Going in circles: returning to the basic physics of rotation

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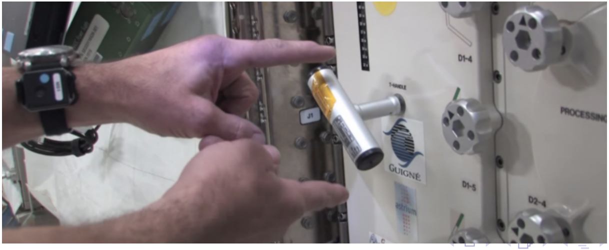
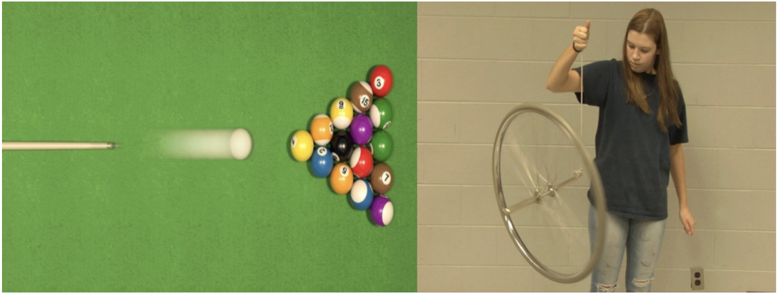
MORIS Meeting



Provocation



- Strange things happen when objects rotate
- Linear momentum is pretty intuitive, angular momentum is not
- **But angular momentum is entirely derivative of linear momentum and is not a separate principle!**

Starting from particles

- Most of mechanics can be derived from the conservation of linear momentum for particles:

$$\mathbf{F} = \frac{\partial}{\partial t}\left(m\frac{\partial \mathbf{x}}{\partial t}\right)$$

- Consider a collection of N particles with masses $\{m_i\}_{i=1}^N$ and positions $\{\mathbf{x}_i(t)\}_{i=1}^N$:

$$\frac{\partial^2}{\partial t^2}\left(\frac{\sum_i m_i \mathbf{x}_i}{\sum_i m_i}\right) = \frac{1}{\sum_i m_i}\sum_i\left(\mathbf{F}_i^{\text{ext}} + \sum_{j\neq i}\mathbf{F}_{ij}\right)$$

- Define $M = \sum_i m_i$ and $\mathbf{X} = \sum_i m_i \mathbf{x}_i / \sum_i m_i$ and note that $\sum_i \sum_{j\neq i} \mathbf{F}_{ij} = \mathbf{0}$ by Newton's third law:

$$M\frac{\partial^2 \mathbf{X}}{\partial t^2} = \sum_i \mathbf{F}_i^{\text{ext}}$$

Interpretation

- External forces cause motion of the center of mass (or vice versa?)
- This result is true for rigid and deformable bodies
- This is a “coarse-graining”—we lose information about the motion of individual particles
- The body can satisfy the coarse-grained linear momentum relation but violate linear momentum conservation at the particle level
- In order to remedy this, we first assume that the collection of particles form a “rigid body,” i.e.

$$\frac{\partial}{\partial t}\frac{1}{2}\|\mathbf{x}_i - \mathbf{x}_j\|^2 = 0, \quad i, j = 1, 2, \dots, N, \quad j \neq i$$
$$\implies (\mathbf{x}_i - \mathbf{x}_j) \cdot (\dot{\mathbf{x}}_i - \dot{\mathbf{x}}_j) = 0$$

- Now decompose motion into (motion of center of mass) + (motion around center of mass)...

Decomposition of motion

- Can be shown that the general form of motion that satisfies the rigid body condition is

$$\dot{\mathbf{x}}_i = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{x}_i$$

- Take time derivative of center of mass to obtain

$$\dot{\mathbf{X}} = \frac{1}{M}\sum_i m_i \dot{\mathbf{x}}_i = \frac{1}{M}\sum_i m_i(\mathbf{V} + \boldsymbol{\omega} \times \mathbf{x}_i) = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{X}$$
$$\implies \mathbf{V} = \dot{\mathbf{X}} - \boldsymbol{\omega} \times \mathbf{X}$$

- Now define center of mass coordinates as $\mathbf{x}_i' = \mathbf{x}_i - \mathbf{X}$ and use the definition of $\mathbf{V}$ to write the velocity of particle i as

$$\dot{\mathbf{x}}_\mathbf{i} = \dot{\mathbf{X}} - \boldsymbol{\omega} \times \mathbf{X} + \boldsymbol{\omega} \times (\mathbf{x}_i' + \mathbf{X}) = \dot{\mathbf{X}} + \boldsymbol{\omega} \times \mathbf{x}_i'$$

- Coarse-grained linear momentum relation governs $\dot{\mathbf{X}}$, what governs the angular velocity $\boldsymbol{\omega}$?

Sidenote: kinetic energy decomposition

- Can use this velocity decomposition in defining the kinetic energy of the (continuous) body Ω:

$$T = \int_\Omega \frac{1}{2}\rho\|\dot{\mathbf{x}}\|^2 d\Omega = \int_\Omega \frac{1}{2}\rho(\|\dot{\mathbf{X}}\|^2 + 2\dot{\mathbf{X}}\cdot(\boldsymbol{\omega}\times\mathbf{x'}) + e_{ijk}\omega_j x'_k e_{i\ell m}\omega_\ell x'_m)d\Omega$$

$$= \frac{1}{2}M\|\dot{\mathbf{X}}\|^2 + 0 + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{I}\boldsymbol{\omega}, \quad \int_\Omega \rho\mathbf{x}' d\Omega = 0, \quad I_{j\ell} := \int_\Omega \rho e_{ijk} e_{i\ell m} x'_k x'_m d\Omega$$

- The kinetic energy of the body decomposes into a purely translational and purely rotational component
- $\mathbf{I}$ is the moment of inertia tensor, which is a property of the body's geometry

Angular momentum of a particle

- Angular momentum equations are obtained by taking the cross product of linear momentum
- For a single particle, this reads

$$\mathbf{x} \times \mathbf{F} = \mathbf{x} \times \frac{\partial}{\partial t}(m\frac{\partial \mathbf{x}}{\partial t}) = \frac{\partial}{\partial t}(\mathbf{x} \times m\frac{\partial \mathbf{x}}{\partial t}) \tag{1}$$

- Define the angular momentum vector for a particle as $\mathbf{L} = \mathbf{x} \times m\frac{\partial \mathbf{x}}{\partial t}$ and the torque as $\boldsymbol{\tau} = \mathbf{x} \times \mathbf{F}$
- This formulation is not useful for a single particle

Angular momentum of body

- The angular momentum about the center of mass of a continuum body is

$$\mathbf{L} = \int_{\Omega} \mathbf{x}' \times \rho \dot{\mathbf{x}}' d\Omega$$

- When the body is rigid, we can use the velocity decomposition to write

$$\mathbf{L} = \int_{\Omega} \mathbf{x}' \times \rho(\boldsymbol{\omega} \times \mathbf{x}') d\Omega = \left(\int_{\Omega} \rho e_{kij} e_{k\ell m} x_j' x_m' d\Omega \right) \omega_\ell = \mathbf{I}\boldsymbol{\omega}$$

- This is the same moment of inertia tensor derived from the rotational kinetic energy

Conservation of angular momentum for a body

- In order for the body to be rigid and in circular motion, there need to be internal "constraint" forces
- Assume particle $\mathbf{x}''$ exerts a force on particle $\mathbf{x}'$ through $\mathbf{c}(\mathbf{x}', \mathbf{x}'')$:

$$\frac{\partial}{\partial t}\mathbf{L} = \frac{\partial}{\partial t}(\mathbf{I}\boldsymbol{\omega}) = \underbrace{\int_{\Omega} \mathbf{x}' \times \mathbf{f}\, d\Omega}_{\text{external torque}} + \underbrace{\int_{\Omega} \mathbf{x}' \times \left(\int_{\Omega'} \mathbf{c}(\mathbf{x}', \mathbf{x}'') d\Omega' \right) d\Omega}_{\text{internal torque}}$$

- The external forces $\mathbf{f}$ are often taken to move with the body and thus the external torque can be time-independent even with rotations

Dealing with the internal torque term

- The net torque from internal forces drops out in certain circumstances
- To see this, consider the discretized form of the internal torque and use Newton's third law:

$$
\begin{aligned}
\boldsymbol{\tau}^{\text{int}} = \sum_i \left(\mathbf{x}_i' \times \sum_j \mathbf{c}_{ij} \right) &= \mathbf{x}_1' \times (\mathbf{c}_{12} + \mathbf{c}_{13} + \dots) + x_2' \times (\mathbf{c}_{21} + \mathbf{c}_{23} + \dots) \\
&= (\mathbf{x}_1' \times \mathbf{c}_{12} + \mathbf{x}_2' \times \mathbf{c}_{21}) + (\mathbf{x}_1' \times \mathbf{c}_{13} + \mathbf{x}_3' \times \mathbf{c}_{31}) \\
&= (\mathbf{x}_1' - \mathbf{x}_2') \times \mathbf{c}_{12}
\end{aligned}
$$

- If the force lies along the relative position vector between the points, each cross product in the sum that forms the internal torque is zero
- This means that the constraint forces which maintain rigidity of the body do not show up in angular momentum conservation

Summary

- Angular momentum conservation for the body is the rotational analogue of the statement that external forces cause motion of the center of mass
- This is the extra requirement needed such that linear momentum at the particle level is satisfied
- This equation governs the angular velocity $\boldsymbol{\omega}$, which fully determines the velocity field of a rigid body
- Conservation of angular momentum reads:

$$\frac{\partial}{\partial t}(\mathbf{I}\boldsymbol{\omega}) = \boldsymbol{\tau}^{\text{ext}} \tag{2}$$

- Most intuitive to the angular momentum and torques about the center of mass of the body

Common simplifications

- Note that an identity for the product of permutation symbols can be used to write

$$I_{i\ell} = \int_\Omega \rho e_{kij} e_{k\ell m} x'_j x'_m d\Omega = \int_\Omega \rho(\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) x'_j x'_m d\Omega$$
$$= \int_\Omega \rho(\delta_{i\ell} x'_m x'_m - x_\ell x_i) d\Omega$$

- The moment of inertia tensor is diagonal when $\int_\Omega \rho x'_\ell x'_i d\Omega = 0$
- Such a coordinate system is found by solving an eigenvalue problem for the "principal axes"
- When the axis of rotation is fixed around x'_3, the coordinate system is aligned with principal axes, and the body is rigid, the governing equation for the angular velocity is

$$I_{33}\dot{\omega}_3 = \tau_3, \quad I_{33} = \int_\Omega \rho(x_1'^2 + x_2'^2) d\Omega$$

Rate of change of angular momentum

- For an arbitrary rotation, both the moment of inertia tensor and angular velocity vector are time-dependent:

$$\frac{\partial}{\partial t}\mathbf{L} = \frac{\partial \mathbf{I}}{\partial t}\boldsymbol{\omega} + \mathbf{I}\frac{\partial \boldsymbol{\omega}}{\partial t} = \boldsymbol{\tau}^{\text{ext}}$$

- The moment of inertia tensor changes in time because the body changes orientation
- Use the Leibniz rule for an integral whose bounds are time-varying:

$$\frac{\partial \mathbf{I}}{\partial t} = \frac{\partial}{\partial t}\left(\int_{\Omega(t)} \rho e_{kij} e_{k\ell m} x'_j x'_m d\Omega\right) = \int_{\partial\Omega} (\rho e_{kij} e_{k\ell m} x'_j x'_m)(\boldsymbol{\omega} \times \mathbf{x}') \cdot \mathbf{n} dS$$

- This requires finding the surface at each point in time and computing integrals over the body...
- The geometry of the body is independent of time if we choose a coordinate system that moves with the body

Time derivative of a quantity in rotating frame

- Define basis vectors $\{\hat{\mathbf{a}}_i\}_{i=1}^3$ which rotate with the body (as opposed to a standard coordinate system is fixed in space)
- Rates of changes of a vector quantity $\mathbf{q}(\mathbf{a})$ arise from movement within the coordinate system and movement of the coordinate system itself

$$\frac{\partial}{\partial t}(q_i \hat{a}_i) = \frac{\partial q_i}{\partial t}\hat{\mathbf{a}}_i + q_i \frac{\partial \hat{\mathbf{a}}_i}{\partial t} = \frac{\partial q_i}{\partial t}\hat{\mathbf{a}}_i + q_i(\boldsymbol{\omega} \times \hat{\mathbf{a}}_i) = \frac{\partial \mathbf{q}}{\partial t} + \boldsymbol{\omega} \times \mathbf{q}$$

- When the body is rigid, the moment of inertia tensor is time-independent in the $\mathbf{a}$ coordinate system, so conservation of angular momentum becomes

$$\frac{\partial}{\partial t}\mathbf{L} = \frac{\partial}{\partial t}(\mathbf{I}\boldsymbol{\omega}) = \mathbf{I}\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{\omega} \times (\mathbf{I}\boldsymbol{\omega}) = \boldsymbol{\tau}^{\text{ext}}$$

- This is the non-linear Euler equation for rotational motion, which predicts interesting behavior such as precession

Deformable bodies

- A rotating body necessarily experiences internal forces to prevent it from flying apart (material points want to move in straight lines)
- These forces will cause deformations, which introduce time-dependence into the moment of inertia tensor even in the body axes
- There is thus a two-way coupling between the equation for the angular velocity and the displacement field
- When the deformations are small, we can solve for the angular velocity independent of the displacement, then compute the displacement with stress equilibrium in a rotating frame:

$$\rho\left(\ddot{\mathbf{u}} + \underbrace{\dot{\boldsymbol{\omega}} \times (\mathbf{a}+\mathbf{u})}_{\text{Euler}} + \underbrace{2(\boldsymbol{\omega} \times \dot{\mathbf{u}})}_{\text{Coriolis}} + \underbrace{\boldsymbol{\omega} \times \boldsymbol{\omega} \times (\mathbf{a}+\mathbf{u})}_{\text{Centripetal}}\right) = \nabla \cdot \boldsymbol{\sigma} + \mathbf{b}(\mathbf{a})$$

- The time derivatives computed in body coordinate system introduce fictitious forces

Euler equation in principal basis

- When the body coordinate system is aligned with principal axes, the moment of inertia tensor is diagonal and the Euler equation in the absence of torques is

$$
\begin{aligned}
I_1\dot{\omega}_1 + \omega_2\omega_3(I_3 - I_2) &= 0 \\
I_2\dot{\omega}_2 + \omega_1\omega_3(I_1 - I_3) &= 0 \\
I_3\dot{\omega}_3 + \omega_1\omega_2(I_2 - I_1) &= 0
\end{aligned}
$$

- Assume that the moments of inertia around the principal axes are ordered so that $I_1 < I_2 < I_3$

Intermediate axis theorem

- Try to spin your phone around each of the three principal axes...it is only possible around the axes with the largest and smallest moments of inertia!
- To see this, consider perturbations $\delta_1(t)$ and $\delta_2(t)$ introduced to angular velocity $\Omega(t)$ around the intermediate axis

$$\omega_1 = \delta_1(t), \quad \omega_2 = \Omega(t), \quad \omega_3 = \delta_3(t)$$

- Plugging into the second equation:

$$I_2\dot{\Omega} = -\delta_1\delta_3(I_1 - I_3) \approx 0 \implies \Omega(t) = \Omega_0 \tag{3}$$

- Plugging into the first and third equations, differentiating the first and substituting the third, we obtain

$$I_1\ddot{\delta}_1 + \frac{\Omega_0^2}{I_3}(I_3 - I_2)(I_1 - I_2)\delta_1 = 0 \tag{4}$$

Intermediate axis theorem (cont.)

- Based on the relative sizes of the moments of inertia, the coefficient on δ_1 is negative
- This corresponds to exponential growth of the perturbation
- Perturbed rotations about the other axes give rise to exponential decay of the perturbation
- Torque free rotation around the intermediate axis is unstable, rotation around the other axes is stable

Momentum conservation in a continuum body

- Conservation of linear momentum of a "chunk" of a continuum is a statement about the motion of the center of mass:

$$\frac{\partial}{\partial t}\int_{\Omega}\rho(\mathbf{x})\mathbf{v}(\mathbf{x})d\Omega = \int_{\Omega}\mathbf{b}(\mathbf{x})d\Omega + \int_{\partial\Omega}\mathbf{t}(\mathbf{x})dS$$

- Center of mass motion is a coarse-grained statement of momentum laws, need angular momentum to ensure that motion is physical:

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\Omega}\mathbf{x}\times\rho(\mathbf{x})\mathbf{v}(\mathbf{x})d\Omega &= \int_{\Omega}\mathbf{x}\times\mathbf{b}(\mathbf{x})d\Omega + \int_{\partial\Omega}\mathbf{x}\times\mathbf{t}(\mathbf{x})dS \\ = \int_{\Omega} e_{ijk}x_j\rho\frac{\partial v_k}{\partial t}d\Omega &= \int_{\Omega} e_{ijk}x_j b_k + e_{ijk}x_j\frac{\partial \sigma_{k\ell}}{\partial x_\ell} + e_{ijk}\sigma_{kj}d\Omega \\ &\implies \int e_{ijk}\sigma_{kj}d\Omega = 0\end{aligned}$$

- This shows that the symmetry of the stress tensor ensures conservation of angular momentum

Outstanding questions

- Angular momentum for deformable bodies
- Make sense of the bike wheel experiment
- Do a precession problem
- Asymmetry of the first Piola-Kirchhoff stress tensor?