KdV Equation

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1 Introduction

The Korteweg-de Vries equation is a third-order nonlinear PDE that can be used to model waves of finite amplitude in shallow water. But it is also interesting for its mathematical properties. For example, it admits a stable traveling wave solution called a "soliton," even though the underlying dynamics are nonlinear. The stability of the soliton is the result of a balance between dispersive and nonlinear effects. The KdV equation is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + k\frac{\partial^3 u}{\partial x^3} = 0$$

We derive a numerical solution to this equation for some initial velocity profile $u(x,0) = u_0(x)$ and zero velocity boundary conditions $u(0,t) = u(L,t) = 0$.

2 Discretization and Weak Form

Since the spatial part of the velocity is one-dimensional, it can easily be discretized with a global basis. The individual basis functions in the set will satisfy the zero displacement boundary conditions by construction. The velocity is discretized as

$$u(x,t) = \sum_i a_i(t) f_i(x)$$

Next, we weaken the spatial part of the governing equation. Integrating against an arbitrary test function w, the weak form is

$$\int_0^L \left(\frac{\partial u}{\partial t}w + u\frac{\partial u}{\partial x}w + k\frac{\partial^3 u}{\partial x^3}w\right)dx = 0$$

The test function is discretized with the same spatial shape functions as the displacement. This reads $w = \sum_i w_i f_i(x)$. Plugging in the discretization of the displacement and test function, and using that the coefficients w_i are arbitrary, we obtain a system of ordinary differential equations:

$$\int_0^L \left[\sum_i \frac{\partial a_i}{\partial t} f_i f_j + \left(\sum_i a_i f_i\right)\left(\sum_k a_k \frac{\partial f_k}{\partial x}\right) f_j + k \sum_i a_i \frac{\partial^3 f_i}{\partial x^3} f_j\right] dx = 0$$

$$= \sum_i \frac{\partial a_i}{\partial t} \int_0^L f_i f_j dx + \sum_i \sum_k a_i a_k \int_0^L f_i f_j \frac{\partial f_k}{\partial x} dx + \sum_i a_i \int k \frac{\partial^3 f_i}{\partial x^3} f_j dx$$

The integral expressions are seen to be matrix quantities. Renaming indices, and using the summation convention, this first order system of ODE's is

$$A_{ij} \frac{\partial a_j}{\partial t} + B_{ikj} a_k a_j + C_{ij} a_j = 0$$

This is the weak form of KdV equations. The displacement boundary conditions are satisfied automatically, and the initial condition becomes

$$a_i^0 = a_i(t=0) = \int_0^L u_0(x) f_i(x) dx$$

3 Time Integration

To solve the weak form of the governing equations, we need a numerical technique for ODE's. Let's use the backward Euler time stepping method to experiment with implicit time integration. Consider the following differential equation:

$$\frac{\partial y}{\partial t} = f(y(t))$$

A forward Euler scheme discretizes the time derivative with

$$\frac{y(t+1) - y(t)}{\Delta t} = f(y(t))$$

whereas backward Euler evaluates the right-hand side at the next time step:

$$\frac{y(t+1) - y(t)}{\Delta t} = f(y(t+1))$$

For the KdV equation equation, the logic is the same but the application a bit messier. Denote the i-th displacement coefficient at time t as a_i^t. Inverting the matrix A off of time derivatives of the displacement coefficients and using the backward Euler scheme, the governing equation reads

$$\frac{a_i^{t+1} - a_i^t}{\Delta t} + A_{ij}^{-1} B_{jk\ell} a_k^{t+1} a_\ell^{t+1} + A_{ij}^{-1} C_{jk} a_k^{t+1} = 0$$

This is a non-linear system of equations for $\underline{a}^{t+1}$ given $\underline{a}^t$. At every time step, the future state is computed through solving this nonlinear system. This shows how the implicit time integration greatly increases the cost of solving this problem. We can use a Newton-Raphson method to solve this system. The residual equation is

$$R_i = a_i^{t+1} + \Delta t A_{ij}^{-1}\big(B_{jk\ell} a_k^{t+1} a_\ell^{t+1} + C_{jk} a_k^{t+1}\big) - a_i^t = 0$$

The Jacboain matrix of this sytem is required for the Newton solve. This reads

$$\frac{\partial R_i}{\partial a_m^{t+1}} = \delta_{im} + \Delta t A_{ij}^{-1}(B_{jm\ell} a_\ell^{t+1} + B_{jkm} a_k^{t+1})$$

Denoting iterations in the Newton solve with the index n, we have that

$$\Delta \underline{a}^{t+1,n} = -\left(\frac{\partial \underline{R}}{\partial \underline{a}^{t+1,n}}\right)^{-1} \underline{R}(\underline{a}^{t+1,n})$$

$$\underline{a}^{t+1,n+1} = \underline{a}^{t+1,n} + \Delta \underline{a}^{t+1,n}$$

4 Results

Don't have time.