Hamiltonian Mechanics

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1 Brief Notes

For a mechanical system with no spatial derivatives (a system of particles) and for the Lagrangian $L = T - V$, Lagrange's equations are

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

where q_i are the generalized displacement degrees of freedom, T is the kinteic energy, and V is the potential. The conjugate momenta are defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

which means that from Lagrange's equations, we have

$$\dot{p}_i = \frac{\partial L}{\partial q_i}$$

These are the necessary preliminary ideas we need to construct the Hamiltonian formulation of mechanics. Note that the given form of Lagrange's equations does not work for continuous elastic systems where spatial derivatives of the displacement variable arise. Spatial derivatives would enter Lagrange's equations in this case. This formulation restricts us to discretized elastic systems. The goal of Hamiltonian mechanics is to formulate a system of governing equations which are first order in time. In the Lagrangian version of mechanics, the state of the system is specified by the n displacement degrees of freedom. Velocities are computed by taking time derivatives of the displacement. But because the governing equations are second order in time, we require $2n$ initial conditions to solve the system. The Hamiltonian formulation seeks a first order formulation, which necessitates increasing the size of the system to $2n$. Each initial condition corresponds to an explicit degree of freedom. It is simple to turn Lagrange's equations into a first order system by introducing a differential equation of the sort $v_i = \dot{q}_i$. This is a common way to carry out numerical integration of second order systems. The new variable v_i represents a new degree of freedom that removes a time derivative from the governing equations. Hamiltonian mechanics seeks to do this, but instead of velocities as the new coordinate, we want the generalized momenta. The state of the system at any instant in time is then the set of position and generalized momenta. Essentially, we need to change coordinates from the Lagrangian construction $L(q_i, \dot{q}_i, t)$ to a new quantity $H(q_i, p_i, t)$ which obeys a new set of $2n$ first order differential equations. To this end, we introduce the Legendre transform. For a function of two variables $f(x, y)$, we have

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = udx + vdy$$

We want to change variables from (x, y) to (u, y). Introduce the new function

$$g = f - ux \implies dg = df - udx - xdu = vdy - xdu$$

Given that the differential of g is also

$$dg = \frac{\partial g}{\partial u}du + \frac{\partial g}{\partial y}dy$$

we have that

$$x = -\frac{\partial g}{\partial u}, \quad v = \frac{\partial g}{\partial y}$$

Apparently, the Legendre transforms use is that taking differentials of the new function g eliminates dependence on the variable x which we are seeking to eliminate. The differential dx is replaced with the differential of the new variable du. Thinking about the meaning of this in the context of the Lagrangian, it is tempting to think that if we know how to write the conjugate momenta in terms of the positions and velocities, we could simply plug this into the Lagrangian. The problem is that we are still taking derivatives with respect to the velocities not the momenta. The Legendre transform replaces derivatives w.r.t. the "velocity" variable x with the new "momentum" variable u. Thus, the problem is fully formulated in terms of position and momentum. Turning to the mechanics problem, we take the negative of the Legendre transform given above to write the Hamiltonian as

$$H(q_i, p_i, t) = \dot{q}_i p_i - L(q_i, \dot{q}_i, t)$$

Here, $\dot{q}_i$ plays the role of x and $\frac{\partial L}{\partial \dot{q}_i} = p_i$ plays the role of $u = \frac{\partial f}{\partial x}$. Using the relations spelled out in the beginning, the differential of the Lagrangian is

$$dL = \dot{p}_i dq_i + p_i d\dot{q}_i + \frac{\partial L}{\partial t}dt$$

This means that the differential of the Hamiltonian is

$$dH = \frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial p_i}dp_i + \frac{\partial H}{\partial t}dt = \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t}dt$$

Differentials with respect to $\dot{q}_i$ have been removed with the Legendre transform. Equating the two forms of the differential, the governing equations are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad -\dot{p}_i = \frac{\partial H}{\partial q_i}, \quad -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$$

It can be shown that in many situations, the Hamiltonian is the total energy of the system. In these cases (and perhaps in others), it is possible to write the Hamiltonian explicitly in terms of the generalized position and momenta without resorting to the Lagrangian. Thus it is clear how to take the derivatives of the Hamiltonian in terms of momenta even when the Hamiltonian is constructed in terms of the Lagrangian where the momenta do not appear. We can consider an example from linear elasticity as an attempt to demonstrate this. Call the discretized displacement degrees of freedom q_i. The Lagrangian is

$$L = T - V = \frac{1}{2} M_{ij} \dot{q}_i \dot{q}_j + F_i q_i - \frac{1}{2} K_{ij} q_i q_j$$

The conjugate momenta are

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = M_{ij} \dot{q}_j$$

Using this definition, the Hamiltonian is then

$$H = \dot{q}_i p_i - \frac{1}{2} \dot{q}_i p_i - F_i q_i + \frac{1}{2} K_{ij} q_i q_j$$

We can also write $\dot{q}_i = M_{ij}^{-1} p_j$ to fully eliminate the velocity degrees of freedom. The Hamiltonian can be written fully in state space as

$$H = \frac{1}{2} M_{ij}^{-1} p_i p_j - F_i q_i + \frac{1}{2} K_{ij} q_i q_j$$

The only explicit time dependence of the problem shows up in the force vector $F_i(t)$. The third of the governing equations then says that

$$\frac{\partial H}{\partial t} = -q_i \frac{\partial F_i}{\partial t}$$

which says that the rate of change of energy of the system is the power input from the forcing. If there were no forcing, or a static forcing, this equation would tell us that energy is conserved. The other two equations for the state are

$$\dot{q}_i = M_{ij}^{-1} p_j, \quad \dot{p}_i = F_i - K_{ij} q_j$$

The second equation is a force equation (rate of change of momentum). As expected, the force is the balance between the applied load and the elastic forces in the system. One benefit of the Hamiltonian formulation is that conservation equations pop out more naturally. Any coordinate (whether position or momentum) ξ_i which does not appear in the Hamiltonian will have a corresponding governing equation

$$\frac{\partial H}{\partial \xi_i} = 0 = \dot{\xi}_i$$

This says that its value is constant in time and it is a conserved quantity. A coordinate which does not appear in the Lagrangian or Hamiltonian is thought of as a symmetry, because changing its value does not change the corresponding energy functional. Thus, symmetries in systems lead to conserved quantities, and the Hamiltonian is particularly convenient for showing this because the problem is formulated explicitly in terms of the position and momenta as independent variables.