Basics of electricity & magnetism

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September 2025

1 Conservation of charge

Consider a charge density $\rho(\mathbf{x}, t)$ such that the total charge in a volume Ω is $Q = \int_\Omega \rho(\mathbf{x}, t) d\Omega$. The time rate of change of the total charge in the volume is driven by currents carrying charges across the boundary of the volume, which we call $\partial\Omega$. Charge is not explicitly created or destroyed inside the volume, it is only transported. This conservation law reads

$$\dot{Q} = \frac{\partial}{\partial t} \int_\Omega \rho(\mathbf{x}, t) d\Omega = -\int_{\partial\Omega} \mathbf{J} \cdot \hat{n} dS, \tag{1}$$

where $\mathbf{J}$ is the current and the minus sign indicates that charge leaves the volume when the current aligns with the normal vector of the surface, which, by construction, is facing outwards. Bringing the time derivative inside the integral and using the divergence theorem on the current term, we have

$$\int_\Omega \frac{\partial \rho}{\partial t} d\Omega + \int_\Omega \nabla \cdot \mathbf{J} d\Omega = 0. \tag{2}$$

Charge conservation applies for any volume Ω, so we can take the limit of Eq. (2) as $|\Omega| \to 0$. This gives the pointwise statement of charge conservation, which is known as the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \tag{3}$$

In fluid mechanics, an analogous equation is derived from the conservation of mass. The analogue of the charge density is the mass density and the analogue of the current is the velocity, which transports mass in and out of the volume.

2 Electrostatics

As the name implies, electricity and magnetism describes how currents and charges give rise to electric and magnetic fields. These interactions are captured in Maxwell's equations, which are a system of partial differential equations which show that the electric and magnetic fields mutually influence each other in their response to charges and currents. In this general setting, where the two fields vary in time, one cannot determine the electric field independent of the magnetic field and vice verse. However, when there is no time dependence, simplifications of Maxwell's equations are possible. Namely, the static assumption implies that the electric and magnetic fields decouple and can be independently determined. We will first work with the electric field (electrostatics).

The electric field $\mathbf{E}(\mathbf{x})$ is the force that a unit point charge experiences at position $\mathbf{x}$. Note that this is a vector-valued quantity, because force is a vector. If one is comfortable with forces, the electric field is rather straightforward to interpret physically. It is directly analogous to a gravitational field $\mathbf{g}(\mathbf{x})$ (from a planet, say) that gives the force a unit point mass experiences at each position. As we have already seen with the continuity equation in Eq. (3), charge and mass seem to behave in similar ways mathematically.

Electric fields arise from distributions of charge. Static electric fields are governed by the Gauss law and the time-independent version of Faraday's law, which read:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad \text{(Gauss)}, \quad \nabla \times \mathbf{E} = \mathbf{0} \quad \text{(Faraday)}. \tag{4}$$

Remember that these are simplifications of the two Maxwell equations which govern the electric field. We present the full set of dynamic Maxwell equations later. As before, $\rho(\mathbf{x})$ is the charge density, and ϵ_0 is the so-called "vacuum permittivity," which is an empirical constant of nature (analogous to the gravitational constant G). Though Eqs. (4) are empirical observations about the electric field (they are not derived from more fundamental principles), we can do some work to interpret what these equations report. Let's start with Faraday's law. Remember from vector calculus that the curl of a gradient is always zero. It is also true that every curl-free field can be written as the gradient of some scalar function. Thus, we define a scalar function Φ such that

$$\mathbf{E}(\mathbf{x}) = -\nabla \Phi(\mathbf{x}), \tag{5}$$

where Φ is called the "electric potential." What is the physical significance of this? Note that the work W done in moving a charge from position $\mathbf{x}_0$ to $\mathbf{x}_1$ along a path parameterized by $\mathbf{x}(t)$ can be written as

$$W = \int_0^T q\mathbf{E}(\mathbf{x}) \cdot \frac{\partial \mathbf{x}}{\partial t} dt, \tag{6}$$

where, by construction of the parameterized path, $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{x}(T) = \mathbf{x}_1$. Eq. (6) computes the work by integrating the power in time. Using the definition of the electric potential, this equation can be rewritten as

$$W = \int_0^T -q\nabla\Phi(\mathbf{x}) \cdot \frac{\partial \mathbf{x}}{\partial t} dt = -q \int_0^T \frac{\partial}{\partial t} \Phi\big(\mathbf{x}(t)\big) dt = -q\Big(\Phi(\mathbf{x}_1) - \Phi(\mathbf{x}_0)\Big). \tag{7}$$

Eq. (7) says that the work done in moving the charge from one position to another is independent of the details of the path taken. Fields that satisfy this property are called "conservative." Any vector field which can be computed as the gradient of a potential function is conservative. For example, the gravitational field is conservative, as there is a gravitational potential function analogous to the electric potential. Thus, the static Faraday law shows that the electric field can be derived from a potential function, which implies that the electric field is conservative.

The Gauss law is easier to interpret in its integral form. The differential form given in Eq. (4) is the pointwise statement of the following:

$$\int_\Omega \nabla \cdot \mathbf{E} d\Omega = \int_{\partial\Omega} \mathbf{E} \cdot \hat{n} dS = \int_\Omega \frac{\rho}{\epsilon_0} d\Omega. \tag{8}$$

The integral statement says that the flux of the electric field through a surface $\partial\Omega$ is a function of the charge enclosed in the volume Ω. The integral statement is the famous Gauss law used in electricity and magnetism to compute the electric field from a point charge with charge q. To illustrate this, say that $\rho(\mathbf{x}) = q\delta(\mathbf{x})$ and Ω is a sphere of radius r centered at the origin. Given the symmetry of the problem, the electric field only depends on the radius r and points radically. Eq. (8) becomes

$$4\pi r^2 E(r) = \frac{q}{\epsilon_0} \implies E(r) = \frac{q}{4\pi\epsilon_0 r^2}. \tag{9}$$

The electric field (vector) from a point charge is thus written in spherical coordinates as $\mathbf{E} = q\hat{\mathbf{r}}/4\pi r\epsilon_0$.

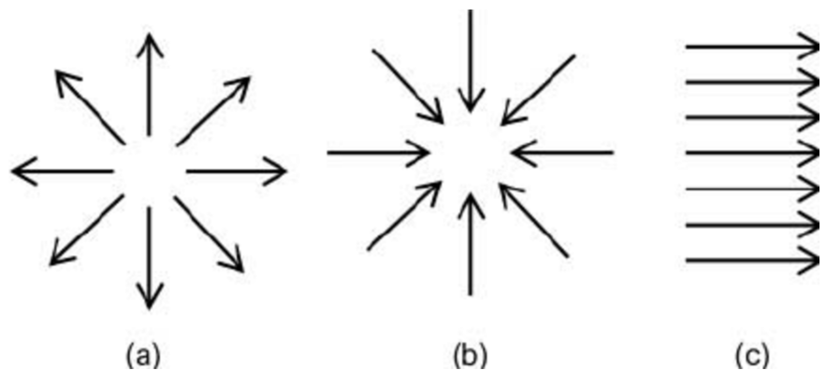

Figure 1: Vector field with positive divergence (source), negative divergence (sink), and zero divergence.

The Gauss law states that charge densities cause divergences of the electric field. Positive divergences of a vector field can be thought of as "sources" of field lines, as shown in Figure 1. Note that, once again, the analogy between the electric and gravitational field is very strong. The equivalent of Gauss' law for gravity is given by

$$\nabla \cdot \mathbf{g} = -4\pi G\rho, \tag{10}$$

where the minus sign arises because the gravitational field is attractive for two masses, as opposed to repulsive for two charges of the same sign, and the factor of 4π is a consequence of how the gravitational constant G is defined. The point is that the laws have the same form. This suggests that we can apply intuition about how gravitational fields arise from mass to understanding how electric fields arise from charges.

We can combine Gauss' law and Faraday's law into a single governing equation for electrostatics. The curl-free condition of the Faraday law implied that the electric field could be written as the gradient of a potential. Plugging this into the Gauss law, we obtain

$$\nabla \cdot \mathbf{E} = \nabla \cdot (-\nabla\Phi) = -\nabla^2\Phi = \frac{\rho}{\epsilon_0}. \tag{11}$$

The electric potential obeys the Poisson equation. Note that in free space, the boundary conditions are that the electric field eventually decays to zero, or that $\Phi \to 0$ as $r \to \infty$. With Eq. (9), we have already found the electric field arising from a single point charge. We can pull some interesting mathematical tricks in order to build up the electric field from more complex distributions of charge. To this end, we define a "Green's function" $\mathcal{G}$ solution to Eq. (11) as

$$\nabla^2\mathcal{G}(\mathbf{x}, \boldsymbol{\xi}) = -\frac{1}{\epsilon_0}\delta(\mathbf{x} - \boldsymbol{\xi}). \tag{12}$$

The function $\mathcal{G}(\mathbf{x}, \boldsymbol{\xi})$ is the distribution of the electric potential in space for a unit point charge placed at position $\boldsymbol{\xi}$. It can be verified that the Green's function of the Poisson problem for the electric potential is

$$\mathcal{G}(\mathbf{x}, \boldsymbol{\xi}) = \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{x} - \boldsymbol{\xi}|}. \tag{13}$$

It is actually not trivial to verify this, though one can by taking the Laplacian of this expression that it is zero when $\mathbf{x} \neq \boldsymbol{\xi}$. Now, note that we can write the charge density in a "delta function basis" as

$$\rho(\mathbf{x}) = \int_\Omega \delta(\mathbf{x} - \boldsymbol{\xi})\rho(\boldsymbol{\xi})d\Omega. \tag{14}$$

In a sense, this is a trivial consequence of the so-called "sifting property" of the delta function. We will take Eq. (12), multiply by $\rho(\boldsymbol{\xi})$ and integrate over the domain. This yields

$$\nabla^2 \int_\Omega \mathcal{G}(\mathbf{x}, \boldsymbol{\xi}) \rho(\boldsymbol{\xi}) d\Omega = -\frac{1}{\epsilon_0} \int \delta(\mathbf{x} - \boldsymbol{\xi}) \rho(\boldsymbol{\xi}) d\Omega = -\frac{1}{\epsilon_0} \rho(\mathbf{x}). \tag{15}$$

We have exchanged integration and differentiation given that the Laplacian acts on the spatial variable, not the dummy integration variable $\boldsymbol{\xi}$. Evidently, whatever quantity the Laplacian is acting on, it satisfies the Poisson problem. We can thus say that

$$\Phi(\mathbf{x}) = \int_\Omega \mathcal{G}(\mathbf{x}, \boldsymbol{\xi}) \rho(\boldsymbol{\xi}) d\Omega, \tag{16}$$

where the Green's function is known through Eq. (13) and the charge density ρ is specified as part of the problem statement. Eq. (16) is the principle of superposition, which, in simple terms, says that if we know the electric potential from a single unit charge, we can compute the electric potential from a distribution of charges by adding up their individual contributions. Note that the Green's function of Eq. (13) is only applicable to the zero far-field boundary conditions. Once the electric potential is computed through superposition, the electric field (the real quantity of interest) is obtained by taking the negative of the gradient of the potential. This reads

$$\mathbf{E}(\mathbf{x}) = -\nabla\Phi(\mathbf{x}) = -\int_\Omega \nabla\mathcal{G}(\mathbf{x}, \boldsymbol{\xi}) \rho(\boldsymbol{\xi}) d\Omega. \tag{17}$$

Solving the two governing equations for electrostatics with a Poisson problem for the electric potential allows all of electrostatics to be reduced to the Green's function of Eq. (13), which can be used to build up the electric field for arbitrarily complex charge distributions with the principle of superposition.

3 Magnetostatics

The magnetic field can be computed independent of the electric field when Maxwell's equations have no time dependence. The governing equations for the magnetic field $\mathbf{B}$ in the static setting are

$$\nabla \cdot \mathbf{B} = 0 \quad \text{(no monopole)}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \text{(Ampere)}, \tag{18}$$

where μ_0 is another empirical constant called the vacuum permeability. The magnetic field does not have such a neat physical interpretation as the electric field. Perhaps the most straightforward way to give it physical meaning is to look at the Lorenz force law, which states that the force $\mathbf{F}$ experienced by a point charge q moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B}). \tag{19}$$

In this interpretation, the magnetic field determines the magnitude of the force experienced by a unit charge moving at unit velocity. Note that the cross product enforces that the Lorenz force is always normal to the velocity. To get a feel for Eq. (19), a particularly simple situation to consider is a uniform magnetic field in the x_3 direction and a charge moving in the x_1-x_2 plane. Newton's second law for the charge (of mass m) states that

$$\frac{m}{q}\ddot{\mathbf{x}} = \dot{\mathbf{x}} \times \mathbf{B} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ B_3 \end{bmatrix} = B_3 \begin{bmatrix} \dot{x}_2 \\ -\dot{x}_1 \\ 0 \end{bmatrix}. \tag{20}$$

Because there are no forces in the x_3 direction, if the initial velocity of the charge is in the x_1-x_2 plane, it will stay in that plane. Collapsing the system to two dimensions, rearranging, taking one time integral, and setting the integration constants to zero for simplicity, we have

$$\frac{\partial}{\partial t} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{qB_3}{m} \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix}. \tag{21}$$

This is the equation for circular motion. The velocity is inversely proportional to the mass of the charge, and proportional to the charge value q and the magnetic field magnitude B_3. Larger magnetic fields will result in larger velocities. This gives some sense of what the magnetic field does physically—it bends the trajectory of moving charges. Note that the moving charge technically gives rise to a current density $\mathbf{J}$, and by Ampere's law, this influences the magnetic field. The above analysis neglects the influence of the moving charge on the magnetic field.

With some sense of what the magnetic field represents physically, we can work to interpret the two governing equations of magnetostatics. The first law, or the divergence-free condition of the magnetic field, is sometimes called the Gauss law of magnetism. In integral form, it states that

$$\int_{\Omega} \nabla \cdot \mathbf{B} = \int_{\partial\Omega} \mathbf{B} \cdot \hat{\mathbf{n}} dS = 0. \tag{22}$$

This says that in any magnetic field, there is no surface $\partial\Omega$ through which there is a non-zero magnetic flux. In other words, whatever field lines enter the surface, they also exit. The divergence free condition says that the magnetic field is like case (c) in Figure 1, where field lines neither begin nor end in the enclosing surface. This is often attributed to the non-existence of magnetic monopoles, meaning that there is no analogue to electric charge for magnetic fields. There are not point sources for magnetic fields in the way that there are for electric or gravitational fields.

The Ampere law shows how magnetic fields arise in the absence of monopoles. Currents $\mathbf{J}$—which are interpreted as charges in motion—introduce curls into the magnetic field. Once again, we can interpret this law with the help of the corresponding integral statement. At this point, we need to be a bit careful. All of the surfaces $\partial\Omega$ we have integrated over up until this point are "closed" surfaces, meaning they represent the boundary of some volume Ω. This is standard fare for someone with a background in continuum mechanics, as it is typically not useful to consider surfaces that do not enclose volumes. This is not the case in electricity and magnetism, as will be seen with the integral statement of Ampere's law. See Figure 2 for an illustration of the differences between surfaces in the divergence theorem (continuum mechanics) and Stokes theorem (used here for Ampere's law). The integral statement of Ampere's law is

$$\int_{\Gamma} (\nabla \times \mathbf{B}) \cdot \hat{\mathbf{n}} d\Gamma = \mu_0 \int_{\Gamma} \mathbf{J} \cdot \hat{\mathbf{n}} d\Gamma, \tag{23}$$

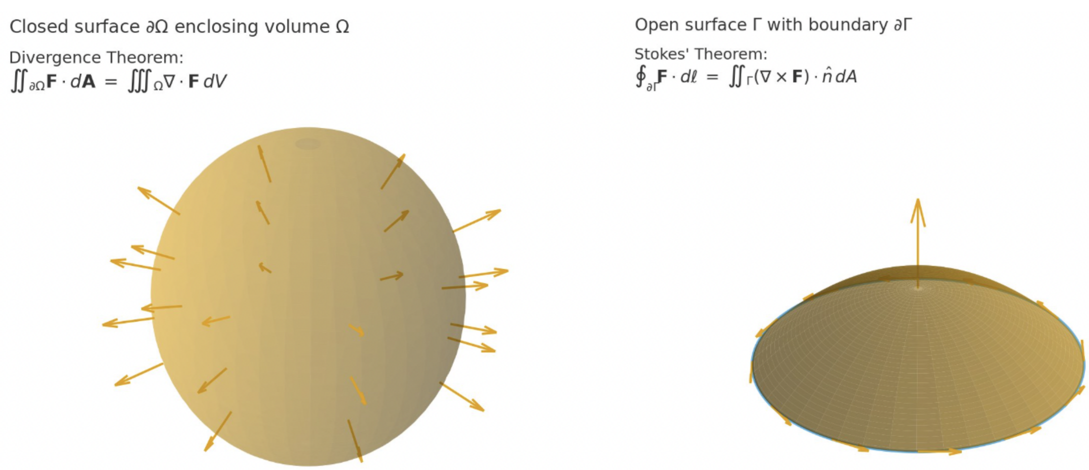

Figure 2: Difference between open and closed surfaces used in the two integral identities of the divergence theorem and Stokes theorem.

where Γ is a surface that explicitly *does not* enclose a volume. The surface Γ has a one-dimensional boundary which we will call $\partial\Gamma$. The right-hand side of Eq. (23) is the amount of current passing through the surface Γ. This is sometimes called I. Using this, and rewriting the left-hand side with Stokes theorem, we have the integral statement of Amperes law as

$$\oint_{\partial\Gamma} \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I. \tag{24}$$

Ampere's law says that currents (charges in motion) cause the magnetic field to circulate. This is perhaps a less satisfying interpretation than that of the electric field. Notice the symmetry between the electric and magnetic fields: the divergence of the electric field is driven by sources (charges), but the electric field is curl-free. The magnetic field is divergence free, but its curl is driven by sources (currents). Just as the curl-free condition on the electric field allowed us to write it with a potential, the divergence-free condition on the magnetic field will allow us to write it with a potential function. Remember from vector calculus that the divergence of a curl is zero. Furthermore, any divergence-free field can be written as the curl of some function. Thus, we can write the magnetic field with a potential function $\mathbf{A}(\mathbf{x})$ as

$$\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}(\mathbf{x}). \tag{25}$$

Note that unlike electrostatics, the potential function is vector-valued. The vector potential formulation of the magnetic field enforces the zero divergence condition automatically. Plugging the potential into Ampere's law, we can combine the two governing equations of magnetostatics into one:

$$\nabla \times \nabla \times \mathbf{A} = \mu_0 \mathbf{J}. \tag{26}$$

It is reasonably straightforward to show (in index notation) that double curls can be rewritten as

$$\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}. \tag{27}$$

Now, note that $\nabla \times \mathbf{A} = \nabla \times (\mathbf{A} + \nabla\chi)$ for any scalar field χ because the curl of a gradient is zero. There is ambiguity as to how to choose the vector potential given that adding in a gradient does not change the resulting magnetic field. A common way to remove this redundancy is the so-called "Coulomb gauge," which enforces that $\nabla \cdot \mathbf{A} = 0$. With the Coulomb gauge, the double curl identity takes a particularly simple form, and the governing equation for magnetostatics becomes

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}, \tag{28}$$

which is analogous to electrostatics except that the Poisson problem applies to each component of the vector potential. Note that the zero far-field boundary conditions, which state the field decays to zero at infinite distances from its source, apply to the magnetic field as well. This means that the Green's function solutions to the Poisson problem from electrostatics can be recycled for the magnetic vector potential. In fact, we can write the vector potential for arbitrary distributions of current with the principle of superposition as

$$\mathbf{A}(\mathbf{x}) = \int_\Omega \mathcal{G}(\mathbf{x}, \boldsymbol{\xi}) \mathbf{J}(\boldsymbol{\xi}) d\Omega. \tag{29}$$

These are the same Green's functions as the case of electrostatics except ϵ_0 is replaced by μ_0. The magnetic field is computed from the vector potential as

$$\mathbf{B}(\mathbf{x}) = \int_\Omega \nabla \times \mathcal{G}(\mathbf{x}, \boldsymbol{\xi}) \mathbf{J}(\boldsymbol{\xi}) d\Omega. \tag{30}$$

4 Maxwell's equations

When the electric and magnetic fields change in time, they are coupled together. The full set of Maxwell's equations is

$$
\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \quad \text{(Gauss)},\\
\nabla \cdot \mathbf{B} &= 0 \quad \text{(monopoles)},\\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(Faraday)},\\
\nabla \times \mathbf{B} &= \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \quad \text{(Ampere)}.
\end{aligned} \tag{31}
$$

Note that the charge density ρ and current $\mathbf{J}$ cannot be independently specified, as they must satisfy the continuity equation from conservation of charge. Similarly, in an Ohmic medium, the current density may be related to the electric field with $\mathbf{J} = \sigma \mathbf{E}$, which says simply that electric field pushes charge around in a medium defined by the parameter σ. An interesting consequence of Maxwell's equation is that they predict the speed of light in free space. In free space, there are no charge densities or currents. Taking the curl of the Faraday law and plugging in the Ampere law, we have

$$
\nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{32}
$$

Using the double curl identity and that the electric field is divergence free, we obtain a wave equation for the electric field:

$$
\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\mu_0 \epsilon_0} \nabla^2 \mathbf{E}. \tag{33}
$$

The speed of light can be read off as $c = 1/\sqrt{\mu_0 \epsilon_0}$. The argument about this coefficient giving the wave speed is really the following: if the Laplacian is transformed to spherical coordinates and the electric field is taken to be purely radial, a one-dimensional wave equation can be derived for its magnitude. The 1D wave equation has an analytic solution in terms of traveling waves, and these waves travel at a rate given by c.

Only the Faraday and Ampere law change when going to the full statement of Maxwell's equations. We can get a feel for what sort of phenomena the coupling introduces by considering simple example problems. A famous example of Faraday's law is a changing magnetic field inducing current in a loop of wire. First, note that, from Stokes theorem, the integral statement of Faraday's law is

$$
\oint_{\partial \Gamma} \mathbf{E} \cdot d\boldsymbol{\ell} = -\int_{\Gamma} \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{\mathbf{n}} d\Gamma. \tag{34}
$$

Now, consider a loop of wire of radius r lying in the x_1-x_2 plane and a magnetic field given by $\mathbf{B}(t) = [0, 0, B_3(t)]$. Given that the line element $d\boldsymbol{\ell}$ is tangent to the circle, the changing magnetic field only induces circumferential components of the electric field. The magnetic field is aligned with the surface defined by the loop of fire, thus Eq. (34) becomes

$$
E_\theta 2\pi r = -\pi r^2 \dot{B}_3 \implies E_\theta = \frac{-r\dot{B}_3}{2}, \tag{35}
$$

where E_θ is the tangential component of the electric field along the loop of wire. The current induced would thus be $J_\theta = \sigma E_\theta$, showing that a changing magnetic field induces currents in a wire. This is called Faraday induction.

Ampere's law states that both currents and changing electric fields induce magnetic fields. This is often illustrated in the context of a charging capacitor, where there is zero "conduction current" flow across the plates (i.e., $\mathbf{J} = \mathbf{0}$), but there is still an induced magnetic field. Running a current through a capacitor causes the voltage across the capacitor to change in time, which causes the electric field to change in time. The general integral statement of Ampere's law with zero conduction current is

$$
\oint_{\partial \Gamma} \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 \epsilon_0 \int_{\Gamma} \frac{\partial \mathbf{E}}{\partial t} \cdot \hat{\mathbf{n}} d\Gamma. \tag{36}
$$

Let's assume that the capacitor comprises two circular disks of radius a that lie in the x_1-x_2 plane, that electric field is in the x_3 direction, and that it is zero outside the two disks. We can take the surface Γ to be a disk of radius r, which, using symmetry, gives an induced magnetic field of

$$B_\theta(r,t) = \frac{\mu_0 \epsilon_0 r}{2} \dot{E}_3, \quad r \leq a. \tag{37}$$

A similar calculation can be carried for the $r > a$, in which case the surface integral of the time derivative of the electric field no longer increases as the radius of the integration surface is expanded. The magnetic field induced by the changing electric field is similar to the case of a magnetic field induced by a current, in the sense that it loops around the direction in which the electric field is changing.

5 Eigenvalue problem

We want to obtain the mode shapes and frequencies of oscillation for the magnetic field. To do this, take the time derivative of the Faraday equation to obtain

$$\nabla \times \frac{\partial \mathbf{E}}{\partial t} = -\frac{\partial^2 \mathbf{B}}{\partial t^2}. \tag{38}$$

Now, use the dynamic Ampere law with $\mathbf{J} = \mathbf{0}$ to write the time derivative of the electric field in terms of the magnetic field. Substituting this, we obtain

$$\nabla \times \nabla \times \left(\frac{1}{\mu_0 \epsilon_0} \mathbf{B} \right) = -\frac{\partial^2 \mathbf{B}}{\partial t^2}. \tag{39}$$

As is standard for linear partial differential equations, we assume that the solution can be represented as a product of a spatial mode $\tilde{\mathbf{B}}(\mathbf{x})$ with a harmonic response in time given by $e^{i\omega t}$. Substituting the assumed solution $\mathbf{B}(\mathbf{x}) = \tilde{\mathbf{B}}(\mathbf{x}) e^{i\omega t}$ into Eq. (39) and canceling the common factor, we obtain a static equation for the spatial component of the solution:

$$\nabla \times \nabla \times \tilde{\mathbf{B}}(\mathbf{x}) = \mu_0 \epsilon_0 \omega^2 \tilde{\mathbf{B}}(\mathbf{x}). \tag{40}$$

This is an eigenvalue partial differential equation with eigenfunction $\tilde{\mathbf{B}}(\mathbf{x})$ and corresponding eigenvalue ω^2. One way to solve this eigenvalue problem is through the Rayleigh quotient. To construct the Rayleigh quotient, we compute the weak form of the eigenvalue problem of Eq. (40) with the test function taken as the solution field. This yields

$$\int_\Omega \left(\nabla \times \nabla \times \tilde{\mathbf{B}} \right) \cdot \tilde{\mathbf{B}} - \mu_0 \epsilon_0 \omega^2 \tilde{\mathbf{B}} \cdot \tilde{\mathbf{B}} d\Omega = 0. \tag{41}$$

We can rearrange this expression to solve for the eigenvalue ω^2. We can also use integration identities to rid of the double curl operation (see final section). The resulting of this is

$$\omega^2 = \frac{\int_\Omega \left(\nabla \times \nabla \times \tilde{\mathbf{B}} \right) \cdot \tilde{\mathbf{B}} d\Omega}{\mu_0 \epsilon_0 \int_\Omega \tilde{\mathbf{B}} \cdot \tilde{\mathbf{B}} d\Omega} = \frac{\int_\Omega \left(\nabla \times \tilde{\mathbf{B}} \right) \cdot \left(\nabla \times \tilde{\mathbf{B}} \right) d\Omega + \int_{\partial\Omega} (\nabla \times \tilde{\mathbf{B}} \times \tilde{\mathbf{B}}) \cdot \hat{\mathbf{n}} dS}{\mu_0 \epsilon_0 \int_\Omega \tilde{\mathbf{B}} \cdot \tilde{\mathbf{B}} d\Omega}. \tag{42}$$

Using index notation, it is clear that the boundary term in the above equation can be rewritten as

$$\int_{\partial\Omega} (\nabla \times \tilde{\mathbf{B}} \times \tilde{\mathbf{B}}) \cdot \hat{\mathbf{n}} dS = \int_{\partial\Omega} (\nabla \times \tilde{\mathbf{B}}) \cdot (\tilde{\mathbf{B}} \times \hat{\mathbf{n}}) dS. \tag{43}$$

If the boundary conditions are such that $\tilde{\mathbf{B}} \times \hat{\mathbf{n}} = \mathbf{0}$, then the Rayleigh quotient is defined as

$$\mathcal{R}'\left(\tilde{\mathbf{B}}(\mathbf{x}) \right) = \frac{\int_\Omega \left(\nabla \times \tilde{\mathbf{B}} \right) \cdot \left(\nabla \times \tilde{\mathbf{B}} \right) d\Omega}{\mu_0 \epsilon_0 \int_\Omega \tilde{\mathbf{B}} \cdot \tilde{\mathbf{B}} d\Omega}. \tag{44}$$

This boundary condition, which states that the magnetic field lines are aligned with the wall normal, is enforced naturally with the Rayleigh quotient given in Eq. (44). These are called the "perfect magnetic conductor" (PMC) boundary conditions. Note that, though we have used both the Faraday and Maxwell equations, we have nowhere enforced that the magnetic field is divergence free. This can be accomplished with the vector potential discussed in the magnetostatics section. Writing the eigenfunction in terms of a vector potential $\tilde{\mathbf{A}}$, the new Rayleigh quotient is

$$\mathcal{R}\Big(\tilde{\mathbf{A}}(\mathbf{x})\Big) = \frac{\int_\Omega \Big(\nabla \times \nabla \times \tilde{\mathbf{A}}\Big) \cdot \Big(\nabla \times \nabla \times \tilde{\mathbf{A}}\Big) d\Omega}{\mu_0 \epsilon_0 \int_\Omega \Big(\nabla \times \tilde{\mathbf{A}}\Big) \cdot \Big(\nabla \times \tilde{\mathbf{A}}\Big) d\Omega}. \tag{45}$$

It turns out that the eigenproblem can be transformed into an optimization problem with the Rayleigh quotient. The eigenfunctions and eigenvalues are given by

$$\tilde{\mathbf{A}}(\mathbf{x}) = \underset{\mathbf{v}(\mathbf{x})}{\operatorname{argmin}}\ \mathcal{R}\Big(\mathbf{v}(\mathbf{x})\Big), \quad \omega^2 = \min_{\mathbf{v}(\mathbf{x})} \mathcal{R}\Big(\mathbf{v}(\mathbf{x})\Big). \tag{46}$$

Consecutive eigenfunctions are found by orthogonalizing the search space with respect to previously obtained eigenfunctions. Note that there are gauge degrees of freedom in the vector potential. The Coulomb gauge may be enforced with a penalty on the divergence of $\tilde{\mathbf{A}}$.

Integration by parts

Formulas for integration by parts are derived from applying the product rule, using integration identities, and rearranging the resulting expressions. We show this here.

5.1 Dot products

$$\begin{aligned} \int_{\partial\Omega} v_{ji} w_i \hat{n}_j dS = \int_\Omega \frac{\partial}{\partial x_j}(v_{ji} w_i) d\Omega = \int_\Omega \frac{\partial v_{ji}}{\partial x_j} w_i d\Omega + \int_\Omega v_{ji} \frac{\partial w_i}{\partial x_j} d\Omega \\ \implies \int_\Omega \mathbf{v} : \nabla \mathbf{w} d\Omega = -\int_\Omega (\nabla \cdot \mathbf{v}) \cdot \mathbf{w} d\Omega + \int_{\partial\Omega} \mathbf{v}\mathbf{w} \cdot \hat{\mathbf{n}} dS. \end{aligned} \tag{47}$$

5.2 Cross products

Using index notation, we write the cross product $\mathbf{v}\times\mathbf{w}$ as $e_{ijk} v_j w_k$ where e_{ijk} is the "permutation symbol."

$$\begin{aligned} \int_{\partial\Omega} e_{ijk} v_j w_k \hat{n}_i dS &= \int_\Omega \frac{\partial}{\partial x_i}(e_{ijk} v_j w_k) d\Omega = \int_\Omega e_{ijk} \frac{\partial v_j}{\partial x_i} w_k d\Omega + \int_\Omega e_{ijk} v_j \frac{\partial w_k}{\partial x_i} d\Omega \\ &= \int_{\partial\Omega} (\mathbf{v} \times \mathbf{w}) \cdot \hat{\mathbf{n}} dS = \int_\Omega \mathbf{w} \cdot (\nabla \times \mathbf{v}) d\Omega - \int_\Omega \mathbf{v} \cdot (\nabla \times \mathbf{w}) d\Omega \\ \implies \int_\Omega \mathbf{w} \cdot (\nabla \times \mathbf{v}) d\Omega &= \int_{\partial\Omega} (\mathbf{v} \times \mathbf{w}) \cdot \hat{\mathbf{n}} dS + \int_\Omega (\nabla \times \mathbf{w}) \cdot \mathbf{v} d\Omega. \end{aligned} \tag{48}$$