Green's functions and linear differential equations

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Introduction

In order to make them a bit more intuitive, we will explore some of the ways that Green's functions connect to concepts from linear algebra. Green's functions are familiar to engineering students from courses that cover the "impulse response" of systems, usually linear time invariant circuits. If they are known, these functions can be used to write down a semi-analytical general solution to an inhomogeneous differential equation. They are also used to showcase the power of linearity in solving differential equations. Over the years, they have remained a bit mysterious to me, and my guess is that few students feel like they have good insight into what is going on with a Green's function solution to a differential equation. Hopefully we can clear some of this up!

Problem setup

To begin, we will work with a generic linear differential equation. Abstractly, we write a 1D boundary value problem as

$$\mathcal{L}\Big(u(x)\Big) = f(x), \quad u(0) = \lambda, \quad \alpha u(1) + \beta \frac{\partial u}{\partial x}(1) = \gamma. \tag{1}$$

The domain is $x \in [0, 1]$, the solution is $u(x)$, and the linear "differential operator" is denoted as $\mathcal{L}$. This is a shorthand for a generic differential equation. Two examples of might be:

$$\begin{aligned} \mathcal{L} = \frac{\partial^2}{\partial x^2} + 1 &\implies \frac{\partial^2 u}{\partial x^2} + u = f(x), \\ \mathcal{L} = \frac{\partial}{\partial x} + c(x) &\implies \frac{\partial u}{\partial x} + c(x)u = f(x). \end{aligned} \tag{2}$$

We assume for simplicity that the left boundary condition only applies to the "displacement" [1]. The right boundary condition we leave more general. Note that by assuming that only the displacement and the derivative of the

[1] We will call the solution $u(x)$ the displacement even though this need not be what it represents. This helps think about the problem if you come from an engineering mechanics background.

displacement show up in the right boundary condition, we have committed ourselves to second order operators, meaning $\mathcal{L}$ contains two spatial derivatives. Second order differential equations are the norm in physics. We will see later how the order of derivative in the boundary condition relates to the order of derivative in the operator.

The Green's functions corresponding to a particular system $\mathcal{L}$ and boundary conditions are defined as

$$\mathcal{L}\Big(G(x,t)\Big) = \delta(x-t), \quad G(0,t) = 0, \quad \alpha G(1,t) + \beta\frac{\partial G}{\partial x}(1,t) = 0. \tag{3}$$

The Green's function $G(x,t)$ stores the spatial response of the system to a point force (delta function, impulse, etc.) at $x = t \in [0,1]$. The Green's functions obey the same form of boundary conditions as Eq. (1), but set equal to zero. Solving Eq. (3) may not be easy. We will not concern ourselves with obtaining the Green's functions, rather we want to understand why functions that obey Eq. (3) are useful. With the Green's functions in hand, a solution to Eq. (1) can be written down explicitly as

$$u(x) = \int_0^1 G(x,t)f(t)dt + u^h(x) \tag{4}$$

where $u^h(x)$ is the "homogeneous" solution, meaning that it satisfies $\mathcal{L}(u^h) = 0$ and the two boundary conditions $u(0) = \lambda$, $\alpha u(1) + \beta\frac{\partial u}{\partial x}(1) = \gamma$. Observe that $u(x)$ necessarily satisfies the boundary conditions. Satisfaction of the left boundary condition is trivial to see, and on the right we have

$$\begin{aligned}\alpha u(1) + \beta\frac{\partial u}{\partial x}(1) &= \alpha\int_0^1 G(1,t)f(t)dt + \alpha u^h(1) + \beta\int_0^1 \frac{\partial G}{\partial x}(1,t)f(t)dt + \beta\frac{\partial u^h}{\partial x}(1)\\ &= \int_0^1 \left(\alpha G(1,t) + \beta\frac{\partial G}{\partial x}(1,t)\right)f(t)dt + \alpha u^h(1) + \beta\frac{\partial u^h}{\partial x}(1)\\ &= \alpha u^h(1) + \beta\frac{\partial u^h}{\partial x}(1) = \gamma.\end{aligned} \tag{5}$$

This is why we chose "homogeneous" boundary conditions on the Green's function—this ensures that all of the work of boundary enforcement is being done by $u^h(x)$. The term in the solution involving the Green's functions does not influence what is going on at the boundaries. But how is it that Eq. (11) is a solution to the original boundary value problem? To see this, we can apply the differential operator on both sides of the equation:

$$
\begin{aligned}
\mathcal{L}\Big(u(x)\Big) &= \mathcal{L}\bigg(\int_0^1 G(x,t)f(t)dt + u^h(x)\bigg) \\
&= \int_0^1 \mathcal{L}\Big(G(x,t)\Big)f(t)dt + u^h(x) \\
&= \int_0^1 \delta(x-t)f(t)dt + \mathcal{L}\Big(u^h(x)\Big) \\
&= f(x) + 0
\end{aligned}
\tag{6}
$$

We plugged in the Green's function solution to the differential equation and obtained $f(x)$ as the right hand side. This is the definition of a solution to a differential equation. The homogeneous solution deals with the boundary conditions, and the term with Green's functions handles the forcing on the system. Eq. (6) may show mathematically why this solution methodology works, but it gives very little insight into what is going on. We can turn to linear algebra in order to make a bit more sense of these concepts.

Analogies to linear algebra

Instead of working with a continuous linear differential equation $\mathcal{L}(u) = f$, we will briefly take a detour to work with a discrete linear system given by $\mathbf{Lu} = \mathbf{f}$. The connection between the continuous and discrete systems is not obvious at this point. That being said, for those familiar with numerical solutions to differential equations, the idea of transforming a continuous differential equation to a discrete system is nothing new. In the discrete linear system, the matrix $\mathbf{L}$ and the vector $\mathbf{f}$ are both known. A solution to this problem involves finding a vector $\mathbf{u}$ such that the linear system holds. Assume for the moment we know nothing about matrix inverses. Let us hypothesize that a solution of the linear system is given by

$$
\mathbf{u} = \mathbf{Gf}. \tag{7}
$$

The matrix $\mathbf{G}$ is not known. We simply assume that the solution is some new linear function of the vector $\mathbf{f}$. What conditions does $\mathbf{G}$ need to obey in order for this expression to constitute a solution to our problem? We can multiply on the left by the linear operator $\mathbf{L}$ (matrices are operators as well) to obtain

$$
\mathbf{Lu} = \mathbf{LGf}. \tag{8}
$$

In order to recover the original equation, we require that $\mathbf{LG} = \mathbf{I}$. This expression provides us sufficient information to determine the entries of the matrix $\mathbf{G}$. Obviously, we have written down an equation for the inverse matrix $\mathbf{G} = \mathbf{L}^{-1}$. But wait! We must remember that linear systems may have a nullspace, which means that there are vectors $\mathbf{Lu}^N = \mathbf{0}$. This means that $\mathbf{Gf} + \mathbf{u}^N$ is also a solution to our problem. If there are a set of nullspace vectors, any vector of the following form satisfies the linear system:

$$\mathbf{u} = \mathbf{G}\mathbf{f} + \sum_i \alpha_i \mathbf{u}_i^N \tag{9}$$

where the coefficients $\boldsymbol{\alpha}$ are totally arbitrary. Notice that the general solution to the linear system of equations looks suspiciously similar to the general solution to the linear differential equation. In both cases, there is a "nullspace" term which is zeroed by the operator. Any linear combination of nullspace elements added into the solution still satisfies the equation unless additional constraints are introduced. This is exactly what boundary conditions do for differential operators—they remove non-uniqueness from the solution by fixing the nullspace vector(s). Differential operators have nullspaces because they zero certain polynomials. The first derivative of a constant is zero, and the second derivative of a linear function is zero. This means that second order differential operators have a nullspace of dimension 2, which is why two boundary conditions are required in order to enforce a unique solution. We cannot impose boundary conditions with derivatives of the same order as the operator because this no longer corresponds to a nullspace element. In the context of linear algebra, "boundary conditions" would be equivalent to putting constraints on the nullspace component of the solution to the linear system.

The analogy between the matrix inverse and the Green's function can be taken even further. We can write the solution to the linear system with the matrix inverse $\mathbf{G} = \mathbf{L}^{-1}$ as

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} — & \mathbf{g}_1 & — \\ — & \mathbf{g}_2 & — \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} | \\ \mathbf{f} \\ | \end{bmatrix}. \tag{10}$$

We are ignoring the nullspace component of the solution and looking only at the "particular solution" (to use common language from differential equations). We see that the dot product of row i of $\mathbf{G}$ with the vector $\mathbf{f}$ gives the i-th component of the solution $\mathbf{u}$. For example, $\mathbf{g}_1 \cdot \mathbf{f} = u_1$. The particular solution to the continuous differential equation is given by the Green's function as

$$u(x) = \int_0^1 G(x,t) f(t) dt. \tag{11}$$

This looks very similar to the continuous version of matrix multiplication $u_i = \sum_j G_{ij} f_j$. Instead of summing over indices, we integrate over coordinates. The dummy variable t serves the role of the index of columns in the matrix, whereas the spatial coordinate does the equivalent of indexing rows. Even the definition of the matrix inverse resembles the definition of a Green's function. Both require that the action of the operator on the desired quantity $G(x,t)$ or $\mathbf{G}$ is to produce zero everywhere except at one select point. This is seen in the two definitions:

$$\sum_k L_{ik}G_{kj} = \delta_{ij}, \quad \mathcal{L}\Big(G(x,t)\Big) = \delta(x-t). \tag{12}$$

We can further strengthen the analogy between the Green's functions and the matrix inverse by showing a particular example where they are equivalent in a more literal sense.

Particular example

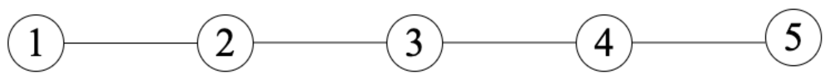

Figure 1: A five node finite difference grid on $[0,1]$. This means that $\Delta x = 1/4$.

We now consider a particular linear differential equation. The following is the 1D equivalent of the Poisson differential equation:

$$\frac{\partial^2 u}{\partial x^2} = f(x), \quad u(0) = u(1) = 0. \tag{13}$$

We again take the domain to be $x \in [0,1]$ but now consider two zero displacement boundary conditions. Given the above discussion, the question we must ask is: how can we make the analogy between inverse matrices and Green's functions more explicit? First, note that the Green's functions must satisfy

$$\frac{\partial^2}{\partial x^2} G(x,t) = \delta(x-t). \tag{14}$$

The Green's function can be computed by integrating twice:

$$G(x,t) = a + bx + I(x-t), \quad G(0) = G(1) = 0 \tag{15}$$

where $I(x-t) = (x-t+|x-t|)/2$, which is simply a way of writing the integral of a unit step function which activates at $x = t$. The constants of integration can be determined with the boundary conditions as $a = 0$ and $b = -I(1-t) = t-1$. The Green's function is a piecewise linear "hat function" which is zero on both sides of the domain. Note that the given boundary conditions with this particular differential operator force the homogeneous solution to be $u^h(x) = 0$. There are no non-zero functions which satisfy the homogeneous differential equation and the boundary conditions. Thus, the solution is given by

$$u(x) = \int_0^1 G(x,t)f(t)dt. \tag{16}$$

We will imagine approximating the solution on the grid shown in Fig 1. This is way of turning the continuous system into a discrete system. We already know that the solution at the two end nodes will be zero. Thus, only the interior nodes 2-4 are true degrees of freedom. Evaluating the Green's function on these nodes, we have

$$G(x_i, t_j) = \begin{bmatrix} -0.1875 & -0.125 & -0.0625 \\ -0.125 & -0.25 & -0.125 \\ -0.0625 & -0.125 & -0.1875 \end{bmatrix}. \tag{17}$$

By discretizing the integral (in a bit of a strange way), we can approximate a solution on the nodes as

$$u(x_i) = \sum_{j=2}^{4} G(x_i, t_j) f(t_j) \Delta x \tag{18}$$

The forcing function $f(x)$ is simply evaluated at the nodes and we pick up an integration element Δx. We can compare this to the system of equations we obtain when discretizing the system with the finite difference method. Remember that a second derivative at a point x_i can be approximated with

$$\frac{\partial^2 u}{\partial x^2}(x_i) \approx \frac{1}{\Delta x^2}(u_{i+1} - 2u_i - u_{i-1}) \tag{19}$$

We note that in the finite difference grid, we take $u_1 = u_5 = 0$ in order to enforce the boundary conditions. This means that the finite difference system be can be written as

$$\frac{1}{\Delta x^2} \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} f_2 \\ f_3 \\ f_4 \end{bmatrix} \tag{20}$$

By doing the simple numerical computation, it can be shown that

$$\left(\frac{1}{\Delta x^2} \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{bmatrix} \right)^{-1} = \Delta x G(x_i, t_j) \tag{21}$$

The Green's function is the continuous version of the matrix that "undoes" the differential operator. This little example suggests that the connection between Green's functions and matrix inverses runs deep!