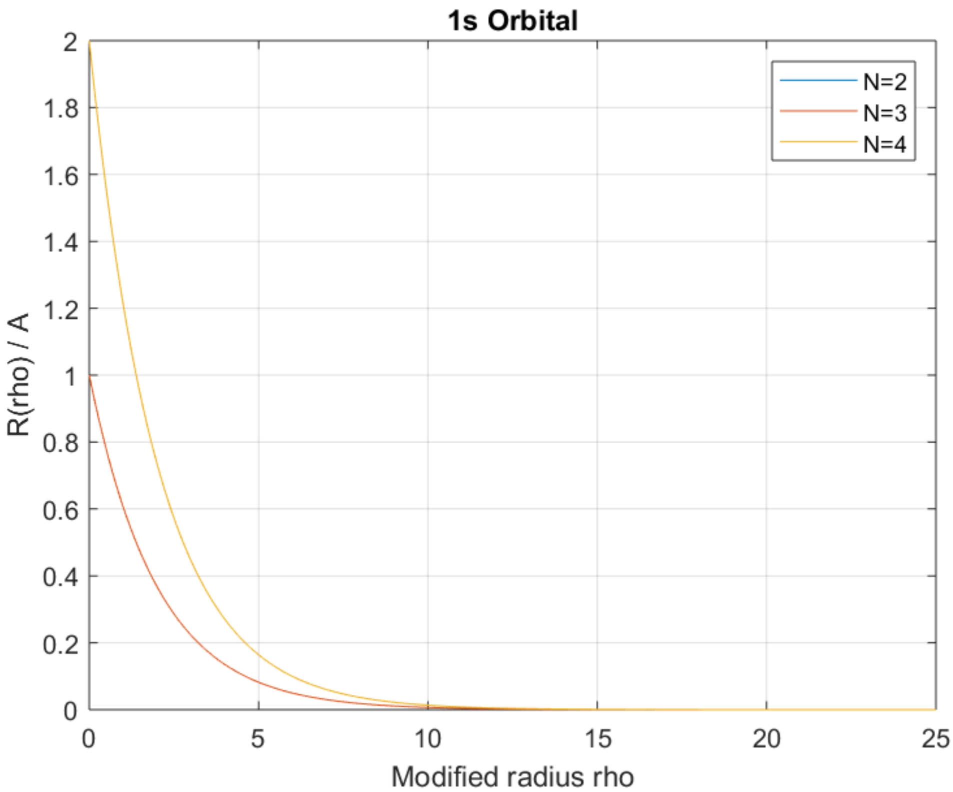


Figure 11: Overlaid plots of 1s orbital shapes for three different dimensionalities

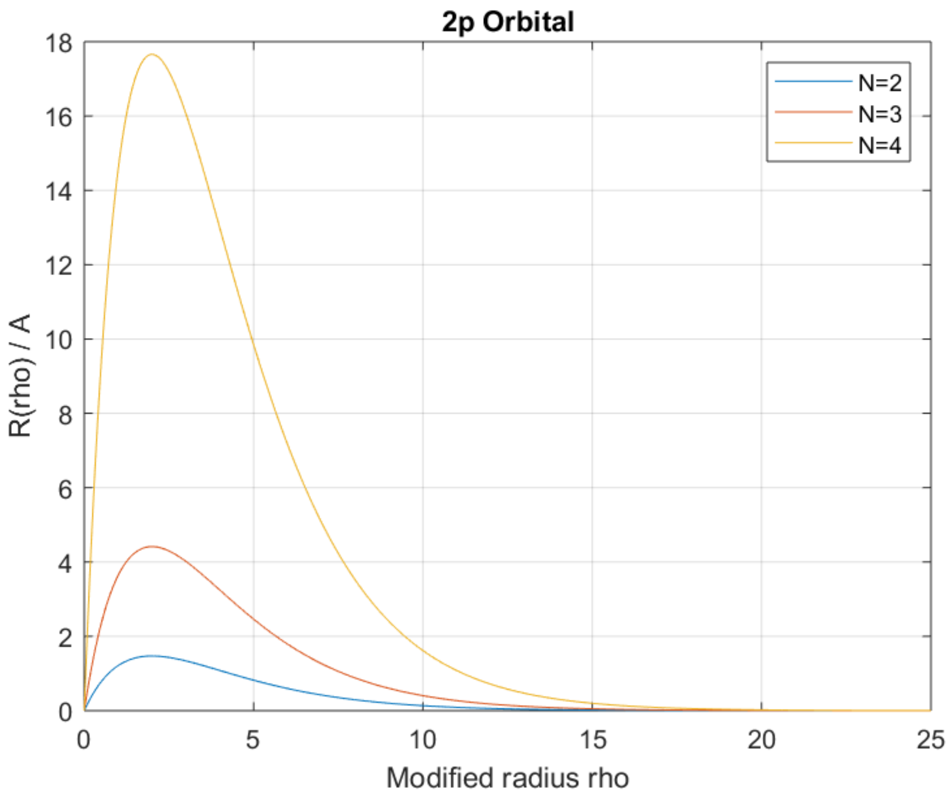


Figure 12: Overlaid plots of 2p orbital shapes

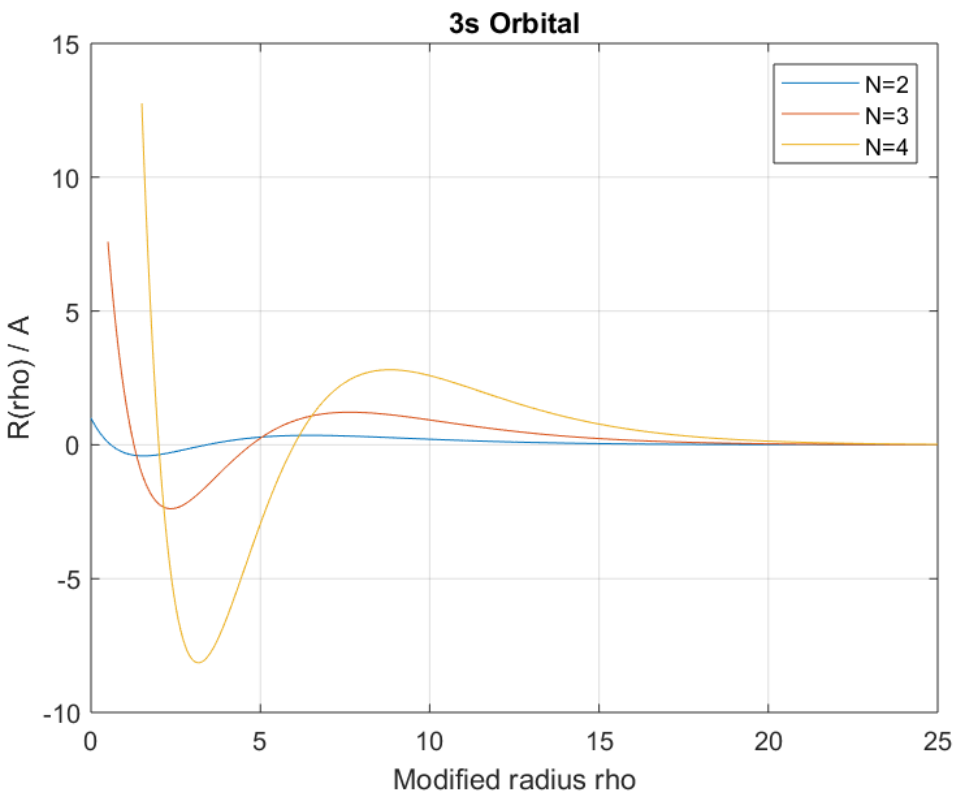


Figure 13: Overlaid plots of 3s orbital shapes

The N-dimensional radial solution reproduces the known shapes of the 3d orbitals. On the first plot, the two and three dimensional solutions are exactly the same and appear as one. Looking at the Laguerre polynomials for two, three and four dimensional 2s orbitals shows that they all have the same shape. These polynomials are the cause of any meaningful change in the nature of the solution, as other appearances of the dimension N simply scale the solution. The derivation of the N-dimensional radial wave function simply amounts to finding a generalized form of the Laguerre polynomials and showing definitively that the rest of the solution is analogous to the 3d case. With a potential energy that is assumed to be independent of dimension and a generalized Laplacian, the orbital's are qualitatively the same in all the dimensions tested, as the overlaid graphs show. The indices on the Laguerre polynomials work out so that, regardless of the dimension, they are first order for 1s, constant for 2p and second order for 3s. Squaring $R(\rho)$ turns probability amplitude into the radial component of the probability distribution $|\psi|^2$.

6 Conclusion

Outside his world . . . all was blank to him; nay not even blank, for blank implies Space; say, rather, all was non-existent.

-Edwin Abbott, Flatland, 1884

We have seen many manifestations of dimensionality from hyperspherical geometry to quantum strangeness. Visualizing and building intuition for higher dimensions is a noble quest, yet usually a futile one–math is clearly the most powerful tool to understanding these worlds. Simple methods of generalizing waves, volumes and surface areas or the laws of physics have fascinating and unexpected consequences. From showing the necessity of three dimensions for orbital and atomic stability to demonstrating the uniquely simple form of the three-dimensional wave equation, it is clear that our universe is a special one. Life as we know it requires a consistent source of energy from a star, stable matter in a variety of states and the ability to transmit and recieve signals, among many other things. If we want to be able to explore the galaxy, we must be permitted escape from our home planet's gravity. None of these things are possible in a universe of different dimensions–we are forced to assume that our three-dimensional world is not just privileged, but essential. In summary, we looked at the behavior of waves, hyperspheres, orbits, rockets and the atom in higher and lower dimensional spaces. It is frustrating and enlightening to imagine the unimaginable; a sort of Zen-like riddle. We challenged the automatic assumption that our universe is the only possible type by investigating phenomena in higher and lower dimensions–a chair could just as well need four legs to stand; perhaps left and right hands aren't that different after all. Curiously, this same investigation showed the necessity of our specific world in order to ask these questions.

7 References

[1] Barrow, J. (2002). The Constants of Nature.

[2] Barrow, J. D., Tipler, F. J., Wheeler, J. A. (2015). The anthropic cosmological principle. Oxford: Oxford University Press.

[3] Abbott, E. A., Lindgren, W. F., Banchoff, T. (2010). Flatland. Cambridge: Cambridge University Press.

[4] Dewdney, A. K. (2001). The planiverse: Computer contact with a two-dimensional world. New York: Copernicus.

[5] Rucker, R. V. (2014). The fourth dimension: Toward a geometry of higher reality. Mineola, NY: Dover Publications.

[6] Jammer, M., Einstein, A. (1993). Concepts of space: The history of theories of space in physics. New York: Dover Publications.

[7] Goldstein, H., Poole, C. P., Safko, J. (2014). Classical mechanics. Harlow: Pearson.

[8] Tangherlini, F. R. (1963). Schwarzschild field in n dimensions and the dimensionality of space problem. Il Nuovo Cimento, 27(3), 636-651. doi:10.1007/bf02784569

[9] Bors, D., Stanczy, R. (2015). Some Non-Linear Second Order Equations Modelling Rocket Motion. Retrieved from https://arxiv.org/pdf/1212.5826.pdf

[10] Huygen's Principle. (n.d.). Retrieved from http://www.mathpages.com/home/kmath242/kmath242.htm

[11] Volume and Surface Area of an n-Dimensional Hypersphere. (2011). Retrieved from http://scipp.ucsc.edu/ haber/ph116A/volume_11.pdf

[12] Feres, R. (2006, September). Geometry in Very High Dimensions. Retrieved from https://www.math.wustl.edu/ feres/highdim

[13] Uva, E. (n.d.). Derivation of Bohr's Radius and Rydberg's Constant. Retrieved from http://www.emsb.qc.ca/laurenhill/science/bohr_radius.pdf

[14] Al-Jaber, S. M. (2016). Uncertainty Relations for Some Central Potentials in N-Dimensional Space. Applied Mathematics, 07(06), 508-517. doi:10.4236/am.2016.76047 Al-Jaber, S. M. (1997). Hydrogen Atom in N Dimensions. International Journal of Theoretical Physics.

[15] Taylor, J. R., Zafiratos, C. D., Dubson, M. A. (n.d.). Modern Physics for Scientists and Engineers(Second ed.).

[16] Laguerre polynomials. (2018, May 28). Retrieved from https://en.wikipedia.org/wiki/Laguerre_polynomials

[17] Miller, J. (n.d.). Laguerre Polynomials. Laguerre differential equation. Recurrence formulas. Generating function. Associated Laguerre polynomials. Retrieved from http://www.solitaryroad.com/c681.

[18] The Hydrogen Atom. (n.d.). Retrieved from http://www.astro.caltech.edu/ srk/Ay126/Lectures/Lecture3/SchrodingerModel.pdf

[19] Centimetre–gram–second system of units. (2018, May 25). Retrieved from https://en.wikipedia.org/wiki/Centimetre–gram–second_system_of_units

[20] McDonald, K. (n.d.). Electrodynamics in 1 and 2 Spatial Dimensions. Retrieved from http://www.physics.princeton.edu/ mcdonald/examples/2dem.pdf

[21] Hydrogen Atom. (2014, January 17). Retrieved from http://web.iitd.ac.in/ nkurur/2013-14/IIsem/cyl100/hatom_radial.pdf

[22] Bohr model. (2018, May 28). Retrieved from https://en.wikipedia.org/wiki/Bohr_model

[23] Relativistic Mechanical Quantities. (n.d.). Retrieved from http://hyperphysics.phy-astr.gsu.edu/hbase/Relativ/relmom.htmlc4

[24] Line element. (2018, April 03). Retrieved from https://en.wikipedia.org/wiki/Line_element

[25] Werbung. (n.d.). Chirality. Retrieved from http://www.chemgapedia.de/vsengine/vlu/vsc/en/ch/12/oc/vlu_organik/stereochemie /chiralitaet.vlu/Page/vsc/en/ch/12/oc/stereochemie/optische_aktivitaet/ optische_aktivitaet.vscml.html

[26] N-sphere. (2018, May 27). Retrieved from https://en.wikipedia.org/wiki/N-sphere

[27] Hypersphere. (n.d.). Retrieved from http://mathworld.wolfram.com/Hypersphere.html

[28] Alpha: Making the world's knowledge computable. (n.d.). Retrieved from http://www.wolframalpha.com/

[29] Dewdney, A. K. (n.d.). Planiverse [Ardeans on a Boat]. Retrieved from https://cs.stanford.edu/people/eroberts/courses/soco/projects/2005-06/planiverse/index.html

[30] Tesseract [Digital image]. (2016, November 11). Retrieved from http://www.supercoloring.com/coloring-pages/tesseract

[31] Chirality [Enantiomers and Handedness]. (n.d.). Retrieved from https://en.wikipedia.org/wiki/Chirality

[32] Properties of n+m Dimensional Spacetime [Digital image]. (n.d.). Retrieved from https://en.wikipedia.org/wiki/Anthropic_principle

[33] Planiverse Engine [Digital image]. (2015, September 20). Retrieved from https://funnyjunk.com/Weeabs/funny-pictures/5689909/88

[34] Planiverse House [Digital image]. (n.d.). Retrieved from http://math2033.uark.edu/wiki/images/5/58/Planiverse.pdf

[35] Derivation of Newton's Shell Theorem [Digital image]. (n.d.). Retrieved from https://en.wikipedia.org/wiki/Shell_theorem/media/File:Shell-diag-1.svg

[36] Virial theorem. (2018, June 12). Retrieved from https://en.wikipedia.org/wiki/Virial_theorem/Generalizations

[37] Isaac Newton. (2018, August 21). Retrieved from https://en.wikipedia.org/wiki/Isaac_Newton_Early_life

[38] May, A. (2017, April 24). How Does Newton Explain Planetary Motion? Retrieved from https://sciencing.com/newton-explain-planetary-motion-20884.html