Rotating Nonlinear Elastic Disk

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Contents

1 Introduction

The purpose of this project is to derive and solve the governing equations of a rotating deformable body. For simplicity, we assume that the body is a two-dimensional circular disk, and that it rotates about an axis of rotation which passes through its center and is perpendicular to its plane. It is convenient to model the displacement response of a body in a coordinate system that rotates

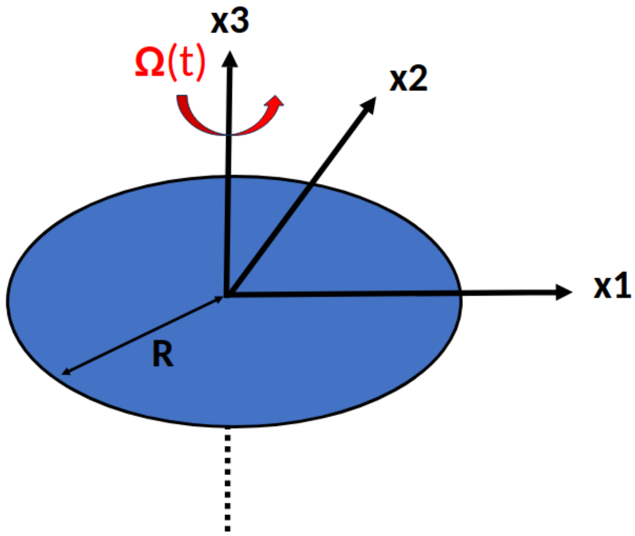

Figure 1: A circular disk of radius R is rotated around the x_3 axis with angular velocity $\Omega_3(t)$. The disk deforms under the inertial forces of rotation and is assumed to be in a 2D stress state.

with the body. Thus, a material point has a fixed spatial position as seen by the rotating coordinate system. But, because Newton's laws apply only in inertial frames, it is necessary to compute time derivatives of the position in reference to "ambient" space. This means that a material point with fixed position in the rotating frame still experiences a time rate of change of its position with respect to ambient space (an inertial frame). Thus, time derivatives of the position vector $\underline{x} = x_i \hat{e}_i$ are due to changes in the coordinates x_i, and due to the motion of the rotating basis vectors through space. In other words, we must be careful to compute inertial accelerations when our coordinate system rotates with the body. The consequence of doing this is that there are additional "fictitious" forces that the continuum experiences, which arise from the prescribed rotation forcing particles in the body to follow curved paths. First, we propose a nonlinear constitutive relation which is used to approximate hardening plasticity without requiring the additional complexity of internal state variables. Next, the strong form of the governing equations in the rotating frame are derived. Afterwards, we proceed to weaken the governing equations and cast them into a form more suitable for a numerical solution. Then, a specific choice of discretization for the two-dimensional displacement field is outlined. The numerical solution is implemented in MATLAB and verified using the method of manufactured solutions. We then simplify the governing equations with various assumptions, and solve them for linear and nonlinear material models.

2 Stress-strain Relation

We will use a stress-strain relation for the 2D solid which approximates hardening plasticity, but without the complexity of dealing with internal state variables. We want a stress-stain relation which is initially linear with slope E_1, and then asymptotically takes on a slope $E_2 < E_1$ at larger values of strain. There will be a region where the slope transitions continuously from E_1 to E_2. A three-parameter stress-strain relation of the following form is capable of capturing this behavior:

$$\sigma = f(\epsilon; a, b, c) = a\epsilon - b(e^{-c\epsilon} - 1), \quad \epsilon \geq 0, \quad a \geq 0, \quad b \geq 0, \quad c \geq a$$

As noted, this stress-strain relation is only valid for $\epsilon \geq 0$. In order to make it anti-symmetric, so that the compression behavior is equivalent up to a minus sign, we can define a regularized indicator function

$$w(x) := \frac{1}{2}\Big(\tanh px + 1\Big)$$

and use this to construct a three-parameter stress-strain relation which approximates hardening plasticity with equivalent behavior in tension and compression as

$$\sigma = w(\epsilon)f(\epsilon; a, b, c) - w(\epsilon)f(-\epsilon; a, b, c)$$

Note that p is a hyperparameter which should be large for the hyperbolic tangent to accurately approximate a step function. See this plot to investigate this relationship further. Note that by varying the parameters, we can see that a controls the slope of the curve at large strains (hardening), b controls the stress at which yielding occurs, and c controls the initial slope. We want to define the three parameters that build this stress-strain relation in terms of parameters with physical meaning. This will be the "elastic modulus" E_1 (initial slope), the plastic modulus E_2 (asymptotic slope), and the yield stress σ_y which controls when the transition occurs. It is not clear from looking at the plot where we should consider yielding to occur. Motivated by true hardening plasticity, we will treat the yield stress as the intersection of lines drawn tangent to the curve at $\epsilon = 0$ and $\epsilon = \infty$. Note that we can compute the initial tangent by linearizing the stress-strain relation around $\epsilon = 0$:

$$f(\epsilon) \approx f(0) + \frac{\partial f}{\partial \epsilon}(0)\epsilon = (a + cb)\epsilon$$

For large strains, the exponential decays to zero and stress-strain relation is

$$f(\epsilon) \approx b + a\epsilon$$

The stress value corresponding to the intersection of these two lines is the yield stress. First, we find the yield strain with

$$(a + cb)\epsilon = b + a\epsilon \implies \epsilon^y = \frac{1}{c}, \quad \sigma^y = b + a\epsilon^y = b + \frac{a}{c}$$

See this plot for an illustration of this procedure. We assume that the constitutive relation is specified with the elastic modulus, plastic modulus, and yield stress. We thus have a system of equations for the parameters:

$$a + cb = E_1, \quad a = E_2, \quad \sigma^y = b + \frac{a}{c}$$

We can solve this system to find the constitutive parameters in terms of the given physical quantities. We have that

$$a = E_2, \quad b = \frac{\sigma_y(E_1 - E_2)}{E_1}, \quad c = \frac{E_2}{\sigma_y - b}$$

See this plot for the stress-strain relation defined in terms of the physical quantities. Now we need to specify how this stress-strain relation interacts with the 2D strain state of the solid. We will assume that the total normal strains are

$$\epsilon_{11}^{tot} = \epsilon_{11} + \nu\epsilon_{22}, \quad \epsilon_{22}^{tot} = \epsilon_{22} + \nu\epsilon_{11}$$

so that the normal stresses are computed with

$$\sigma_{11} = f(\epsilon_{11} + \nu\epsilon_{22}; E_1, E_2, \sigma_y), \quad \sigma_{22} = f(\epsilon_{22} + \nu\epsilon_{11}; E_1, E_2, \sigma_y)$$

Now we deal with the case of shear strains. We assume that the stress-strain relation in shear is equivalent up to a choice of the three parameters. We can write

$$\sigma_{12} = \sigma_{21} = f(2\epsilon_{12}; G_1, G_2, \gamma_y)$$

where only shear strains contribute to shear stresses, per the usual assumptions of an isotropic material. In the computational setting, it is convenient to collapse the stress and strain tensors into vectors. The stress-strain relation in vector form is

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \begin{bmatrix} f(\epsilon_{11} + \nu\epsilon_{22}; E_1, E_2, \sigma_y) \\ f(\epsilon_{22} + \nu\epsilon_{11}; E_1, E_2, \sigma_y) \\ f(2\epsilon_{12}; G_1, G_2, \gamma_y) \end{bmatrix}$$

Note that this constitutive relation assumes a true 2D stress state, as opposed to a plane stress or plain strain simplification of a 3D stress state. We are not arguing that components of the stress and/or strain involving the x_3 index are zero based on the geometry of the plate, as this introduces three-dimensional effects which can only be condensed out of the elastic constitutive relation with the help of linearity. This would become quite complicated in our case of the nonlinear constitutive relation. In other words, the plate is modeled as a true 2D solid.

3 Governing Equations

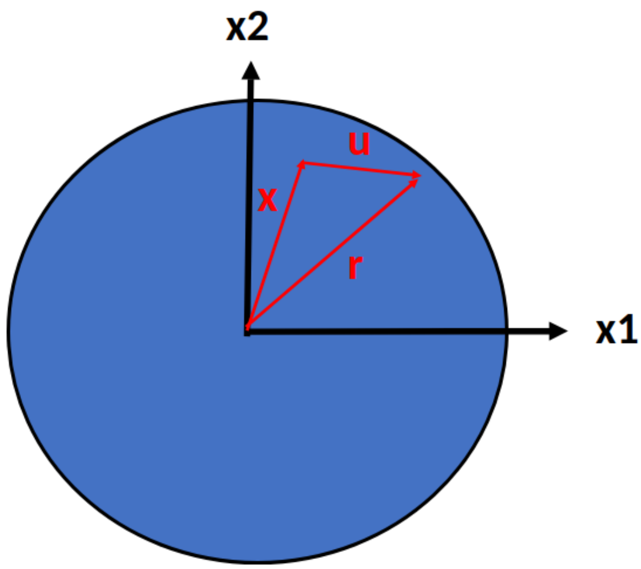

Figure 2: Relationship between the spatial coordinate $\underline{x}$, displacement $\underline{u}(x_1, x_2)$, and position vector $\underline{r}$. For typical solid mechanics problems, it is not necessary to make use of the position vector (on which the balance of momentum technically applies) in computing accelerations because the spatial coordinate/material point $\underline{x}$ is time independent and thus does not appear in the equation of motion. When the coordinate frame defined by axes x_1 and x_2 rotates, the basis vectors have time derivatives which means that the acceleration depends on $\underline{x}$ as well as $\underline{u}$.

We will solve the governing equations for the elastic disk in a coordinate frame that rotates with the body. As shown in Figure 1, we have a disk lying in the x_1-x_2 plane that rotates about its center around the x_3 axis with a prescribed angular velocity $\Omega(t)$. We want to compute the time derivative of the position of a point defined in the rotating frame. Call the basis vectors in the rotating frame $\hat{e}_1$ and $\hat{e}_2$. When computing the velocity and acceleration in the rotating frame, both the components of the position and the basis vectors will have time derivatives:

$$\frac{d}{dt}\underline{r} = \frac{\partial}{\partial t}(r_i \hat{e}_i) = \frac{\partial r_i}{\partial t}\hat{e}_i + r_i \frac{\partial \hat{e}_i}{\partial t}$$

It can be shown that $\partial \hat{e}_i / \partial t = \underline{\Omega} \times \hat{e}_i$, which can be verified for our example by noting that

$$\frac{\partial \hat{e}_1}{\partial t} = \begin{bmatrix} 0 \\ \Omega_3(t) \\ 0 \end{bmatrix} = \underline{\Omega} \times \hat{e}_1, \quad \frac{\partial \hat{e}_2}{\partial t} = \begin{bmatrix} -\Omega_3(t) \\ 0 \\ 0 \end{bmatrix} = \underline{\Omega} \times \hat{e}_2$$

This can be seen by noting that the components of the basis vector $\hat{e}_1$ change when viewed from a stationary frame (non-rotating) frame, and that the basis vector has unit length. When Ω_3 is around the x_3 axis and is positive, the rotation is counter-clockwise, and this means that the instantaneous changes in the components of the basis vector viewed from the stationary frame only have one component. We can write the time derivative of position in the rotating frame as

$$\frac{d}{dt}\underline{r} = \frac{\partial \underline{r}}{\partial t} + \underline{\Omega} \times \underline{r}$$

We need to apply this time derivative again to compute the acceleration. Repeating this process, we can show that the acceleration is

$$\begin{aligned} \frac{d^2}{dt^2}\underline{r} &= \frac{\partial^2 \underline{r}}{\partial t^2} + \frac{\partial}{\partial t}(\underline{\Omega} \times \underline{r}) + \underline{\Omega} \times \frac{\partial \underline{r}}{\partial t} + \underline{\Omega} \times \underline{\Omega} \times \underline{r} \\ &= \frac{\partial^2 \underline{r}}{\partial t^2} + \frac{\partial \underline{\Omega}}{\partial t} \times \underline{r} + 2\left(\underline{\Omega} \times \frac{\partial \underline{r}}{\partial t}\right) + \underline{\Omega} \times \underline{\Omega} \times \underline{r} \end{aligned}$$

Noting that the position vector can be written as $\underline{r} = \underline{x} + \underline{u}$ per Figure 2, and that the time derivative of the spatial coordinate is zero, the acceleration in the rotating frame is

$$\ddot{\underline{r}} = \frac{\partial^2 \underline{u}}{\partial t^2} + \frac{\partial \underline{\Omega}}{\partial t} \times (\underline{x} + \underline{u}) + 2\left(\underline{\Omega} \times \frac{\partial \underline{u}}{\partial t}\right) + \underline{\Omega} \times \underline{\Omega} \times (\underline{x} + \underline{u})$$

We can now make simplifications given that the position and displacement lie in the x_1-x_2 plane and that the angular velocity is around the x_3 axis. First, it can be shown by evaluating the double cross-product with the given geometry that

$$\underline{\Omega} \times \underline{\Omega} \times \underline{r} = -\Omega_3^2 \underline{r} = -\Omega_3^2 (\underline{x} + \underline{u})$$

We see why this term is called centripetal (center-pointing) acceleration: it points from the position to the axis of rotation. This term says that even when the body's position does not change in the rotating frame, by virtue of rotating along with the body, the point is being accelerated off from what would otherwise be a straight line path in the direction of the axis of rotation. Next is the Coriolis force term, which can be simplified to

$$2\left(\underline{\Omega} \times \frac{\partial \underline{u}}{\partial t}\right) = 2\Omega_3 \begin{bmatrix} -\dot{u}_2 \\ \dot{u}_1 \\ 0 \end{bmatrix} = 2\Omega_3 \underline{\underline{Q}}\, \dot{\underline{u}}, \quad \underline{\underline{Q}} := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

where $\underline{\underline{Q}}$ is a rotation matrix which extracts the perpendicular component of the vector it acts on in the plane of the disk. This is equivalent to a counter-clockwise rotation of 90 degrees. We are now neglecting the third component of the displacement because it never appears in these equations. The last term has a similar structure, and can be written as

$$\dot{\underline{\Omega}} \times (\underline{x} + \underline{u}) = \dot{\Omega}_3 \underline{\underline{Q}}\underline{r}$$

Finally, we can write the acceleration as

$$\ddot{\underline{r}} = \ddot{\underline{u}} + \dot{\Omega}_3 \underline{\underline{Q}}\underline{r} + 2\Omega_3 \underline{\underline{Q}}\dot{\underline{u}} - \Omega_3^2 \underline{r}$$

With the acceleration in hand, we can formulate stress equilibrium in the rotating frame. Remember that the displacements are defined in terms of a basis that rotates with the frame. The way in which displacements generate stresses is not influenced by the rotation, only the acceleration term is. Thus we can write the equation of motion as

$$\rho\ddot{\underline{r}} = \rho(\ddot{\underline{u}} + \dot{\Omega}_3 \underline{\underline{Q}}\underline{r} + 2\Omega_3 \underline{\underline{Q}}\dot{\underline{u}} - \Omega_3^2 \underline{r}) = \nabla \cdot \underline{\underline{\sigma}}$$

We have assumed there are no body forces present, meaning that the elastic problem is driven entirely by the inertial forces of rotation. We will assume that the disk has zero traction boundary conditions along its edges. Clearly, the displacement at the origin is zero as this is the axis of rotation, thus removing a rigid body mode which could arise from the zero traction boundary conditions. It will be convenient to separate quantities involving the position vector into a known part involving the spatial position and unknown part involving the displacement. Similarly, we will group terms involving time derivatives of the displacement on one side of the equation. This equation of motion becomes

$$\ddot{\underline{u}} + 2\Omega_3 \underline{\underline{Q}}\dot{\underline{u}} = \nabla \cdot \underline{\underline{\sigma}} + \Omega_3^2 \underline{x} + \Omega_3^2 \underline{u} - \dot{\Omega}_3 \underline{\underline{Q}}\underline{x} - \dot{\Omega}_3 \underline{\underline{Q}}\underline{u}$$

where we are assuming that $\rho = 1$ for simplicity. Define the effective body force as the fictitious force terms which do not depend on the displacement:

$$\underline{b} := \Omega_3^2 \underline{x} - \dot{\Omega}_3 \underline{\underline{Q}}\underline{x}$$

The governing equation takes a slightly simpler form with this convention:

$$\ddot{\underline{u}} + 2\Omega_3 \underline{\underline{Q}}\dot{\underline{u}} = \nabla \cdot \underline{\underline{\sigma}} + \Omega_3^2 \underline{u} - \dot{\Omega}_3 \underline{\underline{Q}}\underline{u} + \underline{b}(t)$$

4 Numerical Solution

We can now devise a method to solve these equations numerically. The first step is to weaken the equation by integrating against a spatial test function $w_i(x_1, x_2)$ over the spatial domain A. Switching to index notation, this reads

$$\int_A \ddot{u}_i w_i + 2\dot{\Omega}_3 Q_{ij}\dot{u}_j w_i dA = \int_A \frac{\partial \sigma_{ij}}{\partial x_j} w_i + \Omega_3^2 u_i w_i - \dot{\Omega}_3 Q_{ij} u_j w_i + b_i w_i dA$$

First, we look at the term involving the divergence of the stress tensor. Noting that the traction boundaries are zero everywhere, we can integrate by parts to find that

$$\int_A \frac{\partial \sigma_{ij}}{\partial x_j} w_i dA = -\int_A \sigma_{ij} \frac{\partial w_i}{\partial x_j} dA = -\int \sigma_{11} w_{1,1} + \sigma_{12} w_{1,2} + \sigma_{21} w_{2,1} + \sigma_{22} w_{2,2} dA$$

$$= -\int_A \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} \cdot \begin{bmatrix} w_{1,1} \\ w_{2,2} \\ w_{1,2} + w_{2,1} \end{bmatrix} dA$$

where we have redefined the stress components $\sigma_1 := \sigma_{11}$, $\sigma_2 := \sigma_{22}$, and $\sigma_{12} = \sigma_{21} := \sigma_3$ for ease of numerical implementation. Note that if we define a matrix of derivatives

$$\underline{\underline{B}} := \begin{bmatrix} \partial/\partial x_1 & 0 \\ 0 & \partial/\partial x_2 \\ \partial/\partial x_2 & \partial/\partial x_1 \end{bmatrix}$$

we can write the above expression as

$$-\int_A \sigma_i B_{ij} w_j dA$$

This will prove to be a useful rearrangement of terms. Plugging this into the weak form of the governing equation, we obtain

$$\int_A \ddot{u}_i w_i + 2\dot{\Omega}_3 Q_{ij}\dot{u}_j w_i dA = \int_A -\sigma_j B_{ji} w_i dA + \Omega_3^2 u_i w_i - \dot{\Omega}_3 Q_{ij} u_j w_i + b_i w_i dA$$

We assume that the test function can be discretized by the product of a matrix of shape functions with a vector of degrees of freedom:

$$w_i = H_{ik}(x_1, x_2) W_k$$

Remember that the test functions in the weak form are arbitrary. When we discretize the test function in this way, we assume that the test functions live in a subspace defined by the shape function matrix. As we will see, we will discretize the displacement with the same shape functions giving rise to a Galerkin method. Plugging in the discretization of the test function and noting that the degrees of freedom W_k are arbitrary, we obtain a system of equations:

$$\int_A \rho \ddot{u}_i H_{ik} + 2\dot{\Omega}_3 Q_{ij}\dot{u}_j H_{ik} dA = \int_A -\sigma_j B_{ji} H_{ik} dA + \Omega_3^2 u_i H_{ik} - \dot{\Omega}_3 Q_{ij} u_j H_{ik} + b_i H_{ik} dA$$

Note that this immediately allows us to define a force vector

$$F_k := \int_A b_i H_{ik} dA$$

We now discretize the displacement in the same manner as the test function, i.e. $u_i = H_{i\ell} U_\ell$. Note, however, that the degrees of freedom on the displacement are time-dependent. Plugging this into the governing equation, we obtain

$$\ddot{U}_\ell \left(\int_A H_{i\ell} H_{ik} dA \right) + \dot{U}_\ell \dot{\Omega}_3 \left(\int_A 2 Q_{ij} H_{ik} H_{j\ell} dA \right) = F_k - \int_A \sigma_j \left(\underline{U} \right) B_{ji} H_{ik} dA$$
$$+ U_\ell \Omega_3^2 \left(\int_A H_{i\ell} H_{ik} dA \right) - U_\ell \dot{\Omega}_3 \left(\int_A Q_{ij} H_{ik} H_{j\ell} dA \right)$$

If we define the usual mass matrix $\underline{\underline{M}}$, and a "rotated" mass matrix $\underline{\underline{T}}$, the system of ODE's governing the dynamics of the rotating plate is

$$\ddot{U}_\ell M_{k\ell} + 2 \dot{U}_\ell \dot{\Omega}_3 T_{k\ell} = F_k - \int_A \sigma_j \left(\underline{U} \right) B_{ji} H_{ik} dA + U_\ell \Omega_3^2 M_{k\ell} - U_\ell \dot{\Omega}_3 T_{k\ell}$$

It is actually convenient for implementation purposes to expand the definition of the force vector in this equation make clear its time dependence:

$$\ddot{U}_\ell M_{k\ell} + 2 \dot{U}_\ell \dot{\Omega}_3 T_{k\ell} = \Omega_3^2 \int_A x_i H_{ik} dA - \dot{\Omega}_3 \int_A Q_{ij} x_j H_{ik} dA - \int_A \sigma_j \left(\underline{U} \right) B_{ji} H_{ik} dA$$
$$+ U_\ell \Omega_3^2 M_{k\ell} - U_\ell \dot{\Omega}_3 T_{k\ell}$$

$$\ddot{U}_\ell M_{k\ell} + 2 \dot{U}_\ell \dot{\Omega}_3 T_{k\ell} = \Omega_3^2 F_k^C - \dot{\Omega}_3 F_k^T - \int_A \sigma_j \left(\underline{U} \right) B_{ji} H_{ik} dA$$
$$+ U_\ell \Omega_3^2 M_{k\ell} - U_\ell \dot{\Omega}_3 T_{k\ell}$$

$$\underline{\underline{M}} \ddot{\underline{U}} + 2 \dot{\Omega}_3 \underline{\underline{T}} \dot{\underline{U}} = \Omega_3^2 \left(\underline{F}^C + \underline{\underline{M}} \underline{U} \right) - \dot{\Omega}_3 \left(\underline{F}^T + \underline{\underline{T}} \underline{U} \right) - \int_A (\underline{\underline{B}} \underline{\underline{H}})^T \underline{\sigma} (\underline{\underline{B}} \underline{\underline{H}} \underline{U}) dA$$

This is the discrete form of the governing equations for the dynamics of the rotating disk under a prescribed angular velocity. All matrix or vector quantities in the above expression are independent of time except for the displacement degrees of freedom. We will assume that the disk starts at rest, thus $\underline{U}(0) = \dot{\underline{U}}(0) = 0$. This equation can be solved with the given initial conditions and an iterative time integration scheme.

5 Displacement Discretization

We can make use of some prior knowledge of the displacement field which will arise from the rotation when considering the discretization. In the absence of body forces, the displacement field should be symmetric under rotations about the x_3 axis. The centrifugal force terms are purely radial. A displacement which is radial and grows linearly with the radius is

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^C = a_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where a_1 controls the magnitude of the displacement and the superscript "C" is used to denote the centrifugal displacement. For our material model which involves yielding, we need to be able to represent centrifugal-type displacements which may not be linear the radius. If we define $r := \sqrt{x_1^2 + x_2^2}$, we can write the centrifugal displacement as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^C = \sum_{i=1}^{N} a_i(t) \left(\frac{r}{R}\right)^{i-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which is a radial displacement whose profile at every angular slice is controlled by a Taylor series in the radius. See this plot for a visualization. There will also be twisting displacements which come from the angular acceleration of the plate and the Coriolis force. The twisting displacement will be similar to the displacements of torsion except that it is not necessarily linear in the radius. The twisting displacement can be written as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^T = \sum_{i=1}^{N} b_i(t) \left(\frac{r}{R}\right)^{i} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

where the superscript "T" is used to denote twisting. Note that the Taylor series in the radius starts with r^1 because the r^0 term is a rigid body rotation that does not generate any stresses. This contrasts with torsion, where shear stresses involving the 3 index are generated even when the disk rotates as a rigid body. By assumption, the total displacement is the sum of the centrifugal and twisting "modes." This ensures certain symmetries that we expect from the fictitious forces. The displacement is then discretized as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \sum_{i} a_i(t) \left(\frac{r}{R}\right)^{i-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \sum_{i} b_i(t) \left(\frac{r}{R}\right)^{i} \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

$$
= \begin{bmatrix} x_1 & x_1\left(\frac{r}{R}\right) & \dots & -x_2\left(\frac{r}{R}\right) & -x_2\left(\frac{r}{R}\right)^2 & \dots \\ x_2 & x_2\left(\frac{r}{R}\right) & \dots & x_1\left(\frac{r}{R}\right) & x_1\left(\frac{r}{R}\right)^2 & \dots \end{bmatrix} \begin{bmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ b_1(t) \\ b_2(t) \\ \vdots \end{bmatrix} := H_{ij}U_j
$$

Note that in deriving the weak form of the governing equations, we assumed that the the boundaries of the disk were traction free. There are no explicit restrictions on the displacement components along the boundaries, so the zero traction boundary condition is enforced weakly. We can also model a "confined" disk, which could arise from the disk being surrounded by a stiff outer ring. This will prove useful in validating the implementation of the nonlinear solution. We still assume a centrifigual and twisting displacement, however we multiply by $\sin\left(\frac{\pi r}{R}\right)$ in order to enforce zero displacement along the boundaries. See this plot for further illustration. Noting that the rigid body mode is now taken out of the first term in the twisting displacement, the displacement approximation for the confined disk is

$$
\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} x_1 \sin\left(\frac{\pi r}{R}\right) & x_1\left(\frac{r}{R}\right)\sin\left(\frac{\pi r}{R}\right) & \dots & -x_2 \sin\left(\frac{\pi r}{R}\right) & -x_2\left(\frac{r}{R}\right)\sin\left(\frac{\pi r}{R}\right) & \dots \\ x_2 \sin\left(\frac{\pi r}{R}\right) & x_2\left(\frac{r}{R}\right)\sin\left(\frac{\pi r}{R}\right) & \dots & x_1 \sin\left(\frac{\pi r}{R}\right) & x_1\left(\frac{r}{R}\right)\sin\left(\frac{\pi r}{R}\right) & \dots \end{bmatrix} \begin{bmatrix} a_1(t) \\ a_2(t) \\ \vdots \\ b_1(t) \\ b_2(t) \\ \vdots \end{bmatrix}
$$

6 Method of Manufactured Solutions

We can validate the numerical implementation of the nonlinear solution using the method of manufactured solutions. Using our nonlinear constitutive model and the vectorized stress tensor, the governing equations for stress equilibrium in an inertial frame are

$$
\sigma_{ij,j} + b_i = 0 \implies b_1 = -\frac{\partial \sigma_1}{\partial x_1} - \frac{\partial \sigma_3}{\partial x_2}, \quad b_2 = -\frac{\partial \sigma_3}{\partial x_2} - \frac{\partial \sigma_2}{\partial x_2}
$$

Note that the stress is a nonlinear function of strain through the constitutive model f, and strain is a linear function of displacement. Thus we have

$$
\underline{\sigma} = \underline{f}\Big(\underline{\underline{B}}\,\underline{\underline{H}}\,\underline{U}\Big)
$$

In the method of manufactured solutions, we assume a displacement field, in this case through the coefficients U and the corresponding basis, then compute the body force consistent with those displacements through gradients of the stress. This body force can then be applied as a load in the numerical solution, and we can verify that we return the same set of displacement coefficients. This is how the code can be verified. Note that we will use the confined disk model for code verification. This allows us to not worry about boundary conditions, as the shape functions enforce the homogeneous Dirichlet boundary automatically.

6.1 Linear Constitutive Relation

When using a linear stress-strain relation, The stress is computed as

$$\underline{\sigma}(x_1, x_2) = \underline{\underline{D}}\,\underline{\underline{B}}\,\underline{\underline{H}}(x_1, x_2)\underline{U}$$

We use the shape functions with zero boundaries outlined above. We assume displacement coefficients and verify that they are returned by the numerical solution. The force vector with a body force computed from stresses is given by

$$F_k = -\int H_{1k}(\sigma_{1,1} + \sigma_{3,2}) + H_{2k}(\sigma_{3,1} + \sigma_{2,2})dA$$

The derivatives of the stress are computed using the constitutive relation, the strain displacement relations, a given set of displacement coefficients, and symbolic differentiation. Running the problem 10 times with randomly initialized manufactured solutions, we obtain an average mean-squared error between the computed and manufactured displacement coefficients of 0.5%.

6.2 Nonlinear Constitutive Relation

When working with the nonlinear constitutive relation, the stresses are computed as

$$\underline{\sigma}(x_1, x_2) = \underline{f}\Big(\underline{\underline{B}}\,\underline{\underline{H}}(x_1, x_2)\underline{U}\Big)$$

The force vector has the same form as that of the linear case given above, except that the spatial derivatives of stress are more complex. We can feed in a strain vector $\underline{\underline{B}}\,\underline{\underline{H}}\,\underline{U}$ with given displacement coefficients but symbolic dependence on the spatial coordinates into a numerical function defining the constitutive relations, and take spatial derivatives symbolically to compute the body force. The body force is then converted to a numerical function and the force vector is formed with numerical integration. Running the problem 10 times with random manufactured coefficients, we obtain an average mean-squared error between the computed and manufactured displacement of 3.2%. This is slightly higher than in the linear case, but still acceptable accuracy. The Newton convergence threshold is set at $1E-9$ and the method typically converges within a few steps.

7 Constant Velocity Steady State Approximation

The steady-state solution of this system is obtained when there are no time derivatives of the displacement and the angular velocity is constant. This implies $\ddot{\underline{U}} = \dot{\underline{U}} = \dot{\Omega}_3 = 0$. The governing equation for the steady state displacement response of the disk is

$$R_k = \int_A \sigma_j\left(\underline{U}\right) B_{ji} H_{ik} dA - \Omega_3^2 M_{k\ell} U_\ell - \Omega_3^2 F_k^C = 0$$

This is a nonlinear system of equations that can be solved with a Newton-Raphson method. For this, we need the Jacobian matrix:

$$\frac{\partial R_k}{\partial U_q} = \int_A \left(\frac{\partial \sigma_j}{\partial \epsilon_s} B_{sr} H_{rq} B_{ji} H_{ik}\right) dA - \Omega_3^2 M_{kq}$$

The derivative $\partial\sigma_j/\partial\epsilon_s$ is evaluated at given strain values, which are computed at each point in the domain using the current value of the displacement coefficients in the Newton solve. This derivative is computed by differentiating the stress-strain relation we have already introduced. The Newton-Raphson method iteratively updates the solution with

$$\Delta\underline{U} = -\left(\frac{\partial \underline{R}}{\partial \underline{U}}(\underline{U}^k)\right)^{-1} \underline{R}(\underline{U}^k), \quad \underline{U}^{k+1} = \underline{U}^k + \Delta\underline{U}$$

and halts once the magnitude of the residual equations becomes sufficiently small. Only the centrifugal force appears in the steady state problem, meaning that we expect displacements which are radially outward.

7.1 Steady State with Linear Material

We can use the case of a steady state rotation and linear material as a first test of the implementation. With a given constitutive matrix $\underline{\underline{D}}$, the residual equations become

$$\underline{R} = \left(\int_A \underline{\underline{H}}^T \underline{\underline{B}}^T \underline{\underline{D}}\,\underline{\underline{B}}\,\underline{\underline{H}} dA\right)\underline{U} - \Omega_3^2 \underline{\underline{M}}\,\underline{U} - \Omega_3^2 \underline{F}^C = 0, \quad \underline{\underline{D}} = E\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1/(1+\nu) \end{bmatrix}$$

$$\implies \left(\underline{\underline{K}} - \Omega_3^2 \underline{\underline{M}}\right)\underline{U} = \Omega_3^2 \underline{F}^C$$

where the elastic stiffness matrix $\underline{\underline{K}}$ is defined in the usual way. We see that the "effective" stiffness matrix $\underline{\underline{K}}^{eff} = \underline{\underline{K}} - \Omega_3^2 \underline{\underline{M}}$ depends on the angular velocity. This makes sense–the disk will experience centrifugal forces that grow linearly with displacement away from the axis of rotation. This translates mathematically a loss of stiffness in the system. The consequence of this is that even in the simple steady state rotation problem with linear material, there are instability phenomena. The effectiveness stiffness matrix can become singular when the angular velocity becomes sufficiently large. The term involving the mass matrix is analogous to a geometric stiffness from the study of buckling. The rotational instability is a buckling-type effect but from dynamic forces. The onset of instability occurs when

$$\Omega_3^2 = \lambda_1$$

where λ_1 is the smallest eigenvalue computed from the generalized eigenvalue problem

$$\underline{\underline{K}}\,\underline{U} = \lambda \underline{\underline{M}}\,\underline{U}$$

This means that the angular velocity cannot be so large as to make the effective stiffness matrix singular. If we apply angular velocities such that $\Omega_3^2 > \lambda_1$, we will still obtain a solution, but it will not be physical. It is necessary to ensure that we do not force the system to the point of instability. We will call the critical angular velocity $\Omega_c = \sqrt{\lambda_1}$. See Figures 3-6 for results of the steady state simulation with linear material pre- and post-instability. For these simulations, we use $E = 1$, $\nu = 0.25$, and $R = 1$. The integration over the circular domain is carried out using a custom integration routine in MATLAB. Note that the traction-free boundary is enforced weakly through the governing equations.

7.2 Steady State with Nonlinear Material

Parameter	Value	Units
ν	0.25	–
E_1	5	Pa
E_2	1	Pa
σ_y	1	Pa
G_1	2	Pa
G_2	0.4	Pa
γ_y	2	Pa

Table 1: Specifying parameters in the nonlinear constitutive relation which approximates hardening plasticity.

The steady state response with the nonlinear material requires a set of seven constitutive parameters to define the stress-strain relation. See the table above for the chosen values of these parameters. A Newton solver is implemented for the nonlinear residual equations. A step size threshold of $1E-8$ is used.

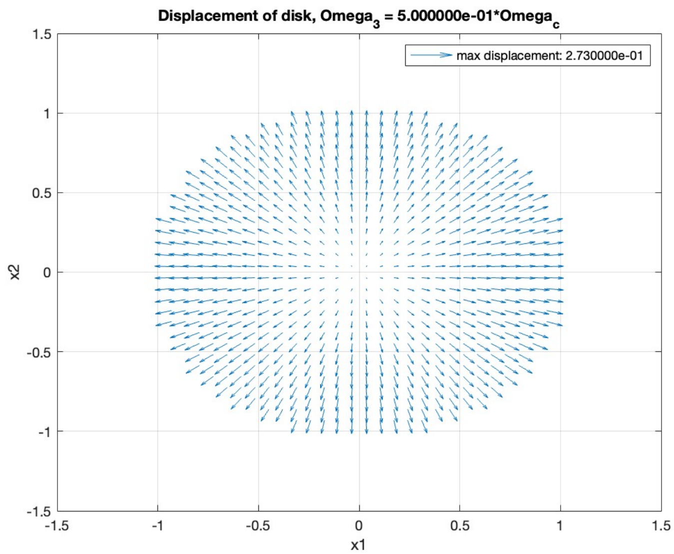

Figure 3: Steady state displacement field for the disk spun at $\Omega_3 = 0.5\Omega_c$. The displacement is radially outward and grows as we approach the edge of the disk under the centrifugal force as expected.

8 Constant Acceleration Steady State Approximation

We can make another steady state approximation for the rotating disk. In this case, we neglect the contribution of the centrifugal terms to the displacement, and justify this by using the confined disk with zero displacement boundary conditions along the edges. Because the radial displacement will be dramatically reduced as a result of these boundary conditions, we argue that the fictitious force involving angular acceleration will dominate the physics of the disk. Note that due to the steady state approximation, the Coriolis force and acceleration of the displacement drop out. In other words, we assume that time derivatives of the displacement are zero, but that the problem is forced by a constant angular acceleration $\dot{\Omega}_3$. Neglecting the centrifugal force allows us to not consider the actual value of the velocity $\Omega_3 = \int \dot{\Omega}_3 dt$. The constant acceleration steady state approximation for the confined disk is

$$R_k = \int_A \sigma_j\left(\underline{U}\right) B_{ji} H_{ik} dA + \dot{\Omega}_3 T_{k\ell} U_\ell + \dot{\Omega}_3 F_k^T = 0$$

This is a nonlinear system of equations that can be solved with a Newton-Raphson method. For this, we need the Jacobian matrix:

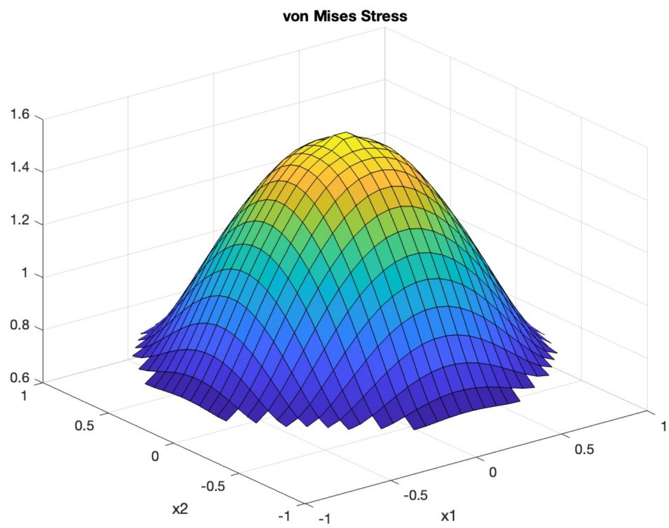

Figure 4: von Mises stress in the disk for linear material and centrifugal forces. The stress field has the expected symmetries, and the stresses are largest at the center of the disk.

$$\frac{\partial R_k}{\partial U_q} = \int_A \left(\frac{\partial \sigma_j}{\partial \epsilon_s} B_{sr} H_{rq} B_{ji} H_{ik} \right) dA + \dot{\Omega}_3 T_{kq}$$

8.1 Linear Material

When the constitutive relation is linear, the governing equation is

$$\left(\underline{\underline{K}} + \dot{\Omega}_3 \underline{\underline{T}} \right) \underline{U} = -\dot{\Omega}_3 \underline{F}^T$$

Note that the effective stiffness matrix depends on the angular acceleration. This suggests that an instability is possible again. The displacement-dependent force arising from the angular acceleration quantifies how a radial displacement acts to increase the effective distance of a material point from the axis of rotation. When the problem is forced by angular acceleration and the centrifugal effects are ignored, there is little reason to think the disk will undergo large radial displacements. Thus, if the displacement $\underline{U}$ is purely angular, there is no increase in the force due to the angular acceleration because the moment arm stays the same. In other words, we do not expect to have to be careful of instabilities for the constant acceleration steady state problem. This can be verified by showing that eigenvalues of the system are extremely large.

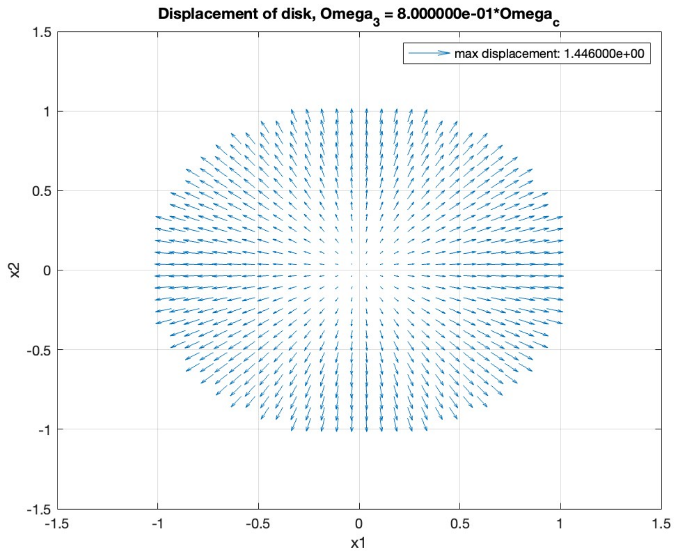

Figure 5: The magnitude of the displacements increases dramatically as the angular velocity approaches the point of instability for the constant velocity problem. The radial form of the displacement is not changed, however.

8.2 Nonlinear Material

We use the same material parameters shown in Table 1 for the nonlinear constitutive response of the confined disk. No instability phenomena are observed with numerical experiments. See Figures 12-14 for results of the simulations.

9 Dynamics

We can time integrate the equations of motion in the rotating frame in order to lift the steady state assumption. However, we will neglect the displacement's contribution to the centrigufal force and the angular acceleration term. We will not neglect the Coriolis effect. The justification for this is that when the displacement is small, the fictitious forces will be dominated by the position of the particle (used to compute the force vectors $\underline{F}^C$ and $\underline{F}^T$) rather than its displacement. Under this assumption, the equations of motion are

$$\underline{\underline{M}}\,\underline{\ddot{U}} + 2\dot{\Omega}_3\underline{\underline{T}}\,\underline{\dot{U}} = \Omega_3^2\underline{F}^C - \dot{\Omega}_3\underline{F}^T - \int_A (\underline{\underline{B}}\,\underline{\underline{H}})^T\underline{\sigma}(\underline{\underline{B}}\,\underline{\underline{H}}\,\underline{U})dA$$

We can solve these equations with a backward Euler time stepping scheme. The nonlinear material model will be used in all simulations. Using finite difference approximations of the time derivatives, the governing equation to advance from t_i to t_{i+1} is

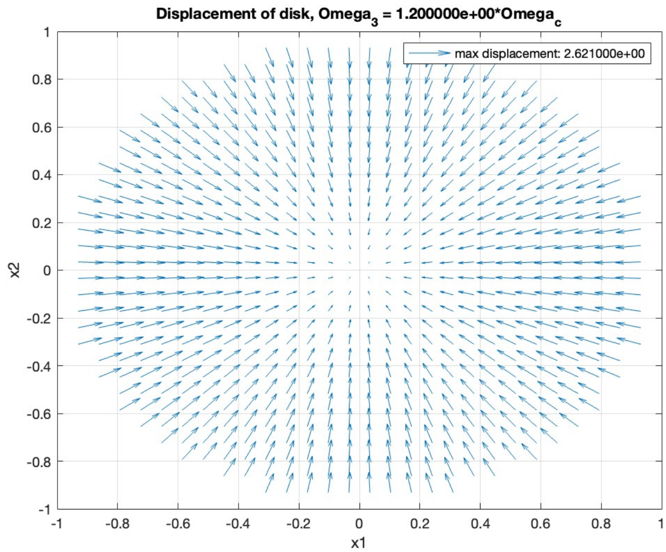

Figure 6: Once the angular velocity exceeds the critical value, the linear system can still be solved but does not produce physically meaningful results. At the critical value of angular velocity, there is no displacement which can equilibrate the applied force. The disk has flown apart.

$$\left(\frac{\underline{U}^{i+1} - 2\underline{U}^{i} + \underline{U}^{i-1}}{\Delta t^2}\right) + 2\dot{\Omega}_3 \underline{\underline{M}}^{-1} \underline{\underline{T}} \left(\frac{\underline{U}^{i+1} - \underline{U}^{i-1}}{2\Delta t}\right) = \underline{\underline{M}}^{-1}\left(\Omega_3^2 \underline{F}^{C} - \dot{\Omega}_3 \underline{F}^{T}\right) - \underline{\underline{M}}^{-1} \underline{F}^{INT}(\underline{U}^{i+1})$$

We have inverted the mass matrix off the second derivative and use two common finite difference approximations of the acceleration and velocity in the Coriolis force term. The force vector from the stresses has been renamed for simplicity and is evaluated at the next time step. This is a nonlinear system of equations for the updated displacement coefficients. If we rename the first term on the right-hand side of the equation $\underline{F}^{EXT}$, the governing equation can be written in residual form as

$$\underline{R} = \underline{U}^{i+1} + \Delta t^2 \underline{\underline{M}}^{-1} \underline{F}^{INT}(\underline{U}^{i+1}) + \Delta t \dot{\Omega}_3 \underline{\underline{M}}^{-1} \underline{\underline{T}} \underline{U}^{i+1} + \left(\underline{U}^{i-1} - 2\underline{U}^{i} - \Delta t^2 \underline{F}^{EXT} - \Delta t \dot{\Omega}_3 \underline{\underline{M}}^{-1} \underline{\underline{T}} \underline{U}^{i-1}\right) = 0 \quad (1)$$

The terms in parentheses are either constant or dependent on past converged values of the displacement. We solve this system with the Newton-Raphson method at each time step. For this, we need the Jacobian matrix given by

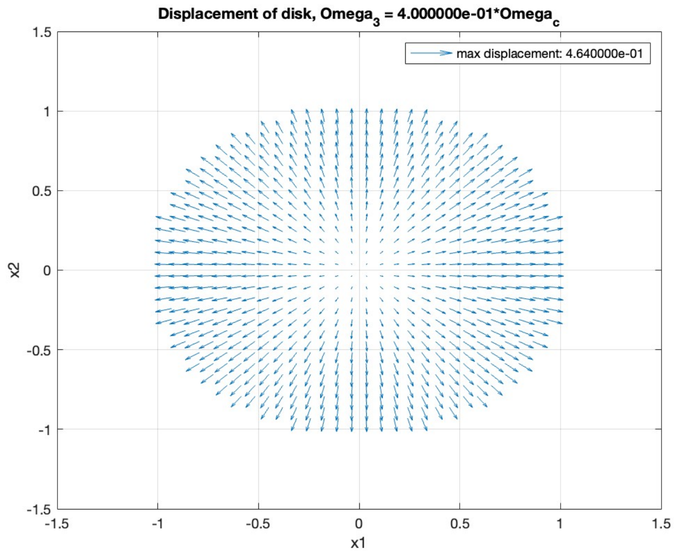

Figure 7: Displacement field for steady state rotation with nonlinear constitutive model. The qualitative form of the displacement is indistinguishable from the linear case. The critical value of the angular velocity is computed using a linearized constitutive relation around zero strain. We expect the nonlinear material to become unstable well before this critical value.

$$\frac{\partial \underline{R}}{\partial \underline{U}^{i+1}} = \underline{\underline{I}} + \Delta t^2 \underline{\underline{M}}^{-1} \frac{\partial \underline{F}^{INT}}{\partial \underline{U}^{i+1}} + \Delta t \underline{\underline{M}}^{-1} \underline{\underline{T}}$$

We have already shown how to compute the derivative of the internal force vector in the steady state problems. A constant angular acceleration is applied, such that $\Omega_3(t) = \Omega_0 t$, and zero initial displacement and velocity are taken as the initial conditions for a disk with traction-free boundaries. See Figure 15 for a plot of the radial and angular displacement at the edge of the plate.

10 Conclusion

A number of cases of the mechanics of a rotating circular plate with free and fixed boundaries were investigated. The governing equations in the rotating frame were derived using the time rate of change of rotating basis vectors in order to compute fictitious forces. A discretization of the displacement field for the 2D solid which was convenient for the dynamics of rotation was introduced and used in the numerical solution. Numerical experiments with linear and nonlinear material models were carried out, with some discussion of dynamic instability. Steady state analyses were conducted first, then the equations of

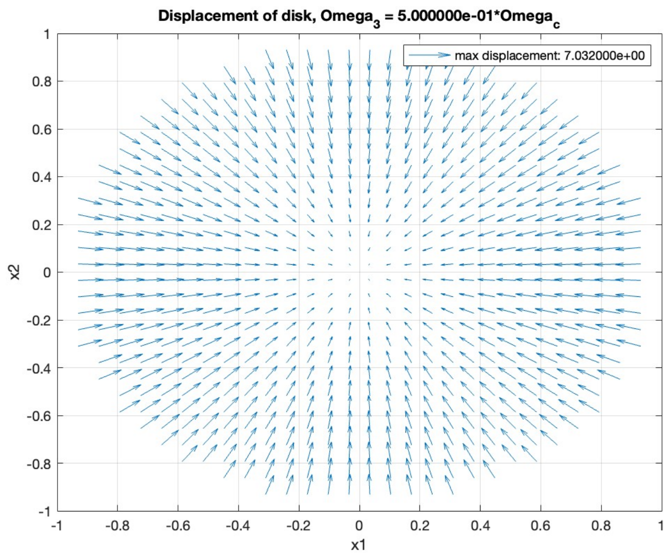

Figure 8: The nonlinear material becomes unstable at half the value of critical angular velocity computed from the linearized constitutive relation. This is due to the softening behavior of the constitutive model.

motion were solved using a backward Euler method. The biggest shortcoming of this work is the nonlinear constitutive model, which is not invariant under rotations to this stress state. Future work might involve the use of a more sophisticated nonlinear constitutive model which does not artificially introduce anistropy into the material, or true elastoplastic analysis which makes use of internal state variables to keep track of yielding.

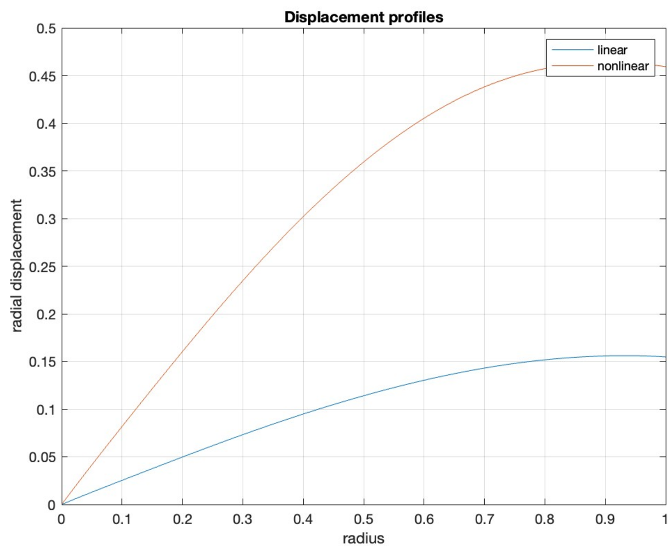

Figure 9: Profiles of the radial displacement comparing the linear and nonlinear material models. The linear model is computed using the initial tangent of the nonlinear model. As expected, the nonlinear model is much less stiff due to the softening behavior.

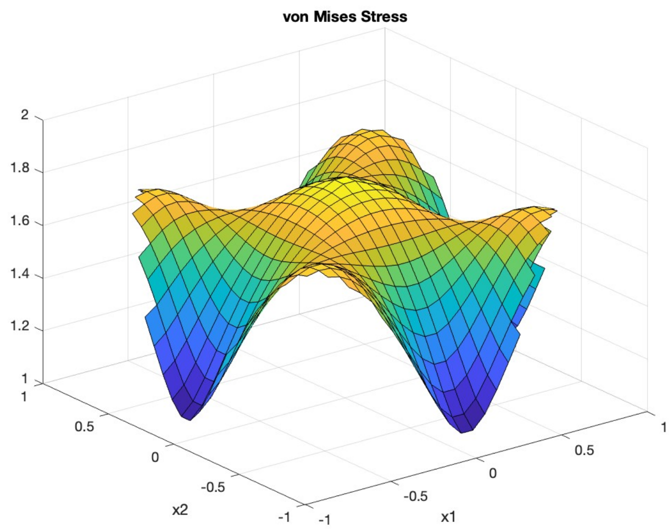

Figure 10: von Mises stress distribution for the rotating disk with the nonlinear constitutive relation. The linear von Mises stress was symmetric under rotations, so we must justify why this one is not. The reason for this is that the nonlinear constitutive relation is not invariant under rotations of the stress state. If we wanted to use an isotropic constitutive relation, it would be necessary to formulate it in terms of invariants of the strain tensor. By neglecting to do this, we have artificially introduced anisotropy into the material. We can either think of this as an oversight with no clear physical basis, or as the constitutive relation implicitly defining some preferred directions in the material. The onset of yielding is sensitive to the stress value, so when the stress state happens to be aligned with the principal directions, we get more yielding than if the stress is in some rotated state where the "total stress" is spread out among the directions. We note that the von Mises stress still has certain symmetries, even if it is not invariant under rotations.

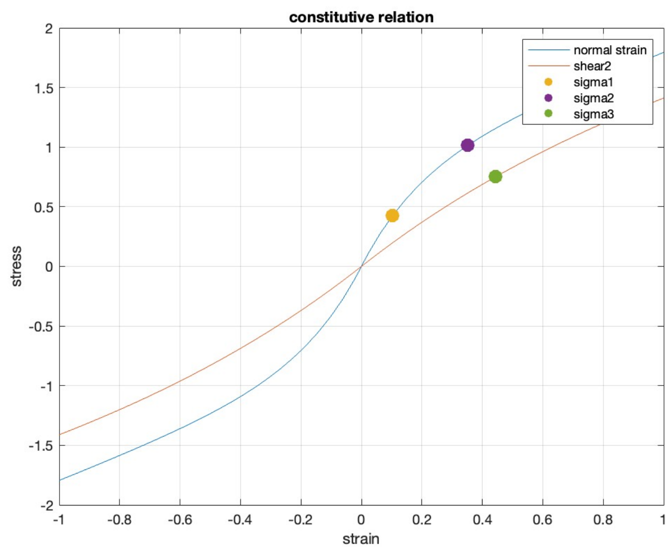

Figure 11: Stress state along edge of disk at position $\underline{x} = [2, 1]/\sqrt{5}$ for $\Omega_3 = 0.4\Omega_c$ indicating that the stress is outside of the initial linear region. Note that because of the geometric instability, the disk often flies apart before advancing well into the hardening region.

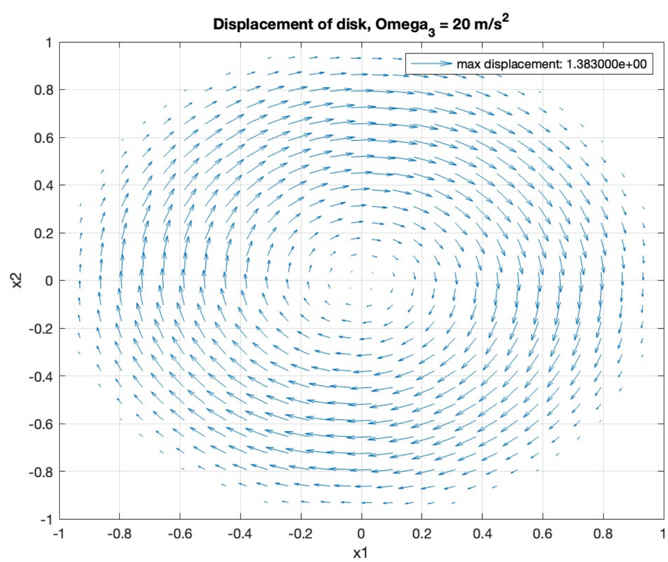

Figure 12: Displacement field for confined disk with centrifugal forces neglected computed using the nonlinear constitutive relation.

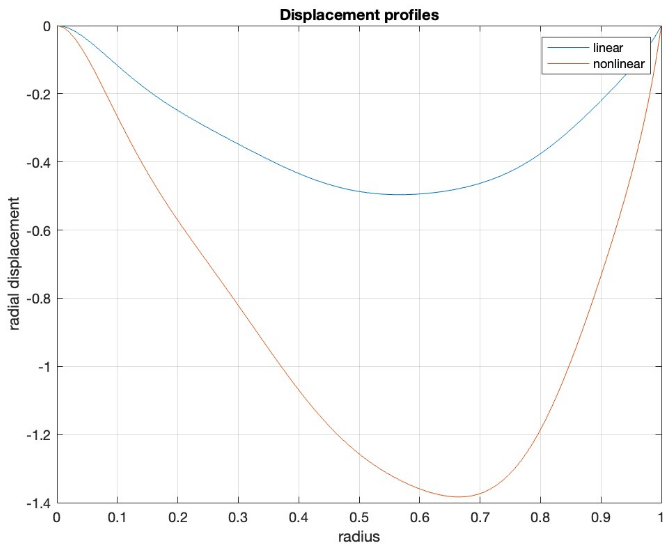

Figure 13: Comparison of linear and nonlinear responses of the confined disk at $\dot{\Omega}_3 = 20m/s^2$ where the linear material model is the initial tangent of the nonlinear model. As expected, the nonlinear model yields and is less stiff.

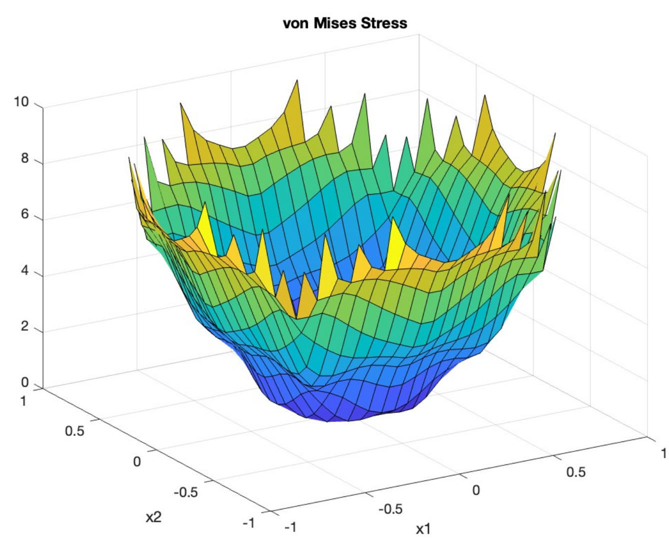

Figure 14: Von Mises stress profile for the confined disk with nonlinear material. The zero displacement boundary conditions cause large displacement gradients and hence large stresses.

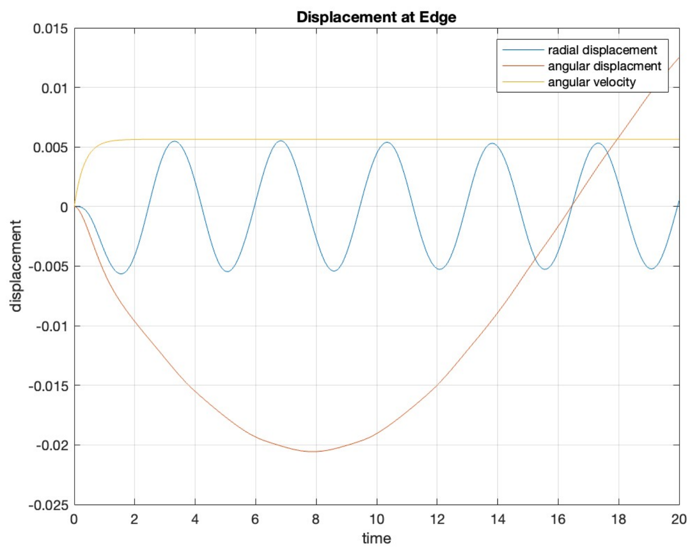

Figure 15: Vibration of a point on the edge of the plate in the radial and angular directions. The applied angular velocity which drives the problem is a saturating exponential. The radial displacement oscillates at a higher frequency than the angular displacement.