Hertzian Contact Mechanics

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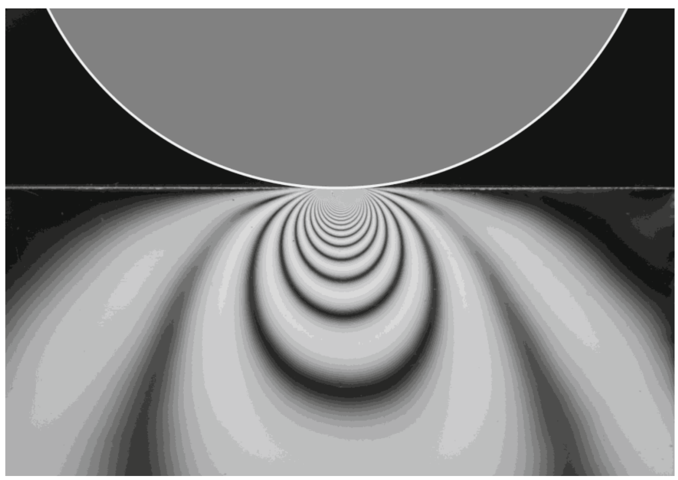

Figure 1: Stress distributions of a sphere in contact with a plane.

Contact mechanics seeks to model elastic deformations caused not by applied forces, rather by contact with other bodies. Hertz developed a theory of contact mechanics in the late 1800's that permitted some analytical solutions. These analytical solutions are mostly useful for benchmarking numerical methods at this point, but they are quite elegant. These notes present some results from Hertzian contact mechanics, and mostly make use of this book. The story of Hertzian contact mechanics begins with the "fundamental" solution of a point force applied to an infinite plane. See Figure 2. A vertical force $F\delta(x-x^*, y-y^*)$ is applied on the supper surface of an elastic body which is defined over the region $[-\infty, \infty] \times [-\infty, \infty] \times [0, \infty]$. The point of application (x^*, y^*) is arbitrary. It is possible to compute the stress and displacement response of the solid to this point force analytically. A simpler case of this is the so-called Flamant solution (outlined here) which finds the response of a semi-infinite solid to an applied line-load using polar coordinates. This is effectively a 2D problem, and the solution is actually quite simple. Note that these solution techniques often produce stresses, which need to be converted to strains and integrated to find displacements. This is a tedious process. We will cite the fundamental solution to the problem shown in Figure 2 without proof. Using the coordinate system specified in the figure, the displacement components for a concentrated force applied at $(x, y, z) = (0, 0, 0)$ are shown in Figure 3. Going forward, we are primarily interested in the displacement of the free surface at $z = 0$, and in particular the vertical component of the displacement, as this will govern the important features of the contact response. In other words, the displacement normal to the surface controls the extent to which the bodies "squish" together. The vertical displacement on the free surface is given by:

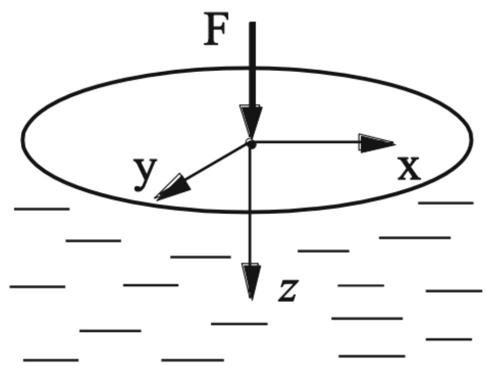

Figure 2: A point force is applied to the semi-infinite elastic solid.

$$u_x = \frac{1+\nu}{2\pi E}\left[\frac{xz}{r^3} - \frac{(1-2\nu)x}{r(r+z)}\right]F_z,$$

$$u_y = \frac{1+\nu}{2\pi E}\left[\frac{yz}{r^3} - \frac{(1-2\nu)y}{r(r+z)}\right]F_z,$$

$$u_z = \frac{1+\nu}{2\pi E}\left[\frac{2(1-\nu)}{r} + \frac{z^2}{r^3}\right]F_z,$$

Figure 3: Displacement components for semi-infinite elastic solid under the action of concentrated force of magnitude F_z applied at the origin.

$$u_z = \frac{1-v^2}{\pi E}\frac{F_z}{\sqrt{x^2+y^2}}$$

Because the response of the solid is linearly elastic, we can compute the displacement from many forces by summing up their individual contributions. In fact, we can use the fundamental solution to find the vertical displacement of the free surface from a pressure distribution $p(x, y)$ through superposition:

$$u_z(x,y) = \frac{1-v^2}{\pi E}\int\int \frac{p(x',y')}{\sqrt{(x-x')^2+(y-y')^2}}dx'dy'$$

This equation states: to find the displacement at position (x, y), sum up the displacement contribution from all the forces $p(x', y')dx'dy'$ scaled by their distance from that point. The bounds of integration are defined by the region over which the pressure is non-zero. This assumes that the pressure distribution is known. This pressure distribution will be used to model the interaction between two bodies in contact. There is no reason to think that we know how to characterize this interaction. If we knew how to do that a priori, there would be no need for contact mechanics! In classic "old school" mechanics form, the approach is to guess a solution and then confirm that the guess was good. Specifically, we will guess pressure distributions and show that their corresponding displacement fields correspond to contact interactions with clear physical interpretations. There are many such cases, but we will show a particularly simple one: a rigid sphere contacting an elastic plane. Consider the following pressure distribution:

$$p(x,y) = p_0\sqrt{1-\left(\frac{x^2+y^2}{a^2}\right)}, \quad \sqrt{x^2+y^2} \le a$$

This is a pressure distribution defined over a circular contact area that is axisymmetric and decreases monotonically as we approach the boundary. Note that the integral which must be computed for the displacement field is very complex. Apparently, this can be accomplished analytically, though in practice it would be reasonable to do numerically. One could do this to verify the analytical expression if desired. Defining $r = \sqrt{x^2+y^2}$, we cite the result of this integration without proof as

$$u_z(r) = \frac{\pi p_0(1-v^2)}{4Ea}(2a^2-r^2), \quad r \le a$$

So the displacement from this contact pressure distribution only depends on the radius and is a paraboloid inside the contact region. Note that this pressure distribution could be used for points outside the contact region $r \le 0$ to compute the displacement. It will have a different functional form than the displacement inside the contact region, but they will be continuous at their interface. Remember that this pressure distribution is essentially pulled out of thin air, and we are studying the corresponding displacement field to see if it has a physical interpretation. The contact region is defined as the region for which the pressure is nonzero. Consider the problem of a rigid sphere indenting an elastic plane shown in Figure 4. As is explained in the figure, we observe that the displacement in contact region is approximately quadratic. This can be visualized with this plot. This means that the pressure distribution we assumed which gave rise to a quadratic displacement behaves like a rigid sphere indenting an elastic plane! The displacement from the spherical indenter is purely geometric, whereas the displacement field computed with the fundamental solution and ansatz pressure distribution involves forces and material parameters. The two expressions can be equated:

$$d - \frac{r^2}{2R} = \frac{\pi p_0 (1-v^2)}{4Ea}(2a^2 - r^2)$$

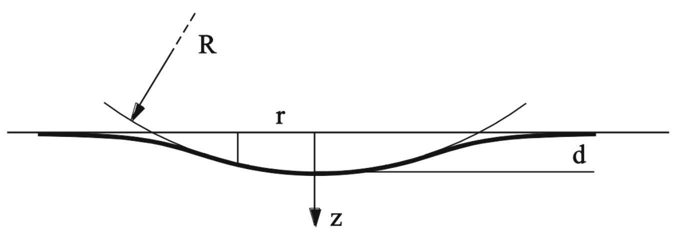

Figure 4: A rigid sphere of radius R is pressed into a deformable plane to a maximum depth of d. Given the z is defined downward, the displacement field can be written for the contact region as $R^2 = x^2 + (z + R - d)^2$, which is approximately $z = d - \frac{x^2}{2R}$. The 3D case is obtained by making the substitution $x \to r$.

Equating the terms multiplying r^2 and r^0 respectively, we find that

$$a = \frac{R(1-v^2)\pi p_0}{2E}, \quad d = \frac{(1-v^2)\pi p_0 a}{2E} = \frac{a^2}{R}$$

The total force applied to the indenter to obtain this contact interaction can be obtained from integrating the assumed pressure distribution:

$$F = \int_0^a p(r) 2\pi r dt = \frac{2}{3} p_0 \pi a^2$$

We can solve for p_0 in terms of other parameters using the above relations:

$$p_0 = \frac{d}{a}\frac{2E}{\pi(1-v^2)} = \sqrt{\frac{d}{R}}\frac{2E}{\pi(1-v^2)}$$

Plugging in for p_0 and $a^2 = Rd$ into the force relation, we obtain

$$F = \frac{4}{3}\frac{E}{1-v^2}R^{1/2}d^{3/2}$$

This is interesting because the force-displacement relationship is nonlinear, even though we are working with linear elasticity. It makes sense that it would require increasing increments of force to obtain the same increment of displacement as the contact area grows and more material is resisting deformation. It is possible to solve more complex contact problems with these methods. They can be extended to model contact problems in which both bodies are elastic and have curvature by defining a fictitious rigid body with an "effective" geometry and an elastic plane with an "effective" modulus. These effective quantities are used to capture the total curvature and total elasticity of the system while pulling back the contact problem to the one we have outlined here.