# Solving Incompressible Navier-Stokes with Chorin's Method

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The incompressible Navier-Stokes equations are

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \nu \frac{\partial u_i}{\partial x_j \partial x_j} - \frac{1}{\rho}\frac{\partial p}{\partial x_i}, \quad \frac{\partial u_i}{\partial x_i} = 0$$

where $u_i$ is the component of velocity in direction $x_i$ and $p$ is the pressure. The first equation expresses the balance of linear momentum, and the second states that the fluid does not change volume as it flows (hence "incompressible"). It is not obvious how to solve these equations with traditional methods such as finite difference or finite elements because the incompressibility constraint needs to be enforced at each point in time. Chorin's method is a popular technique for enforcing the incompressibility constraint. This method is an example of an "operator splitting" approach. We will conceptualize the pressure as acting to enforce incompressibility. Chorin's approach is to first compute a velocity field which does not satisfy incompressibility by ignoring the pressure term in the momentum equation. This velocity field is called the "intermediate" velocity, and can be computed explicitly with a time-stepping scheme. Subsequently, the pressure field is computed in such a way as to restore incompressibility, and is used to correct the intermediate velocity. To see how this works, we discretize the momentum equation in time with a forward difference scheme, and neglect the pressure term

$$\frac{u_i^* - u_i^t}{\Delta t} = -u_j^t \frac{\partial u_i^t}{\partial x_j} + \nu \frac{\partial^2 u_i^t}{\partial x_j \partial x_j}$$

$$\implies u_i^* = \Delta t\Big( - u_j^t \frac{\partial u_i^t}{\partial x_j} + \nu \frac{\partial^2 u_i^t}{\partial x_j \partial x_j}\Big) + u_i^t$$

The superscript "$t$" indicates the solution at the current time, and "$*$" denotes the intermediate velocity. We define the kinematic viscosity $\nu = \mu/\rho$ for convenience. At this point, we do not need to consider discretizing the spatial part of the velocity field. Now, the pressure term is reintroduced with

$$\frac{u_i^{t+1} - u_i^*}{\Delta t} = -\frac{1}{\rho}\frac{\partial p^t}{\partial x_i}$$

This is another forward difference approximation of the time derivative, but this time using the intermediate velocity. It is clear why Chorin's method is an operator split, as the time derivative is divided into one contribution from the velocity-dependent terms, and another contribution from the pressure. This division is the essence of Chorin's method. At face value, it is not clear why this is a fruitful approach, or why it accurately reflects the underlying equations. A satisfying answer to the latter concern is beyond the scope of this report. In response to the former, we can rearrange this equation and write

$$u_i^{t+1} = -\frac{\Delta t}{\rho}\frac{\partial p^t}{\partial x_i} + u_i^*$$

This demonstrates that knowing the pressure allows us to update the velocity field in time. At this point, we have the intermediate velocity but need the updated velocity to be divergence free, which the intermediate velocity is not. The pressure is unknown. Thus, the pressure can be used to correct the intermediate velocity in order to be divergence free. To do this, we can take the divergence of this equation and require that $\nabla \cdot u^{t+1} = 0$. This gives the following governing equation for the pressure

$$\frac{\partial^2 p^t}{\partial x_i \partial x_i} = \frac{\rho}{\Delta t}\frac{\partial u_i^*}{\partial x_i}$$

In order for the velocity at the next time step to be divergence free with the Chorin operator split, the pressure must obey a Poisson equation driven by the failure of the intermediate velocity to satisfy incompressibility. Having discretized in time and used the operator splitting approach, we now have a method to update the velocity field in time. In order to fully specify a numerical solution strategy, we must discuss spatial discretization schemes.

The simplest approach to spatial discretization is to use finite differences for the intermediate velocity, and to solve the Poisson equation for pressure weakly. The problem will be solved in two spatial dimensions. If we store the velocity components on a uniform spatial grid, we can approximate derivatives with

$$\frac{\partial u_i}{\partial x_1}(x_{1n}, x_{2n}) \approx \frac{1}{2\Delta x}\Big(u_i(x_{1n}+\Delta x, x_{2n}) - u_i(x_{1n}-\Delta x, x_{2n})\Big)$$

$$\frac{\partial u_i}{\partial x_2}(x_{1n}, x_{2n}) \approx \frac{1}{2\Delta x}\Big(u_i(x_{1n}, x_{2n}+\Delta x) - u_i(x_{1n}, x_{2n}-\Delta x)\Big)$$

$$\frac{\partial^2 u_i}{\partial x_1^2}(x_{1n}, x_{2n}) \approx \frac{1}{\Delta x^2}\Big(u_i(x_{1n}+\Delta x, x_{2n}) - 2u_i(x_{1n}, x_{2n}) + u_i(x_{1n}-\Delta x, x_{2n})\Big)$$

$$\frac{\partial^2 u_i}{\partial x_2^2}(x_{1n}, x_{2n}) \approx \frac{1}{\Delta x^2}\Big(u_i(x_{1n}, x_{2n}+\Delta x) - 2u_i(x_{1n}, x_{2n}) + u_i(x_{1n}, x_{2n}-\Delta x)\Big)$$

These are all the derivatives that are needed for the finite difference scheme. Noting that this approximation for spatial derivatives is employed, we can write the equation for the intermediate velocity at each grid point as

$$u_i^*(x_{1n}, x_{2n}) = \Delta t \sum_{j=1}^{2} \Big( -u_j^t(x_{1n}, x_{2n}) \frac{\partial u_i^t(x_{1n}, x_{2n})}{\partial x_j} + \nu \frac{\partial^2 u_i^t(x_{1n}, x_{2n})}{\partial x_j^2} \Big) + u_i^t(x_{1n}, x_{2n})$$

Note that the problem must have Dirichlet boundary conditions in order to make use of finite differencing. The grid on which finite differences are computed is "padded" by nodes with specified velocities. Thus, derivatives are only computed on interior points, and the intermediate velocity is given by the velocity boundary conditions at the next time point. On the other hand, derivatives are computed with current values of velocity at the boundary. We will not use velocity boundary conditions that represent an inflow of fluid for simplicity. With the intermediate velocity computed on a grid, we can use this in the Poisson equation for the pressure. This reads

$$\frac{\partial^2 p^t}{\partial x_i \partial x_i} = \frac{\rho}{\Delta t} \frac{\partial u_i^*}{\partial x_i}$$

This is a time-independent Poisson problem that does not lend itself to a solution via finite differencing. Thus, we discretize this problem with a spatial basis functions and solve it weakly. Begin by integrating against a scalar test function $q$ over the domain

$$\int \frac{\partial^2 p^t}{\partial x_i \partial x_i} q dA = \int \frac{\rho}{\Delta t} \frac{\partial u_i^*}{\partial x_i} q dA$$

We will use integration by parts to transfer a derivative from the pressure to the test function. This introduces a boundary term, which forces us to consider boundary conditions on the pressure field. Given that the intermediate velocity respects the Dirichlet boundaries, and the intermediate velocity is corrected with the gradient of the pressure, we can conclude that the pressure gradient is zero on the boundary. Thus the boundary term drops out and we have

$$\int \frac{\partial p^t}{\partial x_i} \frac{\partial q}{\partial x_i} dA = -\frac{\rho}{\Delta t} \int \frac{\partial u_i^*}{\partial x_i} q dA$$

On the spatial domain $[0, L] \times [0, L]$, the pressure and test function can be discretized with typical piecewise linear finite element basis functions. The finite element approximation of the test function can be written as

$$q = \sum_n q_n g_n(x_1, x_2)$$

Plugging the test function into the governing equation and noting that the degrees of freedom $q_n$ are arbitrary, we obtain

$$\int \frac{\partial p^t}{\partial x_i}\frac{\partial g_n}{\partial x_i}dA = -\frac{\rho}{\Delta t}\int \frac{\partial u_i^*}{\partial x_i}g_n dA$$

Now discretize the pressure field in the same way as the test function and plug into the governing equation to obtain

$$\sum_m p_m \int \Big(\frac{\partial g_m}{\partial x_1}\frac{\partial g_n}{\partial x_1} + \frac{\partial g_m}{\partial x_2}\frac{\partial g_n}{\partial x_2}\Big)dA = -\frac{\rho}{\Delta t}\int \Big(\frac{\partial u_1^*}{\partial x_1} + \frac{\partial u_2^*}{\partial x_2}\Big)g_n dA$$

This is a linear system for the pressure degrees of freedom

$$p_m = K_{mn}^{-1}F_n$$

$$K_{mn} := \int \Big(\frac{\partial g_m}{\partial x_1}\frac{\partial g_n}{\partial x_1} + \frac{\partial g_m}{\partial x_2}\frac{\partial g_n}{\partial x_2}\Big)dA$$

$$F_n := -\frac{\rho}{\Delta t}\int \Big(\frac{\partial u_1^*}{\partial x_1} + \frac{\partial u_2^*}{\partial x_2}\Big)g_n dA$$

Because we have computed the intermediate velocity at discrete grid points, it must be interpolated in order to carry out the integral required to form $F_n$. This method allows us to solve the pressure Poisson equation in terms the intermediate velocity. The final step is to use the pressure to correct the intermediate velocity:

$$u_i^{t+1} = -\frac{\Delta t}{\rho}\frac{\partial p^t}{\partial x_i} + u_i^*$$

We take the weak pressure solution and evaluate it on the spatial grid then finite difference to compute its gradient. We have now fully outlined a numerical solution to the 2D incompressible Navier-Stokes with Dirichlet boundary conditions using the Chorin splitting method.