Coupled Viscoelastic Torsion and Heat Conduction

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1 Introduction

A viscoelastic material exhibits explicit time-dependence in its mechanical properties. For example, the stress state in a viscoelastic solid will change with time for constant strain. For many viscoelastic solids, this time dependence might be conceptualized as a "flow" process in which the material generates stresses through both displacement and velocity. This demonstrates that the constitutive behavior of the material has both fluid and solid aspects. Many common engineering materials such as polymers, concrete, and metals exhibit viscoelastic behavior to varying degrees. One interesting feature of viscoelasticity is that deformation of the solid dissipates mechanical energy. In a perfectly elastic material, vibrations will continue indefinitely and the total kinetic energy of the system will not change in time. In contrast, vibrations in a viscoelastic solid will decay over time and kinetic energy will decrease. In the absence of interactions with the environment, the total energy of the system is conserved, thus the dissipative nature of a viscoelastic material acts to convert kinetic energy of the solid to thermal energy. These simple energy considerations establish that viscoelastic materials couple the mechanical and thermal response of a material–in the presence of dissipative constitutive behavior, kinetic energy associated with deformation is converted into heat. Thus, a vibrating viscoelastic body will change temperature. Furthermore, the constitutive response of many viscoelastic materials is dependent on the temperature. These observations motivate a two-way coupled mechanical and thermal problem for a vibrating viscoelastic body–deformations produce heat inputs, these heat inputs change the temperature of the body, the temperature influences the material properties, and the material properties determine the mechanical response. *The goal of this report is to explore computational methods for solving this two-way coupled problem using the example of forced torsional vibrations of a composite shaft.*

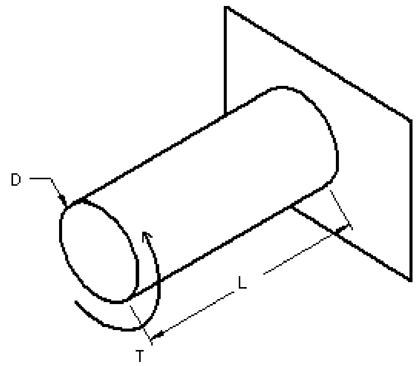

Figure 1: A cylindrical shaft of length L fixed to a wall on one end is driven by a time-varying torque $T(t)$ on the other end. Modeling the shaft as a continuous system, torsional vibrations are described with the angular displacement $\theta(x, t)$.

2 Problem Overview

We want to model the dynamic response of a cylindrical shift to an applied end torque. The usual kinematic assumptions for torsion will be used to reduce this problem to one spatial dimension and one displacement component. See Figure 1 for a schematic of the torsion problem. Similarly, the heat conduction problem will be reduced to one spatial dimension for simplicity. As will be explained below, this will involve integrating the dissipated mechanical energy from the deformation over the cross-section, and using this quantity as a heat input for the one-dimensional heat equation. The shaft will be made of a viscoelastic composite, but the composite nature of the material will only show up in determining an effective (homogenized) thermal conductivity value. The only material property that shows up in the torsional model is the material's shear modulus G, but rigorously determining effective viscoelastic shear properties of a composite microstructure is inherently three-dimensional, and quite a challenging task. That being said, an exploration of periodic homogenization in the context of viscoelasticity will be included in an appendix. Similarly, a derivation of periodic homogenization theory for thermal problems is included in the appendix. This can be used to compute an effective conductivity of a two phase composite for the one-dimensional heat transfer problem. The composite microstructure will be a matrix material with a "stiff" inclusion in the center. See Figure 2 for an example of this type of microstructure.

There are a variety of moving parts to the coupled thermal/structural torsion problem. We need to develop the following theoretical tools with an eye towards numerical implementation:

- Equations of motion for viscoelastic torsion

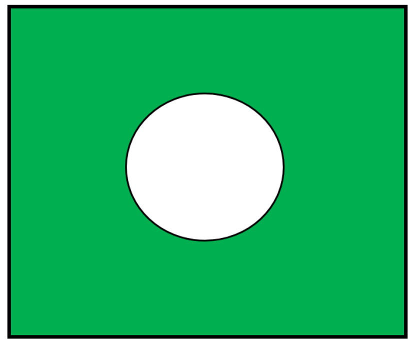

Figure 2: Two-phase composite microstructure with circular inclusion in two spatial dimensions. This will be further simplified to a one-dimensional microstructure with a central inclusion.

- Viscoelastic material model
- Derivation of heat input from dissipation of mechanical energy
- Governing equations for one-dimensional heat transfer
- Effective conductivity for two-phase composite microstructure
- Two-way coupling scheme
- Implementation in MATLAB

The following sections build out the theoretical infrastructure for each of these items. Governing equations are first stated in continuous form, and then discretized. Spectral shape functions which respect the displacement boundary conditions are used to discretize governing equations for simplicity. The goal is to keep numerical implementation as simple as possible while still capturing the complexity of the coupled physics.

3 Viscoelastic Torsion

Consider a cylindrical rod with a constant cross-section undergoing torsion around its long axis. Define this axis to be x_1. By assumption, the displacement field is

$$u_1 = 0, \quad u_2 = -\theta(x_1)x_3, \quad u_3 = \theta(x_1)x_2$$

where $\theta(x_1)$ is the rotation angle of the cross-section at position x_1. Using the infinitesimal strain-displacement relation $\epsilon_{ij} = (u_{i,j} + u_{j,i})/2$, the following strain components are shown to be zero

$$\epsilon_{11} = \epsilon_{22} = \epsilon_{33} = \epsilon_{23} = \epsilon_{32} = 0$$

Using the parameterization of the displacement field in terms of the rotation angle θ, the non-zero components of strain can be written as

$$\epsilon_{12} = \epsilon_{21} = -\frac{1}{2}\frac{\partial \theta}{\partial x_1}x_3, \quad \epsilon_{13} = \epsilon_{31} = \frac{1}{2}\frac{\partial \theta}{\partial x_1}x_2$$

We will assume that the shear modulus is isotropic and does not vary in space. The material is viscoelastic so the stress-strain relation is given by the Boltzmann integral. With a given shear relaxation modulus $G(t)$, the non-zero stress components are

$$\sigma_{12}(t) = \sigma_{21}(t) = \int_0^t 2G(t-\tau)\frac{\partial \epsilon_{12}}{\partial \tau}d\tau = -x_3\int_0^t G(t-\tau)\frac{\partial^2 \theta}{\partial x_1 \partial \tau}d\tau$$

$$\sigma_{13}(t) = \sigma_{31}(t) = \int_0^t 2G(t-\tau)\frac{\partial \epsilon_{13}}{\partial \tau}d\tau = x_2\int_0^t G(t-\tau)\frac{\partial^2 \theta}{\partial x_1 \partial \tau}d\tau$$

The factor of 2 appears because the tensorial shear strain components are half the true strain. The problem is dynamic, so the rotation angle has space and time dependence. We now want to derive the governing equation for the dynamics of viscoelastic torsion. The material is dissipative, so it is not clear whether an energy/variational principle can be used to arrive at a governing equation of motion. Thus, we start with generic three-dimensional stress equilibrium in the absence of body forces

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j}$$

We multiply by a vector-valued test function δu_i and integrate over the volume of the cylindrical shaft to weaken the governing equation:

$$\int_V \left(\rho\frac{\partial^2 u_i}{\partial t^2} - \frac{\partial \sigma_{ij}}{\partial x_j}\right)\delta u_i dV = 0$$

Remembering that the displacement field is parameterized by the scalar rotation angle θ, we can write

$$\delta u_i = \frac{\partial u_i}{\partial \theta}\delta\theta = \begin{bmatrix} 0 \\ -x_3 \\ x_2 \end{bmatrix}\delta\theta$$

Substituting this and the expression for the two non-zero displacement components, the weak form of the governing equations becomes

$$\int_0^L \int_A \left(-\rho x_3\ddot{\theta} - \frac{\partial \sigma_{2j}}{\partial x_j}\right)(-x_3\delta\theta) + \left(\rho x_2\ddot{\theta} - \frac{\partial \sigma_{3j}}{\partial x_j}\right)(x_2\delta\theta)dAdx_1 = 0$$

Define $I := \int_A x_2^2 + x_3^2 dA$ and plug in the definition of the stress components

$$= \int_0^L \rho I \ddot{\theta} \delta\theta dx_1 + \int_0^L \int_A \left(-x_3^2 \int_0^t G(t-\tau) \frac{\partial^3 \theta}{\partial x_1^2 \partial \tau} d\tau - x_2^2 \int_0^t G(t-\tau) \frac{\partial^3 \theta}{\partial x_1^2 \partial \tau} d\tau \right) \delta\theta dA dx_1$$

$$= \int_0^L \rho I \ddot{\theta} \delta\theta dx_1 - \int_0^L I \left(\int_0^t G(t-\tau) \frac{\partial^3 \theta}{\partial x_1^2 \partial \tau} d\tau \right) \delta\theta dx_1$$

Now integrate by parts the spatial derivative onto the test function $\delta\theta$. We assume that the rotation angle is zero at $x_1 = 0$ (wall support) and there is an applied torque at the end of the rod. The weak form of the governing equation for viscoelastic torsion is

$$\int_0^L \rho I \ddot{\theta} \delta\theta dx_1 + \int_0^L I \left(\int_0^t G(t-\tau) \frac{\partial^2 \theta}{\partial x_1 \partial \tau} d\tau \right) \frac{\partial \delta\theta}{\partial x_1} dx_1 = M(t) \delta\theta(L)$$

The applied torque $M(t)$ comes from the boundary term of integration by parts. If we discretize the test function with $\delta\theta = \sum_j w_j f_j(x_1)$ and use the fact that the coefficients w_j are arbitrary, we obtain the following system of equations:

$$\int_0^L \rho I \ddot{\theta} f_j dx_1 + \int_0^L I \left(\int_0^t G(t-\tau) \frac{\partial^2 \theta}{\partial x_1 \partial \tau} d\tau \right) \frac{\partial f_j}{\partial x_1} dx_1 = M(t) f_j(L)$$

We will discretize with global shape functions that respect the wall boundary condition by construction, and enforce the applied torque boundary condition weakly. Thus, we use polynomial shape functions of the form

$$f_j = \left(\frac{x_1}{L} \right)^k$$

The rotation angle has both space and time components. It can be discretized with

$$\theta(x_1, t) = \sum_i \theta_i(t) f_i(x_1)$$

Before we plug this into the weak form, we must remember that the shear relaxation modulus is temperature dependent, and the temperature distribution will vary in space. We must make the substitution

$$G(t) \to G(T(x_1, t), t)$$

The shear relaxation modulus picks up spatial dependence implicitly through the temperature field. We now ask: is the time variable inside the temperature replaced by $t - \tau$ when substituting into the Boltzmann integral? We will argue that this is not the case: the strain at time τ should contribute to the stress at the current time t via the modulus $G(T(x_1,\tau), t-\tau)$. This can be interpreted to mean that a past strain increment contributes to the current stress according to the material properties it saw at the moment of application. Substituting this and the discretization of the rotation angle, we finally obtain

$$\sum_i \frac{\partial^2 \theta_i}{\partial t^2} \int_0^L \rho I f_i f_j dx_1 + \sum_i I \int_0^L \frac{\partial f_i}{\partial x_1} \frac{\partial f_j}{\partial x_1} \left(\int_0^t G(T(x_1,\tau), t-\tau) \frac{\partial \theta_i}{\partial \tau} d\tau \right) dx_1 = M(t) f_j(L)$$

Define the following quantities in order to make this equation more readable:

$$M^\theta_{ij} := \int_0^L \rho I f_i f_j dx_1$$

$$F^\theta_j(t) := M(t) f_j(L)$$

$$S^\theta_j(t) := I \int_0^L \frac{\partial f_j}{\partial x_1} \sum_i \frac{\partial f_i}{\partial x_1} \left(\int_0^t G(T(x_1,\tau), t-\tau) \frac{\partial \theta_i}{\partial \tau} d\tau \right) dx_1$$

Using this notation, the system of equations governing the time evolution of the degrees of freedom is

$$\underline{\underline{M}}^\theta \underline{\ddot{\theta}} + \underline{S}^\theta(\underline{\theta}, t) = \underline{F}(t)$$

We can approximate the second derivative with

$$\ddot{f} = \frac{1}{\Delta t^2} \Big(f(t+\Delta t) - 2f(t) + f(t-\Delta t) \Big)$$

This gives a simple updating scheme to solve the governing system of equations. This reads

$$\underline{\theta}(t+\Delta t) = \Delta t^2 \underline{\underline{M}}^{\theta,-1} \Big(\underline{F}^\theta(t) - \underline{S}^\theta(\underline{\theta}, t) \Big) + 2\underline{\theta}(t) - \underline{\theta}(t-\Delta t)$$

Note that this method of time integration is unstable and requires very small time steps for good results. Unlike in traditional elasticity, forming the viscoelastic "internal force vector" $\underline{\underline{S}}^\theta$ requires a convolution integral and thus makes use of the entire displacement history. Because the shear modulus has spatial variation, a stiffness matrix cannot be pre-computed, further adding to the computational expense of this model.

4 Material Model

We will use a standard linear solid constitutive model for the viscoelastic shear response of the composite rod. This model is simple but reasonably realistic in capturing relaxation and creep behavior. Unfortunately, we neglect the multiscale nature of the composite in modeling its viscoelastic constitutive behavior and opt for doing homogenization on the thermal problem. This could

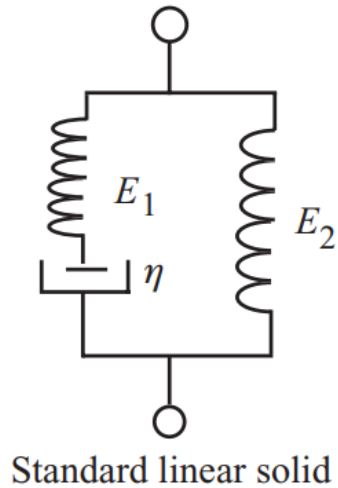

Figure 3: Circuit diagram for the standard linear solid model of a viscoelastic stress-strain relation. Strain is analogous to voltage, and is thus constant across the two branches. Stress is analogous to current, and is divided unequally between the two branches.

be justified on physical grounds by arguing awkwardly that the inclusion of Figure 2 has the same mechanical properties as the matrix but different thermal properties. See Figure 3 for the circuit diagram of the standard linear solid in tension/compression. Departing from the notation of the figure, we will use G to denote shear modulus as is conventional. The (engineering) shear strain is the same across each branch and will be denoted γ. We will call the stress in the left branch σ_1 and the stress in the right branch σ_2. Using the constitutive relations of springs and dampers, the strain rate in the left branch is

$$\frac{\partial \gamma}{\partial t} = \frac{1}{G_1}\frac{\partial \sigma_1}{\partial t} + \frac{\sigma_1}{\eta}$$

In the right branch, the strain rate is

$$\frac{\partial \gamma}{\partial t} = \frac{1}{G_2}\frac{\partial \sigma_2}{\partial t}$$

Multiplying the equations for each branch with their respective moduli and adding them together, we obtain

$$(G_1 + G_2)\frac{\partial \gamma}{\partial t} = \frac{\partial}{\partial t}(\sigma_1 + \sigma_2) + \frac{G_1\sigma_1}{\eta}$$

We can add $G_1 G_2 \gamma/\eta = E_1\sigma_2/\eta$ to both sides of this equation and use that the total stress is $\sigma = \sigma_1 + \sigma_2$ to get

$$(G_1 + G_2)\frac{\partial \gamma}{\partial t} + \frac{G_1 G_2}{\eta}\gamma = \frac{\partial \sigma}{\partial t} + \frac{G_1}{\eta}\sigma$$

We are looking for the relaxation modulus $G(t)$, which describes the time-varying stress response to a strain step function of unit magnitude. The Laplace Transform can be used to compute the relaxation modulus for this system. It can be shown that

$$G(t) = G_2 + G_1 e^{-tG_1/\eta}$$

This is the relaxation modulus for the standard linear solid. We now must incorporate thermal effects into the material's constitutive relation. We will argue that the limiting elastic stiffness G_2 is a decreasing function of the temperature. The viscoelastic component of the material response will be held constant with temperature. The temperature dependence of the material properties introduces the two-way coupling into the viscoelastic torsion problem– energy dissipated to heat is a function of the viscoelastic constitutive model, and the constitutive model is a function of the rod's temperature. The assumption that only the elastic stiffness varies with temperature is motivated by simplicity rather than loyalty to the real phenomena. Real viscoelastic materials will have more complex constitutive models and temperature dependence, but the point of this report is to showcase computational methods, not high-fidelity constitutive modeling. Thus, the viscoelastic material model is

$$G(T,t) = G_2(T) + G_1 e^{-tG_1/\eta}$$

One reasonable choice of temperature dependence for the elastic stiffness is

$$G_2(T) = (G_{2i} - G_{2f})e^{-rT} + G_{2f}$$

where G_{2f} is the asymptotic value of the elastic stiffness and G_{2i} is the initial value for $T = 0$. The parameter r controls the rate at which the asymptotic value is approached. Using this form of the temperature dependence, we have

$$G(T,t) = G_{2f} + (G_{2i} - G_{2f})e^{-rT} + G_1 e^{-tG_1/\eta}$$

5 Heat Input

We will consider all dissipation to come from the damper in the material model of Figure 3. All energy that goes into the damper is not recoverable and is therefore dissipated in the form of heat. The power associated with the damper is

$$p = \sigma_1 \frac{\partial \gamma^d}{\partial t}$$

where σ_1 is the stress in left branch of the circuit and γ^d is the strain associated with the damper in the left branch. This expression comes from the fact that though the stress is constant over the circuit elements in a given branch, the strain varies from one element to the next within a branch. Note that the strain rate through the damper has the simple form of

$$\frac{\partial \gamma^d}{\partial t} = \frac{\sigma_1}{\eta}$$

Given that the total stress is the sum of the stresses from the two branches, we can write

$$\sigma_1 = \sigma - G_2 \gamma$$

For a generic 3D stress state, the stress power is

$$p(x_1, x_2, x_3, t) = \sigma_{ij}^d \dot{\epsilon}_{ij}^d$$

where the superscript "d" indicates the stress and strain associated with the damper. For torsion, the stress power can be written as

$$p(x_1, x_2, x_3, t) = 2\sigma_{13}^d \dot{\epsilon}_{13}^d + 2\sigma_{12}^d \dot{\epsilon}_{12}^d = \frac{2}{\eta}\Big((\sigma_{13}^d)^2 + (\sigma_{12}^d)^2\Big)$$

And given the standard linear solid material model, the stresses associated with the damper can be written as

$$\sigma_{13}^d = \sigma_{13} - 2G_2\epsilon_{13}, \quad \sigma_{12}^d = \sigma_{12} - 2G_2\epsilon_{12}$$

The factor of 2 relates the tensorial shear strains to the engineering shear strains which multiply the shear modulus to give stresses. By integrating over the cross-section, we can compute the heat input to the one-dimensional heat conduction problem:

$$\bar{p}(x_1, t) = \frac{1}{A}\int_A p(x_1, x_2, x_3, t) dA = \frac{2}{A\eta}\int_A (\sigma_{13}^d)^2 + (\sigma_{12}^d)^2 dA$$

We can substitute the definition of the strain components and the Boltzmann integral for the stresses. After some simplification, we arrive at

$$\bar{p}(x_1, t) = \frac{2I}{A\eta}\left(\int_0^t G(t-\tau)\frac{\partial^2 \theta}{\partial x_1 \partial \tau} d\tau - G_2 \frac{\partial \theta}{\partial x_1}\right)^2$$

Using the discretized form of the rotation angle and the fact that the modulus depends on the temperature distribution, this expression can be written in its final form as

$$\bar{p}(x_1, t) = \frac{2I}{A\eta}\left[\sum_j \frac{\partial f_j}{\partial x_1}\left(\int_0^t G(T(x_1, \tau), t-\tau)\frac{\partial \theta_j}{\partial \tau} d\tau - G_2(T(x_1, t))\theta_j(t)\right)\right]^2$$

6 Heat Equation

The governing equation for isotropic heat conduction in three spatial dimensions is

$$\frac{\partial T}{\partial t} = a\nabla^2 T + p$$

We assume the temperature is constant over the cross-sections of the rod, so the temperature only varies in the axial direction. The parameter a is the thermal conductivity and $p(x_1, x_2, x_3)$ is a heat source term supplied by the viscoelastic deformation. This equation reduces to

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x_1^2} + p$$

We can weaken the governing equation by multiplying by an arbitrary test function w and integrating over the volume

$$\int \frac{\partial T}{\partial t} w dV = \int a\frac{\partial^2 T}{\partial x_1^2} w + pwdV$$

We will assume the temperature at the wall support is fixed at zero, and that the rod is insulated at the free end. This means that $T(0) = w(0) = 0$ and $\frac{\partial T}{\partial x}(x = L) = 0$. Additionally, the temperature distribution is constant within the cross-section by assumption. Integrating by parts and computing the area integral, we have

$$\int_0^L \frac{\partial T}{\partial t} w + a\frac{\partial T}{\partial x_1}\frac{\partial w}{\partial x_1} dx_1 = \int_0^L \left(\frac{1}{A}\int_A pdA\right) w dx_1$$

The boundary term from integration by parts drops out because the free end is insulated. The heat supply to the one-dimensional thermal problem comes from integrating the volumetric power generation over each cross-section of the rod. We now see why the dissipative power term was defined as $\bar{p} := (\int_A pdA)/A$. Discretizing the test function with $w(x_1) = \sum_j w_j k_j(x_1)$, we obtain

$$\int_0^L \frac{\partial T}{\partial t} k_j(x_1) + a\frac{\partial T}{\partial x_1}\frac{\partial k_j}{\partial x_1} dx_1 = \int_0^L \bar{p}(x_1, t) k_j(x_1) dx_1$$

We discretize the thermal problem with the same set of shape functions as the torsion problem:

$$k_i(x_1) = \left(\frac{x_1}{L}\right)^i$$

The temperature field has both spatial and time components. Using the same spatial shape functions as the discretization of the test function, the temperature field can be written as

$$T(x_1, t) = \sum_i u_i(t) k_i(x_1)$$

Plugging this in yields a system of ODE's for the evolution of the temperature degrees of freedom in time:

$$\sum_i \frac{\partial u_i}{\partial t} \int_0^L k_i k_j dx_1 + \sum_i u_i \int_0^L a \frac{\partial k_i}{\partial x_1} \frac{\partial k_j}{\partial x_1} dx_1 = \int_0^L \bar{p}(x_1, t) k_j dx_1$$

For simplicity, we can define the following quantities

$$M_{ij}^u := \int_0^L k_i k_j dx_1$$

$$K_{ij}^u := \int_0^L a \frac{\partial k_i}{\partial x_1} \frac{\partial k_j}{\partial x_1} dx_1$$

$$F_j^u(t) := \int_0^L \bar{p}(x_1, t) k_j dx_1$$

The superscript "u" indicates that the quantity corresponds to the thermal problem. Remember that the heat supply term depends on the deformation of the rod. The problem can be written in symbolic notation as

$$\underline{\underline{M}}^u \underline{\dot{a}} + \underline{\underline{K}}^u \underline{a} = \underline{F}^u(t)$$

Using a forward Euler finite difference scheme, the solution at the next time step can be computed in terms of the solution at the previous time step:

$$\underline{a}(t + \Delta t) = \Delta t \underline{\underline{M}}^{u,-1} \Big(\underline{F}^u(t) - \underline{\underline{K}}^u \underline{a}(t) \Big) + \underline{a}(t)$$

7 Effective Conductivity

The expression for the homogenized conductivity tensor is

$$\bar{a}_{im} := \int_\Omega a_{ij} \left(\delta_{mj} + \frac{\partial \chi_m}{\partial y_j} \right) dy$$

See the appendix for a derivation and explanation of this expression. The function $\chi_m(y)$ is the response of the microstructure to a unit temperature gradient in the x_m direction and obeys the following equation

$$\frac{\partial}{\partial y_i} \left(a_{ij} \frac{\partial \chi}{\partial y_j} \right) = -\frac{\partial}{\partial y_i} (a_{ij} \hat{e}_j)$$

In one spatial dimension, there is a single component to the conductivity tensor and the governing equation for χ is particularly simple

$$\bar{a} = \int_\Omega a \left(1 + \frac{\partial \chi}{\partial y} \right) dy$$

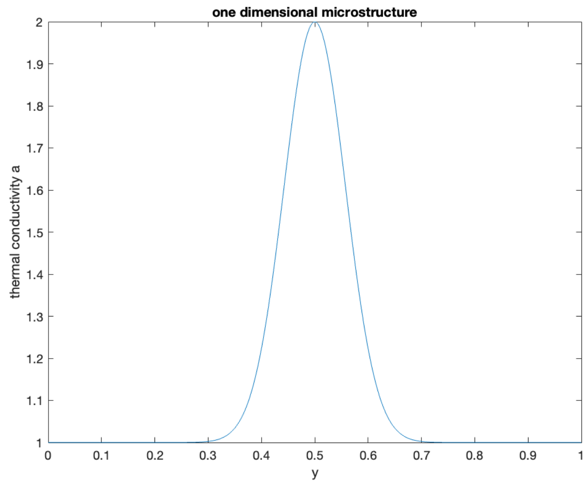

Figure 4: Continuous approximation of two phase composite microstructure in one spatial dimension with a stiff central inclusion. The microstructural domain for the heat transfer problem is a line of unit length over which the thermal conductivity fluctuates.

$$\frac{\partial}{\partial y}\left(a(y)\frac{\partial \chi}{\partial y}\right) = -\frac{\partial a}{\partial y}$$

The function χ has periodic boundary conditions by assumption, and we can see from the expression for the homogenized conductivity that only spatial gradients contribute. Solving this governing equation on the unit domain $\Omega_y = [0, 1]$, the periodic boundary conditions require $\chi(0) = \chi(1)$. This allows us to discretize the solution with

$$\chi(y) = \sum_n \chi_n \sin(n\pi y)$$

This discretization satisfies the periodic boundary conditions by construction, but has no rigid body mode. This is not a problem given that constant shifts in the microstructural temperature response do not contribute to the homogenized conductivity. We will weaken the governing equation and discretize the test function in the same way as the displacement. Using the notation $k_i := \sin(i\pi y)$ and carrying out the same process used to solve the above differential equations, we have

$$\int \frac{\partial}{\partial y}\left(a(y)\frac{\partial \chi}{\partial y}\right) k_j(y)dy = -\int \frac{\partial a}{\partial y} k_j(y)dy$$

Using integration by parts to transfer derivatives onto the test functions and plugging in the discretization of $\chi(y)$, we have

$$\sum_i \chi_i \int a(y) \frac{\partial k_i}{\partial y} \frac{\partial k_j}{\partial y} dy = -\int a(y) \frac{\partial k_j}{\partial y} dy$$

$$\chi_i = \left(\int a(y) \frac{\partial k_i}{\partial y} \frac{\partial k_j}{\partial y} dy \right)^{-1} \left(-\int a(y) \frac{\partial k_j}{\partial y} dy \right)$$

The homogenized conductivity is then

$$\bar{a} = \int_{\Omega_y} a(y) \left(1 + \sum_i \chi_i \frac{\partial k_i}{\partial y} \right) dy$$

Note that for a two-phase composite, one common approach to computing an effective conductivity (or modulus) is to use direct averaging. This gives $a_{mix} = f a_f + (1-f) a_m$ where a_f is the conductivity of the inclusion, a_m is the conductivity of the matrix, and $f = V_f/(V_f + V_m)$ is the volume fraction. The rule of mixtures states that the direct average a_{mix} provides an upper bound on the effective material property. This means that the effective conductivity computed from homogenization should be less than or equal to that of direct averaging. In the case of a continuously varying conductivity, this condition is stated as

$$\bar{a} \leq \frac{1}{L} \int_0^L a(y) dy$$

This condition is interesting to be aware of, and can be used as a check on the value of the homogenized conductivity.

8 Two-way Coupling Scheme

We will use a staggered coupling scheme to integrate the mechanical and thermal problems. At each time step, an update to the displacement will be computed by using the time integration scheme outlined in the section on torsion. The updated displacement field will be used to compute a heat input term for this time step. The heat input is passed to the thermal model, and is used to update the temperature field. The updated temperature field is used to re-compute the material properties of the solid, and the process is repeated. The scheme is called staggered because one problem is solved at a time, and the relevant bits of information are passed from one solver to the other. This contrasts with solving the thermal and mechanical problem simultaneously, which would be a monolithic approach. The basic idea is that at each time step, the output of the mechanical problem is used to compute an input to the thermal problem, and the output of the thermal problem is used to update the mechanical properties at the next time step.

9 Implementation and Results

Parameter	Value	Units
Applied Torque ($M(t)$)	$2500\sin(4\pi t)$	kg∗m
Radius (r)	0.1	m
Length (L)	1	m
Density (ρ)	2700	kg/m^3
Effective Conductivity ($\bar{a}$)	220.67	W/(m∗K)
Damping (η)	3E6	N∗s/m
Stiffness (G_1)	2E9	N/m^2
Stiffness (G_{2i})	2E9	N/m^2
Stiffness (G_{2f})	1E8	N/m^2
Temperature Sensitivity (r_0)	1	–
# Shape Functions (N)	3	–
Duration of Simulation (T)	1	s
Time Steps	1.5E4	–

The method outlined above was implemented in MATLAB. Because the convolution integrals over the entire strain history need to be recomputed at every time step, the method is costly even in one dimension. See Table 1 for the parameters used in the simulation. Depending on the choice of parameters in the material model and the magnitude of the forcing, very large temperatures can be obtained from the viscoelastic dissipation. See Figures 5-7 for results of the model prediction. Five different time steps within the first quarter period of the sinusoidal torque input are displayed to show the spatial distribution of displacement and temperature for the torsion and thermal problems. In looking at the motion of the end of the rod, we see that the displacement lags behind the load as expected from viscoelasticity. Similarly, we see that the first peak in the displacement is smaller than subsequent values, demonstrating the effect of the temperature in softening the material.

10 Conclusion & Future Work

A method to numerically compute the two-way coupled dynamics of heat conduction and viscoelasticity was derived for the case of forced torsional vibrations of a prismatic composite shaft. The composite nature of the material appeared in its thermal properties, but not in its mechanical properties. A standard linear solid model was used for the viscoelastic constitutive behavior, and we hypothesized temperature dependence of the material properties only in the elastic part of the model. The volumetric heat input to the thermal problem was computed using the viscoelastic deformation and the power from the damper in the material model. A staggered scheme was used in the MATLAB implementation, whereby the displacement was first computed at the next time step, and this displacement update used to compute a heat input and solve the thermal problem. Some results for different applied torques and material properties were

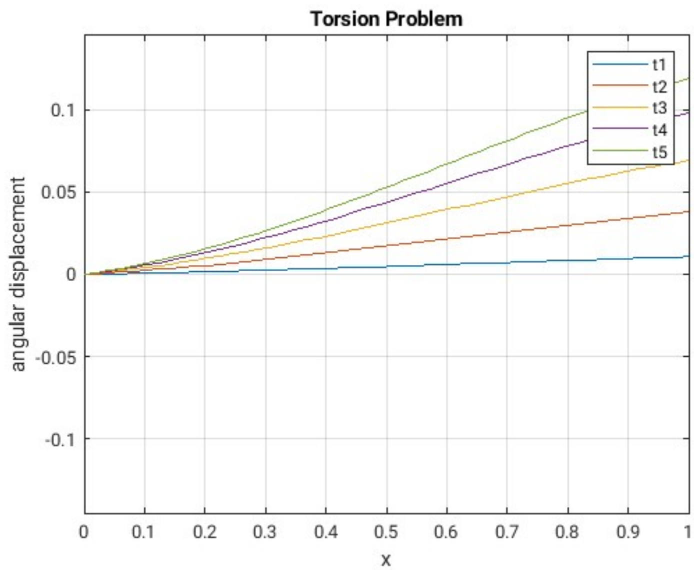

Figure 5: Results for the angular displacement field of the rod as the sinusoidal torque input increases to its peak value at five time steps.

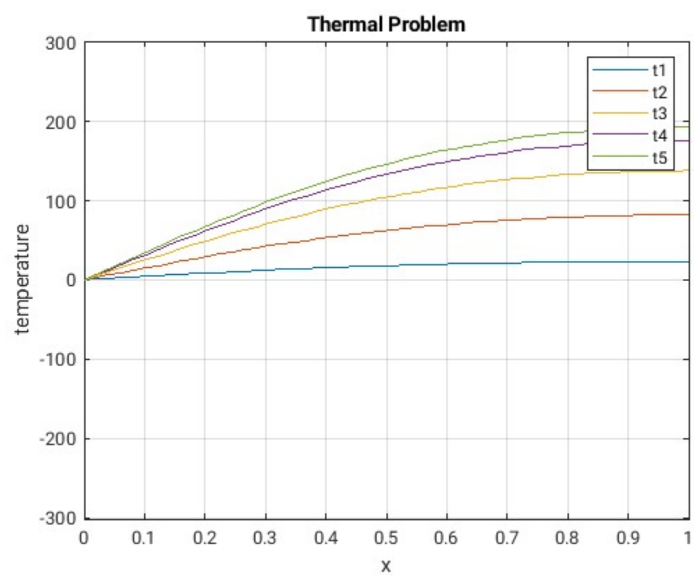

Figure 6: Results for the temperature distribution of the rod driven by the viscoelastic dissipation at five time steps.

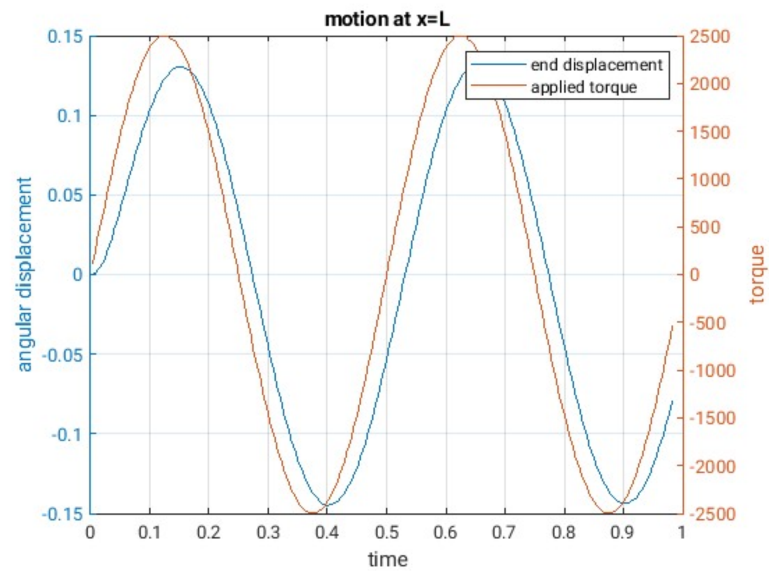

Figure 7: Angular displacement at the end of the rod as a function of time. The displacement lags behind the sinusoidal load as is typical for viscoelastic materials.

provided. Future work should focus on exploring the viscoelastic homogenization procedure outlined in the appendix, and extending the method to finite element discretizations in two and three dimensions.

A Thermal Homogenization

The goal of homogenization in heat transfer problems is to compute an effective conductivity tensor for a material whose properties fluctuate on small scales. To begin, note that Fourier's law furnishes the relation between the heat vector q and gradients of the temperature u:

$$q_i = -a_{ij}\frac{\partial u}{\partial x_j}$$

where a_{ij} is the conductivity tensor. Conservation of energy states that

$$\frac{\partial q_i}{\partial x_i} = f$$

where $f = f(x)$ is a volumetric heat source. Thus, combining Fourier's law and conservation of energy provides a governing equation for heat conduction

$$\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) = -f(x)$$

We use the two-scale expansion from asymptotic homogenization to write

$$x_i^\eta := x_i + \eta y_i$$

where $\eta << 1$ is a scale parameter used to separate the macroscale x from the microscale y. The components of the conductivity tensor will fluctuate periodically on the microscale domain but not vary macroscopically, thus $a_{ij} = a_{ij}(y)$. A superscript of "η" is used to denote a quantity which varies on the two scales. Thus, we also have the following multiscale expansions

$$\frac{\partial}{\partial x_i^\eta} = \frac{\partial}{\partial x_i} + \frac{1}{\eta}\frac{\partial}{\partial y_i}, \quad u^\eta = u_0(x) + \eta u_1(x,y)$$

The governing equation for multiscale heat transfer is

$$\frac{\partial}{\partial x_i^\eta}\left(a_{ij}(y)\frac{\partial u^\eta}{\partial x_j^\eta}\right) = -f(x)$$

Plugging in definitions of the multiscale temperature field and derivative, we obtain

$$\left(\frac{\partial}{\partial x_i} + \frac{1}{\eta}\frac{\partial}{\partial y_i}\right)a_{ij}\left(\frac{\partial u_0}{\partial x_j} + \eta\frac{\partial u_1}{\partial x_j} + \frac{\partial u_1}{\partial y_j}\right) = -f$$

We can distribute the derivative noting that the first order term in the temperature expansion u_0 does not depend on the microscale coordinate, and the conductivity tensor a_{ij} does not depend on the macroscale. Keeping the two lowest order powers of η and arguing that equalities must hold at each order of η independently, we obtain two governing equations

$$\eta^{-1}: \quad \frac{\partial}{\partial y_i}\left(a_{ij}\frac{\partial u_1}{\partial y_j}\right) = -\frac{\partial}{\partial y_i}\left(a_{ij}\frac{\partial u_0}{\partial x_j}\right)$$

$$\eta^0: \quad a_{ij}\frac{\partial^2 u_0}{\partial x_i \partial x_j} + a_{ij}\frac{\partial^2 u_1}{\partial x_i \partial x_j} + \frac{\partial}{\partial y_i}\left(a_{ij}\frac{\partial u_1}{\partial x_j}\right) = -f(x)$$

The first equation shows the microscale problem is driven by macroscopic temperature gradients which are constant over the microstructure. The problem is linear, so it suffices to know the microstructural response to unit temperature gradients in each direction. Call the response of the microstructure to a constant temperature gradient in direction x_m of unit magnitude in direction $\chi_m(y)$. This allows us to write

$$u_1(x,y) = \chi_m(y)\frac{\partial u_0}{\partial x_m}(x)$$

Plugging this into the second governing equation, we can see that this equation cannot be satisfied pointwise. This is because terms involving $\chi(y)$ and its derivatives vary on the microscale, whereas the volumetric heat input only varies macroscopically. Instead we require that the equation is satisfied in an average sense over the microstructure. Call the microstructural domain Ω_y and with no loss of generality, use $|\Omega_y| = 1$. The macroscopic governing equation becomes

$$\left(\int_\Omega a_{ij} dy\right) \frac{\partial^2 u_0}{\partial x_i \partial x_j} + \left(\int_\Omega a_{ij} \frac{\partial \chi_m}{\partial y_j} dy\right) \frac{\partial^2 u_0}{\partial x_m \partial x_i} = -\left(\int_\Omega dy\right) f(x)$$

The divergence term integrates to zero due to periodicity of the microstructural temperature and conductivity tensor. This equation can be re-written as

$$\left[\int_\Omega a_{ij} \left(\delta_{mj} + \frac{\partial \chi_m}{\partial y_j}\right) dy\right] \frac{\partial^2 u_0}{\partial x_m \partial x_i} = -f(x)$$

By analogy to the single-scale governing equation, we can identify the homogenized conductivity tensor as

$$\bar{a}_{im} := \int_\Omega a_{ij} \left(\delta_{mj} + \frac{\partial \chi_m}{\partial y_j}\right) dy$$

B Viscoelastic Homogenization

Using the techniques of asymptotic homogenization, this derivation follows that of linear elasticity but with the modified viscoelastic constitutive relation. As usual, we begin by introducing the two-scale expansion

$$x_i^\eta := x_i + \eta y_i$$

where $\eta << 1$ is a scale parameter. The superscript "η" is meant to indicate a quantity with components on the macroscale x and the microscale y. We begin by writing the multiscale stress equilibrium as

$$\frac{\partial \sigma_{ij}^\eta}{\partial x_j^\eta} + b_i = 0$$

Next, the stress tensor is written using the Boltzmann integral where the relaxation tensor only depends on the microscale y and is periodic on the microscale by assumption:

$$\sigma_{ij}^\eta = \int_0^t C_{ijk\ell}(y, t-\tau) \frac{\partial \epsilon_{k\ell}^\eta}{\partial \tau} d\tau$$

Plugging this into stress equilibrium, we obtain a governing equation for the two-scale viscoelastic solid

$$\frac{\partial}{\partial x_j^\eta} \int_0^t C_{ijk\ell}(y, t-\tau) \frac{\partial^2 u_k^\eta}{\partial x_\ell^\eta \partial \tau} d\tau = -b_i$$

At this point, this equation is not very useful. Note that for a single-scale linear viscoelastic material with no spatial variation, the governing equation is

$$\int_0^t C_{ijk\ell}(t-\tau)\frac{\partial^3 u_k}{\partial x_\ell \partial x_j \partial \tau}d\tau = -b_i$$

This is the viscoelastic equivalent of the Navier equation in linear elasticity. We will attempt to recover an equation of this form (or similar) from the multiscale expansion in order to find the effective viscoelastic properties of the multiscale solid. Turning back to the multiscale problem, we plug in the two scale expansion and definition of the multiscale derivative

$$\left(\frac{\partial}{\partial x_j} + \frac{1}{\eta}\frac{\partial}{\partial y_j}\right)\int_0^t C_{ijk\ell}(y,t-\tau)\frac{\partial}{\partial \tau}\left[\left(\frac{\partial}{\partial x_j} + \frac{1}{\eta}\frac{\partial}{\partial y_j}\right)\left(u_k^0(x) + \eta u_k^1(x,y)\right)\right]d\tau = -b_i$$

The first order term in the expansion of the displacement field does not depend on the microscale by assumption (though this can be proven if not assumed). Multiscale derivatives have components on the macro- and microscale. Distributing derivatives inside the integral, we have

$$= \left(\frac{\partial}{\partial x_j} + \frac{1}{\eta}\frac{\partial}{\partial y_j}\right)\int_0^t C_{ijk\ell}(y,t-\tau)\left[\frac{\partial^2 u_k^0}{\partial x_\ell \partial \tau} + \eta\frac{\partial^2 u_k^1}{\partial x_\ell \partial \tau} + \frac{\partial^2 u_k^1}{\partial y_\ell \partial \tau}\right]d\tau$$

We will keep only the two lowest-order powers of the scale parameter η to simplify the following calculation. This is because higher order terms are neglected per the assumptions of first order homogenization. Passing the derivatives into the integral, we obtain a lengthy expression:

$$= \int_0^t C_{ijk\ell}(t-\tau)\frac{\partial^3 u_k^0}{\partial x_\ell \partial x_j \partial \tau}d\tau + \frac{1}{\eta}\int_0^t \frac{\partial}{\partial y_j}\left(C_{ijk\ell}(t-\tau)\frac{\partial^2 u_k^0}{\partial x_\ell \partial \tau}\right)d\tau + \int_0^t \frac{\partial}{\partial y_j}\left(C_{ijk\ell}(t-\tau)\frac{\partial^2 u_k^1}{\partial x_\ell \partial \tau}\right)d\tau$$
$$+ \int_0^t C_{ijk\ell}(t-\tau)\frac{\partial^3 u_k^1}{\partial x_j \partial y_\ell \partial \tau}d\tau + \frac{1}{\eta}\int_0^t \frac{\partial}{\partial y_j}\left(C_{ijk\ell}(t-\tau)\frac{\partial^2 u_k^1}{\partial y_\ell \partial \tau}\right)d\tau$$

Terms multiplying powers of η are grouped and we argue these equations should be satisfied independently. The order η^{-1} equation describes the response of the microscale:

$$\int_0^t \frac{\partial}{\partial y_j}\left(C_{ijk\ell}(t-\tau)\frac{\partial^2 u_k^1}{\partial y_\ell \partial \tau}\right)d\tau = -\int_0^t \frac{\partial}{\partial y_j}\left(C_{ijk\ell}(t-\tau)\frac{\partial^2 u_k^0}{\partial x_\ell \partial \tau}\right)d\tau$$

This equation describes stress equilibrium for the microstructure driven by a macroscopic strain input. Assume the component of the applied macroscopic strain is a step function $H(t)$ in time so that

$$\frac{\partial^2 u_k^0}{\partial x_\ell \partial \tau} = \frac{\partial}{\partial \tau}\left(\frac{\partial u_k^0}{\partial x_\ell} H(\tau)\right) = \delta(\tau)\frac{\partial u_k^0}{\partial x_\ell}$$

This means that the RHS of the microscale governing equation becomes

$$= -\frac{\partial}{\partial y_j} C_{ijk\ell}(t)\frac{\partial u_k^0}{\partial x_\ell}$$

The equations are linear, so we can write

$$u_i^1 = \chi_{imn}(y,t)\frac{\partial u_m^0}{\partial x_n}$$

$\chi_{imn}(y,t)$ records the i-th time-dependent displacement response at point y in the microstructure for a unit applied strain in direction (m,n). This assumes linear viscoelasticity. We can plug this relation into the macroscale governing equation (order η^0) and integrate over the microscale domain ($|\Omega_y| = 1$). The integration over the microscale domain ensures that the macroscale governing equation is satisfied in an average sense, as it cannot be satisfied pointwise in both the micro- and macroscale domain. The divergence term can be shown to be zero by using the divergence theorem and noting that all functions are periodic. The governing equation for the macroscale is

$$\int_0^t \left(\int_\Omega C_{ijk\ell}(y,t-\tau)d\Omega\right)\frac{\partial^3 u_k^0}{\partial x_\ell \partial x_j \partial \tau}d\tau + \int_0^t \int_\Omega C_{ijk\ell}(y,t-\tau)\frac{\partial}{\partial \tau}\left(\frac{\partial \chi_{kmn}}{\partial y_\ell}\frac{\partial^2 u_m^0}{\partial x_n \partial x_j}\right)d\Omega d\tau$$

We do not recover the Navier equation because the two terms being differentiated in the second integral are time dependent. Thus, the homogenized constitutive relation does not reproduce the physics of a single-scale viscoelastic material with effective properties. Evaluating the time derivative and rearranging, we can see that the governing equation for the macroscale has an additional term that depends on the strain in addition to the time derivative of the strain:

$$\int_0^t \left(\int_\Omega C_{ijk\ell}(y,t-\tau)d\Omega\right)\frac{\partial^3 u_k^0}{\partial x_\ell \partial x_j \partial \tau}d\tau + \int_0^t \int_\Omega C_{ijk\ell}(y,t-\tau)\frac{\partial^2 \chi_{kmn}}{\partial y_\ell \partial \tau}\frac{\partial^2 u_m^0}{\partial x_n \partial x_j}$$
$$+ C_{ijk\ell}(y,t-\tau)\frac{\partial \chi_{kmn}}{\partial y_\ell}\frac{\partial^3 u_m^0}{\partial x_n \partial x_j \partial \tau}d\Omega d\tau$$

This relation can be interpreted as giving two homogenized tensors and written in symbolic notation for readability

$$\underline{\nabla} \cdot \left(\int_0^t \underline{\underline{\underline{\underline{C}}}}^1(t-\tau) : \frac{\partial \underline{\underline{\epsilon}}}{\partial \tau} + \underline{\underline{\underline{\underline{C}}}}^2(t-\tau) : \underline{\underline{\epsilon}}(\tau)d\tau\right) = -\underline{b}$$

The first homogenized tensor is similar to the usual elastic homogenized tensor, which gives the average of microstructure constitutive relation plus flux from unit strains, whereas the second involves time derivatives of the microstructure with respect to the unit strains. Interestingly, this suggests that the effective response of a multiscale viscoelastic solid obeys a different governing equation than a single-scale viscoelastic solid.