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A multi-item inventory model with random replenishment intervals, fuzzy costs and resources under Possibility and Necessity Measure

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Abstract

This paper presents a multi-product multi-period inventory problem with random replenishment intervals and fuzzy costs under space and shortage level constraints. Since the costs (purchasing, holding and backordering) related to inventory system are often imprecise and the replenishment intervals are random in real life, the proposed model is also fuzzy-random Here, the replenishment intervals are taken to be i.i.d random variables and the fuzzy costs are Trapezoidal fuzzy number (TrFN). Using the probability distribution between replenishment epochs the fuzzy-random model is transformed to a fuzzy expected profit model and then using fuzzy arithmetic under function principle the optimistic and pessimistic values of the objective function are obtained. The optimum order quantities for maximum profit are determined with the help of Generalised Reduced Gradient (GRG) method. To illustrate the solution procedure a numerical solution is provided.

Keywords: Replenish-up-to inventory control, Random replenishment interval, Possibility and Necessity measures, m_ρ - measure

1. Introduction

In multi-period inventory control models, continuous review and periodic review are the two main policies. In case of the first policy order can be made at any time depending on the inventory position, and in the second policy an order can be initiated only at the beginning of each period.

Nahmias (1971) considered a ‘periodic review inventory model’ with lost sale, partial backlogging and random lead times under no order crossing assumption. He solved the model by using two heuristics. Donselaar et al. (1996) also suggest another heuristic to find order-up-to level in a periodic review system allowing lost sales. Qu et al. (1999) investigated an integrated inventory-transportation system for multiple products. Downs et al. (2001) developed an inventory problem with multiple items, resource constraints, lags in delivery and lost sales. After showing the convexity of the inventory costs in order-up-to level, they develop a linear programming model based on non parametric estimates of these costs. Chiang (2003) studied a periodic review inventory system with long review periods. He employed a dynamic programming approach to model the problem. Chiang (2006) also considered a periodic review inventory system with replenishment cycles that consists a number of periods. Eynan and Kropp (2007) proposed a periodic review system with the assumption of stochastic demand, variant warehousing cost and safety stock. Teunter et al (2010) proposed a method for determining order-up-to levels under periodic review for compounded binomial demand. Recently, Bijvan and Johansen (2012) proposed a periodic review lost sales inventory models with compounded Poisson demand and constant lead times.

Nahmias and Demmy (1981) were also the first researchers to considered stock rotating in an (s, S) policy under static rotating in continuous review environment. They assume two demand classes with unit Poisson arrivals, constant lead time and full backordering for performance evaluation process. Moon and Kang (1998) considered the compound Poisson demand arrivals and provide a simulation study on the setting of Nahmias and Demmy (1981). Moon and Cha (2005) investigated a continuous review inventory model under the assumption that the replenishment lead time depends on lot size and the production rate of the manufacturer. Jeddi et al. (2004) developed a multi product continuous review system with stochastic demand and shortages under budgetary constraint. Mohebbi and Posner (2002) considered a continuous review inventory system for multiple replenishment orders with lost sales. Taleizadeh (2008) developed a multi product, multi constraints inventory model with stochastic replenishment. They showed that the model to be an integer non-linear programming and proposed a Simulated Annealing to solve it. Chiang (2010) considered an order expediting policy for continuous review systems with manufacturing lead time. Recently, Axsater and Viswanthan (2012) proposed a continuous review inventory problem of an independent supplier to evaluate the value of information about the customer's inventory level.

In most of the existing literature, inventory related costs are assume to be deterministic and represented as real numbers. But, in real situation the inventory costs are usually imprecise in nature due to the influence of various uncontrollable factors. For example, costs may depend on some foreign monetary unit. In such a case, due to exchange rates, the costs are often not known precisely. Inventory carrying cost may also dependent on some parameters like interest rate and stock keeping unit's market price, which are not precise. Also the shortage cost is often difficult to determine precisely in the case when it reflects not just 'lost sale' but also 'a loss of customers will'. Therefore, these cost parameters are described as "approximately equal some certain amount" and so it is more reasonable to characterize these parameters as fuzzy.

Since such type of uncertainties cannot be measured properly using the concept of probability theory, fuzzy set theory has been used to model the real uncertain inventory situation. Park (1987) applied fuzzy set theory to classical EOQ model by representing ordering and holding costs with fuzzy numbers and solved by fuzzy arithmetic operation based extension principle. Chen and Wang (1996) fuzzified the demand, ordering cost, carrying cost and backorder cost into Trapezoidal fuzzy numbers in EOQ model with backorder. Petrovic et al. (1996) developed a newsboy problem in fuzzy environment where uncertain demand was represented by a discrete fuzzy set and inventory cost was given as triangular fuzzy number. Yao and Lee (1999), Yao and Su (2000) and Yao and Chiang (2003) discussed various inventory problems without and with backorder and production inventory control. Besides, some researchers incorporate chance constraint programming introduced by Liu and Iwamura (1998) in inventory models. Maiti and Maiti (2006) extended this work where pessimistic return of the objective function is optimized using necessity measure of fuzzy event and they used to solve a two-warehouse fuzzy inventory model. Wang et al. (2007) proposed fuzzy dependent chance programming model to find the optimal order quantity for maximizing the credibility of an event such that total cost in playing periods does not exceed a certain budget level. Chiang (2010) developed a single item continuous review order expediting inventory policy with manufacturing lead time. Dey and Chakraborty (2012) proposed a periodic review inventory system with variable lead time and negative exponential crashing cost in fuzzy-random environment. Recently, Wang et al (2012) considered two continuous review inventory models with backorders and lost sales under fuzzy demand and different demand situations.

To the best of our knowledge the past works on fuzzy inventory model considered either optimistic or pessimistic approach. If the decision maker (DM) is optimistic, he/she may choose possibility measure. According to Gao and Liu (2001) a fuzzy event may fail even though its possibility achieves 1, and hold even though its necessity is zero. Consequently, high level of confidence in possibility measure does not guarantee the occurrence of fuzzy event. However, the fuzzy event must hold if its credibility is 1, and fail if its credibility is zero. The viewpoint of this research work is different which considers the DM to be eclectic. Therefore we need to make use a combination of both possibility and necessity measure.

In this paper, a multi-product inventory model with space constraint and shortage level constraint is formulated in random fuzzy environment. Here the time periods between replenishments are stochastic variables and follows exponential distribution with a known mean and inventory costs are imprecise and represented by trapezoidal fuzzy numbers (TrFN). The rest of the paper is organized as follows. Section 2 provides a brief introduction to the possibility and necessity measures. Section 3 presents notation, problem assumptions and the proposed problem formulation. In section 4, considering optimistic and pessimistic values of the objective function two methods are suggested for solving the problem. Sections 5 provide a numerical example and section 6 discuss the results. The conclusion and future scope is given in section 7.

2. Basic concept and methodology

In this section, we introduce some basic concepts of possibility, necessity measures of a fuzzy event.

To measure the possibility that a fuzzy set belongs to another fuzzy set, we need to introduce the definition of possibility and necessity measures. The definitions are given as follows:

Definition 2.1: Suppose ξ is restricted by a fuzzy set $\tilde{A}$ in the universe X. Further suppose that the possible distribution of ξ, π_ξ is taken to be equal to the membership function $\mu_{\tilde{A}}(x)$. Then the possibility of the fuzzy event $\{\xi \in \tilde{B}\}$ can be defined by

$$\text{Pos}\{\xi \in \tilde{B}\} = \sup_{x \in X} \min\left\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right\}.$$

The dual measure of possibility, i.e. the necessity measure of the event $\{\xi \in \tilde{B}\}$ is defined as

$$\text{Nec}\{\xi \in \tilde{B}\} = \inf_{x \in X} \max\left\{1 - \mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\right\}.$$

Suppose b is a crisp number, then Pos$\{\tilde{A} \le b\}$ represent the maximum likelihood of the event that $\tilde{A}$ is less than b and Nes$\{\tilde{A} \le b\}$ estimates the minimum likelihood of the event that $\{\tilde{A} \le b\}$ will occur. By definition, we have

$$\text{Pos}\{\tilde{A} \le b\} = \text{Pos}\{\xi \in (-\infty, b]\} = \sup_{x \le b}\left\{\mu_{\tilde{A}}(x)\right\}$$

$$\text{Nes}\{\tilde{A} \le b\} = \text{Nes}\{\xi \in (-\infty, b]\} = \inf_{x > b}\left\{1 - \mu_{\tilde{A}}(x)\right\}$$

$$\text{Pos}\{\tilde{A} \ge b\} = \text{Pos}\{\xi \in [b, \infty)\} = \sup_{x \ge b}\left\{\mu_{\tilde{A}}(x)\right\}$$

$$\text{Nes}\{\tilde{A} \ge b\} = \text{Nes}\{\xi \in [b, \infty)\} = \inf_{x < b}\left\{1 - \mu_{\tilde{B}}(x)\right\}.$$

Example 2.1: A trapezoidal fuzzy variable $\tilde{A}$ determined by quadruplet (a_1, a_2, a_3, a_4) of crisp numbers with $a_1 < a_2 < a_3 < a_4$, whose membership function is given by (cf. Fig.-1)

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1} & \text{if } a_1 \le x < a_2 \\ 1 & \text{if } a_2 \le x \le a_3 \\ \dfrac{a_4 - x}{a_4 - a_3} & \text{if } a_3 \le x \le a_4 \\ 0 & \text{otherwise} \end{cases}$$

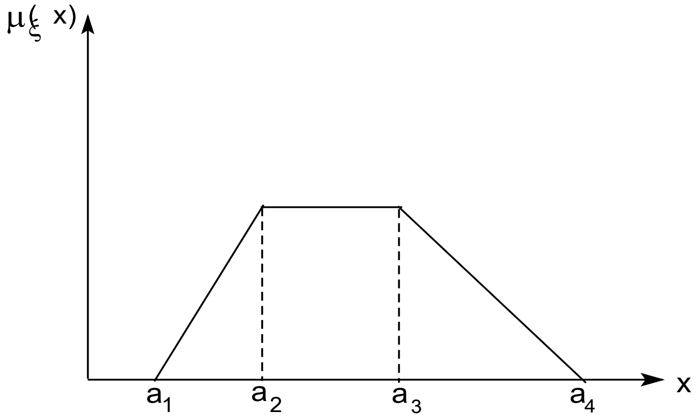

Figure 1. Membership function of trapezoidal fuzzy (TrFN) number

According to Definition 2.1, we can easily obtain the possibility and necessity of fuzzy event $\{\tilde{A} \geq x\}$ as

$$\text{Pos}\{\tilde{A} \geq x\} = \begin{cases} 1 & \text{if } x \leq a_3 \\ \dfrac{a_4 - x}{a_4 - a_3} & \text{if } a_3 < x \leq a_4 \\ 0 & \text{if } x > a_4, \end{cases}$$

$$\text{Nec}\{\tilde{A} \geq x\} = \begin{cases} 0 & \text{if } x \geq a_2 \\ \dfrac{a_2 - x}{a_2 - a_1} & \text{if } a_1 \leq x < a_2 \\ 1 & \text{otherwise,} \end{cases}$$

In fuzzy inventory models, possibility and necessity measures are employed by many researchers. Since fuzzy estimates are based on human judgement, however, they should reflect some assessment of whether the DM tends towards a 'looser' interpretation of fuzzy estimate (possibility) or a 'tighter' one (necessity). Actually, for the optimistic DM, the possibility measure is much suitable where as if the DM is pessimistic; he may use the necessity measure as a tool to make the decision.

According to Yang and Iwamura [2008], if a DM wants to seek the best decision to maximize the chance of fuzzy event $\{f(\mathbf{x}, \tilde{\mathbf{A}}) \in \mathbf{B}\}$, $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is the decision variables vector, $\tilde{\mathbf{A}}$ is fuzzy parameter vector and $\mathbf{B} \in R^n$ (n-dimensional real space). If the DM use possibility measure as a chance measure then a decision $\mathbf{x^*}$ will be recognised as the best and it satisfies $\text{Pos}\{f(\mathbf{x}, \tilde{\mathbf{A}}) \in \mathbf{B}\} = 1$. In fact, a fuzzy event may fail though its possibility may achieve 1. This implies that for some realization value $\mathbf{y}$

with $\mu_{\tilde{A}}(\mathbf{y}) > 0$, the event { f(**x***, **y**) $\in$ **B**} may not appear. So depending upon the nature of the DM, he/she may choose the possibility measure if he/she is optimistic and does not care about the potential risk otherwise he/she may choose necessity measure as a chance measure. In practice, the decision **x*** is not necessarily the best decision for the necessity measure since the corresponding objective value is less than or equal to 1. Thus the DM may select a better solution $\bar{\mathbf{x}}^*$ as the optimal decision. If the necessity measure of fuzzy event { f($\bar{\mathbf{x}}^*, \tilde{\mathbf{A}}$) $\in$ **B**} achieves 1, the realization value $\bar{\mathbf{x}}$ of $\tilde{\mathbf{A}}$ with $\mu_{\tilde{A}}(\bar{\mathbf{x}}) > 0$, the event { f($\bar{\mathbf{x}}^*, \tilde{\mathbf{A}}$) $\in$ **B**} must hold.

In practice, most DMs are neither absolutely optimistic nor absolutely pessimistic. Accordingly, a DM attitude factor ρ ($0 \le \rho \le 1$) was introduced in the decision process by Wang and Shu [2005], which produce a balance between the optimistic and the pessimistic.

Suppose, $\bar{m}$ = Pos{ $\tilde{A} \le b$} and $\underline{m}$ = Nes{ $\tilde{A} \le b$}; then the weighted DM's judgement of the event { $\tilde{A} \le b$} is given by

$$m_\rho = \rho\,\bar{m} + (1 - \rho)\,\underline{m},$$

where ρ is predetermined by the DM according to his nature. Further larger value of the parameter ρ, the DM is more optimistic and if ρ = 1, then m_ρ - measure degenerates to possibility measure. If ρ = 0, then m_ρ - measure degenerates to necessity measure. If ρ = 0.5, then m_ρ - measure degenerates to credibility measure by Liu and Liu [2002].

Definition 2.2: If $\tilde{A}$ be a fuzzy variable and $\alpha \in (0, 1]$, then

$$\tilde{A}_{\inf}(\rho,\alpha) = \inf\left\{b : m_\rho\{\tilde{A} \le b\} \ge \alpha\right\} \text{ and}$$

$$\tilde{A}_{\sup}(\rho,\alpha) = \sup\left\{b : m_\rho\{\tilde{A} \ge b\} \ge \alpha\right\}$$

are respectively called the (ρ, α) - pessimistic and (ρ, α) – optimistic values of $\tilde{A}$.

Lemma 2.1: If a TrFN $\tilde{A}$ determined by quadruplet (a_1, a_2, a_3, a_4) then we have

$$\tilde{A}_{\inf}(\rho,\alpha) = \begin{cases} \dfrac{\alpha(a_2 - a_1)}{\rho} + a_1 & \text{if } \alpha \le \rho \\ \dfrac{(1-\alpha)(a_3 - a_4)}{(1-\rho)} + a_4 & \text{if } \alpha > \rho \end{cases}$$

and

$$\tilde{A}_{\sup}(\rho,\alpha) = \begin{cases} \dfrac{\alpha(a_3 - a_4)}{\rho} + a_4 & \text{if } \alpha \le \rho \\ \dfrac{(1-\alpha)(a_2 - a_1)}{(1-\rho)} + a_1 & \text{if } \alpha > \rho \end{cases}$$

Proof:

$$m_\rho\{\tilde{A} \le b\} = \rho\, Pos\{\tilde{A} \le b\} + (1-\rho)\, Nes\{\tilde{A} \le b\}$$

$$= \rho Pos\{\tilde{A} \le b\} + (1-\rho)\left(1 - Pos\{\tilde{A} \ge b\}\right)$$

$$= \begin{cases} 0 & if\ b < a_1 \\ \rho\left(\dfrac{b-a_1}{a_2-a_1}\right) & if\ a_1 \le b < a_2 \\ \rho & if\ a_2 \le b < a_3 \\ \rho + (1-\rho)\left(\dfrac{a_4-b}{a_4-a_3}\right) & if\ a_3 \le b \le a_4 \\ 1 & if\ a_4 < b \end{cases}$$

This is easy to see that

$$\tilde{A}_{\inf}(\rho,\alpha) = \begin{cases} \dfrac{\alpha(a_2-a_1)}{\rho} + a_1 & if\ \alpha \le \rho \\ \dfrac{(1-\alpha)(a_3-a_4)}{(1-\rho)} + a_4 & if\ \alpha > \rho \end{cases}$$

and

$$\tilde{A}_{\sup}(\rho,\alpha) = \begin{cases} \dfrac{\alpha(a_3-a_4)}{\rho} + a_4 & if\ \alpha \le \rho \\ \dfrac{(1-\alpha)(a_2-a_1)}{(1-\rho)} + a_1 & if\ \alpha > \rho \end{cases}$$

3. Constraint Fuzzy-Random Inventory Model

The mathematical model in this paper is developed on the basis of the following assumptions and notations:

3.1 Assumptions

To describe the problem we introduce the following assumptions:

(1) The times between replenishments are i.i.d random variables.
(2) Demand rate is known and constant.
(3) Inventory costs (purchasing, holding and shortage) are not known precisely and represents as Trapezoidal fuzzy numbers (TrFN).
(4) Lead time is zero.
(5) Shortages are allowed, but backlogged partially.

3.2 Notations

The following notations are employed throughout this paper to develop this model

PF   total expected profit
W   total available warehouse space

For i^{th} (i = 1, 2, ….., n) product

D_i:	demand rate
Q_i:	initial inventory level
R_i:	expected amount order in each cycle
$\hat{T}_i$:	time period between two replenishment (a random variable)
T_{Maxi}	upper limit of the probability distribution of $\hat{T}_i$
T_{Mini}	lower limit of the probability distribution of $\hat{T}_i$
$f_{Ti}(t_i)$	probability density function of $\hat{T}_i$
T_{0i}:	time at which inventory level reaches zero
β_i:	fraction of unmet demand backordered
s_i:	sales price per unit item
$\tilde{h}_i$:	holding cost per unit quantity per unit time (a fuzzy variable)
$\tilde{\pi}_i$:	shortage cost per unit quantity per unit time (a fuzzy variable)
$\tilde{p}_i$:	purchase cost per unit quantity of material (a fuzzy variable)
S_i:	lower limit of the service level
w_i:	required warehouse space per unit item.

3.3 Model formulation

In the development of the inventory model of the i^{th} product, we assume the time periods between replenishments are stochastic variables. According to Ertogal and Rahim (2005) two cases may occur, in the first case the time between replenishments is less than the amount of time required for the inventory level depleted completely (Fig.-2) and in the second case, the time between replenishment exceeds the period in which the inventory level depletes zero and shortage occurs (Fig.-3) which are backlogged partially at the beginning of each period.

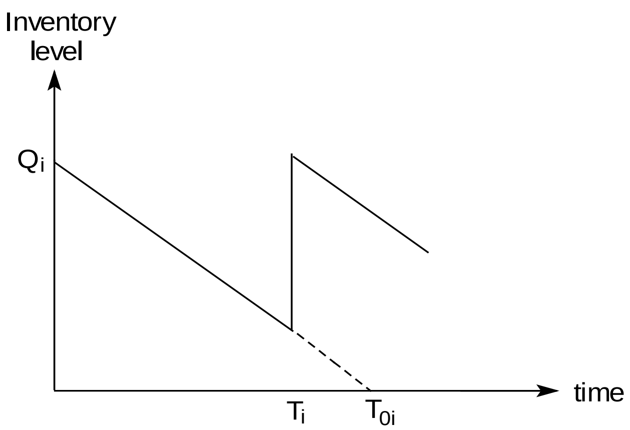

Figure 2. Presenting the inventory cycle for no backorder case

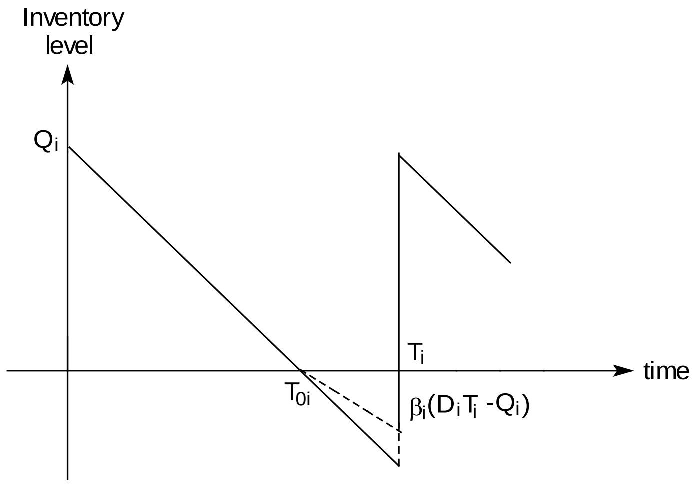

Figure 3. Presenting the inventory cycle for backorder case

To calculate the expected profit per cycle for the i[th] product in fuzzy random environment, we need to evaluate the following:

$$\text{PF}(Q_i, \hat{T}_i, \tilde{p}_i, \tilde{h}_i, \tilde{\pi}_i) = (s_i - \tilde{p}_i)R_i - \tilde{h}_i I_i - (s_i - \tilde{p}_i)L_i - \tilde{\pi}_i B_i$$

where

$$R_i = \int_{T_{Min_i}}^{t_{0i}} D_i T_i f_{T_i}(t_i)dt_i + \int_{t_{0i}}^{T_{Max_i}} \left\{ Q_i + \beta_i D_i \left(T_i - \frac{Q_i}{D_i} \right) \right\} f_{T_i}(t_i)dt_i$$

The expected average inventory in a cycle is

$$I_i = \int_{T_{Min_i}}^{t_{0i}} \left(Q_i T_i + \frac{D_i T_i^2}{2} \right) f_{T_i}(t_i)dt_i + \int_{t_{0i}}^{T_{Max_i}} \left(\frac{Q_i^2}{2D_i} \right) f_{T_i}(t_i)dt_i$$

The expected total unmet demand in a cycle is

$$B_i = \beta_i \int_{t_{0i}}^{T_{Max_i}} (D_i T_i - Q_i) f_{T_i}(t_i)dt_i$$

and the expected lost demand in a cycle is

$$L_i = (1-\beta_i) \int_{t_{0i}}^{T_{Max_i}} (D_i T_i - Q_i) f_{T_i}(t_i)dt_i$$

Then the total expected profit for all products is as follows

$$\text{PF}(\boldsymbol{Q}, \boldsymbol{\hat{T}}, \boldsymbol{\tilde{p}}, \boldsymbol{\tilde{h}}, \boldsymbol{\tilde{\pi}}) = \sum_{i=1}^{n} \text{PF}(Q_i, \hat{T}_i, \tilde{p}_i, \tilde{h}_i, \tilde{\pi}_i)$$

Therefore the complete mathematical model of the multi-product inventory system with random replenishment under space and shortage level constraints is

Maximize $\text{PF}(\boldsymbol{Q}, \boldsymbol{\hat{T}}, \boldsymbol{\tilde{p}}, \boldsymbol{\tilde{h}}, \boldsymbol{\tilde{\pi}})$

Subject to

$$\sum_{i=1}^{n} w_i Q_i \le W$$

$$\Pr(\hat{T}_i > T_{0i}) = \int_{\frac{Q_i}{D_i}}^{T_{Max_i}} f_{T_i}(t_i)\,dt_i \le 1 - S_i$$

$Q_i \ge 0$, i = 1, 2, …., n.

(Since the shortages occur only when the cycle time is greater than T_{0i}, and that the lower limit of the service level is S_i.)

4. Model with an exponential distribution for $\hat{T}_i$ and TrFN for $\tilde{p}_i$, $\tilde{h}_i$ and $\tilde{\pi}_i$

In this subsection we discuss the random replenishment model in the fuzzy sense.

If we assume that the time between replenishments is exponentially distributed with λ_i arrival rate, fuzzy expected profit in this model is given by

$$\text{Maximize PF}(\boldsymbol{Q}, \hat{\boldsymbol{T}}, \tilde{\boldsymbol{p}}, \tilde{\boldsymbol{h}}, \tilde{\boldsymbol{\pi}}) = \sum_{i=1}^{n}\left\{(s_i - \tilde{p}_i)R_i - \tilde{h}_i I_i - (s_i - \tilde{p}_i)L_i - \tilde{\pi}_i B_i\right\}$$

$$= \sum_{i=1}^{n}\left\{\frac{1}{\lambda_i}[D_i(1-\beta_i)(\tilde{p}_i - s_i) - \tilde{\pi}_i\beta_i D_i]e^{-\left(\frac{Q_i}{D_i}\right)\lambda_i} + \frac{1}{\lambda_i}[D_i(\tilde{p}_i - s_i) - \tilde{h}_i Q_i] + \frac{\tilde{h}_i Q_i}{\lambda_i^2}[1 - e^{-\left(\frac{Q_i}{D_i}\right)\lambda_i}]\right\} \quad (1)$$

Subject to

$$\sum_{i=1}^{n} w_i Q_i \le W$$

$$e^{-\left(\frac{Q_i}{D_i}\right)\lambda_i} \le 1 - S_i$$

$Q_i \ge 0$, i = 1, 2, …., n.

Now, let us consider the inventory costs, $\tilde{p}_i$, $\tilde{h}_i$ and $\tilde{\pi}_i$ are imprecise in nature and expressed by trapezoidal fuzzy numbers (TrFNs). A TrFN, for example, $\tilde{p}_i = (p_{i1}, p_{i2}, p_{i3}, p_{i4})$, satisfying the condition $0 \le p_{i1} \le p_{i2} \le p_{i3} \le p_{i4}$ and has the following membership function:

$$\mu_{\tilde{p}_i}(x) = \begin{cases} \dfrac{x - p_{i1}}{p_{i2} - p_{i1}} & \text{if } p_{i1} \le x < p_{i2} \\ 1 & \text{if } p_{i2} \le x \le p_{i3} \\ \dfrac{p_{i4} - x}{p_{i4} - p_{i3}} & \text{if } p_{i3} < x \le p_{i4} \\ 0 & \text{otherwise} \end{cases}$$

Hence by using the fuzzy arithmetic operations by function principle, the fuzzy expected profit reduces to a trapezoidal fuzzy number

$\tilde{PF} = (PF_1, PF_2, PF_3, PF_4)$.

Here PF_1, PF_2, PF_3, PF_4 are all positive real valued function of Q_i (i = 1, 2, …, n) satisfying the conditions $PF_1 \le PF_2 \le PF_3 \le PF_4$. Using the functional principle the expressions of PF_r (r = 1, 2, 3, 4) are as follows:

$PF_r =$

$$\sum_{i=1}^{n}\left\{\frac{1}{\lambda_i}[D_i(1-\beta_i)(p_{ri}-s_i)-\pi_{(4-r+1)i}\beta_i D_i]e^{-\left(\frac{Q_i}{D_i}\right)\lambda_i}+\frac{1}{\lambda_i}[D_i(p_{ri}-s_i)-h_{(4-r+1)i}Q_i]+\frac{h_{ri}Q_i}{\lambda_i^2}[1-e^{-\left(\frac{Q_i}{D_i}\right)\lambda_i}]\right\}.$$

Case 1: In this case we maximize the optimistic value of $\tilde{PF}$ with predefined value α_1. Then the problem reduces to

$$\underset{Q}{Max}\ Max\, X_1$$

Subject to $m_{\rho_1}\left(\tilde{PF} \ge X_1\right) \ge \alpha_1$

Using lemma 2.1 the above fuzzy constraint optimization problem can be transformed to the equivalent crisp problem as

$$\underset{Q}{Max}\begin{cases}\dfrac{\alpha_1(PF_3-PF_4)}{\rho_1}+PF_4 & if\ \ \alpha_1\le\rho_1\\[2ex] \dfrac{(1-\alpha_1)(PF_2-PF_1)}{(1-\rho_1)}+PF_1 & if\ \ \alpha_1>\rho_1\end{cases}$$

Subject to

$$\sum_{i=1}^{n} w_i Q_i \le W$$

$$e^{-\left(\frac{Q_i}{D_i}\right)\lambda_i} \le 1 - S_i$$

$Q_i \ge 0$, i = 1, 2, …., n.

Case 2: In this case we maximize the pessimistic value of $\tilde{PF}$ with predefined value α_2. Then the problem reduces to

$$\underset{Q}{Max}\ Min\, X_2$$

Subject to $m_{\rho_2}\left(\tilde{PF} \le X_2\right) \ge \alpha_2$

Using lemma 2.1 the above fuzzy constraint optimization problem can be transformed to the equivalent crisp problem as

$$\underset{Q}{Min}\begin{cases}\dfrac{\alpha_2(PF_2-PF_1)}{\rho_2}+PF_1 & if\ \ \alpha_2\le\rho_2\\[2ex] \dfrac{(1-\alpha_2)(PF_3-PF_4)}{(1-\rho_2)}+PF_4 & if\ \ \alpha_2>\rho_2\end{cases}$$

Subject to

$$\sum_{i=1}^{n} w_i Q_i \le W$$

$$e^{-\left(\frac{Q_i}{D_i}\right)\lambda_i} \le 1 - S_i$$

$$Q_i \ge 0,\ i = 1, 2, \ldots, n.$$

5. Numerical Illustration

For the illustration purpose we present a multi-product inventory problem with three products and the data are given in table 5.1. Here we consider the imprecise purchase cost inventory holding cost and shortage cost as trapezoidal fuzzy numbers.

Table 5.1. Input data

Items	**D_i**	**β_i**	**λ_i**	**s_i**	**p_i**	**h_i**	**π_i**	**w_i**	**S_i**
1	30	0.5	1/25	125	(82, 85, 90, 98)	(2, 2.2, 2.5, 2.7)	(5, 6, 8, 9)	3	0.55
2	25	0.6	1/40	140	(94, 97, 100, 102)	(2.5, 2.8, 3, 3.2)	(6, 8, 9, 10)	4	0.5
3	20	0.55	1/30	120	(90, 93, 95, 98)	(1.7, 2, 2.2, 2.5)	(4, 5, 7, 8)	3	0.6
W = 6000									

Using theses values, the problem (I) has been solved using a non-linear optimization technique (GRG method) for different values of α_j and ρ_j (j = 1, 2) and the results are presented in Table 5.2 to Table 5.5.

Table 5.2. Variations in ρ_1 and α_1 ($\alpha_1 \le \rho_1$)

ρ_1 / ↓ α_1		0.25	0.5	0.75	1
0.2	X_1	64306.2	76372.7	80394.8	82405.9
0.5			**58273.1**	68328.4	73356.1
0.7				60284.1	67322.9
0.95					**59781.3**

Table 5.3. Variations in ρ_1 and α_1 ($\alpha_1 > \rho_1$)

ρ_1 / ↓ α_1		0	0.2	0.5	0.75
0.2	X_1	25964.0			
0.5		17194.1	20848.2		
0.7		11347.6	13539.9	20117.3	
0.95		**4039.1**	4404.5	5500.7	8424.1

Table 5.4. Variations in ρ_2 and α_2 ($\alpha_2 \leq \rho_2$)

ρ_2 / $\downarrow \alpha_2$		0.25	0.5	0.75	1
0.2	X_2	25964.0	14270.7	10372.9	8424.1
0.5			**31810.7**	22066.2	17194.0
0.7				29861.8	23040.7
0.95					**30349.1**

Table 5.5. Variations in ρ_2 and α_2 ($\alpha_2 > \rho_2$)

ρ_2 / $\downarrow \alpha_2$		0.25	0.5	0.75	1
0.2	X_2	64306.2			
0.5		73356.1	69585.3		
0.7		79389.3	77126.8	70339.5	
0.95		**86930.8**	86553.8	85422.5	82405.9

6. Discussion

Table 5.2 and Table 5.3 show the maximum optimistic value of the objective function measured in method 1. It is observed from the tables that if α_1 increases for each fixed value of ρ_1, the value of the objective function decreases while increasing the value of ρ_1 for each fixed value of α_1, the value of the objective function increases.
Since for the parameter $\rho_1 = 1$ and $\rho_1 = 0$, m_{ρ_1} - measure degenerates to the possibility and necessity measures respectively, we have the optimal values of the objective function at different confidence level α_1. For example, at $\alpha_1 = 0.95$ the optimistic and pessimistic values of the objective function are 59781.3 and 4039.1.

Similarly, Table 5.4 and Table 5.5 show the minimum optimistic value of the objective function measured in method 2. It is observed from the tables that if α_2 increase for each fixed value of ρ_2, the value of the objective function increases while increasing the value of ρ_2 for each fixed value of α_2, the value of the objective function decreases.
At $\alpha_2 = 0.95$ the optimistic and pessimistic values of the objective function are 86930.8 and 30349.1.

7. Conclusion and Future Scope

Now-a-days, to tackle the real world uncertain inventory systems the fuzzy set theory and the probability theory has been used very nicely. In this paper, a fuzzy stochastic inventory problem proposed for the random replenishment intervals and imprecise inventory costs using fuzzy set theory and probability theory complementary. After de-randomization fuzzy arithmetic operation under function principle, the optimistic and pessimistic values of the objective function are obtained. A possibility and necessity measure produces a balance between optimism and pessimism. Finally, the critical values of fuzzy objective function with respect to m_ρ - measure are obtained and the

result indicates that the confidence interval is wide in optimistic case as compare to pessimistic case.
There are several possible directions for future research. This work can be directly extended to consider some other distributions like uniform, normal etc. for replenishment intervals. It is also interesting to consider the problems with variable demand or the items received are not all perfect.

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Appendix

Let $\tilde{A} = (a_1, a_2, a_3, a_4)$ and $\tilde{B} = (b_1, b_2, b_3, b_4)$ be two trapezoidal fuzzy numbers then the fuzzy arithmetical operations under function principle are as follows:

1. Addition: $\tilde{A} + \tilde{B} = \tilde{C}$, where the membership function of $\tilde{C}$ is

$$\mu_{\tilde{C}}(z) = \begin{cases} \dfrac{z - (a_1 + b_1)}{(a_2 + b_2) - (a_1 + b_1)} & \text{if } (a_1 + b_1) \le z < (a_2 + b_2) \\ 1 & \text{if } (a_2 + b_2) \le z < (a_3 + b_3) \\ \dfrac{(a_4 + b_4) - z}{(a_4 + b_4) - (a_3 + b_3)} & \text{if } (a_3 + b_3) \le z < (a_4 + b_4) \\ 0 \quad \text{otherwise} & \end{cases}$$

where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are any real numbers.

2. Multiplication: $\tilde{A}.\tilde{B} = \tilde{C}$, where the membership function of $\tilde{C}$ is

$$\mu_{\tilde{C}}(z) = \begin{cases} \dfrac{z - a_1b_1}{a_2b_2 - a_1b_1} & \text{if } a_1b_1 \le z < a_2b_2 \\ 1 & \text{if } a_2b_2 \le z < a_3b_3 \\ \dfrac{a_4b_4 - z}{a_4b_4 - a_3b_3} & \text{if } a_3b_3 \le z < a_4b_4 \\ 0 & \text{otherwise} \end{cases}$$

where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are all non zero positive real numbers.

3. Subtraction: $\tilde{A} - \tilde{B} = \tilde{C}$, where the membership function of $\tilde{C}$ is

$$\mu_{\tilde{C}}(z) = \begin{cases} \dfrac{z - (a_1 - b_4)}{(a_2 - b_3) - (a_1 - b_4)} & \text{if } (a_1 - b_4) \le z < (a_2 - b_3) \\ 1 & \text{if } (a_2 - b_3) \le z < (a_3 - b_2) \\ \dfrac{(a_4 - b_1) - z}{(a_4 - b_1) - (a_3 - b_2)} & \text{if } (a_3 - b_2) \le z < (a_4 - b_1) \\ 0 & \text{otherwise} \end{cases}$$

where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are any real numbers.

4. Division: $\tilde{A} / \tilde{B} = \tilde{C}$, where the membership function of $\tilde{C}$ is

$$\mu_{\tilde{C}}(z) = \begin{cases} \dfrac{z - (a_1 / b_4)}{(a_2 / b_3) - (a_1 / b_4)} & \text{if } (a_1 / b_4) \le z < (a_2 / b_3) \\ 1 & \text{if } (a_2 / b_3) \le z < (a_3 / b_2) \\ \dfrac{(a_4 / b_1) - z}{(a_4 / b_1) - (a_3 / b_2)} & \text{if } (a_3 / b_2) \le z < (a_4 / b_1) \\ 0 & \text{otherwise} \end{cases}$$

where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are all non zero positive real numbers.

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A New Method for Rank, Mode, Divergence and Spread on Generalized Exponential Trapezoidal Fuzzy Numbers

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Abstract

In this paper, we want proposed a new method for ranking of generalized exponential trapezoidal fuzzy numbers based on rank, mode, divergence and spread. For the validation the results of the proposed approach are compared with different existing approaches. The main advantage of the proposed approach is that the proposed approach provides the correct ordering of generalized and normal trapezoidal fuzzy numbers and also the proposed approach is very simple and easy to apply in the real life problems.For the validation, the results of the proposed approach are compared with different existing approaches.

Keywords: Ranking function, Generalized triangular fuzzy numbers.

1. Introduction

In most of cases in our life, the data obtained for decision making are only approximately known. (Zadeh 1965) introduced the concept of fuzzy set theory to meet those problems. (Dubois and Prade 1978) defined any of the fuzzy numbers as a fuzzy subset of the real line. Fuzzy numbers allow us to make the mathematical model of linguistic variable or fuzzy environment. Ranking fuzzy numbers were first proposed by (Jain 1976) for decision making in fuzzy situations by representing the ill-defined quantity as a fuzzy set. (Bortolan and Degani 1985) reviewed some of these ranking methods for ranking fuzzy subsets. (Chen 1985) presented ranking fuzzy numbers with maximizing set and minimizing set. (C. Liang 2006) and (Wang and Lee 2008) also used the centroid concept in developing their ranking index. (Chen and Chen 2007) presented a method for ranking generalized trapezoidal fuzzy numbers. (Abbasbandy and Hajjari 2009) introduced a new approach for ranking of trapezoidal fuzzy numbers based on the left and right spreads at some α-levels of trapezoidal fuzzy numbers. (Chen and Chen 2009) presented a method for fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and different spreads. (Rezvani 2010, 2011) presented a method for ranking generalized fuzzy numbers. Moreover, (Rezvani

2012) proposed a new method for ranking in perimeters of two generalized trapezoidal fuzzy numbers.
(Pushpinder Singh 2010) proposed a method for ranking of generalized trapezoidal fuzzy numbers based on rank, mode, divergence and spread. In this paper, we want proposed a new method for ranking of generalized exponential trapezoidal fuzzy numbers based on rank, mode, divergence and spread. For the validation the results of the proposed approach are compared with different existing approaches. The main advantage of the proposed approach is that the proposed approach provides the correct ordering of generalized and normal trapezoidal fuzzy numbers and also the proposed approach is very simple and easy to apply in the real life problems.For the validation, the results of the proposed approach are compared with different existing approaches.
The rest of this paper is organized as follows. In section 2, some basic definitions are reviewed. In section 3, the value of $\Re(A)$, *Mode(A), Divergence(A), Left and Right spead* of generalized trapezoidal fuzzy numbers proved. In section 4, some important results are proved that are useful for the proposed approach. In section 5, a new approach is proposed for the ranking of generalized trapezoidal fuzzy numbers. In section 6, the ranking results of the proposed approach are compared with different existing approaches. The conclusion is discussed in section 7.

2. Preliminaries

Generally, a generalized fuzzy number A is described as any fuzzy subset of the real line R, whose membership function μ_A satisfies the following conditions,
(i): μ_A is a continuous mapping from R to the closed interval $[0,1]$,
(ii): $\mu_A(x) = 0,\ -\infty < x \le c$,
(iii): $\mu_A(x) = L(x)$ is strictly increasing on $[c,a]$,
(iv): $\mu_A(x) = w,\ a < x \le b$,
(v): $\mu_A(x) = R(x)$ is strictly decreasing on $[b,d]$,
(vi): $\mu_A(x) = 0,\ d \le x < \infty$,
Where $0 < w \le 1$ and a,b,c and d are real numbers. We call this type of generalized fuzzy number a trapezoidal fuzzy number, and it is denoted bye

$$A = (c,a,b,d;w)_{LR} \ .$$

When $w = 1$, this type of generalized fuzzy number is called normal fuzzy number and is represented by

$$A = (c,a,b,d)_{LR} \ .$$

However, these fuzzy numbers always have a fix range as $[c,d]$. Here, we define its general from as follows:

$$f_A(x) = \begin{cases} we^{-[(a-x)/\alpha]} & x \le a \ , \\ w & a \le x \le b \ , \\ we^{-[(x-b)/\beta]} & x \le a \ , \end{cases} \qquad (1)$$

Where $0 < w \le 1$ and a, b are real numbers, and α, β are positive real numbers. We denote this type of generalized exponential fuzzy number as

$$A = (a, b, \alpha, \beta; w)_E \ .$$

Especially, when $w = 1$, we denote it as

$$A = (a, b, \alpha, \beta)_E \ .$$

integral value of graded mean h-level as follow. Let the generalized exponential fuzzy number $A = (a, b, \alpha, \beta)_E$, Where $0 < w \le 1$ and α, β are positive real numbers, a, b are real numbers as in formula (1). Now, let two monotonic functions be

$$L(x) = we^{-[(a-x)/\alpha]} \ , R(x) = we^{-[(x-b)/\beta]} \qquad (2)$$

then the inverse functions of function L and R are L^{-1} and R^{-1} respectively. the hlevel graded mean value of generalized exponential fuzzy number $A = (a, b, \alpha, \beta; w)_E$ can be express as

$$h[L^{-1}(h) + R^{-1}(h)]/2 \qquad (3)$$

Definition 1. Let $A = (a, b, \alpha, \beta; w)_E$ **,** be generalized exponential number, then the graded mean integration representation of A is define by

$$P(A) = \int_0^w h \ (\frac{L^{-1}(h) + R^{-1}(h)}{2} dh) / \int_0^w h \ dh \qquad (4)$$

Theorem 1. Let $A = (a, b, \alpha, \beta; w)_E$ **,** be generalized exponential number with $0 < w \le 1$ and α, β are positive real numbers, a, b are real numbers. then the graded mean integration representation of A is

$$P(A) = \frac{a+b}{2} + \frac{\beta - \alpha}{4} \ . \qquad (5)$$

Proof:

$$L^{-1}(h) = a - \alpha(\ln \frac{w}{h}) \, ,$$

$$R^{-1}(h) = b + \beta(\ln \frac{w}{h}) \ .$$

$$P(A) = \frac{1}{2}\int_0^w h[a + b + \beta(\ln \frac{w}{h}) - \alpha(\ln \frac{w}{h})] \ dh \, / \frac{1}{2} w^2$$

$$= \frac{a+b}{2} + \frac{\beta-\alpha}{2}\int_0^w h(\ln \frac{w}{h}) \ dh = \frac{a+b}{2} + \frac{\beta-\alpha}{2}[\int_0^w h \ln(w) - \int_0^w h \ln(h)] \ dh$$

$$= \frac{a+b}{2} + \frac{\beta-\alpha}{2}\int_0^w h[\ln(w) - \int_0^w h \ln(h)] \ dh = \frac{a+b}{2} + \frac{\beta-\alpha}{4} \ .$$

Remark 1. When $\alpha = \beta$, $P(A) = \dfrac{a+b}{2}$.

3. Find value of $\Re(A)$, Mode(*A*), Divergence(*A*), Left and Right spead

Theorem 2. Let $A=(a,b,\alpha,\beta;w)_E$ **,** be any generalized exponential Trapezoidal fuzzy number, Then

(1) $\Re(A)=\frac{1}{2}\int_0^w [L^{-1}(h)+R^{-1}(h)]\,dh$,

(2) $Mo\,\mathrm{d}e(A)=\frac{w\,(b+\alpha)}{2}$,

(3) $Divergence(A)=w\,(\beta-a)$,

(4) $Left\ \ spead\,(A)=w\,(b-a)$ **,**

(5) $Right\ \ spead\,(A)=w\,(\beta-\alpha)$ **.**

Proof: We have

(1)

$$L^{-1}(h)=a-\alpha(\ln\frac{w}{h})$$

$$R^{-1}(h)=b+\beta(\ln\frac{w}{h})$$

$$\Re(A)=\frac{1}{2}\int_0^w [(a-\alpha(\ln\frac{w}{h}))\ dh+(b+\beta(\ln\frac{w}{h}))\ dh]=\frac{1}{2}\int_0^w [a+b+(\beta-\alpha)\ln\frac{w}{h}]\ dh$$

$$=\frac{w\,(a+b)}{2}+\frac{(\beta-\alpha)}{2}\int_0^w \ln\frac{w}{h}\ \ dh=\frac{w\,(a+b)}{2}+\frac{(\beta-\alpha)}{2}[\int_0^w \ln w\ \ dh-\int_0^w \ln h\ \ dh]$$

$$=\frac{w\,(a+b)}{2}+\frac{(\beta-\alpha)}{2}[w\ \ \ln w\ -[w\ \ \ln w\ -w\ ]]=\frac{w\,(a+b)}{2}+\frac{w\,(\beta-\alpha)}{2}=\frac{w\,(a+b+\beta-\alpha)}{2}$$

So

$$\Re(A)=\frac{w\,(a+b+\beta-\alpha)}{2}\ .$$

(2)

$$Mo\,\mathrm{d}e(A)=\frac{1}{2}\int_0^w b\ \ dh+\frac{1}{2}\int_0^w \alpha\ \ dh=\frac{w\,(b+\alpha)}{2}.$$

4. Some property for generalized exponential trapezoidal fuzzy numbers

Theorem 3. Let $A=(a_1,b_1,\alpha_1,\beta_1;w_1)$ and $B=(a_2,b_2,\alpha_2,\beta_2;w_2)$ be two generalized exponential Trapezoidal fuzzy number, Then

(i): $\Re(A)=\Re(B)$, (6)

(ii): $\mathrm{mod}e(A)=\mathrm{mod}e(B)$, (7)

(iii): $divergence(A)=divergence(B)$. (8)

Proof:

(i): $\Re(A)=\Re(B)\Rightarrow\frac{w_1(a_1+b_1+\beta_1-\alpha_1)}{2}=\frac{w_2(a_2+b_2+\beta_2-\alpha_2)}{2}$

$$\Rightarrow w_1(a_1+b_1+\beta_1-\alpha_1)=w_2(a_2+b_2+\beta_2-\alpha_2) \; .$$

(ii): $\operatorname{mod} e(A)=\operatorname{mod} e(B)\Rightarrow \dfrac{w_1(b_1+\alpha_1)}{2}=\dfrac{w_2(b_2+\alpha_2)}{2}\Rightarrow w_1(b_1+\alpha_1)=w_2(b_2+\alpha_2)$.

(iii): $divergence(A)=divergence(B)\Rightarrow w_1(\beta_1-\alpha_1)=w_2(\beta_2-\alpha_2)$.

Also of theorem 3. we can say

$$w_1\alpha_1=w_2\alpha_2 \; , \tag{9}$$

$$w_1\beta_1=w_2\beta_2 \; , \tag{10}$$

$$w_1(b_1+\alpha_1)=w_2(b_2+\alpha_2) \; . \tag{11}$$

Theorem 4. Let $A=(a_1,b_1,\alpha_1,\beta_1;w_1)$ and $B=(a_2,b_2,\alpha_2,\beta_2;w_2)$ be two generalized exponential Trapezoidal fuzzy number, Then

(i): $Left\ \ spead(A)>Left\ \ spead(B)\ \ iff\ \ w_1b_1>w_2b_2$,

(ii): $Left\ \ spead(A)<Left\ \ spead(B)\ \ iff\ \ w_1b_1<w_2b_2$,

(iii): $Left\ \ spead(A)=Left\ \ spead(B)\ \ iff\ \ w_1b_1=w_2b_2$.

Proof: [1].

Theorem 5. Let $A=(a_1,b_1,\alpha_1,\beta_1;w_1)$ and $B=(a_2,b_2,\alpha_2,\beta_2;w_2)$ be two generalized exponential Trapezoidal fuzzy number, Then

(i): $Right\ \ spead(A)>Rightt\ \ spead(B)\ \ iff\ \ w_1\beta_1>w_2\beta_2$,

(ii): $Right\ \ spead(A)<Right\ \ spead(B)\ \ iff\ \ w_1\beta_1<w_2\beta_2$,

(iii): $Right\ \ spead(A)=Right\ \ spead(B)\ \ iff\ \ w_1\beta_1=w_2\beta_2$.

Proof: [1].

Theorem 6. Let $A=(a_1,b_1,\alpha_1,\beta_1;w_1)$ and $B=(a_2,b_2,\alpha_2,\beta_2;w_2)$ be two generalized exponential Trapezoidal fuzzy number, Then

(a): $\Re(A)=\Re(B)$,

(b): $\operatorname{mod} e(A)=\operatorname{mod} e(B)$,

(c): $divergence(A)=divergence(B)$.

Then

(i): $Left\ \ spead(A)>Left\ \ spead(B)\ \ iff\ \ \ Right\ \ spead(A)>Rightt\ \ spead(B)$,

(ii): $Left\ \ spead(A)<Left\ \ spead(B)\ \ iff\ \ \ Right\ \ spead(A)<Rightt\ \ spead(B)$,

(iii): $Left\ \ spead(A)=Left\ \ spead(B)\ \ iff\ \ \ Right\ \ spead(A)=Rightt\ \ spead(B)$.

Proof: [1].

5. Proposed approach for ranking of generalized trapezoidal fuzzy numbers

Let $A=(a_1,b_1,\alpha_1,\beta_1;w_1)$ and $B=(a_2,b_2,\alpha_2,\beta_2;w_2)$ be two generalized exponential Trapezoidal fuzzy number, Then use the following steps to compare A,B

* step 1: Find $\Re(A)$ and $\Re(B)$

Case (i) If $\Re(A) > \Re(B)$ *then* $A > B$

Case (ii) If $\Re(A) < \Re(B)$ *then* $A < B$

Case (iii) If $\Re(A) = \Re(B)$ then go to step 2.

* step 2: Find $\text{mod}e(A)$ and $\text{mod}e(B)$

Case (i) If $\text{mod}e(A) > \text{mod}e(B)$ *then* $A > B$

Case (ii) If $\text{mod}e(A) < \text{mod}e(B)$ *then* $A < B$

Case (iii) If $\text{mod}e(A) = \text{mod}e(B)$ then go to step 3.

* step 3: Find $divergence(A)$ and $divergence(B)$

Case (i) If $divergence(A) > divergence(B)$ *then* $A > B$

Case (ii) If $divergence(A) < divergence(B)$ *then* $A < B$

Case (iii) If $divergence(A) = divergence(B)$ then go to step 4.

* step 4: Find $left\ \ speard\,(A)$ and $left\ \ speard\,(B)$

Case (i) If $left\ \ speard\,(A) > left\ \ speard\,(B)$

i.e., $w_1 b_1 > w_2 b_2$ then $A > B$ (from theorem 3.)

Case (ii) If $left\ \ speard\,(A) < left\ \ speard\,(B)$

i.e., $w_1 b_1 < w_2 b_2$ then $A < B$ (from theorem 3.)

Case (iii) If $left\ \ speard\,(A) = left\ \ speard\,(B)$

i.e., $w_1 b_1 = w_2 b_2$ then go to step 5.

* step 5: Find w_1 and w_2

Case (i) If $w_1 > w_2$ then $A > B$

Case (ii) If $w_1 < w_2$ then $A < B$

Case (iii) If $w_1 = w_2$ then $A = B$.

6. Results and discussion

In this section, seven sets of fuzzy numbers are compared using the proposed approach and existing approaches. The results are shown in Table 1.

Example 1. Let $A = (0.2, 0.4, 0.6, 0.8; 0.35)$ and $B = (0.1, 0.2, 0.3, 0.4; 0.7)$ be two generalized Trapezoidal fuzzy number, Then

* step 1

$\Re(A) = 0.14$ and $\Re(B) = 0.14$. Since $\Re(A) = \Re(B)$, so go to step 2

* step 2

$\text{mode}(A) = 0.175$ and $\text{mode}(B) = 0.175$. Since $\text{mode}(A) = \text{mode}(B)$, so go to step 3

* step 3
$divergence(A) = 0.21$ and $divergence(B) = 0.21$. Since $divergence(A) = divergence(B)$ so go to step 4

* step 4
$left\ \ spread(A) = 0.07$ and $left\ \ spread(B) = 0.07$. Since
$left\ \ spread(A) = left\ \ spread(B)$, so go to step 5

* step 5
$w_1 = 0.35$ and $w_2 = 0.7$, Since $w_1 < w_2 \Rightarrow A < B$.

Example 2. Let $A = (0.1, 0.2, 0.4, 0.5; 1)$ and $B = (0.1, 0.3, 0.3, 0.5; 1)$ be two generalized Trapezoidal fuzzy number, Then
* step 1
$\Re(A) = 0.2$ and $\Re(B) = 0.3$. So $\Re(A) < \Re(B) \Rightarrow A < B$.

Example 3. Let $A = (0.1, 0.2, 0.4, 0.5; 1)$ and $B = (1, 1, 1, 1; 1)$ be two generalized Trapezoidal fuzzy number, Then

* step 1
$\Re(A) = 0.2$ and $\Re(B) = 1$. So $\Re(A) < \Re(B) \Rightarrow A < B$
.
Example 4. Let $A = (-0.5,-0.3,-0.3,-0.1;1)$ and $B = (0.1, 0.3, 0.3, 0.5; 1)$ be two generalized Trapezoidal fuzzy number, Then

* step 1
$\Re(A) = -0.3$ and $\Re(B) = 0.3$. So $\Re(A) < \Re(B) \Rightarrow A < B$.

Example 5. Let $A = (0.3, 0.5, 0.5, 1;1)$ and $B = (0.1, 0.6, 0.6, 0.8; 1)$ be two generalized Trapezoidal fuzzy number, Then

* step 1
$\Re(A) = 0.65$ and $\Re(B) = 0.45$. So $\Re(A) > \Re(B) \Rightarrow A > B$.

Example 6. Let $A = (0, 0.4, 0.6, 0.8;1)$ and $B = (0.2, 0.5, 0.5, 0.9; 1)$ and $C = (0.1, 0.6, 0.7, 0.8; 1)$ be two generalized Trapezoidal fuzzy number, Then

* step 1
$\Re(A) = 0.3$ and $\Re(B) = 0.55$ and $\Re(C) = 0.4$. So
$\Re(A) < \Re(C) < \Re(B) \Rightarrow A < C < B$.

Example 7. Let $A = (0.1, 0.2, 0.4, 0.5;1)$ and $B = (-2, 0, 0, 2; 1)$ be two generalized Trapezoidal fuzzy number, Then

* step 1
$\Re(A)=0.2$ and $\Re(B)=0$. So $\Re(A)>\Re(B)\Rightarrow A>B$.

It is clear from Table 1 that the results of the proposed approach are same as obtained by using the existing approach . The main advantage of the proposed approach is that the proposed approach provides the correct ordering of generalized and normal trapezoidal fuzzy numbers and also the proposed approach is very simple and easy to apply in the real life problems.

Table 1. A comparison of the ranking results for different approaches

Approaches	Example 1	Example 2	Example 3	Example 4	Example 5	Example 6	Example 7
Cheng (1998)	A<B	A~B	Not Comparable	A~B	A>B	A<B<C	Not Comparable
Chu and Tsao (2002)	A<B	A~B	Not Comparable	A<B	A>B	A<B<C	Not Comparable
Chen and Chen (2007)	A<B	A<B	A<B	A<B	A>B	A<C<B	A>B
Abbasbandy and Hajjari(2009)	Not Comparable	A~B	A<B	A~B	A<B	A<B<C	A>B
Chen and Chen (2009)	A<B	A<B	A<B	A<B	A>B	A<B<C	A>B
Kumar et al. (2010)	A>B	A~B	A<B	A<B	A>B	A<B<C	A>B
Pushpinder Singh et al. (2010)	A<B	A<B	A<B	A<B	A>B	A<B<C	A>B
S. rezvani (2012)	A<.B	A>B	A>B	A~B	A~B	A>C>B	A<B
Proposed approach	A<B	A<B	A<B	A<B	A>B	A<C<B	A>B

7. Conclusions

(Pushpinder Singh 2010) proposed a method for ranking of generalized trapezoidal fuzzy numbers based on rank, mode, divergence and spread. For the validation the results of the proposed approach are compared with different existing approaches. The main advantage of the proposed approach is that the proposed approach provides the correct ordering of generalized and normal trapezoidal fuzzy numbers and also the proposed approach is very simple and easy to apply in the real life problems.

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On the Assessment of Treatment Levels in Relation to Body Mass Index Value through Fuzzy Mathematical Modelling

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Abstract

The objective of this paper is to propose a model for the assessment of treatment levels in relation to body mass index (BMI) value, using a form of partial differential equation of parabolic type, in which BMI diffusivity under treatment and BMI acceleration are modelled using a fuzzy parameter. We apply Fourier's law to express the equation of BMI value, which is known as diffusion equation. The chosen partial differential equation is solved using boundary conditions. The BMI diffusivity and BMI acceleration are determined with the use of fuzzy rule based system. The calculated solution is compatible with the behaviour of obesity under treatment, as reported in the medical literature.

Keywords: Body mass index; Diffusion equation; Fuzzy logic

1. Introduction

Obesity is defined as an abnormal growth of adipose tissues. Over-weight is usually due to obesity. The obesity is a key risk factor in the natural history of non-communicable diseases (Park, 2002). This paper proposes a model for the assessment of treatment levels in relation to BMI value using a form of diffusion partial differential equation of parabolic type to model BMI diffusivity under treatment and BMI acceleration using a fuzzy parameter. The mathematical model for diffusion phenomena such as BMI diffusivity and acceleration is based on the conduction of heat in solids and diffusion of voracity of viscous fluid flow past a body (Howison, 2005). In this case, the Fourier heat conduction equation or diffusion equation is solved with a set of initial and boundary conditions. In this work, BMI diffusivity and BMI acceleration depends on the BMI value & degree of obesity and the characteristics of induction over which they occurs. The BMI diffusivity and BMI acceleration are determined by means of a system based on fuzzy rules. All the rule bases are processed using Mamdani's inference method with centre of gravity defuzzification. The other inference method such as Takagi-Sugeno was not used because the FRBS output variables are not precise and they cannot be written as input variables. The model is constructed using expert medical knowledge on behaviors of obesity under treatment. In the past, mathematical modeling on uncertainty was made especially in system modeling, optimization control and pattern recognition to mention just a few. In this paper, we use fuzzy set theory to deal with the uncertain nature of BMI diffusivity under treatment and BMI acceleration. The next section presents data. In section 3, we present a partial differential equation with fuzzy parameter for the assessment of treatment levels in relation to BMI value.

2. Data

Applied data in this research for the assessment of treatment levels in relation to BMI value have been chosen with the help of medical experts and from the medical literature. A large body data exists in the occurrence of obesity. There is no natural dividing line between high and normal obesity. The acceleration of obesity is complex and is one of multiple causations. The core part involves the body mechanism of each individual, such as greater absorption of food and greater conversion of food into fat. Although obesity can be easily identified at first sight, a precise assessment requires measurements and reference standard. For assessing the obesity, the reference table for BMI was framed (Park, 2002). The degree of obesity is expressed in terms of BMI value. The calculated BMI value is corresponding to cut-offs. The various recommended cut-off values for BMI correspond to the weights for the heights. The world health organization (WHO) gives the much needed reference table for BMI, which can be used internationally as reference standard for assessing the obesity. The standard reference table states as BMI 18.5 – BMI 24.99 is the normal range for an individual and BMI value more than the normal range indicates the degree of obesity. For modeling the assessment of obesity, we developed the following formula with the use of above data.

$$BMI\ (U) = \begin{cases} 0, & BMI\ 18.5 \le U \le BMI\ 24.99 \\ U, & BMI\ 25 \le U \end{cases}$$

The control of obesity centers round weight reduction. The weight reduction can be achieved by various levels of treatment. The levels of treatment with indication are described as follow: *Level 1* indicates awareness creation. *Level 2* indicates adopting behavioral changes + level 1. *Level 3* indicates following dietary changes + level 2. *Level 4* indicates increased physical activity + level 3. *Level 5* indicates taking drugs for emotional disturbances + level 4. *Level 6* indicates taking appetite suppressing drugs + level 5. *Level 7* indicates surgical treatment, if necessary + level 6. *Level 8* indicates recourse of treatment from level 1 to level 7. *Level 9* indicates repeat level 8. *Level 10* indicates repeat level 9 etc…. All observations are subjected to variation. The observations depends upon three major factors namely observer variation, biological variation and variations related to technical methods. Observer variations may be minimized by taking the average of several repeated measurements at the same time. The BMI variability associated with many physiological variables such as height, weight etc. The fluctuation in the variant measures in the same individual is due to changes in the parameter observed, variations in the way patients perceive their symptoms & answer and regression to the mean. Technical method variation is due to the defective instruments etc. The BMI variability may be tackled by repeat measurements and by replacement of defective instruments etc.

3. Fuzzy mathematical model

Fuzzy set theory is a mathematical tool for modeling uncertain phenomena. The nature of obesity involves several uncertain variables, such as BMI value in different individual, BMI diffusivity with a complex of interventions, BMI variability and BMI acceleration. Thus, the aim of the paper is to find a solution for modeling the assessment of treatment levels in relation to BMI value treating uncertain variables as a fuzzy parameter. The model proposed for the assessment of treatment levels in relation to BMI value will be studied by means of diffusion partial differential equation of parabolic type, given by

$$\frac{\partial U}{\partial t} - \tilde{\alpha}(U, Ind\,T_1)\cdot\frac{\partial^2 U}{\partial x^2} = 0 \qquad \dots (1)$$

The functional variable $U = U(x, t)$ indicates the BMI value at the instant $t \in [0, T]$ and at the point x indicates the treatment. Thus, we assume that the fuzzy parameter $\tilde{\alpha}(U, Ind\,T_1) > 0$ represent the BMI diffusivity where the fuzzy rule based system (FRBS) determine the total induction ($Ind\,T_1$) that depends on input variable Ind_P (*Thermogenics*) and Ind_Q (*Nutritive values*). The BMI diffusivity depends on the BMI value and on its induction levels. Here we assume that x indicate the treatment where $x = 0$ indicates no treatment and the treatment expressed in terms of treatment levels. We have to specify the degree of obesity, when there is no treatment by means of fuzzy differential equation given by

$$\frac{d\,V(t)}{d\,t} - \tilde{\beta}(V(t),\ Ind\,T_2)\,V(t) = 0, \quad V(t_0) = v_0 \qquad \dots (2)$$

The functional variable $V(t)$ indicates the degree of obesity at the instant $t \in [0, T]$ and v_0 is the initial degree of the obesity. The coefficient proportionality $\tilde{\beta}(V(t),\ Ind\,T_2) > 0$ is the fuzzy parameter. We represent the acceleration of the degree of the obesity by the parameter $\tilde{\beta}(V(t),\ Ind\,T_2)$, where the acceleration $\tilde{\beta}(V(t),\ Ind\,T_2)$ determined by fuzzy rule based system (FRBS) and the total induction $Ind\,T_2$ depends on the fuzzy input variables Ind_R (*absorption of food*) and Ind_S (*conversion of food into fat*). The acceleration of the degree of the obesity depends on the degree of obesity and on its induction levels. Thus, we get degree of obesity at time t is $V(t) = v_0 \exp(t\cdot\tilde{\beta}(V(t),\ Ind\,T_2))$.

3.1. Boundary conditions for the partial differential equation

Using Fourier's law, we can write the boundary condition of the individual at no treatment, ($x = 0$), as

$$- \gamma \frac{\partial U}{\partial x} = v_0 \; e^{\, t \cdot \tilde{\beta}(V(t), \; Ind \, T_2)} \; \text{at} \; x = 0 \qquad \ldots (3)$$

where γ is the BMI variability. The second boundary condition states that the BMI variation goes to normal at $x \to \infty$, that is ,

$$- \gamma \frac{\partial U}{\partial x} \to 0 \; \text{as} \; x \to \infty . \qquad \ldots (4)$$

Hence, the boundary value problem we now solve is

$$\frac{\partial U}{\partial t} - \tilde{\alpha} \frac{\partial^2 U}{\partial x^2} = 0 , \quad - \gamma \frac{\partial U}{\partial x} = v_0 \; e^{\tilde{\beta} t} \; \text{at} \; x = 0 , \; - \gamma \frac{\partial U}{\partial x} \to 0 \; \text{as} \; x \to \infty .$$

3.2. Analytic solution of the boundary value problem

Assume the solution of BMI value can be written as

$$U(x, t) = W(x) \exp(\tilde{\beta} t) . \qquad \text{...} (5)$$

Now substitution of the equation (5) into the equation (1) we obtain

$$\frac{\partial^2 W}{\partial x^2} \exp(\tilde{\beta} t) = \frac{\tilde{\beta}}{\tilde{\alpha}} W \exp(\tilde{\beta} t) . \qquad \ldots (6)$$

Note that the derivative is an ordinary derivative, which is the case since W is a function of x-alone. Dividing each equation by $\exp(\tilde{\beta} t)$, we obtain an ordinary differential equation for $W(x)$,

$$\frac{d^2 W}{d x^2} - \frac{\tilde{\beta}}{\tilde{\alpha}} W = 0 . \qquad \ldots (7)$$

Hence, the general solution of the equation (7) is

$$W(x) = A e^{x \sqrt{\frac{\tilde{\beta}}{\tilde{\alpha}}}} + B e^{- x \sqrt{\frac{\tilde{\beta}}{\tilde{\alpha}}}} \qquad \ldots (8)$$

where A and B are arbitrary constants.
Using the equation (5) and the equation (8) the BMI value is given by

$$U(x, t) = \left(A e^{x\sqrt{\frac{\tilde{\beta}}{\tilde{\alpha}}}} + B e^{-x\sqrt{\frac{\tilde{\beta}}{\tilde{\alpha}}}} \right) e^{\tilde{\beta} t} \qquad \ldots (9)$$

The arbitrary constants A and B are found by applying the boundary conditions in the equation (3). Applying the boundary condition at $x = 0$,

$$-\gamma \sqrt{\frac{\tilde{\beta}}{\tilde{\alpha}}} \, (A - B) \exp(\tilde{\beta} t) = v_0 \exp(\tilde{\beta} t)$$

and hence

$$A - B = -\frac{v_0}{\gamma} \sqrt{\frac{\tilde{\alpha}}{\tilde{\beta}}}$$

Applying the other boundary condition, as $x \to \infty$, noting that $\exp\left(-x \sqrt{\frac{\tilde{\beta}}{\tilde{\alpha}}} \right) \to 0$ and $\exp\left(x \sqrt{\frac{\tilde{\beta}}{\tilde{\alpha}}} \right) \to \infty$, the condition can only be satisfied if $A = 0$. Hence, we obtain

$$B = \frac{v_0}{\gamma} \sqrt{\frac{\tilde{\alpha}}{\tilde{\beta}}} .$$

Substituting for A and B back into the equation (9), the BMI value $U(x, t)$ is now given by

$$U(x, t) = \frac{v_0}{\gamma} \sqrt{\frac{\tilde{\alpha}}{\tilde{\beta}}} \cdot \exp\left(\tilde{\beta} t - \sqrt{\frac{\tilde{\beta}}{\tilde{\alpha}}} \, x \right)$$

Assume that $\tilde{\alpha} = \gamma^2 \tilde{\beta}$ and the initial BMI value is written as

$$U(x_0, t_0) = u_0 = \frac{v_0}{\gamma} \sqrt{\frac{\tilde{\alpha}}{\tilde{\beta}}}$$

Thus the BMI value is

$$U(x) = u_0 \exp\left(-\sqrt{\frac{\tilde{\beta}}{\tilde{\alpha}}}\; x \right).$$

From the above, we obtain the BMI value. Otherwise, we can say that the analysis is capable to express what happens over treatment, in this case such as $x \to \infty$ then $U \to 0$.

4. Linguistic variables and rule base

Fuzzy sets are a way to represent imprecise information and knowledge (Zimmermann, 1996). The values of induction along Ind_P & Ind_Q are expressed as {very low, low, medium, high, very high} while those of the BMI value (U) are expressed, the term sets {very low, low, medium, high, very high}. The membership functions that specify the meaning of the linguistic values are shown in the figure-4, 5, 6 and 7 for inductions Ind_P & Ind_Q**,** total induction $Ind\,T_1$, BMI value U and BMI diffusivity $\tilde{\alpha}$ respectively. Total induction is the output variable in FRBS-1 that depends on the variables of induction along Ind_P & Ind_Q. BMI diffusivity is the output variable in FRBS-2 that depends on the variables of total induction $Ind\,T_1$ and BMI value U (*Figure-2*). Total induction is expressed in a term set {very little, little, intermediate, great, very great}. The membership functions that designate the meaning of the linguistic values are given in figure-5&7 for total induction and BMI diffusivity, respectively. The rule base that encodes the relationship of inductions along Ind_P & Ind_Q and total induction $Ind\,T_1$ is summarized in *table-1*. The rule base that encodes the relationship of total induction $Ind\,T_1$, BMI value U and BMI diffusivity $\tilde{\alpha}$ is shown *table-2*. The values of induction along Ind_R & Ind_S are expressed as {poor, medium, moderate, high} while those of the degree of obesity (V) are expressed, the term sets {very low, low, medium, high, very high}. The membership functions that specify the meaning of the linguistic values are shown in the figure-8, 9, 10 and 11 for inductions Ind_R & Ind_S**,** total induction $Ind\,T_2$, degree of obesity (V) and BMI acceleration $\tilde{\beta}$, respectively. Total induction is the output variable in FRBS-I that depends on the variables of induction along Ind_R & Ind_S. BMI acceleration is the output variable in FRBS-II that depends on the variables of total induction $Ind\,T_2$ and degree of obesity (V) (*Figure-3*). Total induction is expressed in a term set {very small, small, intermediate, large}. The membership functions that designate the meaning of the linguistic values are given in figure-9&11 for total induction and BMI acceleration $\tilde{\beta}$, respectively. The rule base that encodes the relationship of inductions along Ind_R & Ind_S and total induction $Ind\,T_2$ is summarized in *table-3*. The rule base that encodes the relationship of total induction $Ind\,T_2$, degree of obesity (V) and BMI acceleration $\tilde{\beta}$ is shown *table-4*.

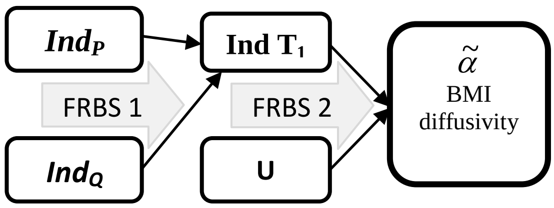

Figure 2. Fuzzy rule based system for BMI diffusivity

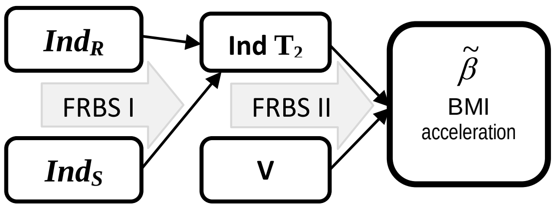

Figure 3. Fuzzy rule based system for BMI acceleration

Table 1. Fuzzy rules for total induction ($IndT_1$)

Ind_P / Ind_Q	*Very low*	*Low*	*Medium*	*High*	*Very high*
Very low	***Very little***	***Little***	***Little***	***Intermediate***	***Intermediate***
Low	***Little***	***Little***	***Intermediate***	***Intermediate***	***Great***
Medium	***Little***	***Intermediate***	***Intermediate***	***Great***	***Great***
High	***Intermediate***	***Intermediate***	***Great***	***Great***	***Very great***
Very high	***Intermediate***	***Great***	***Great***	***Very great***	***Very great***

Table 2. Fuzzy rules for BMI diffusivity ($\tilde{\alpha}$)

U / $IndT_1$	*Very low*	*Low*	*Medium*	*High*	*Very high*
Very little	***Very small***	***Small***	***Small***	***Medium***	***Medium***
Little	***Small***	***Small***	***Medium***	***Medium***	***Large***
Intermediate	***Small***	***Medium***	***Medium***	***Large***	***Very large***
Great	***Medium***	***Medium***	***Large***	***Large***	***Very large***
Very great	***Medium***	***Large***	***Large***	***Very large***	***Very large***

Table 3. Fuzzy rules for total induction ($IndT_2$)

Ind_R / Ind_S	*Poor*	*Medium*	*Moderate*	*High*
Poor	***Very small***	***Small***	***Small***	***Intermediate***
Medium	***Very small***	***Small***	***Intermediate***	***Intermediate***
Moderate	***Small***	***Intermediate***	***Intermediate***	***Large***
High	***Intermediate***	***Intermediate***	***Large***	***Large***

Table 4. Fuzzy rules for BMI acceleration ($\tilde{\beta}$)

V / $IndT_2$	*Very low*	*Low*	*Medium*	*High*	*Very high*
Very small	***Very little***	***Little***	***Little***	***Medium***	***Medium***
Small	***Little***	***Little***	***Medium***	***Medium***	***Great***
Intermediate	***Little***	***Medium***	***Medium***	***Great***	***Great***
Large	***Medium***	***Medium***	***Great***	***Great***	***Very great***

5. Fuzzy computational simulation

In order to define the type of induction as a function of the type of individual, we allot a value [0, 1] for induction variables Ind_P and Ind_Q. The chosen trapezoidal membership function which associated to a fuzzy set $\tilde{F}$, $\mu_{\tilde{F}}(y)$ including four parameters (Yen & Langari, 2002) (y_1, y_2, y_3, y_4), is given by (see Figure. 1)

$$\mu_{\tilde{F}}(y) = \begin{cases} \dfrac{y - y_1}{y_2 - y_1}, & if \quad y_1 \le y \le y_2 \\ 1, & if \quad y_2 \le y \le y_3 \\ \dfrac{y_4 - y}{y_4 - y_3}, & if \quad y_3 < y \le y_4 \end{cases}$$

where the membership function of a fuzzy set $\tilde{F}$ is denoted as $\mu_{\tilde{F}}$, and the membership value of y (scale of parameter) in $\tilde{F}$ is denoted as $\mu_{\tilde{F}}(y)$.

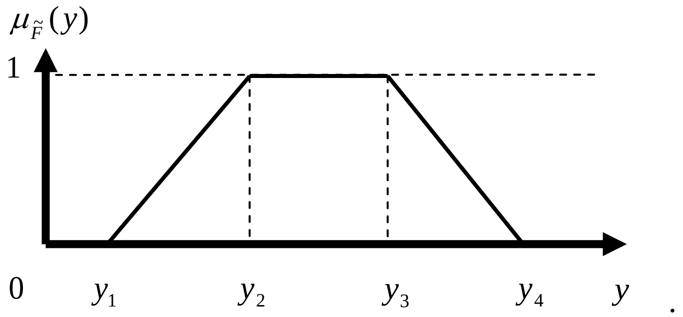

Figure1. Fuzzy trapezoidal number $\tilde{F} = (y_1, y_2, y_3, y_4)$

If the positions are located in the area of *very small*, the value of $\tilde{\alpha}$ in those positions is calculated by $y_1 + (y_4 - y_1) * R$, where (R) is equal to the random value of y in interval $(0, 1)$, *y_1 = 0.0 and y_4 = 0.12.* If the positions are located in the area of *small*, the value of $\tilde{\alpha}$ in those positions is calculated by $y_1 + (y_4 - y_1) * R$, where (R) is equal to the random value of y in interval $(0, 1)$, *y1 = 0.06 and y_4 = 0.26.* If the positions are located in the area of *medium*, the value of $\tilde{\alpha}$ in those positions is calculated by $y_1 + (y_4 - y_1) * R$, where (R) is equal to the random value of y in interval $(0, 1)$, *y_1 = 0.2 and y_4 = 0.5.* If the positions are located in the area of *large*, the value of $\tilde{\alpha}$ in those positions is calculated by $y_1 + (y_4 - y_1) * R$, where (R) is equal to the random value of y in interval $(0, 1)$, *y_1 = 0.4 and y_4 = 0.8.* If the positions are located in the area of *very large*, the value of $\tilde{\alpha}$ in those positions is calculated by $y_1 + (y_4 - y_1) * R$, where (R) is equal to the random value of y in interval $(0, 1)$, *y1 = 0.7 and y_4 = 1.0.* Similarly, the BMI acceleration is represented by trapezoidal membership function. To obtain the fuzzy output for FRBS, we use BMI value and degree of obesity.

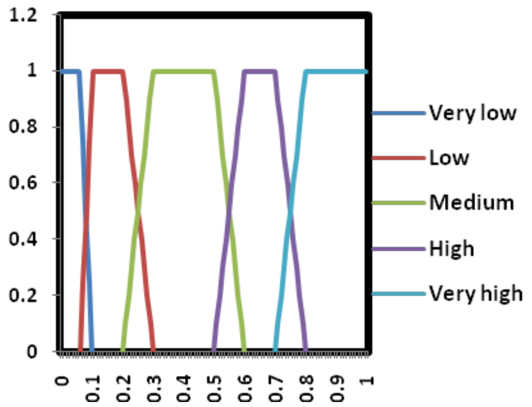

Figure 4. Membership function for inductions Ind_P & Ind_Q

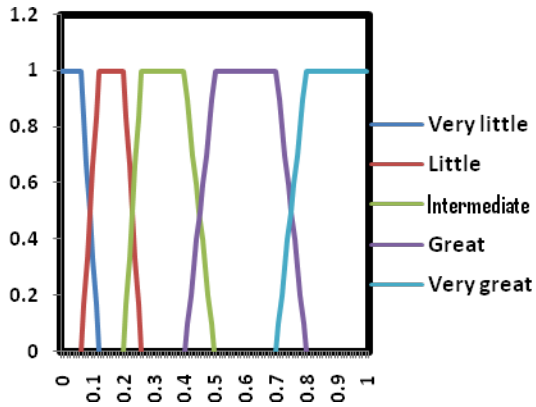

Figure 5. Membership function for total induction $Ind\,T_1$

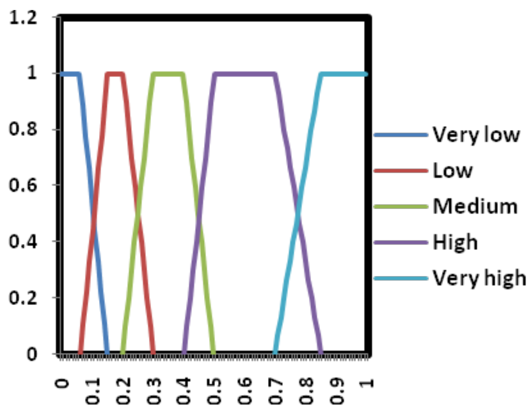

Figure 6. Membership function for U

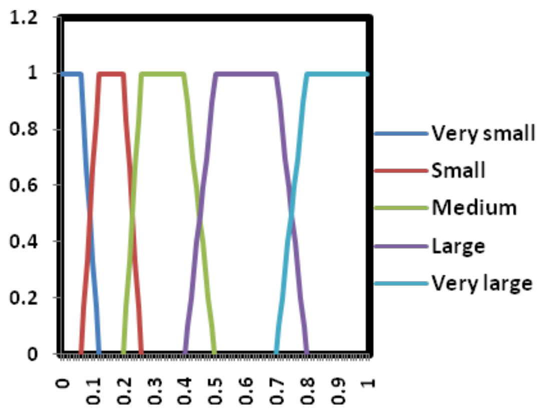

Figure 7. Membership function for $\tilde{\alpha}$

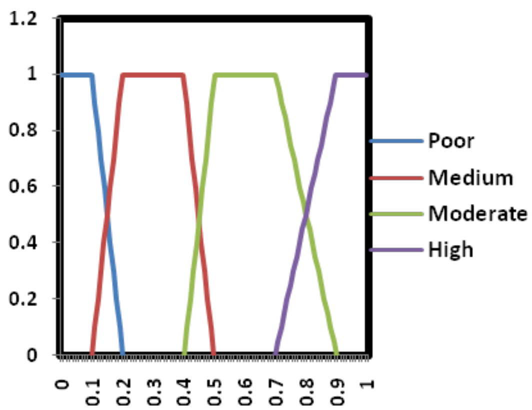

Figure 8. Membership function for inductions Ind_R & Ind_S

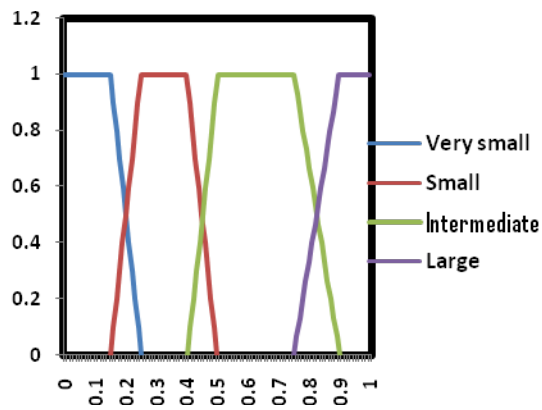

Figure 9. Membership function for total induction $Ind\,T_2$

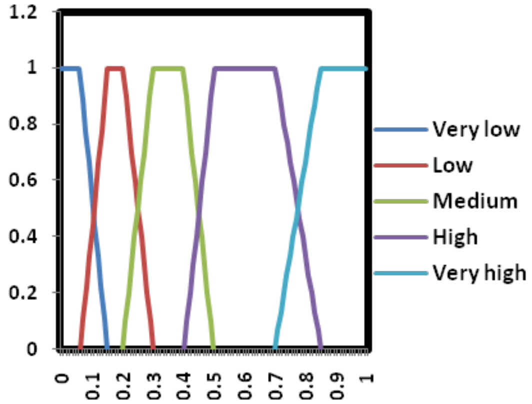

Figure 10. Membership function for V

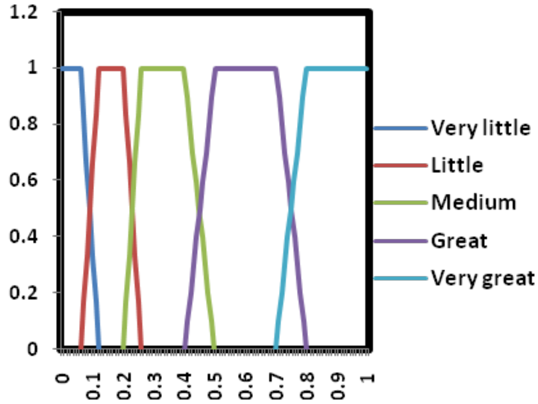

Figure 11. Membership function for $\tilde{\beta}$

6. Conclusion

This paper proposes a model for the assessment of treatment levels in relation to BMI value using the diffusion partial differential equation of parabolic type that models the BMI diffusivity and BMI acceleration using a fuzzy parameter. The fuzzy model use uncertainty parameter. Thus, we conclude that, due to uncertainty obesity phenomena, the combination of differential equation and fuzzy set theory enables our computational simulation to portray the phenomena under study much more faithfully.

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The Forecasting of Istanbul Stock Market with a High Order Multivariate Fuzzy Time Series Forecasting Model

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Abstract

The fuzzy time series approaches, which recently are intensively considered by the researchers, consist of three stages of fuzzification, determination of fuzzy relations and defuzzification. Several studies using different approaches in these steps have been conducted in literature. In most of the studies related fuzzy time series, the membership degrees of belonging to every fuzzy set of each observation are ignored in the stages. This conflicts the fuzzy sets theory and causes the loss of information. The fuzzy set theory is required to regard these membership degrees. In that case, how the membership degrees are defined is considered as an important issue. On the other hand, there is another problem that many fuzzy time series are necessarily modeled by a multivariate fuzzy time series forecasting method to discover the fuzzy relations between that time series and the others. In accordance with this purpose, Yolcu (2011) present a new multivariate fuzzy time series procedure in which the method of fuzzy C-means is used to define the membership degrees and a neural network with the input and output, which are composed of the membership degrees, is used to define fuzzy relations. In this study, we intend to introduce this high-order multivariate fuzzy time series forecasting model and it was applied to 100 in stocks and bonds exchange market of Istanbul which is frequently used in the literature.

Keywords: Multivariate fuzzy time series, membership degree, forecasting, fuzzy C-means, artificial neural network

1. Introduction

Fuzzy time series procedures, attracted attention of many researchers in recent years, have quite a wide area of use, such as information technology, economy, environmental sciences and hydrology. The reason for this, since the data in almost all these areas contain uncertainty, taking into consideration them as fuzzy time series will appear to be more appropriate.

Fuzzy time series was put forward firstly by Song and Chissom (1993a). Fuzzy time series methods depend on the theory of fuzzy set introduced firstly by Zadeh (1965). Fuzzy time series forecasting procedures consist of three fundamental stages; these are fuzzification, determination of fuzzy relations and defuzzification.

The length of interval which is required for partitioning the universe of discourse has a significant impact on the forecasts. For this reason, many studies in the literature on the determination of interval length are available. Song and Chissom (1993a,b, 1994) and Chen (1996, 2002) determine fixed length arbitrarily. Huarng (2001) determines it by depending on the mean and the distribution based approaches. Egrioglu et al. (2010, 2011) identify it by solving an optimization problem. More recent studies state that the interval length is not necessarily fixed. Kuo et al. (2009, 2010), Davari et al. (2009), Park et al. (2010) and Hsu et al. (2010) use the particle swarm optimization technique and Chen and Chung (2006) and Lee et al. (2007, 2008) use the genetic algorithm to find the changeable length. In the approach which has been proposed by Huarng and Yu (2006a) the length was not fixed and Yolcu et al. (2009) used optimization with univariate-constrained and they analyzes time series-with trend. Cheng et al. (2008) and Li et al. (2007) used fuzzy C-means (FCM) procedure for fuzzification.

In fuzzy time series analyses, the determination of fuzzy relations is also an important issue since it affects the forecasting performance. In the literature, there exist some studies which contribute to that stage. For this purpose, Song and Chissom (1993a,b, 1994) used the matrix algebra. Chen (1996) used the fuzzy logic group table. This technique has been used in subsequent studies. The artificial neural networks (ANN) have been also used for this purpose, recently. The studies by Huarng and Yu (2006b), Aladag et al. (2009), Egrioglu et al. (2009a,b,c) and Yu and Huarng (2008, 2010) can be given as examples. In the defuzzification stage the centroid method is frequently used.

For many data sets encountered in real life, a high-order fuzzy time series forecasting model would be more appropriate to be analyzed while first-order fuzzy time series forecasting model can be enough to fit to some fuzzy time series data. Moreover sometimes a univariate fuzzy time series forecasting model can be required, some other time bivariate or multivariate fuzzy time series forecasting models can be needed for analyzing.

In fuzzy time series procedures, the fuzzification stage plays an important role on the forecasting performance. Hence, the subjective approaches, which could be preferred in some cases, should be avoided. In addition, in the determination of fuzzy relations, fuzzy logic group relation tables and ANN generally have been used. However, in all of these approaches the fuzzy set indices have been used. It should not be forgotten that fuzzy observation is belonging to the each fuzzy sets with a certain membership degrees. However, taking into account of all membership degrees associated with all fuzzy sets of each observation could prevent loss of information so forecasting performance can be improved. This brings a question to the mind that how can be the membership degrees determined?

Yu and Huarng (2010) used an ANN, of which the input and output are the membership degrees, for determining the fuzzy relations. However, in this study, the membership degrees were arbitrarily defined. Moreover, Yolcu et al. (2011) proposed a new fuzzy time series forecasting model which takes into account all memberships which are not arbitrarily defined. On the other hand, this method involved a first-order univariate model.

Another issue to consider is that the model frequently has necessarily got multivariate and autoregressive structure for most cases in real life problems. Therefore bivariate or multivariate fuzzy time series forecasting model with high-order is mostly required to discover the fuzzy relationships between the fuzzy time series. Hsu et al. (2010) and Lee et al. (2006, 2007, 2008) used two factor high-order fuzzy time series forecasting methods. Moreover, Jilani and Burney (2007, 2008), Jilani et al. (2007) and Egrioglu et al. (2009a) used multivariate high-order fuzzy time series forecasting method. However, in none of these studies membership degrees were taken into account.

Yolcu (2011) proposed a new approach in which there is no need to partition the universe of discourse and determine the interval length. In the Yolcu's (2011) approach, the membership degrees are determined by using the procedure FCM and the fuzzy relations are determined by ANN. The inputs and the outputs of this ANN consist of the membership degrees. Finally, the model based on this approach is a high-order multivariate fuzzy time series forecasting model. In this study, the approach proposed by Yolcu (2011) was applied to 100 in stocks and bonds exchange market of Istanbul (IMKB).

The remainder of this paper is organized as follows: the fuzzy C-means clustering and artificial neural networks are briefly presented in the Section 2 and 3, respectively. Fuzzy time series and its basic definitions are given in Section 4. Section 5 introduces the Yolcu's method. In the Section 6, the method is applied to IMKB and the obtained results are comparatively presented. Finally, conclusions and discussions are given.

2. The fuzzy C-means clustering method

The fuzzy C-means (FCM) clustering technical method is first introduced by Bezdek (1981). This is a most widely used fuzzy clustering algorithm. FCM partitions set of n observations c $(1 < c < n)$ fuzzy cluster with v_j cluster center. The fuzzy clustering of o is described by a fuzzy matrix μ with n rows and c columns in which n is the number of data objects and c is the number of clusters. μ_{ij}, the element in the ith row and jth column in μ, indicates the degree of association or membership function of the ith object with the jth cluster. The characters of μ are as follows:

$$\mu_{ij} \in [0,1] \quad \forall\, i = 1,2,\cdots,n \,;\, \forall\, j = 1,2,\cdots,c \tag{1}$$

$$\sum_{j=1}^{c} \mu_{ij} = 1 \quad \forall\, i = 1,2,\cdots,n \tag{2}$$

$$0 < \sum_{i=1}^{n} \mu_{ij} < n \quad \forall\, j = 1,2,\cdots,c \tag{3}$$

The objective function of FCM algorithm is to minimize the equation (4).

$$J_\beta(X,V,\mu) = \sum_{j=1}^{c}\sum_{i=1}^{n}\mu_{ij}^{\beta}d_{ij} \quad (4)$$

where,

$$d_{ij} = \|x_i - v_j\| \quad (5)$$

in which, $\beta\ (\beta > 1)$ is a scalar termed the weighting exponent and controls the fuzziness of the resulting clusters and d_{ij} is the Euclidian distance from object x_i to the cluster center v_j. In this method, minimizing is done by an iterative algorithm. In each repetition the values of μ_{ij} and v_j are updated by the formulas given in equation (6) and equation (7).

$$v_j = \frac{\sum_{i=1}^{n}\mu_{ij}^{\beta}x_i}{\sum_{i=1}^{n}\mu_{ij}^{\beta}} \quad (6)$$

$$u_{ij} = \frac{1}{\sum_{k=1}^{c}\left(\frac{d_{ij}}{d_{ik}}\right)^{2/(\beta-1)}} \quad (7)$$

3. Artificial neural network

Artificial neural networks (ANN) can be defined as the mathematical algorithm that is inspired by the biological neural networks (Gunay et al., 2007). Artificial neural networks are much more different than biological ones in terms of their structure and ability (Zurada, 1992). Artificial neural networks compose of a mathematical model (Zhang, 1998). The learning capability of an artificial neuron is achieved by adjusting the weights in accordance to the chosen learning algorithm. The basic architecture consists of three types of neuron layers: input, hidden, and output layers. In feed-forward networks, the signal flow is from input to output units, strictly in a feed-forward direction. Artificial neural network architectures are characterized by the following attributes:

Number of Layers: The artificial neurons are arranged in an input layer, one or more hidden layers, and an output layer.

Number of Neurons: The artificial neural network has to learn the features of the series for the analysis and forecasting of a fuzzy time series. As the number of neurons in the input and output layers are determined by the training patterns, the number of neurons in the hidden layers can then be chosen arbitrarily (see Figure 1). More artificial neurons imply more weighting matrices. Thus, from classical fields of application of artificial neural networks (e.g., pattern recognition), the well-known problem of over fitting must be considered.

Activation Function: The proper selection of activation function that enables curvilinear matching between input and output units, significantly affect the performance of the network.

Method of Training: The learning situations in neural networks may be classified into three distinct sorts. These are supervised learning, unsupervised learning, and reinforcement learning. In supervised learning, an input vector is presented at the inputs together with a set of desired responses, one for each node, at the output layer. The most widely used one is Back Propagation algorithm which updates weights based on the difference between available data and the output of the network. Learning parameter which is used in back propagation algorithm and which can be taken fixedly or updated in the algorithm dynamically, plays an important role in reaching optimal results.

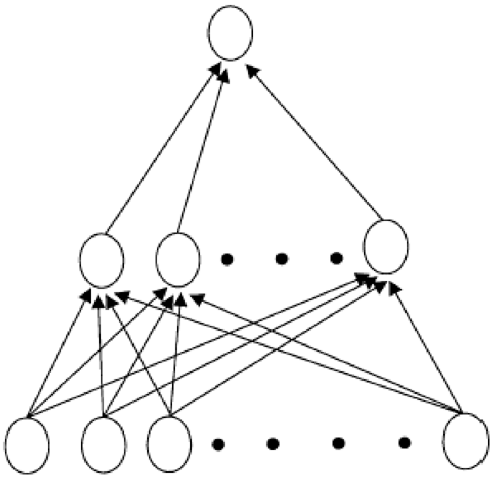

Figure 1. Architecture of multilayer feed forward neural network

4. Fuzzy time series

The definition of fuzzy time series was first introduced by Song and Chissom (1993a,b, 1994). In contrast to conventional time series methods, various theoretical assumptions do not need to be checked in fuzzy time series approaches. The most important advantage of the fuzzy time series approaches is to be able to work with a very small set of data. The definition of fuzzy time series is given as follows:

Let U be the universe of discourse, where $U = \{u_1, u_2, \cdots, u_n\}$. A fuzzy set A_i of U can be defined as,

$$A_i = \frac{\mu_{A_i}(u_1)}{u_1} + \frac{\mu_{A_i}(u_2)}{u_2} + \cdots + \frac{\mu_{A_i}(u_n)}{u_n} \tag{8}$$

Where μ_{A_i} is the membership function of the fuzzy set A_i and $\mu_{A_i}; U \to [0,1]$. In addition to, $\mu_{A_i}(u_j), j = 1,2, \cdots, n$ denotes is a generic element of fuzzy set $A_i; \mu_{A_i}(u_j)$ is the degree of belongingness of u_1 to $A_i; \mu_{A_i}(u_j) \in [0,1]$.

Definition 1. Let $Y(t)$ $(t = \cdots, 0, 1, 2, \cdots)$, a subset of real numbers, be the universe of discourse on which fuzzy sets $f_i(t)$ are defined. If $F(t)$ is a collection of $f_1(t), f_2(t), \cdots$ then $F(t)$ is called a fuzzy time series defined on $Y(t)$.

Definition 2. Fuzzy time series relationships assume that $F(t)$ is caused only by $F(t-1)$, then the relationship can be expressed as: $F(t) = F(t-1) * R(t, t-1)$, which is the fuzzy relationship between $F(t)$ and $F(t-1)$, where $*$ represents as an operator. To sum up, let $F(t-1) = A_i$ and $(t) = A_j$. The fuzzy logical relationship between $F(t)$ and $F(t-1)$ can be denoted as $A_i \rightarrow A_j$ where A_i (current state) refers to the left-hand side and A_j (next state) refers to the right-hand side of the fuzzy logical relationship. Furthermore, these fuzzy logical relationships can be grouped to establish different fuzzy relationship.

Definition 3. Let $F(t)$ be a fuzzy time series. If $F(t)$ is a caused by$(t-1), F(t-2), \cdots$ $F(t-m)$, then this fuzzy logical relationship is represented by

$$F(t-m), F(t-m+1), \cdots\ F(t-2), F(t-1) \rightarrow F(t) \tag{9}$$

and it is called the m^{th} order fuzzy time series forecasting model. Where "$F(t-m),\ F(t-m+1), \cdots\ F(t-2), F(t-1)$" refers to the current state and $F(t)$ refers to the next state.

Definition 4. Let $F_1(t)$ and $F_2(t)$ be two fuzzy time series. Suppose that $F_1(t-1) \rightarrow A_i$, $F_2(t-1) \rightarrow B_k$, and $F_1(t) \rightarrow A_j$. A bivariate fuzzy logic relationship is defined as $A_i, B_k \rightarrow A_j$, where A_i, B_k are referred to as the left hand side and A_j as the right hand side of the bivariate fuzzy logical relationship. Therefore, first order bivariate fuzzy time series forecasting model is as follows:

$$F_1(t-1), F_2(t-1) \rightarrow F_1(t) \tag{10}$$

Where $F_1(t)$ and $F_2(t)$ are called the main factor fuzzy time series and the secondary factors fuzzy time series, respectively$(t = \cdots, 0, 1, 2, \cdots)$.

Definition 5. Let $F_1(t)$ and $F_2(t)$be two fuzzy time series. If $F_2(t)$ is caused by $(F_1(t-1), F_2(t-1)), (F_1(t-2), F_2(t-2)), \cdots, F_1(t-m), F_2(t-m)$then this fuzzy logical relationship is represented by

$$\left.\begin{array}{c} (F_1(t-m), F_2(t-m)), \\ \vdots \\ \vdots \\ (F_1(t-2), F_2(t-2)), \\ \\ (F_1(t-1), F_2(t-1)) \end{array}\right\} \rightarrow F_1(t) \tag{11}$$

and it is called the two-factors m^{th} order fuzzy time series forecasting model, where $F_1(t)$ and $F_2(t)$ are called the main factor fuzzy time series and the second factor fuzzy time series, respectively $(t = \cdots, 0, 1, 2, \cdots)$.

Definition 6. Let $F_1(t), F_2(t), \cdots, F_k(t)$ be m fuzzy time series. If $F_1(t)$ is caused by $(F_1(t-1), F_2(t-1), \cdots, F_k(t-1)), (F_1(t-2), F_2(t-2), \cdots, F_k(t-2)), \cdots, (F_1(t-m), F_2(t-m), \cdots, F_k(t-m))$ then this fuzzy logical relationship is represented by

$$\left.\begin{array}{c} (F_1(t-m), F_2(t-m), \cdots F_k(t-m)), \\ \vdots \\ \vdots \\ (F_1(t-2), F_2(t-2), \cdots F_k(t-2)), \\ \\ (F_1(t-1), F_2(t-1), \cdots F_k(t-1)) \end{array}\right\} \rightarrow F_1(t) \tag{12}$$

and it is called the k-factors m^{th} order fuzzy time series forecasting model, where $F_1(t)$ and $F_2(t), F_3(t), \cdots, F_k(t)$ are called the main factor fuzzy time series and the secondary factors fuzzy time series, respectively $(t = \cdots, 0, 1, 2, \cdots)$.

5. The high-order multivariate fuzzy time series forecasting model

In this study, we focused on high-order multivariate fuzzy time series forecasting model introduced firstly by Yolcu (2011). In the most of fuzzy time series procedures in the literature the partitioning of the universe of discourse in the fuzzification stage is based on subjective decisions. As we pointed out many times this is a crucial decision in terms of forecasting performance. Therefore Yolcu (2011) preferred a more systematic procedure by using one of fuzzy sampling techniques. The method which is proposed by Yolcu (2011) uses the FCM in the fuzzification stage, takes into account the membership degrees in the determination stage of fuzzy relations and constructs a multivariate fuzzy time series forecasting model. This method has some important advantages which can be summarized as follows:

- In this approach, the number of intervals, the interval length, and the degrees of memberships are not arbitrarily determined in the fuzzification phase so the approach is more systematic method than the others.

- Membership values of observations are employed when fuzzy relations are defined. Therefore, information loss is prevented and also explanatory power of the model increase.

- In lots of encountered problems in the real life, a forecasting method must be set to reveal fuzzy relations between fuzzy time series and some other fuzzy time series. This method overcomes this problem via its multivariate structure.

- The method proposed by Yolcu (2011) is the first multivariate high-order fuzzy time series forecasting method in the literature in which membership degrees are taken into account to define fuzzy relations.

- In addition to the advantages of this method is given above, it has high forecasting accuracy.

The algorithm of the method is given below.

Step 1 Determine the model order and construct the lagged variable.

The model order d which is planed as high-order and with k variables is defined. Then $(d-1)$ lagged variables are constructed.

Step 2 Time series are fuzzified.

The number of fuzzy sets c is defined with the number of observations, n and the constraint $2 \leq c \leq n$. The data consisted of the observations of each $k \times d$ variables in the multivariate structure model is applied FCM and the fuzzy sets are determined. Then the centers of the sets are calculated and the values of the centers of the fuzzy sets $v_r,\ r = 1,2,\cdots,c$ are ordered ascending. These ordered sets are symbolized by $L_r,\ r = 1,2,\cdots,c$. Finally the membership degrees of belonging to the fuzzy sets of each observation are obtained.

Step 3 The fuzzy relations are determined by feed forward artificial neural network .

The architecture of a FFANN with 2 neurons in the hidden layer for the model with 3 sets is given Fig. 2. In Fig. 2 $\mu_{L_i}(DS(t))$ represents the membership degrees for the data set at time t. The membership degrees in the left side of the figure are inputs of the FFANN, in the right side are the target values of FFANN. The outputs of the system are forecasts at time t.

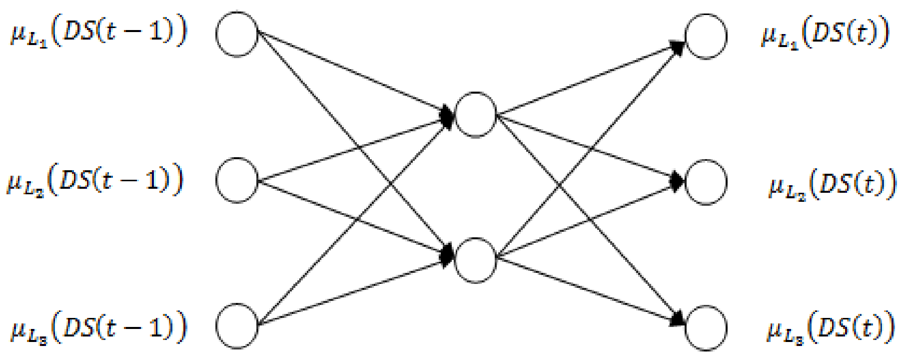

Figure 2. The architecture of an FFANN with c fuzzy sets.

Step 4 Fuzzy forecasts are defuzzified.

In order to get the fuzzy forecast at time t, first of all we have to define the membership degrees of the observation at time $t-1$; the observation means actually $(t-1)^{th}$ row of the data set, which is based on the centers of the fuzzy sets, $v_r, \ r = 1,2,\cdots,c$, defined by FCM. These membership degrees are given as the inputs of FFANN then the output are requested from the system. When the outputs are generated by the system these outputs are going to be the membership degrees of the fuzzy forecast at time t. We want to be noticed that the sum of the membership degrees are not equal to 1 in contrast to FCM. In the defuzzification stage the membership degrees are converted to the weights as in (13). Then we get the defuzzyfiyed forecast are obtained as in (14).

$$w_{it} = \frac{\hat{u}_{it}}{\hat{u}_{1t}+\hat{u}_{2t}+\cdots+\hat{u}_{ct}} \tag{13}$$

$$\hat{X}_t = \sum_{i=1}^{c} w_{it} v_i \tag{14}$$

Here, $\hat{u}_{it}$ is the membership degree of belonging to the i^{th} fuzzy set for the forecast at time t obtained from the output of the system and w_{it} is i^{th} weight which is obtained as in (13).

6. The application

The focused on method in this paper has been applied to three different data sets which are called IMKB. Three different data sets for IMKB time series were used to compare the results obtained from the focused on method to those produced by other methods. Data set 1 includes observations between 03.10.2008 – 31.12.2008, data set 2 includes observations between 01.10.2009 – 31.12.2009 and data set 3 includes observations between 01.10.2010 – 23.12.2010, respectively. The time-series graphs of the three data sets are shown in Figure 3–5, respectively. The last seven and fifteen observations of every data set are used for test set. In this application the main factor has been determined as "IMKB" $(X(t))$ and "TL/ USD" exchange rate $(Y_1(t))$ and "TL/ EURO" exchange rate $(Y_2(t))$ as the secondary factors. In addition to the features of the parameters of all applications are specified as follows:

- The model order (d) has been tried between 1 and 10. This means that each data set has been analyzed for 10 distinct forecasting models.
- The number of fuzzy sets (c) is tried between 5 and 15.
- For the system FFANN the number of units in the hidden layers has been changed between 1 and 15.

In all layer of FFANN the following logistic activation function is used.

$$f(x) = \left(1 + exp(-x)\right)^{-1} \tag{15}$$

The learning of FFANN is generalized by the algorithm of Levenberg-Marquardt on the training data. Then FFANN would have been trained about the relationships between the membership degrees for sequential observations.

The assessments of the results have been made by the following measures. These are the root of mean square error (RMSE), the mean absolute percentage error (MAPE) and the direction accuracy (DA).

$$RMSE = \sqrt{\frac{\sum_{t=1}^{n}(X_t - \hat{X}_t)^2}{n}} \tag{16}$$

$$MAPE = \frac{1}{n}\sum_{t=1}^{n}\left|\frac{X_t - \hat{X}_t}{X_t}\right| \tag{17}$$

$$DA = \frac{1}{n-1}\sum_{t=1}^{n-1}\begin{cases}1, & (X_{t+1} - X_t)(\hat{X}_{t+1} - X_t) > 0 \\ 0, & o.w.\end{cases} \tag{18}$$

The results were compared to the results obtained from the some other methods in the literature. The performance measures and the forecasts belonging to the best case of each method are summarized Table 1-6.

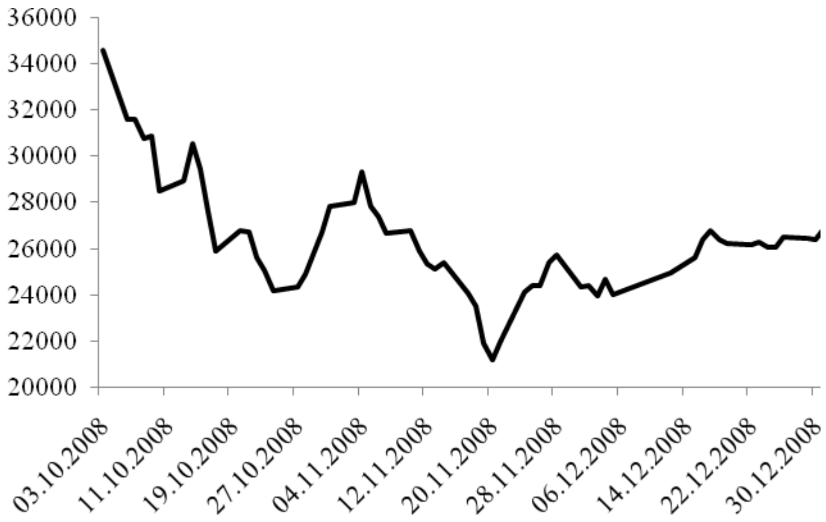

Figure 3. The data set 1.

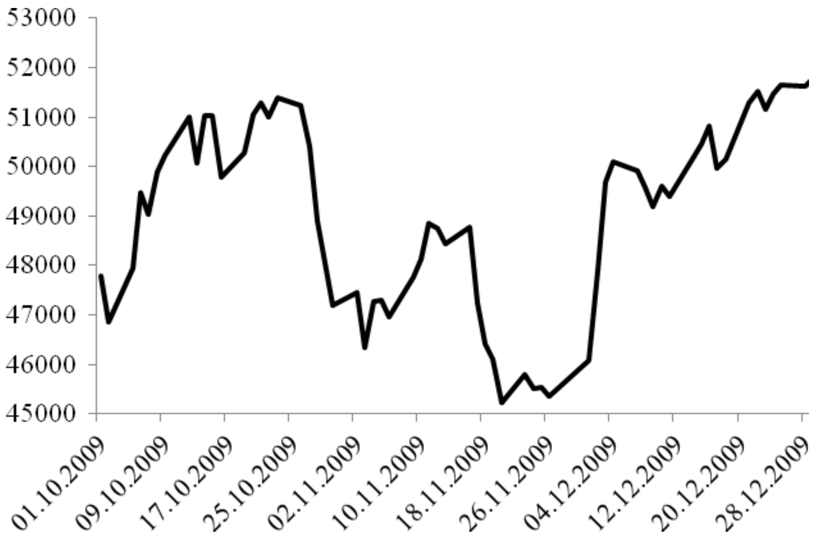

Figure 4. The data set 2.

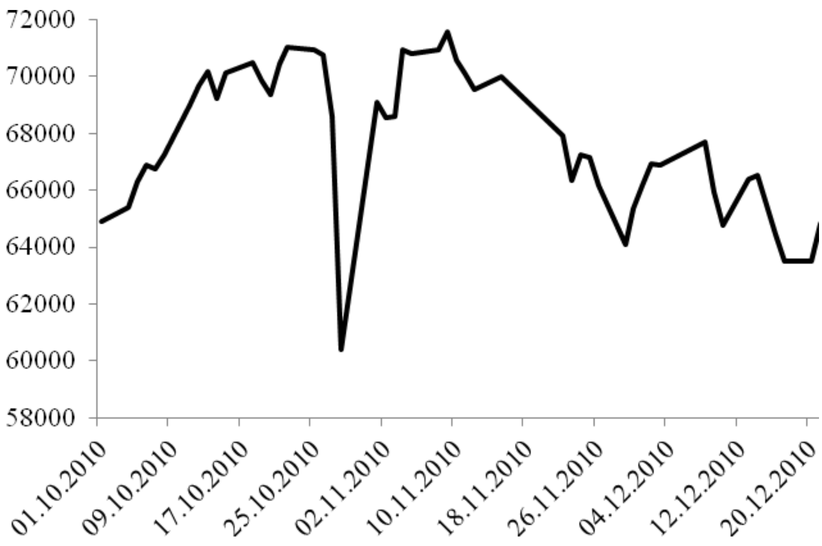

Figure 5. The data set 3.

Table 1. The results of data set 1, with sample size 7 for the test data

Date	IMKB	Song-Chissom (1993b)	Chen (1996)	Huarng (2001[a])	Huarng (2001[b])	Huarng and Yu (2006a)	Cheng et al. (2008)	Yolcu et al. (2011)	Yolcu (2011)
23.12.2008	26294	26410	26400	26200	26100	26091	26390	26274	26415.22
24.12.2008	26055	26410	26400	26200	26367	26091	26390	26273	25881.81
25.12.2008	26059	26410	26400	26200	26100	26091	26390	26339	26246.13
26.12.2008	26499	26410	26400	26200	26100	26091	26390	26337	26646.42
29.12.2008	26424	26410	26400	26200	26500	26608	26390	26565	26332.73
30.12.2008	26411	26410	26400	26200	26500	26608	26390	26429	26401.48
31.12.2008	26864	26410	26400	26200	26500	26091	26390	26460	26705.26
RMSE		261.01	259.76	310.47	251.24	354.72	258.87	219.27	138.90
MAPE		0.75%	0.75%	0.96%	0.80%	0.98%	0.76%	0.67%	0.48%
DA		66.67%	66.67%	83.33%	50.00%	50.00%	66.67%	66.67%	100.00%

a. Distribution-Based **b.** Average-Based

Table 2. The results of data set 1, with sample size 15 for the test data

Date	IMKB	Song-Chissom (1993b)	Chen (1996)	Huarng (2001[a])	Huarng (2001[b])	Huarng and Yu (2006a)	Cheng et al. (2008)	Yolcu et al. (2011)	Yolcu (2011)
05.12.2008	24035	24337	24750	25200	24700	25598	25541	24355	24317.16
12.12.2008	24937	24337	24750	23533	24100	24597	25406	24279	25362.67
15.12.2008	25598	25670	24750	25200	26700	25598	25406	25371	25105.15
16.12.2008	26396	25670	26250	24600	24100	25426	25406	25704	26332.66
17.12.2008	26765	26319	26250	26200	26300	26255	25406	26348	26443.36
18.12.2008	26396	26558	26250	26600	26550	26733	25406	26344	26469.05
19.12.2008	26205	26319	26250	26200	26300	26255	25406	26348	26471.74
22.12.2008	26199	26319	26250	26200	26300	26255	25406	26315	26380.10
23.12.2008	26294	26319	26250	26200	26100	26255	25406	26316	26334.62
24.12.2008	26055	26319	26250	26200	26300	26255	25406	26313	26350.68
25.12.2008	26059	26319	26250	26200	26100	26255	25406	26330	26391.44
26.12.2008	26499	26319	26250	26200	26100	26255	25406	26330	26392.34
29.12.2008	26424	26319	26250	26200	26500	26255	25406	26419	26394.20
30.12.2008	26411	26319	26250	26200	26500	26255	25406	26365	26390.80
31.12.2008	26864	26319	26250	26200	26500	26255	25406	26357	26353.89
RMSE		338.96	378.54	718.66	743.54	544.14	987.04	337.07	282.01
MAPE		1.03%	1.11%	1.90%	1.83%	1.41%	3.54%	1.00%	0.89%
DA		71.43%	71.43%	71.43%	50.00%	71.43%	57.14%	71.43%	71.43%

a. Distribution-Based **b.** Average-Based

Table 3. The results of data set 2, with sample size 7 for the test data

Date	IMKB	Song-Chissom (1993b)	Chen (1996)	Huarng (2001[a])	Huarng (2001[b])	Huarng and Yu (2006a)	Cheng et al. (2008)	Yolcu et al. (2011)	Yolcu (2011)
23.12.2009	51162	51137	52150	51900	51573	51033	50872	51317	51341
24.12.2009	51461	51137	50850	50700	50373	51033	50763	51317	51336
25.12.2009	51661	51137	50850	50700	51240	51033	50763	51317	51341
28.12.2009	51619	51137	52150	51900	51573	51033	50763	51317	51341
29.12.2009	51786	51137	52150	51900	51573	51033	50763	51317	51341
30.12.2009	51668	51137	52150	51900	51773	52004	50763	51317	51341
31.12.2009	52825	51137	52150	51900	51573	51033	50763	51317	51341
RMSE		771.02	666.50	659.95	671.48	830.44	1084.21	640.41	625.26
MAPE		1.16%	1.23%	1.11%	0.97%	1.28%	1.85%	0.90%	0.86%
DA		33.33%	33.33%	33.33%	33.33%	16.67%	33.33%	50.00%	50.00%

a. Distribution-Based **b.** Average-Based

Table 4. The results of data set 2, with sample size 15 for the test data

Date	IMKB	Song-Chissom (1993b)	Chen (1996)	Huarng (2001[a])	Huarng (2001[b])	Huarng and Yu (2006a)	Cheng et al. (2008)	Yolcu et al. (2011)	Yolcu (2011)
11.12.2009	49386	49872	50250	49500	49100	49748	49984	49516	49507.18
14.12.2009	50198	48606	48750	49500	49300	49316	49984	50064	49609.84
15.12.2009	50450	49872	50250	49900	50500	50405	49984	50942	49673.15
16.12.2009	50817	50294	50250	49900	48900	48886	49984	50217	50713.53
17.12.2009	49963	50294	50250	50300	50900	48886	49984	49641	50178.52
18.12.2009	50138	49872	50250	49900	49900	49748	49984	49619	49847.44
21.12.2009	51281	49872	50250	49900	50500	50405	49984	50932	50847.57
22.12.2009	51533	51137	51000	50300	50967	50625	49984	50817	50840.97
23.12.2009	51162	51137	51000	51900	51500	51065	49984	51100	51094.21
24.12.2009	51461	51137	51000	50300	50550	50625	49984	50646	51063.35
25.12.2009	51661	51137	51000	50300	51500	51065	49984	51073	51055.12
28.12.2009	51619	51137	51000	51900	51700	51065	49984	51117	51053.60
29.12.2009	51786	51137	51000	51900	51700	51065	49984	51114	51048.78
30.12.2009	51668	51137	51000	51900	51700	51963	49984	51119	51048.90
31.12.2009	52825	51137	51000	51900	51700	51065	49984	51117	51047.61
RMSE		810.85	820.57	815.99	760.77	917.11	1381.68	662.35	668.72
MAPE		1.28%	1.33%	1.34%	1.10%	1.47%	2.25%	1.06%	1.03%
DA		28.57%	50.00%	28.57%	50.00%	35.71%	42.86%	50.00%	50.00%

a. Distribution-Based **b.** Average-Based

Table 5. The results of data set 3, with sample size 7 for the test data

Date	IMKB	Song-Chissom (1993b)	Chen (1996)	Huarng (2001[a])	Huarng (2001[b])	Huarng and Yu (2006a)	Cheng et al. (2008)	Yolcu et al. (2011)	Yolcu (2011)
15.12.2010	65499	65355	65500	66167	66500	67254	65992	65999	66059.58
16.12.2010	64429	65355	65500	65500	66300	66035	65992	64813	65832.91
17.12.2010	63524	65974	65500	66000	64500	65435	65992	64864	64250.22
20.12.2010	63502	64736	64950	63500	63500	63668	65992	64871	64250.22
21.12.2010	64820	64736	64950	63500	63500	63668	65992	64858	64250.22
22.12.2010	65440	65974	65500	66000	65500	66038	65992	65522	64250.22
23.12.2010	66219	65355	65500	65500	66300	66035	65992	64813	66062.72
RMSE		1161.21	1047.84	1212.81	1014.73	1256.90	1544.90	929.82	856.66
MAPE		1.39%	1.20%	1.51%	1.17%	1.63%	2.00%	1.13%	1.18%
DA		50.00%	50.00%	50.00%	50.00%	50.00%	50.00%	50.00%	50.00%

a. Distribution-Based **b.** Average-Based

Table 6. The results of data set 3, with sample size 15 for the test data

Date	IMKB	Song-Chissom (1993b)	Chen (1996)	Huarng (2001[a])	Huarng (2001[b])	Huarng and Yu (2006a)	Cheng et al. (2008)	Yolcu et al. (2011)	Yolcu (2011)
01.12.2010	66156	65974	65500	66500	65300	66035	65776	66421	66155.17
02.12.2010	66939	66163	65500	66167	64100	66048	65600	65817	65204.42
03.12.2010	66860	66163	67517	66167	66700	66946	65776	66776	66775.52
08.12.2010	67705	66163	67517	66167	66700	66946	65600	66752	66775.77
09.12.2010	65914	66206	66325	67833	67700	66035	65776	66783	66777.94
10.12.2010	64759	65974	65500	66500	65900	66048	65600	65023	64938.77
13.12.2010	66380	65974	65500	65500	64700	65435	65776	66005	65037.52
14.12.2010	66510	66163	65500	66167	67100	66946	65600	66774	66758.95
15.12.2010	65499	66163	65500	66167	66500	66946	65776	66775	66772.94
16.12.2010	64429	65974	65500	66500	66300	66035	65600	65064	65154.58
17.12.2010	63524	65277	65500	65500	64500	65435	65776	65019	64959.36
20.12.2010	63502	65277	64950	63500	63500	63668	65600	65037	64960.35
21.12.2010	64820	65277	64950	63500	63500	63668	65776	65035	64960.35
22.12.2010	65440	65974	65500	65500	65500	65435	65600	65599	65073.31
23.12.2010	66219	65974	65500	66500	66300	66035	65776	66416	66095.58
RMSE		998.30	1047.84	1200.93	1283.18	961.50	1197.78	816.92	931.96
MAPE		1.27%	1.20%	1.49%	1.56%	1.14%	1.50%	0.99%	1.11%
DA		57.14%	50.00%	50.00%	57.14%	50.00%	57.14%	64.29%	64.29%

a. Distribution-Based **b.** Average-Based

7. Conclusions and discussion

Most of the real time series data can be evaluated by fuzzy time series approaches since the uncertainty in these time series can be considered as uncertainty defined in fuzziness.

In almost every study in fuzzy time series literature, the partitioning the universal discourse, which is a subjective approach, has been used. Then fuzzy clustering techniques has been began to use instead. The tables of group relations have been used in the stage of defining the fuzzy relations in many researches. Alternatively artificial neural networks are begun to use for the same purpose. In all of mentioned researches using artificial neural networks, the inputs of the network are the index number of the fuzzified observations since the original observations are fuzzified by the index number of the fuzzy set with the highest membership degree. Yolcu (2011) proposed to regard all of the membership degrees of observations. First of alli it is necessary to find all membership degrees by the way which is not arbitrary. Yolcu (2011) did this by using FCM to find membership degrees of belonging to every fuzzy set of any observation. After that, all memberships degrees of an observation are given to the network as inputs and the outputs are requested as membership degrees. This approach uses all membership degrees in the defuzzificaion stage. Finally the approach proposed by Yolcu (2011) was a high-order multivariate fuzzy time series forecasting model with all these things.

In this study, this method suggested by Yolcu (2011) was illustrated by applying it to three different time series data. And the results were compared with those from other approaches previously used in literature. According to these results, it was observed that the Yolcu's (2011) method provides superior forecasting performance in almost all cases.

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