Variational Multiscale Method

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1 Brief Notes

To illustrate the basic ideas of VMS, assume we have a multiscale energy for a one dimensional solid such as

$$\Pi = \int \left(\frac{1}{2} k(x;\epsilon) u_x^2 - fu \right) dx$$

where the material has some multiscale properties such as

$$k(x;\epsilon) = k_0(1 + a\sin(x/\epsilon))$$

The condition for an extremum of the energy is

$$\delta\Pi = \frac{\partial \Pi}{\partial u_x} w_x + \frac{\partial \Pi}{\partial u} w = 0$$

$$0 = \int \left(k(x) u_x w_x - fw \right) dx$$

The VMS framework decomposes the solution and test function into a coarse and fine component:

$$u(x) = \bar{u} + u', \quad w(x) = \bar{w} + w'$$

Over bars indicate the coarse (large scale) component, and primes indicate fine scale fluctuations. Plugging this into the weak form:

$$= \int k(x)(\bar{u}_x + u'_x)(\bar{w}_x + w'_x) - f(\bar{w} + w') dx$$

The coarse and fine components of the test function are both arbitrary, so this gives rise to two separate equations:

$$\int k(x) \Big(\bar{u}_x + u'_x \Big) \bar{w}_x dx = \int f \bar{w} dx$$

$$\int k(x)\Big(\bar{u}_x + u'_x\Big)w'_x dx = \int fw' dx$$

These are coarse and fine equations respectively. The fine scale equation can be rearranged to read

$$\int k(x)u'_x w'_x dx = \int fw' - k(x)\bar{u}_x w'_x dx = \int \Big(k(x)\bar{u}_{xx} - f\Big)w' dx$$

The fine scale correction is a function of the residual of the coarse solution (failure of coarse solution to satisfy equilibrium). If we integrate by parts derivatives off of the test functions and rid of the integrals, we have that

$$\mathcal{L}u' = -\Big(\mathcal{L}\bar{u} - f\Big)$$

where the differential operator is defined as

$$\mathcal{L} := -\frac{\partial}{\partial x}\left(k(x)\frac{\partial}{\partial x}\right)$$

We now assume that we have some way of finding or approximating the inverse of the operator in the fine scale setting:

$$u' = \mathcal{L}^{-1}(f - \mathcal{L}\bar{u}) \approx \mathcal{M}(f - \mathcal{L}\bar{u})$$

Thus the fine scale correction can be written as a function of the coarse solution. This is what people refer to as "modeling" the fin escale. It is not clear how the inverse operator $\mathcal{M}$ is obtained–this is the trick of VMS. However, once a suitable representation of the inverse operator is in hand, we can substitute this expression for the fine scale solution into the coarse scale equation:

$$\int k(x)\left(\bar{u}_x + \frac{\partial}{\partial x}\mathcal{M}(f - \mathcal{L}\bar{u})\right)\bar{w}_x dx = \int f\bar{w} dx$$

This weak form equation is fully defined in terms of the coarse solution $\bar{u}(x)$. One technique for obtaining the inverse operator is to use "fine scale green's functions" denoted as $g'(x, y)$. If the fine scale green's functions are known, then the fine scale solution can be written explicitly as a function of the coarse residual:

$$u'(x) = \int g'(x, y)\Big(f - \mathcal{L}\bar{u}\Big)(y) dy$$

If the material is constant but the force has multiscale behavior, there will be analytical expressions available for the green's functions. However, when the material is variable, the fine scale green's functions can be approximated numerically. Note that VMS does not require assumptions on scale separation. In spirit, it seems similar to multiscale finite elements, as they both allow finite sub-grid corrections to the finite element solution. The multiscale finite element method may require assumptions on sub-grid correction being zero along element boundaries, which is not necessary in VMS.