Weird Physics

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1 Rotating Beam

1.1 Governing equation

We want to model the dynamics of an Euler-Bernoulli beam that is undergoing time varying rotation about a fixed axis. The axis will be perpendicular to the span of the beam such that beam traces out a plane as it rotates. The rotation axis will be defined as x_3, the length of the beam as x_1, and x_2 will be in the direction of counterclockwise rotation w.r.t. positive x_1 as seen looking down from the x_3 axis. Because the unit vectors $\hat{e}_1$ and $\hat{e}_2$ rotate with the beam's angular velocity vector $\Omega_3(t)\hat{e}_3$, there will be fictitious forces in the equation of motion. These arise from time derivatives of the unit vectors. In order to compute the equation of motion in the rotating frame, we take derivatives of the position vector in the rotating frame. The time derivative of the position in a frame rotating with beam is

$$\frac{d}{dt}\underline{r} = \frac{\partial}{\partial t}(r_i\hat{e}_i) = \frac{\partial r_i}{\partial t}\hat{e}_i + r_i\frac{\partial \hat{e}_i}{\partial t}$$

$$\frac{d^2}{dt^2}\underline{r} = \frac{\partial}{\partial t}\left(\frac{\partial r_i}{\partial t}\hat{e}_i + r_i\frac{\partial \hat{e}_i}{\partial t}\right) = \frac{\partial^2 r_i}{\partial t^2}\hat{e}_i + \frac{\partial r_i}{\partial t}\frac{\partial \hat{e}_i}{\partial t} + \frac{\partial r_i}{\partial t}\frac{\partial \hat{e}_i}{\partial t} + r_i\frac{\partial^2 \hat{e}_i}{\partial t^2}$$

Note that Newton's law operates on the position vector not the displacement. The relation between the two in solid mechanics is

$$\underline{r}(t) = \underline{x} + \underline{u}(t)$$

In inertial frames, there is no distinction made from the standpoint of accelerations between the position and the displacement because the spatial position $\underline{x}$ is independent of time. But for rotating frames, we will see that it is important to distinguish between position and displacement, as the fictitious forces do not depend only on derivatives of $\underline{r}$. Thus, the spatial coordinate $\underline{x}$ makes an explicit appearance in the equations of motion. Using the relation between position and displacement, along with the definition of the time derivative of rotating unit vectors, we have that

$$\frac{d^2}{dt^2}\underline{r} = \underline{\ddot{u}} + 2\underline{\Omega} \times \underline{\dot{u}} + \frac{\partial \underline{\Omega}}{\partial t} \times (\underline{x} + \underline{u}) + \underline{\Omega} \times (\underline{\Omega} \times (\underline{x} + \underline{u}))$$

The governing equations for elasticity then involve multiple fictitious forces from the rotating coordinate system:

$$\rho\underline{\ddot{u}} = \nabla \cdot \underline{\underline{\sigma}} + \underline{b} - 2\rho(\underline{\Omega} \times \underline{\dot{u}}) - \rho\Big(\underline{\dot{\Omega}} \times (\underline{x} + \underline{u})\Big) - \rho\underline{\Omega} \times (\underline{\Omega} \times (\underline{x} + \underline{u}))$$

1.2 Kinematic model of 3D beam

The displacement field for the beam is

$$\begin{aligned} u_1 &= \bar{u}_1(x_1) + x_3\Phi_2(x_1) - x_2\Phi_3(x_1) \\ u_2 &= \bar{u}_2(x_1) \\ u_3 &= \bar{u}_3(x_1) \end{aligned}$$

The axial displacement is arises from uniform tension/compression and rotations of the cross sections. These cross-sectional rotations are related to the bending displacements $\bar{u}_2$ and $\bar{u}_3$. Note that Φ_2 corresponds to a positive rotation (by the right-hand rule) around the x_2 axis, and Φ_3 corresponds to a positive rotation around the x_3 axis. In this axis system, positive rotations around x_2 correspond to positive displacements in x_1 (tension) when $x_3 > 0$, and similarly, positive values of x_2 and positive Φ_3 correspond to compression. The only non-zero strain component from the stress field for Euler-Bernoulli beam bending is

$$\epsilon_{11} = \frac{\partial \bar{u}_1}{\partial x_1} + x_3k_2 - x_2k_3$$

$$\sigma_{11} = E\bigg(\frac{\partial \bar{u}_1}{\partial x_1} + x_3k_2 - x_2k_3\bigg)$$

$$\Phi_2 = \frac{\partial \bar{u}_3}{\partial x_1}, \quad \Phi_3 = \frac{\partial \bar{u}_2}{\partial x_1}, \quad k_2 := \frac{\partial \Phi_2}{\partial x_1}, \quad k_3 := \frac{\partial \Phi_3}{\partial x_1}$$

1.3 Variational Formulation

Treating the fictitious forces from rotations of body forces, we can write the total potential energy of the dynamics problem:

$$\begin{aligned} \Pi = \int_V \frac{1}{2}\rho\dot{u}_i\dot{u}_i - \frac{1}{2}\sigma_{ij}\epsilon_{ij} + b_iu_i - 2\rho e_{ijk}\Omega_k\dot{u}_ku_i - \rho e_{ijk}\dot{\Omega}_jx_ku_i \\ - \rho e_{ijk}\dot{\Omega}_ju_ku_i - \rho e_{ijk}\Omega_je_{k\ell m}\Omega_\ell x_mu_i - \rho e_{ijk}\Omega_je_{k\ell m}\Omega_\ell u_mu_idV \end{aligned}$$

1.4 Kinetic Energy

We can simplify each term one-by-one. Start with the kinetic energy:

$$\int_V \frac{1}{2}\rho \dot{u}_i \dot{u}_i dV = \int_V \frac{\rho}{2}\Big((\dot{\bar{u}}_1 + x_3\dot{\Phi}_2 - x_2\dot{\Phi}_3)^2 + \dot{\bar{u}}_2^2 + \dot{\hat{u}}_3^2\Big)dV$$

$$= \int_0^L \int_A \frac{\rho}{2}\Big(\dot{\bar{u}}_1^2 + x_3^2\dot{\Phi}_2^2 + x_2^2\dot{\Phi}_3^2 + 2\dot{\bar{u}}_1 x_3 \Phi_2 - 2\dot{\bar{u}} x_2 \dot{\Phi}_3 - 2x_2 x_3 \dot{\Phi}_2 \dot{\Phi}_3 + \dot{\bar{u}}_2^2 + \dot{\bar{u}}_3^2\Big)dA dx_1$$

If the $x_2 - x_3$ axes are centered at the centroid of the section and are aligned with principal axes, the area integral zeroes three of the terms from the u_1 kinetic energy. Evaluating the area integral, we obtain

$$\int_0^L \frac{\rho}{2}\Big(A\dot{\bar{u}}_1^2 + I_3\dot{\Phi}_2^2 + I_2\dot{\Phi}_3^2 + \dot{\bar{u}}_2^2 + \dot{\bar{u}}_3^2\Big)dx_1$$

1.5 Strain Energy

Now we can look at the strain energy term. There is only one non-zero stress and strain component, so we have

$$\int_V \frac{1}{2}\sigma_{ij}\epsilon_{ij} dV = \int_V \frac{E}{2}(\bar{\epsilon}_1 + x_3 k_2 - x_2 k_3)^2 dV$$

$$= \int_0^L \int_A \frac{E}{2}(\bar{\epsilon}_1^2 + x_3^2 k_2^2 + x_2^2 k_3^2 + 2\bar{\epsilon}_1 x_3 k_2 - 2\bar{\epsilon}_1 x_2 k_3 - 2x_3 x_2 k_2 k_3) dA dx_1$$

Similar to above, evaluating the area integral cancels many of the terms. We obtain

$$\int_0^L \frac{E}{2}\Big(A\bar{\epsilon}_1^2 + I_3 k_2^2 + I_2 k_3^2\Big)dx_1$$

1.6 Fictitious Forces

We will assume there are no body forces, so this term in the energy is zero. We can turn to the first fictitious force of rotation term:

$$-2\int_V \rho e_{ijk}\Omega_j \dot{u}_k u_i dV$$

The angular velocity $\Omega_k = \Omega_3(t)$ has a fixed orientation but varies in time. Thus, using the properties of the permutation tensor, we can write

$$-2\rho \int_V e_{132}\Omega_3 \dot{u}_2 u_1 + e_{231}\Omega_3 \dot{u}_1 u_2 dV = 2\rho\Omega_3 \int_V -\dot{u}_1 u_2 + \dot{u}_2 u_1 dV$$

$$= 2\rho\Omega_3 \int_0^L \int_A \dot{\bar{u}}_2(\bar{u}_1 + x_3\Phi_2 - x_2\Phi_3) - (\dot{\bar{u}}_1 + x_3\dot{\Phi}_2 - x_2\dot{\Phi}_3)\bar{u}_2 dAdx_1$$

Evaluating the area integral once again zeros some of these terms. We are left with

$$= 2\rho A\Omega_3 \int_0^L \Big(\dot{\bar{u}}_2\bar{u}_1 - \dot{\bar{u}}_1\bar{u}_2\Big)dx_1$$

We can move on to the next rotational term from the energy above:

$$-\int_V \rho e_{ijk}\dot{\Omega}_j u_k u_i dV = -\int_V \rho e_{i3k}\dot{\Omega}_3 u_k u_i dV = \int_V -\rho\dot{\Omega}_3(e_{132}u_2u_1+e_{231}u_1u_2)dV = 0$$

The term involving the time derivative of the angular velocity vector drop outs. This is probably because the orientation of the angular velocity does not change. The next fictitious force term is

$$-\int \rho e_{ijk}\dot{\Omega}_j x_k u_i dV = -\rho\dot{\Omega}_3 \int_0^L \int_A e_{132}x_2u_1 + e_{231}x_1u_2 dAdx_1$$

$$= \rho A\dot{\Omega}_3 \int_0^L x_1\bar{u}_2 dx_1$$

The first of centripetal force terms is

$$-\int_V \rho e_{ijk}\Omega_j e_{k\ell m}\Omega_\ell x_m u_i dV = -\rho \int_V e_{i3k}\Omega_3 e_{k3m}\Omega_3 x_m u_i dV$$

$$= -\rho\Omega_3^2 \int_V (e_{132}e_{231}x_1u_1 + e_{231}e_{132}x_2u_2)dV = \rho\Omega_3^2 \int_V x_1u_1 + x_2u_2 dV$$

$$= \rho A\Omega_3^2 \int_0^L x_1\bar{u}_1 dx_1$$

And the second centripetal force term is

$$-\int_V \rho e_{ijk}\Omega_j e_{k\ell m}\Omega_\ell u_m u_i dV = -\rho \int_V e_{i3k}\Omega_3 e_{k3m}\Omega_3 u_m u_i dV$$

$$= \rho\Omega_3^2 \int_0^L \int_A (\bar{u}_1 + x_3\Phi_2 - x_2\Phi_3)^2 + \bar{u}_2^2 dAdx_1$$

$$\rho\Omega_3^2 \int_0^L A\bar{u}_2^2 + A\bar{u}_1^2 + I_3\Phi_2^2 + I_2\Phi_3^2 dx_1$$

1.7 Putting it all together

The total potential energy written using the kinematic assumptions of 3D beam theory is

$$\Pi = \int_0^L \frac{\rho}{2}\Big(A\dot{\bar{u}}_1^2 + I_3\dot{\Phi}_2^2 + I_2\dot{\Phi}_3^2 + \dot{\bar{u}}_2^2 + \dot{\bar{u}}_3^2\Big)dx_1 - \int_0^L \frac{E}{2}\Big(A\bar{\epsilon}_1^2 + I_3k_2^2 + I_2k_3^2\Big)dx_1$$
$$2\rho A\Omega_3 \int_0^L \Big(\dot{\bar{u}}_2\bar{u}_1 - \dot{\bar{u}}_1\bar{u}_2\Big)dx_1 + \rho A\dot{\Omega}_3 \int_0^L x_1\bar{u}_2 dx_1 +$$
$$\rho A\Omega_3^2 \int_0^L x_1\bar{u}_1 dx_1 + \rho\Omega_3^2 \int_0^L A\bar{u}_2^2 + A\bar{u}_1^2 + I_3\Phi_2^2 + I_2\Phi_3^2 dx_1$$

Using the definitions of the rotation angles and curvatures, the energy can be written explicitly in terms of the displacement field. Bars over the displacement components will be dropped for readability.

$$\Pi = \int_0^L \frac{\rho}{2}\Big(A\Big(\frac{\partial u_1}{\partial t}\Big)^2 + I_3\Big(\frac{\partial^2 u_3}{\partial x_1\partial t}\Big)^2 + I_2\Big(\frac{\partial^2 u_2}{\partial x_1\partial t}\Big)^2 + A\Big(\frac{\partial u_2}{\partial t}\Big)^2 + A\Big(\frac{\partial u_3}{\partial t}\Big)^2\Big)dx_1$$
$$-\int_0^L \frac{E}{2}\Big(A\Big(\frac{\partial u_1}{\partial x_1}\Big)^2 + I_3\Big(\frac{\partial^2 u_3}{\partial x_1^2}\Big)^2 + I_2\Big(\frac{\partial^2 u_2}{\partial x_1^2}\Big)^2\Big)dx_1 + 2\rho A\Omega_3 \int_0^L \Big(\frac{\partial u_2}{\partial t}u_1 - u_2\frac{\partial u_1}{\partial t}\Big)dx_1$$
$$\rho A\dot{\Omega}_3 \int_0^L x_1u_2dx_1 + \rho A\Omega_3^2 \int_0^L x_1u_1dx_1 + \rho\Omega_3^2 \int_0^L Au_2^2 + Au_1^2 + I_3\Big(\frac{\partial u_3}{\partial x_1}\Big)^2 + I_2\Big(\frac{\partial u_2}{\partial x_1}\Big)^2 dx_1$$

We will assume that the problem is driven by the time varying angular velocity $\Omega_3(t)$. We have assumed that there are no tractions or body forces by not including these as external forces in the energy. Notice that the x_3 problem is totally decoupled from the x_1 and x_2 problems. This is because fictitious forces are perpendicular to the rotation axis, and we expect a decoupling of the coordinate directions for linear elasticity (this is the case in usual 3D bending analysis). There is a new centripetal force term in the x_3 problem, though it seems that it will be small compared to elastic forces (rotation speed squared almost certainly small compared to the modulus). Probably allowing the orientation of the angular velocity to vary in time would have fully coupled all of the directions. It is perhaps interesting to note that the x_1 and x_2 problems are coupled as a result of the rotation, which does not happen in an inertial frame. Let's look at the $x_1 - x_2$ problem by ignoring terms involving u_3:

$$\Pi = \int_0^L \frac{\rho}{2}\Big(A\Big(\frac{\partial u_1}{\partial t}\Big)^2 + I_2\Big(\frac{\partial^2 u_2}{\partial x_1 \partial t}\Big)^2 + A\Big(\frac{\partial u_2}{\partial t}\Big)^2\Big)dx_1$$
$$-\int_0^L \frac{E}{2}\Big(A\Big(\frac{\partial u_1}{\partial x_1}\Big)^2 + I_2\Big(\frac{\partial^2 u_2}{\partial x_1^2}\Big)^2\Big)dx_1 + 2\rho A\Omega_3 \int_0^L \Big(\frac{\partial u_2}{\partial t}u_1 - u_2\frac{\partial u_1}{\partial t}\Big)dx_1$$
$$+\rho A\dot{\Omega}_3 \int_0^L x_1 u_2 dx_1 + \rho A\Omega_3^2 \int_0^L x_1 u_1 dx_1 + \rho\Omega_3^2 \int_0^L A u_2^2 + A u_1^2 + I_2\Big(\frac{\partial u_2}{\partial x_1}\Big)^2 dx_1$$

Discretize the two displacements with

$$u_1 = \sum_i a_i f_i(x_1), \quad u_2 = \sum_i b_i f_i(x_1)$$

$$\Pi = \int_0^L \frac{\rho}{2}\Big(A\sum_i\sum_j \dot{a}_i\dot{a}_j f_i f_j + I_2\sum_i\sum_j \dot{b}_i\dot{b}_j \frac{\partial f_i}{\partial x_1}\frac{\partial f_j}{\partial x_1} + A\sum_i\sum_j \dot{b}_i\dot{b}_j f_i f_j\Big)dx_1$$
$$-\int_0^L \frac{E}{2}\Big(A\sum_i\sum_j a_i a_j \frac{\partial f_i}{\partial x_1}\frac{\partial f_j}{\partial x_1} + I_2\sum_i\sum_j b_i b_j \frac{\partial^2 f_i}{\partial x_1^2}\frac{\partial^2 f_j}{\partial x_1^2}\Big)dx_1$$
$$+ 2\rho A\Omega_3 \int_0^L \Big(\sum_i\sum_j \dot{b}_i a_j f_i f_j - \sum_i\sum_j b_i \dot{a}_j f_i f_j\Big)dx_1$$
$$+ \rho A\dot{\Omega}_3 \int_0^L x_1 \sum_i b_i f_i dx_1 + \rho A\Omega_3^2 \int_0^L x_1 \sum_i a_i f_i dx_1$$
$$+\rho\Omega_3^2 \int_0^L \left(A\sum_i\sum_j b_i b_j f_i f_j + A\sum_i\sum_j a_i a_j f_i f_j + I_2\sum_i\sum_j b_i b_j \frac{\partial f_i}{\partial x_1}\frac{\partial f_j}{\partial x_1}\right)dx_1$$

There are four fundamental quantities to define:

$$M_{ij} := \int_0^L f_i f_j dx_1$$
$$K_{ij} := \int_0^L \frac{\partial f_i}{\partial x_1}\frac{\partial f_j}{\partial x_1} dx_1$$
$$H_{ij} := \int_0^L \frac{\partial^2 f_i}{\partial x_1^2}\frac{\partial^2 f_j}{\partial x_1^2} dx_1$$
$$F_i := \int_0^L x_1 f_i dx_1$$

With the spatial part of the energy integrated out, we can write it as

$$\Pi = \frac{\rho A}{2}\dot{a}_i\dot{a}_j M_{ij} + \frac{\rho I_2}{2}\dot{b}_i\dot{b}_j K_{ij} + \frac{\rho A}{2}\dot{b}_i\dot{b}_j M_{ij} - \frac{EA}{2}a_i a_j K_{ij} - \frac{EI_2}{2}b_i b_j H_{ij} \\ + 2\rho A\Omega_3\dot{b}_i a_j M_{ij} - 2\rho A\Omega_3 b_i\dot{a}_j M_{ij} + \rho\dot{\Omega}_3 F_i b_i + \rho A\Omega_3^2 F_i a_i + \rho\Omega_3^2 A b_i b_j M_{ij} \\ + \rho\Omega_3^2 A a_i a_j M_{ij} + \rho\Omega_3^2 I_2 b_i b_j K_{ij}$$

The governing equations for this problem can be obtained by using the Euler-Lagrange equations for the multiple degree of freedom system. When only the time variable is present (as is the case with the spatially discretized energy), these equations read

$$\frac{\partial}{\partial t}\frac{\partial\Pi}{\partial\dot{q}} - \frac{\partial\Pi}{\partial q} = 0$$

Thus, the two governing equations are

$$\rho A M_{ij}\ddot{a}_j + EAK_{ij}a_j - 2\rho A\Omega_3 M_{ij}\dot{b}_j - 2\rho A M_{ij}\frac{\partial}{\partial t}(b_j\Omega_3) - \rho A\Omega_3^2 F_i - 2\rho\Omega_3^2 A M_{ij} a_j = 0$$

$$\implies \rho M_{ij}\ddot{a}_j + (EK_{ij} - 2\rho\Omega_3^2 M_{ij})a_j - 2\rho\dot{\Omega}_3 M_{ij} b_j - 4\rho\Omega_3 M_{ij}\dot{b}_j = \rho\Omega_3^2 F_i$$

$$\rho I_2 K_{ij}\ddot{b}_j + \rho A M_{ij}\ddot{b}_j + EI_2 H_{ij} b_j + 2\rho A M_{ij}\frac{\partial}{\partial t}(\Omega_3 a_j) + 2\rho A\Omega_3 M_{ij}\dot{a}_j \\ - \rho\dot{\Omega}_3 F_i - 2\rho\Omega_3^2 A M_{ij} b_j - 2\rho\Omega_3^2 I_2 K_{ij} b_j = 0$$

$$\implies (\rho I_2 + \rho A M_{ij})\ddot{b}_j + (EI_2 H_{ij} - 2\rho\Omega_3^2 A M_{ij} - 2\rho\Omega_3^2 I_2 K_{ij})b_j \\ + 2\rho A\dot{\Omega}_3 M_{ij} a_j + 4\rho A\Omega_3 M_{ij}\dot{a}_j = \rho\dot{\Omega}_3 F_i$$

It is interesting to note that the rotations introduce a forcing term to the u_1 and u_2 displacement problems. Centripetal forces are proportional to the position and the square of the angular velocity. These appear as a force for displacement in the x_1 direction. There is a Coriolis-type force for displacements in the x_2 direction, where a force arises from angular acceleration. The typical Coriolis forces are a function of the velocity and appear in the left side of the equation. In block matrix form, this problem can be written in terms of constant and time-dependent matrices defined from the above equations:

$$\begin{bmatrix} \underline{\underline{M}}^1 & 0 \\ 0 & \underline{\underline{M}}^2 \end{bmatrix} \begin{bmatrix} \underline{\ddot{a}} \\ \underline{\ddot{b}} \end{bmatrix} + \begin{bmatrix} 0 & \underline{\underline{M}}^3(t) \\ \underline{\underline{M}}^4(t) & 0 \end{bmatrix} \begin{bmatrix} \underline{\dot{a}} \\ \underline{\dot{b}} \end{bmatrix} + \begin{bmatrix} \underline{\underline{M}}^5 & \underline{\underline{M}}^6(t) \\ \underline{\underline{M}}^7(t) & \underline{\underline{M}}^6 \end{bmatrix} \begin{bmatrix} \underline{a} \\ \underline{b} \end{bmatrix} = \begin{bmatrix} \underline{F}^1(t) \\ \underline{F}^2(t) \end{bmatrix}$$

Define new matrices once again and finite difference the derivatives

$$\underline{\underline{A}}\left(\frac{\underline{x}(t+\Delta t)-2\underline{x}(t)+\underline{x}(t-\Delta t)}{\Delta t^2}\right)+\underline{\underline{B}}\left(\frac{\underline{x}(t+\Delta t)-\underline{x}(t-\Delta t)}{2\Delta t}\right)+\underline{\underline{C}}\underline{x}(t)=\underline{F}(t)$$

This can be written as

$$\frac{\underline{\underline{A}}}{\Delta t^2}\underline{x}(t+\Delta t)+\frac{\underline{\underline{B}}(t)}{2\Delta t}\underline{x}(t+\Delta t)=\underline{F}(t)-\underline{\underline{C}}(t)\underline{x}(t)+\frac{\underline{\underline{A}}}{\Delta t^2}\Big(2\underline{x}(t)-\underline{x}(t-\Delta t)\Big)+\frac{\underline{\underline{B}}(t)}{2\Delta t}\underline{x}(t-\Delta t)$$

$$\underline{x}(t+\Delta t)=\left(\frac{\underline{\underline{A}}}{\Delta t^2}+\frac{\underline{\underline{B}}(t)}{2\Delta t}\right)^{-1}\left(\underline{F}(t)-\underline{\underline{C}}(t)\underline{x}(t)+\frac{\underline{\underline{A}}}{\Delta t^2}\Big(2\underline{x}(t)-\underline{x}(t-\Delta t)\Big)+\frac{\underline{\underline{B}}(t)}{2\Delta t}\underline{x}(t-\Delta t)\right)$$

This is an interesting problem because the matrices are time dependent. What time step do we evaluate them at? We cannot pre-compute an inverse because the entries of these quantities change with time. Do normal mode methods work with time-dependent matrices? This is also an interesting problem because there is a damping-type term present.

2 Fractional Derivative

In this paper, fractional derivatives are used for a viscoelastic constitutive model. This inspires the idea of killing two birds with one stone in doing a non-linear elastic problem and fractional derivatives in one. We will use a potentially made-up model of a Saint-Venant constitutive relation but in the viscoelastic context. A one-dimensional problem will be solved. In this case, the Green-Lagrange strain is

$$E=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2$$

For a fixed left boundary $u(0)=0$ and an applied traction on the right boundary, the weak form of the static governing governing equation is

$$\int_0^L\frac{\partial\Psi}{\partial E}\delta E-t\delta u(L)=0$$

But the viscoelastic material response ensures that the problem is dynamic, thus we must include inertial effects. Using that the second Piola-Kirchoff stress is the derivative of the energy w.r.t. the Green-Lagrange strain, the governing equation is

$$\int_0^L\left(S(t)\delta E+\rho\frac{\partial^2 u}{\partial t^2}\delta u\right)dV-t(t)\delta u(L)=0$$

We will argue that the stress is computed from the strain via a fractional derivative in time of order α:

$$S(x,t) = D^{\alpha} E(x,t) = \frac{\partial^a}{\partial t^a} I^b(E(x,t))$$

where we have $\alpha = a - b$ and a is computed from rounding α up to the nearest integer. The expression $I^b(\cdot)$ indicates the Cauchy repeated integral b times.

$$I^b(f(t)) = \frac{1}{\Gamma(b)} \int_a^t f(\tau)(t-\tau)^{b-1} d\tau$$

Plugging this in, we obtain

$$S(x,t) = \frac{1}{\Gamma(b)} \left(\Pi_{i=1}^{a}(b-i)\right) \int_a^t E(x,\tau)(t-\tau)^{b-a-1} d\tau$$

$$S(x,t) = f(a,b) \int_a^t \left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 \right) (t-\tau)^{b-a-1} d\tau$$

Discretizing with $u(x,t) = \sum_i u_i(t) N_i(x)$, this becomes

$$S(x,t) = f(a,b) \int_a^t \left(\sum_i u_i(\tau) \frac{\partial N_i}{\partial x} + \frac{1}{2}\left(\sum_j u_j(\tau) \frac{\partial N_j}{\partial x} \right)^2 \right) (t-\tau)^{b-a-1} d\tau$$

Simplifying the first term and leaving it "undiscretized" for now, but using the discretization for the other terms, the weak form of the governing equation becomes

$$\int_0^L S(x,t) \frac{\partial E}{\partial(\frac{\partial u}{\partial x})} \frac{\partial \delta u}{\partial x} dx + \left(\rho \int_0^L N_i N_j dx \right) \ddot{u}_j = t(t) N_i(L)$$

The chain rule is used on the Green-Lagrange strain to test against linear degrees of freedom in the virtual displacement. Renaming the mass matrix and force vector in the usual way, we can plug in to simplify the internal force vector term:

$$M_{ij}\ddot{u}_j + \int_0^L f(a,b) \int_a^t \left(\sum_i u_i(\tau) \frac{\partial N_i}{\partial x} + \frac{1}{2}\left(\sum_j u_j(\tau) \frac{\partial N_j}{\partial x} \right)^2 \right) (t-\tau)^{b-a-1} d\tau$$
$$* \left(1 + \sum_\ell u_\ell \frac{\partial N_\ell}{\partial x} \right) \frac{\partial N_i}{\partial x} dx = F_i(t)$$

Note that in this derivation we have assumed that the material properties are all unity (elastic stiffness and damping parameters) as it is not clear exactly how they show up with the fractional derivative. For example, they scale with the order of the fractional derivative. The point of this derivation is to illustrate the kind of numerical problems one obtains when using fractional derivatives. These problems are "non-local" in the sense that the displacement history is used in computing the stress. This is similar to typical viscoelastic constitutive relations, where convolution integrals appear.

3 Pre-twist Torsion

Figure 1: Example of rod generated from a cross-sectional extrusion that rotates as it is extruded. This geometry is interesting because it is an odd and more complex case of St. Venant torsion. The twist couples the tension and torsion response of the rod.

We will derive the governing equations for the static response of a "twisted" rod. The cross-section has the same shape at each axial position along the rod, but this shape rotates with the length of the rod. Call x_1 the coordinate axis going down the length of the rod. Under the usual assumptions of St. Venant torsion, the displacement components are

$$u_1 = u_1(x_1) + \frac{\partial \theta}{\partial x_1} \Psi(x_1, x_2, x_3)$$

$$u_2 = -x_3 \theta_2(x_1)$$

$$u_3 = x_2 \theta(x_1)$$

The displacement field is parameterized in terms of the rotation angle $\theta(x_1)$ in the usual way. As a result of the twist in the rod, the warping function Ψ depends on the axial coordinate x_1. The exact nature of this dependence is discussed later. Furthermore, we hypothesize an explicit axial displacement $u_1(x_1)$

which is not the result of the warping. This axial displacement will arise from the geometry of the rod, and is used to model the coupled tension/torsion response. Intuitively, this could arise from the twisted rod having an“unwinding” response under the action of an applied torque. The strain components from this displacement field are

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} + \frac{\partial \theta}{\partial x_1}\frac{\partial \Psi}{\partial x_1} + \frac{\partial^2 \theta}{\partial x_1^2}\Psi$$

$$\epsilon_{13} = \epsilon_{31} = \frac{1}{2}\frac{\partial \theta}{\partial x_1}\left(\frac{\partial \Psi}{\partial x_3} + x_2\right)$$

$$\epsilon_{12} = \epsilon_{21} = \frac{1}{2}\frac{\partial \theta}{\partial x_1}\left(\frac{\partial \Psi}{\partial x_2} - x_3\right)$$

The rest of the strain components are zero. Assuming isotropic linear elastic stress-strain relation, the stress components are simply

$$\sigma_{11} = E\epsilon_{11}$$

$$\sigma_{12} = 2G\epsilon_{12}$$

$$\sigma_{13} = 2G\epsilon_{13}$$

We will use a variational approach to derive the governing equations for this problem. The energy for the static torsion problem with an applied end torque is

$$\Pi = \int_V \frac{1}{2}\sigma_{ij}\epsilon_{ij}dV - M\theta(L) = \int_V \frac{1}{2}\sigma_{11}\epsilon_{11} + \sigma_{12}\epsilon_{12} + \sigma_{13}\epsilon_{13}dV - M\theta(L)$$

$$= \int_V \frac{1}{2}E\epsilon_{11}^2 + 2G\epsilon_{12}^2 + 2G\epsilon_{13}^2 dV - M\theta(L)$$

We can expand and simplify these terms individually. First start with the energy from tension:

$$\frac{E}{2}\int_V \epsilon_{11}^2 dV = \frac{E}{2}\int_V \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial \theta}{\partial x_1}\frac{\partial \Psi}{\partial x_1} + \frac{\partial^2 \theta}{\partial x_1^2}\Psi\right)^2 dV$$

$$= \frac{E}{2}\int_V \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial \theta}{\partial x_1}\frac{\partial \Psi}{\partial x_1}\right)^2 + \left(\frac{\partial^2 \theta}{\partial x_1^2}\Psi\right)^2 + 2\frac{\partial u_1}{\partial x_1}\frac{\partial \theta}{\partial x_1}\frac{\partial \Psi}{\partial x_1}$$
$$+ 2\frac{\partial u_1}{\partial x_1}\frac{\partial^2 \theta}{\partial x_1^2}\Psi + 2\frac{\partial \theta}{\partial x_1}\frac{\partial \Psi}{\partial x_1}\frac{\partial^2 \theta}{\partial x_1^2}\Psi dV$$

We can evaluated the area integral by noting that Ψ is the only the, that has dependence on the cross-sectional position variables x_2 and x_3.

$$= \frac{E}{2}\int_0^L \left(\frac{\partial u_1}{\partial x_1}\right)^2 \left(\int dA\right) + \left(\frac{\partial \theta}{\partial x_1}\right)^2 \left(\int \left(\frac{\partial \Psi}{\partial x_1}\right)^2 dA\right) + \left(\frac{\partial^2 \theta}{\partial x_1^2}\right)^2 \left(\int \Psi^2 dA\right)$$

$$+2\frac{\partial u_1}{\partial x_1}\frac{\partial \theta}{\partial x_1}\left(\int \frac{\partial \Psi}{\partial x_1} dA\right) + 2\frac{\partial u_1}{\partial x_1}\frac{\partial^2 \theta}{\partial x_1^2}\left(\int \Psi dA\right) + 2\frac{\partial \theta}{\partial x_1}\frac{\partial^2 \theta}{\partial x_1^2}\left(\int \Psi \frac{\partial \Psi}{\partial x_1} dA\right) dx_1$$

If the cross-section is generated by extruding and rotating at a constant rate, the derivatives of the warping function with the axial coordinate will be constant, thus their integrals will be independent of x_1. Thus, we can define the following geometric quantities to simplify this expression:

$$A_1 := \int dA$$

$$A_2 := \int \left(\frac{\partial \Psi}{\partial x_1}\right)^2 dA$$

$$A_3 := \int \Psi^2 dA$$

$$A_4 := \int \frac{\partial \Psi}{\partial x_1} dA$$

$$A_5 := \int \Psi dA$$

$$A_6 := \int \Psi \frac{\partial \Psi}{\partial x_1} dA$$

The energy from tension is then

$$= \frac{E}{2}\int_0^L A_1\left(\frac{\partial u_1}{\partial x_1}\right)^2 + A_2\left(\frac{\partial \theta}{\partial x_1}\right)^2 + A_3\left(\frac{\partial^2 \theta}{\partial x_1^2}\right)^2 + 2A_4\frac{\partial u_1}{\partial x_1}\frac{\partial \theta}{\partial x_1}$$
$$+ 2A_5\frac{\partial u_1}{\partial x_1}\frac{\partial^2 \theta}{\partial x_1^2} + 2A_6\frac{\partial \theta}{\partial x_1}\frac{\partial^2 \theta}{\partial x_1^2} dx_1$$

Now we can look at the energy contribution from the shear strains. First, we have

$$2G\int_V \epsilon_{12}^2 dV = 2G\int_V \left(\frac{1}{2}\frac{\partial \theta}{\partial x_1}\left(\frac{\partial \Psi}{\partial x_2} - x_3\right)\right)^2 dV$$
$$= \frac{G}{2}\int_V \left(\frac{\partial \theta}{\partial x_1}\right)^2 \left(\left(\frac{\partial \Psi}{\partial x_2}\right)^2 + x_3^2 - 2x_3\frac{\partial \Psi}{\partial x_2}\right) dV$$

Distributing the area integral again, we have

$$= \frac{G}{2}\int_0^L A_7(x_1)\left(\frac{\partial \theta}{\partial x_1}\right)^2 dx_1$$

The coefficient A_7 depends on the axial coordinate because the x_2 derivative of the warping function will vary when the section is at different angles from its twist. Thus, this is not a geometric quantity.

$$A_7(x_1) := \int \left(\frac{\partial \Psi}{\partial x_2}\right)^2 + x_3^2 - 2x_3\frac{\partial \Psi}{\partial x_2} dA$$

Similarly for the other shear strain component, we have

$$2G\int_V \epsilon_{13}^2 dV = 2G\int_V \left(\frac{1}{2}\frac{\partial \theta}{\partial x_1}\left(\frac{\partial \Psi}{\partial x_3} + x_2\right)\right)^2 dV$$

$$= \frac{G}{2}\int_V \left(\frac{\partial \theta}{\partial x_1}\right)^2 \left(\left(\frac{\partial \Psi}{\partial x_3}\right)^2 + x_2^2 + 2x_2\frac{\partial \Psi}{\partial x_3}\right) dV$$

$$= \frac{G}{2}\int_0^L A_8(x_1)\left(\frac{\partial \theta}{\partial x_1}\right)^2 dx_1$$

The total potential energy is then

$$\Pi = -M\theta(L) + \frac{E}{2}\int_0^L A_1\left(\frac{\partial u_1}{\partial x_1}\right)^2 + A_2\left(\frac{\partial \theta}{\partial x_1}\right)^2 + A_3\left(\frac{\partial^2 \theta}{\partial x_1^2}\right)^2 + 2A_4\frac{\partial u_1}{\partial x_1}\frac{\partial \theta}{\partial x_1}$$
$$+ 2A_5\frac{\partial u_1}{\partial x_1}\frac{\partial^2 \theta}{\partial x_1^2} + 2A_6\frac{\partial \theta}{\partial x_1}\frac{\partial^2 \theta}{\partial x_1^2}dx_1 + \frac{G}{2}\int_0^L \Big(A_7(x_1) + A_8(x_1)\Big)\left(\frac{\partial \theta}{\partial x_1}\right)^2 dx_1$$

We will derive the governing equations in strong form, as a contrast to always building weak forms for the sake of numerical solutions. For an energy functional of the form

$$\Pi = \int f(u, u_x, u_{xx}, x)dx$$

the corresponding Euler-Lagrange equations provide a PDE whose solution is a minimizer. The Euler-Lagrange equations in the presence of second derivatives are

$$\frac{\partial^2}{\partial x^2}\frac{\partial f}{\partial u_{xx}} - \frac{\partial}{\partial x}\frac{\partial f}{\partial u_x} + \frac{\partial f}{\partial u} = 0$$

It can be shown that in the case of the energy functional for torsion of the twisted rod, the two governing equations are

$$A_3\frac{\partial^4 \theta}{\partial x_1^4} + A_5\frac{\partial u_1^3}{\partial x_1^3} - A_2\frac{\partial^2 \theta}{\partial x_1^2} - A_4\frac{\partial^2 u_1}{\partial x_1^2} + \frac{G}{E}\frac{\partial}{\partial x}\left((A_7(x_1) + A_8(x_1))\frac{\partial \theta}{\partial x_1}\right) = 0$$

$$A_1 \frac{\partial^2 u_1}{\partial x_1^2} + A_4 \frac{\partial^2 \theta}{\partial x_1^2} + A_5 \frac{\partial^3 \theta}{\partial x_1^3} = 0$$

The torsion and tension problems are coupled, as shown by the presence of u_1 terms in the governing equation for the rotation angle θ, and vice verse. We have not yet explored how the quantities A_i can be computed, or how exactly the warping function depends on x_1. For an elliptical cross-section with major and minor axis lengths b and a respectively, the warping function is known to be

$$\Psi = \frac{b^2 - a^2}{b^2 + a^2} x_2 x_3$$

We assume that this same warping function applies for the cross-sections of the twisted rod, except that x_2 and x_3 are aligned with local major and minor axes of the ellipse. Call the local axes x_2' and x_3'. For the twisted rod, the warping function is

$$\Psi(x_1, x_2, x_3) = \frac{b^2 - a^2}{b^2 + a^2} x_2' x_3'$$

The relation between the local (primed) position in the cross section and the global coordinate system has the form

$$\begin{bmatrix} x_2' \\ x_3' \end{bmatrix} = \underline{\underline{R}}(\alpha x_1) \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$$

We have already assumed that the twist rate of the rod's cross-section is constant. Thus, we can say that the angle at which the cross-section at position x_1 is rotated is αx_1 where α is a constant twist rate. The parameter α is geometric in that it determines how twisted the rod is. The quantity $\underline{\underline{R}}$ is a rotation matrix relating the global coordinates to the local ones in terms of the angle of rotation. Thus, the x_1 derivative of the warping function comes from the angle used to build the rotation matrix which transforms from global to local coordinates. There would be no x_1 dependence of the warping function if it weren't for the twist of the rod. Computing all of the parameters involving integrals of the warping function and its derivatives over the cross-section is a cumbersome task. We are somewhat brief in deriving the form of the warping function for the twisted rod, but this sketch should suffice to illustrate the moving parts of this problem. This problem ends up being surprisingly complex in spite of its proximity to the usual St. Venant torsion analysis.

4 Analysis of Pre-stressed Structures

When solving a solid mechanics problem, it is almost universally assumed that the reference configuration, or undeformed state, is stress-free. The idea is that only subsequent deformations of the structure generate stresses. But what if there are known to be existing internal stresses which are not caused by deformations from the reference state? These could arise from manufacturing processes such as heat treatment. It could be that a part's shape was permanently altered in a metal forming process, meaning that plastic strains hold the part in some deformed configuration, even when external loads are removed. The internal stress state in a plastically deformed part is non-trivial, though it may be tempting if any subsequent stress analysis is performed on this part to treat this as the stress-free reference configuration. Let's call the pre-stress state $\tilde{\sigma}_{ij}$. This needs to be experimentally or computationally determined, but it is treated as a known quantity throughout the structure. When starting with a stress or strain formulation of a mechanics problem, it is necessary to check the compatibility condition. When we start from the perspective of displacements, it is enforced naturally that stress and strain fields arise from gradients of the displacement. However, not all symmetric tensors (strain tensors) can be obtained by taking gradients of the displacement. This is because there are six independent components of a symmetric 3D tensor, but only three displacement components. Thus, a generic strain field could over-determine the displacement unless it chosen properly. The same logic applies for the stress tensor, which is obtained as a linear combination of strains. The question we can ask if: what condition on the strains will ensure that they can be computed from gradients of the displacement? This is easiest to see in two dimensions. The three independent strain components are

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \frac{\partial u_2}{\partial x_2} = \epsilon_{22}, \quad \epsilon_{12} = \epsilon_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

By taking derivatives of these strain components, we can see that

$$\frac{\partial^2 \epsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \epsilon_{22}}{\partial x_1^2} = 2\frac{\partial^2 \epsilon_{12}}{\partial x_1 \partial x_2}$$

only if the strain components are derived from the strain-displacement relations. This relation does not hold for a generic symmetric tensor. This is called the compatibility condition. Note that if a stress field is specified, we can write

$$\underline{\epsilon} = \underline{\underline{C}}^{-1}\underline{\sigma}$$

and then verify that the stress field satisfies compatibility. So the pre-stress field $\tilde{\sigma}$ must be compatible in order to be physical. I see a few options for how to deal with the pre-stress problem. The first is to use a linearity argument to say that the stresses generated by deformation will not be influenced by the pre-stress if the material behaves elastically, i.e. there is no yielding or other nonlinear phenomena. If we want to compute the total stress state, we simply compute the stress field from external loads and add the pre-stress state. If there is some kind of material nonlinearity, such as yielding at a specified stress level, we might ignore the pre-stresses by correcting the yield criteria with the stresses at zero deformation. For example, if the onset of plasticity occurs in a bar at $\sigma_y = 2$, and there is a uniform pre-stress state of $\sigma = 1$, the effective yield criteria becomes $\tilde{\sigma}_y = 1$. A slightly bizarre third method would be to compute a fictitious reference configuration which is stress-free. This might look like the following: characterize the pre-stress state $\tilde{\sigma}_{ij}(x)$. Use a material model to convert from a stress field to a strain field $\tilde{\epsilon}_{ij}$. We then seek a displacement field which gives rise to these strains. Assuming the strains cannot be integrated analytically, this displacement field must be parameterized in some way, perhaps with a neural network or some other flexible approximation framework. Working in two dimensions, we call the displacement field $\underline{u}(x_1, x_2; \underline{\theta}) = [u_1(x_1, x_2; \underline{\theta}), u_2(x_1, x_2; \underline{\theta})]^T$. With a given "pre-strain" field, we solve the following optimization problem

$$\underset{\theta_1,\ldots,\theta_N}{argmin}\left[\int \left(\epsilon_{11} - \frac{\partial u_1}{\partial x_1}\right)^2 + \left(\epsilon_{22} - \frac{\partial u_2}{\partial x_2}\right)^2 + \left(\epsilon_{12} - \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)\right)^2 d\Omega\right]$$

The strain components are known and the displacement components depend on the parameters, which are determined by minimizing the mismatch between the true strain, and that which is computed from the displacement field. Once the displacement field is known, we can map the pre-stressed structure back to a fictitious reference configuration which is stress-free. Note that the displacement field can then be plugged into the governing equations to find a corresponding volumetric force required to produce this displacement. This is essentially using the method of manufactured solutions. There are nuances with boundary conditions and what not, but the idea seems to be generally sound. To be clear, it is not necessary to compute the displacement in order to obtain this body force, as the stress could have been used as well. But it is nice to have the ability to visualize the geometry of this fictitious reference configuration through the displacement. Having this body force allows us to solve a problem in the fictitious reference configuration. Call the body force obtained from the method of manufactured solutions with the displacement matched to the pre-strain field $\tilde{b}_i(x_1, x_2)$. We can then solve problems defined in the pre-strain field by superimposing this body force. The governing equations would be

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \tilde{b}_i = 0$$

Imposing the body force leads to the pre-stress configuration even in the absence of other external loads. If there is nonlinearity in the material, this method appears to handle the correction to the yield criteria, where smaller loads than expected may be required to produce yielding because of the existing stresses.

5 Blasius Equation

It is known from that the 99% boundary layer for an infinite flat plate in free stream flow is

$$\delta(x) = \sqrt{\frac{\nu x}{U}}$$

This is the height above the plate at which the x-velocity is 99% of the free stream value. For a steady, incompressible flow, the x-component of the governing equations are

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

where the second term in the Laplacian is neglected by assumption. The boundary conditions are

$$u(x,0) = v(x,0) = 0, \quad u(x,\infty) = U$$

Assuming that the flow is self-similar down the length of the plate, we can introduce the similarity variable

$$\eta = \frac{y}{\delta(x)}$$

which implies that the flow is the same at each percent height of the boundary layer. By assumption, we can write

$$u(x,y) = Ug(\eta)$$

and use the definition of the stream function ($u = \partial\Psi/\partial y$, $v = -\partial\Psi/\partial x$) to write

$$\Psi = \int u dy = \delta(x)\int u d\eta = \delta(x)U\int g(\eta)d\eta = \delta(x)Uf(\eta)$$

Plugging the stream function into the x-momentum equation, we get

$$\frac{\partial\Psi}{\partial y}\frac{\partial^2\Psi}{\partial x\partial y} - \frac{\partial\Psi}{\partial x}\frac{\partial^2\Psi}{\partial y^2} = \nu\frac{\partial^3\Psi}{\partial y^3}$$

It can be shown that by plugging the definition of the stream function with the similarity variable into this expression, we obtain the Blasius equation

$$\frac{\partial^3 f}{\partial\eta^3} + \frac{1}{2}f\frac{\partial^2 f}{\partial\eta^2} = 0$$

The boundary conditions on the Blasius equation are

$$f'(0) = f(0) = 0, \quad f'(\infty) = 1$$

There are two boundary conditions on the left end and one asymptotic condition on the right end. The typical approach to solve this problem is to choose some "large" value for η and enforce the boundary condition at that finite value. Shooting methods of time integration, whereby the initial condition $f''(0)$ is iterated until the desired end condition is obtained, seem to be the standard approach. This is somewhat odd, because when the end condition at ∞ is replaced as a boundary condition at some finite η, this is a standard boundary value problem. Thus, we can use the weak form of the governing equation to explore numerically solving the Blasius equation as a boundary value problem. Our right boundary condition will be $f'(L) = 1$. We multiply the governing equation by an arbitrary test function $w(\eta)$ and integrate over the domain:

$$\int_0^L f'''w + \frac{1}{2}ff''wd\eta = 0$$

The boundary condition $f(0) = 0$ can be enforced strongly by choosing a basis for f and w that satisfy this condition automatically. The other two Neumann-type boundary conditions need to be enforced weakly. We can integrate the first term by parts to obtain

$$\int_0^L -f''w' + \frac{1}{2}ff''wd\eta + f''(L)w(L)$$

where only one boundary term appears from the fact taht $w(0) = 0$. We have not yet made use of the other two boundary conditions, so we integrate the first term by parts one more time to expose terms involving f':

$$\int_0^L f'w'' + \frac{1}{2}ff''wd\eta + f''(L)w(L) - f'(L)w'(L) + f'(0)w'(0)$$

Note that it not advantageous to integrate the second term from the governing equation by parts because it is non-linear and will lead to a more complex weak form. Using the two boundary conditions on f', the weak form becomes

$$\int_0^L f'w'' + \frac{1}{2}ff''wd\eta + f''(L)w(L) - w'(L) = 0$$

To solve this numerically, we can discretize the test function with

$$w(\eta) = \sum_i w_i g_i(\eta)$$

where $g_i(0) = 0$ in order to strongly enforce the Dirichlet boundary. Because the test function is arbitrary, the coefficients w_i are arbitrary, thus the discretized weak form becomes a system of equations

$$\int_0^L f'g_i'' + \frac{1}{2}ff''g_id\eta + f''(L)g_i(L) - g_i'(L) = 0$$

Now we discretize the solution in the same way:

$$f(\eta) = \sum_j f_j g_j(\eta)$$

$$\implies \sum_j f_j \int_0^L g_j' g_i'' d\eta + \sum_j \sum_k f_j f_k \int_0^L \frac{1}{2} g_j g_k'' g_i d\eta + \sum_j f_j g_j''(L) g_i(L) - g_i'(L) = 0$$

Defining the following quantities allows us to write this in a simpler form:

$$\chi_{ij} := \int_0^L g_i'' g_j' d\eta$$

$$\kappa_{ijk} := \int_0^L \frac{1}{2} g_i g_j g_k'' d\eta$$

$$\Gamma_{ij} := g_i(L) g_j''(L)$$

$$\Phi_i := g_i'(L)$$

$$\implies \chi_{ij} f_j + \kappa_{ijk} f_j f_k + \Gamma_{ij} f_j - \Phi_i = 0 := R_i$$

We need to solve the non-linear system of governing equations R_i. This can be done with Newton's method, for which a tangent stiffness matrix is needed. Because the non-linearity is only quadratic, this is a reasonably simple quantity. It can be written as

$$\frac{\partial R_i}{\partial f_\ell} = \chi_{i\ell} + \kappa_{i\ell k} f_k + \kappa_{ij\ell} f_j + \Gamma_{i\ell}$$

This term is used in a Newton solve algorithm to iteratively find coefficient $\underline{f}$ that satisfy the system of equations $\underline{R}$. This is how the Blasius equation could be solved as a boundary value problem.