A fuzzy membership approach to obtain efficient solutions of the interval valued bicriteria shortest path problem

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Abstract
Optimal solutions of the interval valued bicriteria shortest path problem are obtained by using a new fuzzy membership approach. The two criteria of cost and duration of travel being not precise in real life problems are represented by interval numbers. Ordering between all types of interval numbers is defined by a fuzzy membership approach. Incorporating the newly defined ordering, Dijkstras algorithm is modified and extended to get the set of efficient solutions of the problem. The method is explained by implementing it on a numerical.

Keywords: Interval number, bicriteria, membership function, shortest path

1. Introduction

The techniques of mathematical optimization are applied across various applications in the areas of business and engineering. The computational methods have served as tools to solve problems of signal processing, mechanical designing, revenue management, facility allocation problems etc. Limited availability of valuable resources and the objective of attainment of desirable outcomes have developed a very systematic approach to the process of decision making. In the field of mathematical programming at times it is either not possible or not necessary to quantify precisely all the decision variables and parameters. This imprecision of facts is dealt with the tool of fuzzy optimization which works on systems with uncertain data and vague objectives. Many such problems have been solved by representing the imprecise parameters with the help of fuzzy sets. Some of the pioneers in this field have been Zadeh (1965), Delgado et al (1994), Zimmermann (1997) and Lodwick et al (2010). In these techniques the fuzzy sets were defined using membership functions which were used to convert the problems to crisp ones. Choosing the appropriate membership function is the main hindrance for these methods. Another method of dealing with such situations is to replace the fuzzy
parameters by intervals whose end points signify the range within which the parameters can take values. Many techniques have been employed to compare these intervals to suit the realistic problems. Moore (1979) first proposed two transitive order relations of interval numbers. But these definitions failed to compare all types of intervals. Ishibuchi and Tanaka (1990) improved these definitions and gave new ones separately for maximization and minimization problems which apart from using the end points of the intervals also used centre and radius.

Chanas and Kuchta (1996) extended the technique given by Ishibuchi and Tanaka (1990) by modifying the ranking definitions and applying the concept of \( t_0, t_1 \) – cut of the intervals. The drawbacks of this approach were further rectified by Hu and Wang (2006) who used the center as well as the radius of the interval numbers for comparison. But this technique did not consider the say of the decision maker. To overcome this shortcoming Mahato and Bhunia (2006) proposed a new method which gave more weightage to the acceptance level of the decision maker. A different approach of comparing intervals was based on calculations of some special indices or functions. Such techniques were proposed by Sengupta and Pal (2000) who gave two ranking definitions using a specific index known as value judgement index. Their method used the center and radius of interval numbers to determine the index. Later Levin (2004) proposed another interval comparison technique using the remoteness function.

Some researchers used probabilistic approach to compare intervals. One such approach was given by Kundu (1997) which applied the concept of fuzzy leftness relation between intervals for comparison. However the method was unable to rank intervals with the same mid value. This shortcoming was removed by making some modifications which were given by Hu and Wang (2006). Sengupta and Pal (2000) gave another approach to compare intervals which used the membership functions of the rejection set of a given set. This definition was proposed from the point of view of a pessimistic decision maker. But this technique would give varying results if the pessimible level was changed.

Later Sevastjanov and Rog (2006) defined ranking techniques which used special functions to compare intervals. But this technique had a drawback of being unable to compare intervals with same mid value. Yet another approach given by Yu et al (2015) considers multiple objective transportation problems with the multiple objectives being maximization of profits, minimization time of delivery etc. The problems considered are characterized by interval valued coefficients in the objective function and source and destination parameters. The method interactively solved them by converting them into single objective problems.

The shortest path problem (SPP) deals with finding the path between two vertices with the objective being to optimize total sum of weights of corresponding edges. Dijkstra (1959), Floyd (1962), Papadimitriou and Steiglitz (2006) and Taha (2008) to name a few have developed algorithms to solve such problems. The Dijkstra’s algorithm named after Dijkstra is used to find the shortest path between two nodes for both directed and undirected connected graphs with edges having non-negative weights. In the implementation of this algorithm, starting from the source vertex the shortest path is traced reaching the destination vertex by considering all vertices reachable from the
source vertex. Vertices are added in the shortest path depending upon their distance from the source vertex. First the vertex closest to the source vertex is added, then among the remaining vertices the next vertex closest to the source vertex is added and the process continues till the destination vertex is reached. In Floyd’s algorithm, developed by Floyd, the network is represented by a matrix and shortest path is found between every pair of vertices. The time complexity of Dijkstra’s algorithm is quadratic in the number of vertices as compared to the time complexity of Floyd’s algorithm which is cubic in the number of vertices and as such Dijkstra’s algorithm gives a result faster than the Floyd’s algorithm in case shortest path from one vertex to another is to be found. However Floyd’s algorithm gives the shortest path between every pair of vertices simultaneously which is not obtained in Dijkstra’s algorithm. Papadimitriou and Steiglitz (2006) have applied the primal dual approach and branch and bound approach to develop two algorithms to solve the SPP and Taha (2008) has solved the SPP by using a dynamic programming approach.

In real life shortest path problems the weights associated with edges signifying cost and time are not precise. To overcome this difficulty numerous algorithms have been developed with the weights on edges of network being fuzzy numbers. Dubois and Prade (1980) were the first researchers to solve the fuzzy SPP (FSPP) with weights of edges being fuzzy numbers. By their proposed method the shortest distance between two vertices could be determined, however shortest path could not be found in all cases. Many workers have since then solved the FSPP using various techniques each having its advantages and limitations. Lin and Chen (1994) have used a fuzzy linear programming approach to solve FSPP. Okada (2001) defined a “degree of possibility” to solve the FSPP. Klien (1991) and Mahdavi et al (2009) have used dynamic programming to solve FSPP. Li et al (1996) have used neural networks to solve the FSPP. Gent et al (1997), Takahashi and Yamakami (2005) and Lin and Shih (2011) have proposed Genetic algorithms to solve FSPP. Anusuya and Kavitha (2015) have solved the FSPP by applying a hybrid genetic ant (HGA) algorithm. Ebrahimnejad et al (2015) have applied a particle swarm optimization algorithm to solve the FSPP. Hernandes et al (2007) have used a generic ranking technique to get optimal solutions of FSPP. The multiobjective FSPP has been considered by Shih and Lee (1999), Okada and Sopar (2000) and Yu and Wei (2007). Kumar and Kaur (2011) have proposed an algorithm to obtain both shortest path and shortest distance in a FSPP. Prakash and Lakshmi (2015) have solved the FSPP with weights on edges being trapezoidal fuzzy numbers by using the graded mean integration of Pascal’s triangle.

Gupta and Pal (2006) have considered FSPP with weights on edges being interval fuzzy numbers. They have found optimal solutions from optimistic and pessimistic points of view. Nayeeem and Pal (2005) have considered the arc lengths of FSPP as interval numbers and triangular fuzzy numbers and developed a common algorithm to find the shortest path from a vertex to every other vertex for both types of numbers. Meenakshi and Kaliraja (2012) have considered the Interval valued fuzzy networks and have found the shortest path by considering lower and upper limit fuzzy networks associated with fuzzy weights. Okada and Gen (1993) have also considered arc lengths as interval numbers. They have defined a partial ordering between interval numbers as follows:
(i) For a maximization problem
\[ A \preceq_\beta B \iff B^c - A^c \geq (1 - \beta) (B^w - A^w) \]
and \[ B^c - A^c \geq \beta (B^w - A^w), \] for \( \beta \in [0, 0.5] \)

(ii) For a minimization problem
\[ A \preceq_\beta B \iff B^c - A^c \geq (1 - \beta) (A^w - B^w) \]
and \[ B^c - A^c \geq \beta (A^w - B^w), \] for \( \beta \in [0, 0.5] \)

They have shown that the relation \( \preceq_\beta \) is a partial ordering and at \( \beta = 0.5 \) it satisfies comparability. In this paper a technique is proposed which solves the bicriteria shortest path problem with the weights on edges being interval numbers. A set of efficient solutions of the problem is obtained by modifying and extending Dijkstra’s algorithm wherein comparison is based on the membership function associated with each interval. Finally the method is explained by implementing it on an example. The results obtained are compared with the results obtained by the method proposed by Okada and Gen (1993) and it is observed that the newly developed membership function approach gives better efficient solutions. The proposed technique is also easier to understand and implement in comparison to the existing techniques and less time consuming.

Definition 1: Interval number representation and ordering relation
An interval number \( A \) is defined as \( A = [a^L, a^R] = \{ x: a^L \leq x \leq a^R \} \) with \( A^w = (a^R - a^L)/2 \) and \( A^c = (a^R + a^L)/2 \) denoting the width (radius) and center of the interval respectively. Such numbers are frequently used in real life problems to represent imprecise quantities such as cost and time which cannot be exactly determined.
Ordering approaches between pairs of interval numbers have been discussed by many workers. Given two intervals \( A = [a^L, a^R] \) and \( B = [b^L, b^R] \) the different combinations in terms of ordering are

- (a) \( a^L \leq b^L \) and \( a^R \geq b^R \)
- (b) \( a^L \leq b^L \) and \( a^R \geq b^R \)
- (c1) \( a^L \leq b^L \) and \( a^R \leq b^R \)
- (c2) \( a^L \leq b^L \) and \( a^R \leq b^R \)
- (c3) \( a^L \leq b^L \) and \( a^R \geq b^R \)

For intervals of the type (a) \( , (b) \) \( , (c1) \) and (c2) ,"Interval A is less than Interval B" iff \( a^L \leq b^L \) and \( a^R \leq b^R \). This ordering is denoted by \( A \preceq_{LR} B \) and was proposed by Ishibuchi and Tanaka (1990). For overlapping intervals of type (c3) it becomes difficult for the DM to decide as to which interval to be taken as minimum. To overcome this
difficulty an ordering for overlapping intervals of types (c1), (c2) and (c3) is defined in
the present paper by using a fuzzy membership approach.
For two overlapping intervals $A$ and $B$ with $a^w > b^w$ we define a function

$$f(A,B) = \frac{(b^l - a^l)}{2(a^w - b^w)}.$$  

For interval of type (c1) $b^L = a^L$, so $f(A, B) = 0$ ;
For interval of type (c2) $b^R = a^R$, so $f(A, B) = 1$;
For all other overlapping intervals of type (c3), $0 < f(A, B) < 1$.

Thus when interval type changes from (c1) to (c2) it is observed that $f(A,B)$ changes
continuously from 0 to 1.

Definition 2:
Given two overlapping intervals $A$ and $B$, $a^w > b^w$. The ordering $A \preceq B$ (Interval $A$ is
less than Interval $B$) is defined by the fuzzy membership function $f(A,B)$ given by

$$f(A,B) = \begin{cases} 
0, & b^L = a^L, b^R < a^R \\
\frac{(b^L - a^L)}{2(a^w - b^w)}, & a^L < b^L < b^R < a^R \\
1, & a^L < b^L, b^R = a^R
\end{cases}$$

When $f(A,B) = 1$, $A \preceq B$ with degree of acceptability 1.
When $f(A,B) = 0$, $A \preceq B$ with degree of acceptability 0 and hence $B \preceq A$ with degree of acceptibility 0-1.
For all other values of $f(A,B)$ between 0 to 1 degree of acceptability of $A \preceq B$ is
obtained by the ratio $\frac{(b^L - a^L)}{2(a^w - b^w)}$.  

For example: If $f(A,B) = \frac{(b^L - a^L)}{2(a^w - b^w)} = 0.2$ it implies $A \preceq B$ with degree of acceptability
0.2 and $B \preceq A$ with a degree of acceptability 1-0.2 = 0.8. Hence interval B is smaller
than interval A.
If $f(A,B) = \frac{(b^L - a^L)}{2(a^w - b^w)} = 0.7$ it implies $A \preceq B$ with degree of acceptability 0.7 or $B \preceq A$
with a degree of acceptability 1-0.7 = 0.3. Hence interval A is smaller than interval B.
If $f(A,B) = \frac{(b^L - a^L)}{2(a^w - b^w)} = 0.5$ it implies $A \preceq B$ with degree of acceptability 0.5 or $B \preceq A$
with a degree of acceptability 1-0.5 = 0.5. In this case either of the two intervals can be
taken as the smaller one depending upon the nature of the problem.

Definition 3:
For the pair of interval numbers $A=[a^L, a^R]$ and $B=[b^L, b^R]$
$A + B = [a^L, a^R] + [b^L, b^R] = [a^L + b^L, a^R + b^R]$, $kA = k[a^L, a^R] = [ka^L, ka^R]$, $k \geq 0$

Definition 4:
Given two intervals $A=[a^L, a^R] = <a^C, a^W>$ and $B=[b^L, b^R] = <b^C, b^W>$,
(i) For intervals of Type (a), (b), (c1) and (c2) “$A \preceq B$ ” iff $a^L \leq b^L$ and $a^R \leq b^R$,
denoted by $A \preceq_{LR} B$.  

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(ii) For intervals of type (c3) “A ≤ B” iff $f(A, B) = \frac{(b^L - a^L)}{2(a^U - b^U)} \geq 0.5$ denoted by $A \leq B$ with degree of acceptability $f(A, B)$.

In the present paper a bicriteria shortest path problem is considered. Objective of the problem is to minimize the total cost and duration of travel between two specified nodes in the network. The two entities of cost and duration are taken as interval numbers and the shortest path is obtained by applying an extended Dijkstra’s algorithm incorporating the orderings between intervals defined above in Definition 4. A set of efficient solutions $(C_{\text{min}}^i, T_{\text{min}}^i)$ is obtained, $C_{\text{min}}^i$ denoting the total minimum cost of travel and $T_{\text{min}}^i$ denoting the total minimum duration of travel $(i=1,2,…,n)$. The two entities obtained in the set of efficient solutions satisfy

$$C_{\text{min}}^1 \leq C_{\text{min}}^2 \leq C_{\text{min}}^3 \leq \cdots \leq \cdots \leq C_{\text{min}}^n$$

$$T_{\text{min}}^n \geq T_{\text{min}}^{n-1} \geq \cdots \leq T_{\text{min}}^3 \leq T_{\text{min}}^2 \leq T_{\text{min}}^1.$$

Since each solution is better than another with respect to one objective and worse with respect to the other, such a set of solutions provides the decision maker (DM) more flexibility in his choice depending upon his priority objective. Pareto (1971), Chen, Huang and Yang (2005), Chankong and Haimes (1983), Koopmans (1951) and Kuhn and Tucker (1951) are a few workers who have discussed efficient solutions.

2. Formulation of the problem

A simple planar weighted graph is considered with $n$ nodes labeled 1,2,…n. Let $C_{ij}$ be the cost of travel from node $i$ to node $j$, $T_{ij}$ be the duration of travel from node $i$ to node $j$ is not traversed and $x_{ij} = 0$, if path from node $i$ to node $j$ is traversed and $x_{ij} = 1$, if path from node $i$ to node $j$ is traversed, $(C_{ij}, T_{ij}) = ([C_{ij}^L, C_{ij}^R], [T_{ij}^L, T_{ij}^R])$ is the weight of an edge between the $i$th and $j$th node.

Objective of the problem is to find the shortest path from node 1 to node n which will

Minimize $z = \sum_{j=1}^{n} \sum_{i=1}^{n} (x_{ij}[C_{ij}^L, C_{ij}^R], x_{ij}[T_{ij}^L, T_{ij}^R])$ (1)

Subject to the constraints:

$$\sum_{j=1}^{n} x_{i1} = 0, \ldots (2), \sum_{j=1}^{n} x_{i1} = 0, \ldots (3)$$

$$\sum_{i=1}^{n} x_{ij} \leq 1, \ldots (4), \sum_{j=1}^{n} x_{ij} \leq 1, \ldots (5)$$

$$C_{ij}^R, C_{ij}^L, T_{ij}^L, T_{ij}^R \geq 0$$

Constraint (2) and (3) suggest that node 1 and node 3 are the source and destination nodes respectively and constraints (4) and (5) suggest that the graph is simple.

3. Solution procedure

Let $P$ be the set of nodes selected to find the shortest path from Node 1 to Node n and $Q$ be the set of nodes not selected in the shortest path.
3.1 Procedure to obtain the 1st efficient Solution

Initially P = \{1\} and Q = \{2, 3, 4, 5, ..., n\}. Dijkstra’s algorithm is applied to the weighted graph focusing on minimizing the first entity in the ordered pair which is the cost of travel.

For all nodes adjacent to node 1 in the set P, C_{1j} is found, j = 2,3,...,n. If there is no edge between node 1 and node k ( k = 2,3,...,n), then C_{1k} = \infty. Minimum of all finite interval valued C_{1j}’s is found by applying the orderings A \leq_{LR} B for two intervals A and B of type (a),(b),(c1) and (c2) and A \leq B for intervals of type (c3).

Let C_{1m} be the minimum interval. Then node m enters the set P. This process is repeated with the two nodes which are in set P by finding the minimum cost of travel between the nodes in set P and adjacent nodes in the set Q, and the node corresponding to the minimum value of cost of travel at this step enters the set P. The process is repeated till the total minimum cost of travel from Node 1 to Node n is obtained. Variables x_{ij}’s at level 1 are obtained and the total duration of travel corresponding to the minimum total cost is calculated. The 1st efficient solution obtained is denoted by \( (C_{1\text{min}}^{1}, T_{1\text{min}}^{1}) \).

3.2 Procedure to obtain the 2nd efficient Solution

To obtain the 2nd efficient solution Dijkstra’s algorithm is extended by selecting a node i corresponding to the total minimum cost in the set P only if the sum of total duration of travel up to node i and duration of travel from node i to an adjacent node j belonging to the set Q is less than \( T_{\text{1min}}^{2} \). Proceeding in the same way as for obtaining the 1st efficient solution, the 2nd efficient solution obtained is denoted by \( (C_{\text{min}}^{2}, T_{\text{min}}^{2}) \).

3.3 Procedure to obtain the 3rd and subsequent efficient Solutions

To obtain the 3rd efficient solution Dijkstra’s algorithm is extended by selecting a node i corresponding to the total minimum cost in the set P only if the sum of total duration of travel up to node i and duration of travel from node i to an adjacent node j belonging to the set Q is less than \( T_{\text{1min}}^{3} \). Proceeding in the same way as for obtaining the 1st efficient solution, the 3rd efficient solution obtained is denoted by \( (C_{\text{min}}^{3}, T_{\text{min}}^{3}) \).

Subsequent efficient solutions are obtained in the same manner with the process terminating when no further efficient solutions can be obtained.

Since Dijkstra’s algorithm at each step to obtain the 1st efficient solution is applied on the interval number representing the cost of travel so the 1st efficient solution denoted by \( (C_{\text{min}}^{1}, T_{\text{min}}^{1}) \) gives the minimum cost and corresponding time. To obtain the 2nd and subsequent efficient solutions, Dijkstra’s algorithm is in each step applied on the interval number representing the cost subject to restrictions explained above and in the subsequent solutions \( (C_{\text{min}}^{2}, T_{\text{min}}^{2}) \), \( (C_{\text{min}}^{3}, T_{\text{min}}^{3}) \), etc cost increases whereas as time decreases. So the first efficient solution is the one with minimum cost whereas the last one is the minimum duration of travel.

Thus \( C_{\text{min}}^{1} \leq C_{\text{min}}^{2} \leq C_{\text{min}}^{3} \leq C_{\text{min}}^{4} \leq \ldots \leq C_{\text{min}}^{n} \) and \( T_{\text{min}}^{1} \leq T_{\text{min}}^{2} \leq T_{\text{min}}^{3} \leq \ldots \leq T_{\text{min}}^{n} \).
It is required to find the set of efficient solutions which minimize the total cost and duration of travel from node 1 to node 6 in the above weighted network.

4.1 Steps to obtain the 1st efficient solutions of the numerical problem

Step 1: Initially $P=\{1\}$, $Q=\{2,3,4,5,6\}$.
- $C_{12} = [30,40]$, $C_{14} = [59,67]$, $C_{13} = [60,80]$, $C_{15} = \infty$, $C_{16} = \infty$.
- $C_{12} \leq_{LR} C_{13}$ and $C_{12} \leq_{LR} C_{14}$.
- $\text{Min}(C_{12}, C_{13}, C_{14}) = C_{12}$. Therefore Node 2 enters the set $P$.

Step 2: $P=\{1,2\}$, $Q=\{3,4,5,6\}$.
- $C_{15} = C_{12} + C_{25} = [60,79]$, $C_{14} = [59,67]$, $C_{13} = [60,80]$, $C_{16} = \infty$.
- $C_{14} \leq_{LR} C_{13}$ and $C_{14} \leq_{LR} C_{15}$.
- $\text{Min}(C_{13}, C_{14}, C_{15}) = C_{14}$. Therefore node 4 enters the set $P$.

Step 3: $P=\{1,2,4\}$, $Q=\{3,5,6\}$.
- $C_{13} = \text{Min}(C_{13}, C_{14} + C_{43}) = \text{Min}([60,80], [64,74])$.
- $[64,74] \preceq [60,80]$ with degree of acceptability 0.6.
- Therefore $C_{13} = \text{Min}([60,80], [64,74]) = [64,74]$.
- $C_{15} = \text{Min}(C_{12} + C_{25}, C_{14} + C_{45}) = \text{Min}([60,79], [64,73])$.
- $[64,73] \preceq [60,79]$ with degree of acceptability 0.6.
- Therefore $C_{15} = \text{Min}([60,79], [64,73]) = [64,73]$.
- $C_{16} = C_{14} + C_{46} = [87,127]$.
- $C_{15} \leq_{LR} C_{13}$ and $C_{15} \leq_{LR} C_{16}$.
- $\text{Min}(C_{13}, C_{15}, C_{16}) = C_{15}$. Therefore node 5 enters the set $P$. 

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Step 4: \( P = \{1, 2, 4, 5\} , Q = \{3, 6\} \)

\[ C_{16} = \min (C_{14} + C_{46}, C_{14} + C_{45} + C_{56}, C_{12} + C_{25} + C_{56}) = \min(87, 127, 96, 112, 92, 118) \]

with degree of acceptability 0.625 and \( 96, 112 \leq 92, 118 \) with degree of acceptability 0.6.

Therefore \( C_{16} = [96, 112] \)

Hence variables \( x_{ij} \)'s at level 1 are \( x_{14}, x_{45} \) and \( x_{56} \). The optimal path is 1-4-5-6 and 1st efficient solution is \((C_{min}^1, T_{min}^1) = ([96, 112], [38, 44])\)

4.2 Steps to obtain the 2nd efficient solution

Step 1: Initially \( P = \{1\} , Q = \{2, 3, 4, 5, 6\} \).

\[ C_{12} = [30, 40] , T_{12} + T_{25} = [10, 59] \]

with degree of acceptability 0.65. Therefore node 2 can enter set \( P \) at this stage.

\[ C_{14} = [59, 67], T_{14} + T_{43} = [14, 21] \]

Therefore node 4 can enter set \( P \) at this stage.

\[ C_{13} = [60, 80], T_{13} + T_{36} = [30, 50] \]

Therefore node 3 can enter set \( P \) at this stage.

\[ C_{15} = \infty, C_{16} = \infty \]

\[ C_{12} \leq L_R, C_{13} \text{ and } C_{12} \leq L_R, C_{14} \]

Min\((C_{12}, C_{13}, C_{14}) = C_{12} \). Therefore Node 2 enters the set \( P \).

Step 2:

\[ P = \{1, 2\} , Q = \{3, 4, 5, 6\} \]

\[ C_{14} = [59, 67], T_{14} + T_{43} = [14, 21] \]

Therefore node 4 can enter set \( P \) at this stage.

\[ C_{13} = [60, 80], T_{13} + T_{36} = [30, 50] \]

Therefore node 3 can enter set \( P \) at this stage.

\[ C_{15} = \infty \]

\[ C_{15} = C_{12} + C_{25} = [60, 79], T_{12} + T_{25} + T_{56} = [20, 70] \]

with degree of acceptability 0.59. Therefore node 5 cannot enter set \( P \) at this stage.

\[ C_{14} \leq L_R, C_{13} \]

Therefore Node 4 enters the set \( P \).

Step 3:

\[ P = \{1, 2, 4\} , Q = \{3, 5, 6\} \]

\[ C_{13} = \min (C_{13}, C_{14} + C_{43}) = \min([60, 80], [64, 74]) \]

\[ [64, 74] \leq [60, 80] \] with degree of acceptability 0.6. Therefore \( C_{13} = [64, 74] \)

\[ T_{14} + T_{43} + T_{36} = [34, 44], [34, 44] \leq L_R T_{min}^1 \]

Therefore node 3 can enter set \( P \) at this stage.

\[ C_{15} = \min(C_{12} + C_{25}, C_{14} + C_{45}) = \min([60, 79], [64, 73]) \]

\[ [64, 73] \leq [60, 79] \] with degree of acceptability 0.6. Therefore \( C_{15} = [64, 73] \).

\[ T_{14} + T_{45} + T_{56} = [38, 44] \] which is the 1st efficient solution. Therefore node
5 cannot enter set P at this stage.

\[ C_{16} = C_{14} + C_{46} = [87,127], T_{14} + T_{46} = [22, 58], T_{14} + T_{46} \preceq T_{min} \] with degree of acceptability 0.53. Therefore node 6 can enter set P at this stage.

\[ C_{13} \preceq C_{16}. \] Therefore Node 3 enters the set P.

**Step 4:**

\[ P = \{1, 2, 4, 3\}, Q = \{5, 6\} \]

\[ C_{15} = \text{Min}(C_{12} + C_{25}, C_{14} + C_{45}) = \text{Min}([60,79], [64,73]) = [64,73] \] with degree of acceptability 0.6. Therefore \( C_{15} = [64,73] \).

\[ T_{14} + T_{45} + T_{56} = [38,44] \] which is the 1st efficient solution. Therefore node 5 cannot enter set P at this stage.

\[ C_{16} = \text{Min}(C_{14} + C_{43} + C_{36}, C_{14} + C_{46}, C_{13} + C_{36}) \]

\[ = \text{Min}([84,134], [87,127],[80,140]) \]

\[ [87,127] \preceq [84,134] \] with degree of acceptability 0.7, 

\[ [87,127] \preceq [80,140] \] with degree of acceptability 0.65

Therefore \( C_{16} = [87,127], T_{14} + T_{46} = [22, 58], T_{14} + T_{46} \preceq T_{min} \).

Hence variables \( x_{ij} \)'s at level 1 are \( x_{14} \) and \( x_{46} \). The optimal path is 1-4-6 and 2nd efficient solution is \( (C_{min}^2, T_{min}^2) = ([87,127], [22,58]) \).

Since \([96,112] \preceq [87,127] \) with degree of acceptability 0.625 and 
\([22,58] \preceq [38,44] \) with degree of acceptability 0.53, \( C_{min}^2 \preceq C_{min}^2 \) and 
\( T_{min}^2 \preceq T_{min}^1 \).

**4.3 Steps to obtain the 3rd efficient solution**

**Step 1:** Initially \( P = \{1\}, Q = \{2,3,4,5,6\} \).

\[ C_{12} = [30,40], T_{12} + T_{25} = [10,59], [10,59] \preceq T_{min}^2 \] with degree of acceptability 0.92. Therefore node 2 can enter set P at this stage.

\[ C_{14} = [59,67], T_{14} + T_{43} = [14,21], [14,21] \preceq_{LR} T_{min}^4. \] Therefore node 4 can enter set P at this stage.

\[ C_{13} = [60,80], T_{13} + T_{36} = [30,50], [30,50] \preceq T_{min}^2 \] with degree of acceptability 0.5. Therefore node 3 can enter set P at this stage.

\[ C_{15} = \infty, C_{16} = \infty \]

\[ C_{12} \preceq_{LR} C_{13} \] and \( C_{12} \preceq_{LR} C_{14} \),

\( \text{Min}(C_{12}, C_{13}, C_{14}) = C_{12} \). Therefore Node 2 enters the set P.

**Step 2:**

\( P = \{1,2\}, Q = \{3,4,5,6\} \)

\[ C_{14} = [59,67], T_{14} + T_{43} = [14,21], [14,21] \preceq_{LR} T_{min}^4. \] Therefore node 4 can enter set P at this stage.

\[ C_{13} = [60,80], T_{13} + T_{36} = [30,50], [30,50] \preceq T_{min}^2 \] with degree of acceptability 0.5. Therefore node 3 can enter set P at this stage.
\[ C_{16} = \infty \]
\[ C_{15} = C_{12} + C_{25} = [60, 79], T_{12} + T_{25} + T_{36} = [20, 70], T_{min}^2 \leq [20, 70] \] with degree of acceptability 0.72. Therefore node 5 cannot enter set P at this stage.

\[ C_{14} \leq LR C_{13}, \text{ Therefore Node 4 enters the set } P. \]

**Step 3:**

\[ P = \{1, 2, 4\}, Q = \{3, 5, 6\} \]

\[ C_{13} = \text{Min}(C_{13}, C_{14} + C_{43}) = \text{Min}(60,80, 64,74) \]

\[ [64,74] \leq [60,80] \] with degree of acceptability 0.6. Therefore, \( C_{13} = [64,74] \).

\[ T_{14} + T_{43} + T_{36} = [34, 44], [34, 44] \leq T_{min}^2 \] with degree of acceptability 0.54. Therefore node 3 can enter set P at this stage.

\[ C_{15} = \text{Min}(C_{12} + C_{25}, C_{14} + C_{45}) = \text{Min}(60,79, 64,73) \]

\[ [64,73] \leq [60,79] \] with degree of acceptability 0.6. Therefore \( C_{15} = \text{Min}(60,79, 64,73) = [64,73] \).

\[ T_{14} + T_{45} + T_{56} = [38, 44] \] which is the 1st efficient solution. Therefore node 5 cannot enter set P at this stage.

\[ C_{16} = C_{14} + C_{46} = [87,127], T_{14} + T_{46} = [22, 58], \text{ which is the 2nd efficient solution. Therefore node 6 cannot enter set P at this stage.} \]

Therefore Node 3 enters the set P.

**Step 4:**

\[ P = \{1, 2, 4, 3\}, Q = \{5,6\} \]

\[ C_{16} = \text{Min}(C_{14} + C_{43} + C_{36}, C_{13} + C_{36}) = \text{Min}(84,134, 80,140) \]

\[ [84,134] \leq [80,140] \] with degree of acceptability 0.6. Therefore \( C_{16} = [84,134], T_{14} + T_{43} + T_{46} = [34, 44], T_{14} + T_{43} + T_{46} \leq T_{min}^2 \).

Hence variables \( x_{ij} \)'s at level 1 are \( x_{14}, x_{43}, \text{ and } x_{46} \). The optimal path is 1-4-3-6 and 3rd efficient solution is \( (C_{min}^3, T_{min}^3) = ([84,134], [34,44]). \)

Since \([87,127] \leq [84,134] \) with degree of acceptability 0.7 and \([34,44] \leq [22,58] \) with degree of acceptability 0.54, \( C_{min}^2 \leq C_{min}^2 \leq C_{min}^3 \) and \( T_{min}^3 \leq T_{min}^2 \leq T_{min}^1 \).

**4.4 Steps to obtain the 4th efficient solution**

**Step 1:**

Initially \( P = \{1\}, Q = \{2,3,4,5,6\}. \)

\[ C_{12} = [30,40], T_{12} + T_{25} = [10,59], [10,59] \leq T_{min}^3 \] with degree of acceptability 0.61. Therefore node 2 can enter set P at this stage.

\[ C_{14} = [59,67], T_{14} + T_{43} = [14,21], [14,21] \leq LR T_{min}^3. \] Therefore node 4 can enter set P at this stage.
\[ C_{13} = [60,80], \quad T_{13} + T_{36} = [30,50], \quad T_{36}^{\min} \leq [30,50] \text{ with degree of acceptability } 0.6. \quad \therefore \text{node 3 cannot enter set } P \text{ at this stage.} \]

\[ C_{15} = \infty, \quad C_{16} = \infty \]

\[ C_{12} \leq \ell_R C_{14}, \quad \text{Min}(C_{12}, C_{14}) = C_{12}. \quad \therefore \text{Node 2 enters the set } P. \]

**Step 2:**

\[ P = \{1,2\}, \quad Q = \{3,4,5,6\} \]

\[ C_{14} = [59, 67], \quad T_{14} + T_{43} = [14, 21], \quad [14, 21] \leq \ell_R T_{14}^{\min}. \quad \therefore \text{node 4 can enter set } P \text{ at this stage.} \]

\[ C_{13} = [60,80], \quad T_{13} + T_{36} = [30,50], \quad T_{36}^{\min} \leq [30,50] \text{ with degree of acceptability } 0.6. \quad \therefore \text{node 3 cannot enter set } P \text{ at this stage.} \]

\[ C_{16} = \infty \]

\[ C_{15} = C_{12} + C_{25} = [60, 79], \quad T_{12} + T_{25} + T_{56} = [20, 70], \quad T_{56}^{\min} \leq [20,70] \text{ with degree of acceptability } 0.65. \quad \therefore \text{node 5 cannot enter set } P \text{ at this stage.} \]

\[ \therefore \text{Node 4 enters the set } P. \]

**Step 3:**

\[ P = \{1, 2, 4\}, \quad Q = \{3,5,6\} \]

\[ C_{13} = \text{Min}(C_{13}, C_{14} + C_{43}) = \text{Min}( [60,80], [64,74]) \]

\[ [64,74] \leq [60,80] \text{ with degree of acceptability } 0.6. \quad \therefore \quad C_{13} = [64,74]. \]

\[ T_{14} + T_{43} + T_{36} = [34, 44] \text{ which is the } 3^{rd} \text{ efficient solution. Therefore node } 3 \text{ cannot enter set } P \text{ at this stage.} \]

\[ C_{15} = \text{Min}(C_{12} + C_{25}, C_{14} + C_{45}) = \text{Min}( [60,79], [64,73]) \]

\[ [64,73] \leq [60,79] \text{ with degree of acceptability } 0.6. \]

\[ \therefore \quad C_{15} = \text{Min}( [60,79], [64,73]) = [64,73]. \]

\[ T_{14} + T_{45} + T_{56} = [38, 44] \text{ which is the } 1^{st} \text{ efficient solution. Therefore node } 5 \text{ cannot enter set } P \text{ at this stage.} \]

\[ C_{16} = C_{14} + C_{46} = [87,127], \quad T_{14} + T_{46} = [22, 58], \text{ which is the } 2^{nd} \text{ efficient solution. Therefore node } 6 \text{ cannot enter set } P \text{ at this stage.} \]

\[ \therefore \text{Hence the process terminates and there are no more efficient solutions.} \]

**Table 1. Efficient solutions of the formulated problem**

<table>
<thead>
<tr>
<th>Sl. No.</th>
<th>Variables (x_{ij})’s at level 1</th>
<th>Efficient solutions ((C_{min}^{i}, T_{min}^{i}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(x_{14}, x_{45}) and (x_{56})</td>
<td>((C_{min}^{1}, T_{min}^{1}) = ([96,112], [38,44]))</td>
</tr>
<tr>
<td>2</td>
<td>(x_{14}) and (x_{46})</td>
<td>((C_{min}^{2}, T_{min}^{2}) = ([87,127], [22,58]))</td>
</tr>
<tr>
<td>3</td>
<td>(x_{14}, x_{43}) and (x_{36})</td>
<td>((C_{min}^{3}, T_{min}^{3}) = ([84,134], [34,44]))</td>
</tr>
</tbody>
</table>
Table 1 shows the efficient solutions of the formulated problem. We can observe that the 1st efficient solution is associated with minimum cost and the 3rd efficient solution is associated with minimum duration of travel.

The numerical is also solved by using the method proposed by Okada and Gen (1993) and results are shown in Table 2. It can be observed that the proposed method giving three efficient solutions gives more choice to the decision maker in comparison to the two efficient solutions obtained by the method proposed by Okada and Gen (1993).

Table 2. Efficient solutions obtained by applying the method of Okada and Gen (1993)

<table>
<thead>
<tr>
<th>Sl. No.</th>
<th>Efficient solutions $\left( C_{\min}^i, T_{\min}^i \right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\left( C_{\min}^1, T_{\min}^1 \right) = ( [96,112], [38,44])$</td>
</tr>
<tr>
<td>2</td>
<td>$\left( C_{\min}^2, T_{\min}^2 \right) = ( [84,134], [34,44])$</td>
</tr>
</tbody>
</table>

5. Conclusions and future work

The order relation between overlapping intervals is defined in terms of a fuzzy membership function and efficient solutions of the bicriteria shortest path problem are obtained. The two criteria of cost and time are taken as interval numbers as in real life these values are not precise. It has been observed on comparing with other methods that the proposed fuzzy membership approach method apart from being very easy to understand and implement, also gives more efficient solutions to the decision maker. The set of efficient solutions obtained provides more flexibility to the decision maker who can select the solution according to his priority. For example if perishable goods like dairy products have to be transported then the decision maker would consider the efficient solution with minimum duration and in other cases where there are no time restrictions he would consider the efficient solution with minimum cost. The method discussed can be extended to problems with larger number of nodes and to optimize other criteria also.

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Classical Modeling of HIV Virus Infected Population in Imprecise Environments

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Abstract

In this paper we discuss the behavior of classical HIV virus infected population model in imprecise environment. Two different types of imprecise environment namely fuzzy and interval environments are taken. How the behaviors can changes of the model in different imprecise environment is discuss briefly by theoretically and by numerically.

Keywords: Classical HIV model, Fuzzy differential equation, Interval differential equation, Stability analysis.
1. Introduction

1.1 Modeling with impreciseness

The aim of mathematical modeling is to imitate some real world problems as far as possible. The presence of imprecise variable and parameters in practical problems in the field of bio-mathematical modeling became a new area of research in uncertainty modeling. So the solution procedure of such problems is very important. Once the mathematical programming technique i.e., the solution of said problems with uncertainty are developed then many real real life model in different fields with imprecise variable which can be formulated and solved easily and accurately.

1.2 Imprecise differential equation

Differential equations may arise in the mathematical modeling of real world problems. But when the impreciseness comes into it, the behavior of the differential equation is changed. The solution procedures are taken in different way. In this paper we take two types of imprecise environments: Fuzzy and Interval and find their exact solution.

1.2.1 Fuzzy differential equation

The topic “fuzzy differential equation”(FDE) has been speedily developing in recent years. The appliance of fuzzy differential equations is an inherent way to model dynamic systems under possibilistic uncertainty (see Zadeh (2005)). The concept of the fuzzy derivative was first intiated by Chang and Zadeh (1972). It was followed up by Dubois and Prade (1982). Other methods have been smeared by Puri and Ralescu (1983) and Goetschel and Voxman (1986). The concept of differential equations in a fuzzy environment was first formulated by Kaleva (1987). In fuzzy differential equation all derivative is deliberated as either Hukuhara or generalized derivatives. The Hukuhara differentiability has a deficiency (see Diamond and Kloeden (1994)). The solution turns fuzzier as time goes by. Bede (2006) exhibited that a large class of BVPs has no solution if the Hukuhara derivative is applied. To exceeds this difficulty, the concept of a generalized derivative was developed (see Bede snd Gal (2005) and Cano and Flores (2008)) and fuzzy differential equations were smeared using this concept (see Bede et al. (2007), Cano et al. (2007, 2008), Stefanini and Bede (2009)). Khastan and Nieto (2010) set up the solutions for a large enough class of boundary value problems using the generalized derivative. Obviously the disadvantage of strongly generalized differentiability of a function in comparison H-differentiability is that, a fuzzy differential equation has no unique solution (see Bede and Gal (2005)). Recently, Stefanini and Bede (2008) by the concept of generalization of the Hukuhara difference for compact convex set, introduced generalized Hukuhara differentiability (see Stefanini and Bede (2009)) for fuzzy valued function and they displayed that, this concept of differentiability have relationships with weakly generalized differentiability and strongly generalized differentiability.
There are many approaches for solving FDE. Some researchers transform the FDE into equivalent fuzzy integral equation and then solve this (see Allahviranloo et al. (2011), Chen et al. (2008), Regan et al. (2003)). Another one is Zadeh extension principle method. In this method first solve the associated ODE and lastly fuzzify the solution and check whether it is satisfied or not. For details see Buckley and Feuring (2000, 2001). In the third approach, the fuzzy problem is converted to a crisp problem. H"ullermeier (1997), uses the concept of differential inclusion. In this way, by taking an $\alpha$-cut of the initial value and the solution, the given differential equation is converted to a differential inclusion and the solution is accepted as the $\alpha$-cut of the fuzzy solution. Laplace transform method is use many where in linear FDE (see Allahviranloo and Ahmadi (2010), Toloui and Barkhordary (2010)). Recently, Mondal and Roy (2013) solve the first order Linear FDE by Lagrange multiplier method. Using generalized Hukuhara differentiability concept we transform the given FDE into two ODEs and this ODEs also a differential equation involving the parametric form of a fuzzy number.

1.2.2 Interval differential equation

An interval number is itself a imprecise parameter. Because the value is not a crisp number, the value is lie between two crisp number. When we take any parameter may be coefficients or initial condition or both of a differential equation then the interval differential equation comes. The basic behaviors of that number are different from a crisp number. So the calculus of that types numbers valued functions are different. So we need to study the differential equation in these environments.

1.3 Motivation for taking imprecise parameter

Impreciseness comes in every model for biological system. The necessity for taking some parameter as imprecise in a model is important topic today. There are so many works done on biological model with imprecise data. Some-times parameters are taken as fuzzy sometime it is interval. Our main aim is modeled a biological problem associated with differential equation with some imprecise parameters. Thus fuzzy differential equation and imprecise differential equation are important. Now we can concentrate some previous work on biological modeling in imprecise environments:

1.3.1 Work done using fuzzy differential equation on bio mathematical problem


1.3.2 Work done using interval differential equation in bio mathematical problem

Many authors consider interval parameter with differential equation in bio mathematical model. For presence of interval parameter the equation become interval differential equation. Using the property of interval number they solve the concerned model and discuss the behavior. Pal et. al. (2014) considered a bioeconomic modeling of two-prey and one-predator fishery model with optimal harvesting policy through hybridization approach in interval environment. Whereas optimal harvesting of prey–predator system with interval biological parameters is discussed by Pal et al. (2013c). S.Sharma and G.P.Samanta (2014) consider optimal harvesting of a two species competition model with imprecise biological parameters in.

1.4 Novelties

Although some developments are done but some new interest and new work have done by our self which is mentioned bellow:

(i) The basic HIV dynamics model is solved in imprecise environments.
(ii) Differential equation is solved in imprecise environment.
(iii) Consider fuzzy and interval both case as imprecise environments.
(iv) Find the exact solution of said types of differential equation.
(v) The fuzzy differential equation is solved by fuzzy differential equation approach.
(vi) How the model behaves in different imprecise environments is numerically illustrated.
(vii) The fuzzy stability and interval stability concept is addressed here.

Moreover we can say that all developments can help for the researchers who engage with uncertainty modeling, differential equation and mathematical biology. One can
modeled and stability analysis on any biological model with uncertainty and differential equation by same approach.

1.5 Structure of the paper

In first section we give introduction on our related work. Second section belongs to preliminaries. In third section we consider basic HIV population dynamics model. Fourth section belongs to the HIV modeling in fuzzy environment whereas fifth section belongs to HIV model in interval environment. Section six we give numerical exam in fuzzy and interval environment separately and discuss the behavior of solution separately. In section seven fuzzy and interval stability analysis are addressed for the model. Section eight is for conclusion and future research.

2. Preliminaries

2.1 Basic concept on fuzzy set theory

Definition: Fuzzy Set: A fuzzy set \( \tilde{A} \) is defined by
\[
\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in A, \mu_{\tilde{A}}(x) \in [0,1]\}.
\]
In the pair \((x, \mu_{\tilde{A}}(x))\) the first element \(x\) belong to the classical set \(A\), the second element \(\mu_{\tilde{A}}(x)\), belong to the interval \([0,1]\), called Membership function.

Definition: \(\alpha\)-cut of a fuzzy set: The \(\alpha\)-level set (or interval of confidence at level \(\alpha\) or \(\alpha\)-cut) of the fuzzy set \(\tilde{A}\) of \(X\) is a crisp set \(A_{\alpha}\) that contains all the elements of \(X\) that have membership values in \(A\) greater than or equal to \(\alpha\) i.e.
\[
\tilde{A} = \{x : \mu_{\tilde{A}}(x) \geq \alpha, x \in X, 0 < \alpha \leq 1\}.
\]

Definition: Fuzzy Number: The basic definition of fuzzy number is [30]: If we denote the set of all real numbers by \(\mathcal{R}\) and the set of all fuzzy numbers on \(\mathcal{R}\) is indicated by \(\mathcal{R}_{\tilde{F}}\) then a fuzzy number is a mapping such that \(u: \mathcal{R} \rightarrow [0,1]\), which satisfies the following four properties

(1) \(u\) is upper semi continuous.
(2) \(u\) is a fuzzy convex i.e., \(u(\lambda x + (1-\lambda)y) \geq \min \{u(x), u(y)\}\) for all \(x, y \in \mathcal{R}, \lambda \in [0,1]\).
(3) \(u\) is normal, i.e., \(\exists x_{0} \in \mathcal{R}\) for which \(u(x_{0}) = 1\).
(4) \(\text{supp} u = \{x \in \mathcal{R} | u(x) > 0\}\) is support of \(u\) and the closure of \(\text{supp} u\) is compact.

Definition: Parametric form of fuzzy number: [31] A fuzzy number is represented by an ordered pair of functions \((u_{1}(\alpha), u_{2}(\alpha))\), \(0 \leq \alpha \leq 1\), that satisfy the following condition:
(1) \(u_{1}(\alpha)\) is a bounded left continuous non decreasing function for any \(\alpha \in [0,1]\).
(2) \(u_{2}(\alpha)\) is a bounded left continuous non increasing function for any \(\alpha \in [0,1]\).
(3) \(u_{1}(\alpha) \leq u_{2}(\alpha)\) for any \(\alpha \in [0,1]\).

Note: If \(u_{1}(\alpha) = u_{2}(\alpha) = \alpha\), then \(\alpha\) is a crisp number.
**Definition: Triangular Fuzzy Number:** A Triangular fuzzy number (TFN) denoted by $\tilde{A}$ is defined as $(a, b, c)$ where the membership function

$$
\mu_{\tilde{A}}(x) = \begin{cases} 
0, & x \leq a \\
\frac{x-a}{b-a}, & a < x \leq b \\
1, & x = b \\
\frac{c-x}{c-b}, & b \leq x < c \\
0, & x \geq c
\end{cases}
$$

**Definition:** $\alpha$-cut of a fuzzy set $\tilde{A}$: The $\alpha$-cut of $\tilde{A} = (a, b, c)$ is given by $A_\alpha = [a + \alpha(b-a), c - \alpha(c-b)], \forall \alpha \in [0,1]$

### 2.2 Basic concepts on fuzzy calculus

**Definition:** Let $x, y \in E^1$. If there exists $z \in E^1$ such that $x = y + z$, then $z$ is called the Hukuhara-difference of fuzzy numbers $x$ and $y$, and it denoted by $z = x \ominus y$.

Remark that $x \ominus y \neq x + (-1)y$.

**Definition:** Let $f: [a, b] \to E^1$ and $t_0 \in [a, b]$. We say that $f$ is Hukuhara differential at $t_0$, if there exist an element $f'(t_0) \in E^1$ such that for all $h > 0$ sufficiently small, there exists $f(t_0 + h) \ominus f(t_0), f(t_0) \ominus f(t_0 - h)$ and the limits exists in metric $D$

$$
\lim_{h \to 0} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \to 0} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0)
$$

**Definition:** Let $f: (a, b) \to E$ and $x_0 \in (a, b)$. We say that $f$ is strongly generalized differential at $x_0$ (Bede-Gal differential) if there exists an element $f'(x_0) \in E$, such that

(i) for all $h > 0$ sufficiently small, there exist $f(x_0 + h) - h f(x_0)$ and $f(x_0) - h f(x_0 - h)$ and the limits exist in the metric $D$

$$
\lim_{h \to 0} \frac{f(x_0 + h) - h f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0) - h f(x_0 - h)}{h} = f'(x_0)
$$

Or

(ii) for all $h > 0$ sufficiently small, there exist $f(x_0) - h f(x_0 + h)$ and $f(x_0 - h) - h f(x_0 - h)$ and the limits exist in the metric $D$

$$
\lim_{h \to 0} \frac{f(x_0) - h f(x_0 + h)}{-h} = \lim_{h \to 0} \frac{f(x_0 - h) - h f(x_0)}{-h} = f'(x_0)
$$

Or

(iii) for all $h > 0$ sufficiently small, there exist $f(x_0 + h) - h f(x_0)$ and $f(x_0 - h) - h f(x_0 - h)$ and the limits exist in the metric $D$
\[
\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{f(x_0 - h) - f(x_0)}{-h} = f'(x_0)
\]

Or (iv) for all \( h > 0 \) sufficiently small, there exist \( f(x_0) - h f(x_0 + h) \) and \( f(x_0) - h f(x_0 - h) \) and the limits exists in the metric \( D \)

\[
\lim_{h \to 0} \frac{f(x_0) - h f(x_0 + h)}{-h} = \lim_{h \to 0} \frac{f(x_0) - h f(x_0 - h)}{h} = f'(x_0)
\]

\( (h \text{ and } -h \text{ at denominators mean } \frac{1}{h} \text{ and } \frac{-1}{h}, \text{ respectively).} \)

2.3 Basic concepts on interval number and calculus

**Definition:** Interval number: An interval number \( I \) is represented by closed interval \([I_l, I_u]\) and defined by \( I = [I_l, I_u] = \{x: I_l \leq x \leq I_u, x \in R\} \), where \( R \) is the set of real numbers and \( I_l \) and \( I_u \) are the left and right limit of the interval number respectively.

**Lemma:** The interval \([I_l, I_u]\) can also be represented as \( h(p) = (I_l)^{1-p} (I_u)^p \) for \( p \in [0,1] \).

**Properties:** If \( I = [I_l, I_u] \) and \( J = [J_l, J_u] \) be two interval can be written as interval-valued function as \( h(p) = (I_l)^{1-p} (J_u)^p \) and \( k(p) = (I_l)^{1-p} (J_u)^p \) for \( p \in [0,1] \) then

1. \( m(p) = (I_l + J_l)^{1-p} (I_u + J_u)^p \)
2. \( n(p) = (I_l - J_l)^{1-p} (I_u - J_u)^p \)
3. \( d(p) = (\min\{I_l, J_l, I_u, J_u\})^{1-p} (\max\{I_l, J_l, I_u, J_u\})^p \)
4. \( e(p) = (k(I_l)^{1-p} (I_u)^p \) if \( k > 0 \)
5. \( q(p) = (k(I_u)^{1-p} (I_u)^p \) if \( k < 0 \)

Where, \( m(p), n(p), d(p), e(p), q(p) \) are interval valued function for \( I + J, I - J, IJ, kI, I/K \) where \( k \) is constant and \( p \in [0,1] \).

**Theorem:** The differential equation with interval valued coefficient and initial condition \( x'(t) = f(t, k, x(t)), \hat{x}(t_0) = x_0, \) where \( \hat{x} \in [x_{0l}, x_{0u}] \) and \( \hat{k} \in [k_l, k_u] \) (coefficients)(are all >0) are also written as interval-valued functional form as \( x'(t; p) = f(t, (k_t)^{1-p} (k_u)^p, x(t; p)) \) with initial condition \( x(t_0; p) = (x_{0l})^{1-p} (x_{0u})^p \) for \( p \in [0,1] \).

**Proof:** The differential equation can be written as \( x'(t) = f(t, [k_l, k_u], x(t)) \) with \( x(t_0) = [x_{0l}, x_{0u}] \)

Let \( k \in [k_l, k_u] \) and \( x_0 \in [x_{0l}, x_{0u}] \) respectively.

Then the differential equation becomes (using interval arithmetic operation and property)

\( x'(t) = f(t, k, x(t)) \) with \( x(t_0) = x_0 \)
For a fixed \( n \), let us consider the interval valued function \( h_n(p) = a_n^{(1-p)} b_n^p \) for \( p \in [0,1] \) for an interval \( \delta_n \in [a_n, b_n] \).

Though \( h_n(p) \) is continuous and strictly increasing function, then

\[
x'(t) = f(t, k', x(t)) \quad \text{with} \quad x(t_0) = x_0' \quad \text{where} \quad k' \in (k_l)^{1-p} (k_u)^p \quad \text{and} \quad x_0' \in (x_{0l})^{1-p} (x_{0u})^p\]

Hence the parametric form of the above differential equation is

\[
x'(t; p) = f(t, (k_l)^{1-p} (k_u)^p, x(t; p)) \quad \text{with initial condition} \quad x(t_0; p) = (x_{0l})^{1-p} (x_{0u})^p \quad \text{for} \quad p \in [0,1].
\]

**Lemma:** The condition for existence of the solution of the interval differential equation is

\[
x(t; p = 0) \geq x(t; p = 1)
\]

Where, \( x(t, p) \) be the solution of interval-valued differential equation.

3. HIV Virus Infected Population model

Here we are interested in the development of the AIDS model in fuzzy environment. The classical Anderson’s model \([51]\) is a macroscopic model for AIDS. Is given by:

\[
\frac{dx(t)}{dt} = -\lambda(t) x(t)
\]

\[
\frac{dy(t)}{dt} = \lambda(t) x(t) = \lambda(t)(1 - y(t))
\]

with initial condition \( x(0) = 1 \) and \( y(t) = 0 \).

Here \( \lambda(t) \) is the transference rate between infected susceptible individuals and infected individuals that develop AIDS, \( x(t) \) is the proportion of infected susceptible population that does not have AIDS symptoms yet (asymptomatic), and \( y(t) \) is the proportion of the population that has developed AIDS symptoms (symptomatic). Anderson assumes \( \lambda(t) = \alpha t, \alpha > 0 \). Thus the model becomes

\[
\frac{dx(t)}{dt} = -\alpha t x(t)
\]

\[
\frac{dy(t)}{dt} = \alpha t (1 - y(t))
\]

4. Modeling of HIV Virus Infected Population in Fuzzy environment

For this model we consider the three cases

**Case 4.1:** The infected populations i.e., initial condition is Fuzzy number.

**Case 4.2:** The transference rate i.e., coefficients is Fuzzy number.

**Case 4.3:** The infected populations and the transference rate both i.e., initial Number and coefficients are both fuzzy number.

Now we briefly discuss the case by mathematical illustration.
Solution of Case 4.1: The problem is
\[
\begin{align*}
\frac{dx_1(t, \alpha)}{dt} &= -at x_2(t, \alpha) \\
\frac{dx_2(t, \alpha)}{dt} &= -at x_1(t, \alpha) \\
\frac{dy_1(t, \alpha)}{dt} &= at(1 - y_2(t, \alpha)) \\
\frac{dy_2(t, \alpha)}{dt} &= at(1 - y_1(t, \alpha))
\end{align*}
\]
With initial conditions \([x_1(t_0, \alpha), x_2(t_0, \alpha)] = [x_{01}(\alpha), x_{02}(\alpha)], [y_1(t_0, \alpha), y_2(t_0, \alpha)] = [0,0]\)

This equation can be written as
\[
\begin{bmatrix}
\frac{dx_1(t, \alpha)}{dt} \\
\frac{dx_2(t, \alpha)}{dt} \\
\frac{dy_1(t, \alpha)}{dt} \\
\frac{dy_2(t, \alpha)}{dt}
\end{bmatrix} =
\begin{bmatrix}
0 & -at & 0 & 0 \\
-at & 0 & 0 & 0 \\
0 & 0 & 0 & -at \\
0 & 0 & -at & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t, \alpha) \\
x_2(t, \alpha) \\
y_1(t, \alpha) \\
y_2(t, \alpha)
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
at \\
0
\end{bmatrix}
\]

Solution: The solutions are given by
\[
\begin{align*}
x_1(t, \alpha) &= \left(\frac{x_{01}(\alpha) + x_{02}(\alpha)}{2}\right) e^{-at^2} + \left(\frac{x_{01}(\alpha) - x_{02}(\alpha)}{2}\right) e^{at^2} \\
x_2(t, \alpha) &= \left(\frac{x_{01}(\alpha) + x_{02}(\alpha)}{2}\right) e^{-at^2} - \left(\frac{x_{01}(\alpha) - x_{02}(\alpha)}{2}\right) e^{at^2} \\
y_1(t, \alpha) &= 1 - e^{-\frac{a(t^2 + t^2)\alpha^2}{2}} \\
y_2(t, \alpha) &= 1 - e^{-\frac{a(t^2 - t^2)\alpha^2}{2}}
\end{align*}
\]

Solution of Case 4.2: The problem is
\[
\begin{align*}
\frac{dx_1(t, \alpha)}{dt} &= -ta_2(\alpha) x_2(t, \alpha) \\
\frac{dx_2(t, \alpha)}{dt} &= -ta_1(\alpha) x_1(t, \alpha) \\
\frac{dy_1(t, \alpha)}{dt} &= ta_1(\alpha) - ta_2(\alpha) y_2(t, \alpha) \\
\frac{dy_2(t, \alpha)}{dt} &= ta_2(\alpha) - ta_1(\alpha) y_1(t, \alpha)
\end{align*}
\]
With initial conditions \([x_1(t_0, \alpha), x_2(t_0, \alpha)] = [x_0, x_0], [y_1(t_0, \alpha), y_2(t_0, \alpha)] = [0,0]\)

This equation can be written as
Solution: The solution are written as

\[
x_1(t, \alpha) = \frac{\sqrt{a_1(\alpha)} + \sqrt{a_2(\alpha)} a_2(\alpha)}{a_1(\alpha)} \ e^{\frac{-a_2(\alpha) t}{a_1(\alpha)}} + \frac{\sqrt{a_1(\alpha)} - \sqrt{a_2(\alpha)} a_2(\alpha)}{a_1(\alpha)} \ e^{\frac{-a_1(\alpha) t}{a_2(\alpha)}}
\]

\[
x_2(t, \alpha) = \frac{\sqrt{a_1(\alpha)} - \sqrt{a_2(\alpha)} a_2(\alpha)}{a_1(\alpha)} \ e^{\frac{-a_1(\alpha) t}{a_2(\alpha)}} - \frac{\sqrt{a_1(\alpha)} + \sqrt{a_2(\alpha)} a_2(\alpha)}{a_1(\alpha)} \ e^{\frac{-a_2(\alpha) t}{a_1(\alpha)}}
\]

\[
y_1(t, \alpha) = \frac{a_2(\alpha)}{a_1(\alpha)} - \frac{\sqrt{a_1(\alpha)} a_2(\alpha)}{a_1(\alpha)} \left( \frac{a_2^2(\alpha) - \sqrt{a_1(\alpha)} a_2(\alpha)}{2a_2(\alpha) \sqrt{a_1(\alpha)} a_2(\alpha)} \right) e^{\frac{-a_1(\alpha) t}{a_2(\alpha)}}
\]

\[
y_2(t, \alpha) = \frac{a_1(\alpha)}{a_2(\alpha)} + \frac{a_2^2(\alpha) - \sqrt{a_1(\alpha)} a_2(\alpha)}{2a_2(\alpha) \sqrt{a_1(\alpha)} a_2(\alpha)} e^{\frac{-a_2(\alpha) t}{a_1(\alpha)}}
\]

Solution of Case 4.3: The problem is

\[
\frac{dx_1(t, \alpha)}{dt} = -ta_2(\alpha) x_2(t, \alpha)
\]

\[
\frac{dx_2(t, \alpha)}{dt} = -ta_1(\alpha) x_1(t, \alpha)
\]

\[
\frac{dy_1(t, \alpha)}{dt} = ta_1(\alpha) - ta_2(\alpha) y_2(t, \alpha)
\]

\[
\frac{dy_2(t, \alpha)}{dt} = ta_2(\alpha) - ta_1(\alpha) y_1(t, \alpha)
\]

With initial conditions \([x_1(t_0, \alpha), x_2(t_0, \alpha)] = [x_{01}(\alpha), x_{02}(\alpha)], [y_1(t_0, \alpha), y_2(t_0, \alpha)] = [0, 0]\).

This equation can be written as
Solution: The solution is written as

\[
\begin{bmatrix}
\frac{dx_1(t, \alpha)}{dt} \\
\frac{dx_2(t, \alpha)}{dt} \\
\frac{dy_1(t, \alpha)}{dt} \\
\frac{dy_2(t, \alpha)}{dt}
\end{bmatrix} =
\begin{bmatrix}
0 & -\alpha_2(\alpha) & 0 & 0 \\
-\alpha_1(\alpha) & 0 & 0 & 0 \\
0 & 0 & 0 & -\alpha_2(\alpha) \\
0 & 0 & -\alpha_1(\alpha) & 0
\end{bmatrix}
\begin{bmatrix}
x_1(t, \alpha) \\
x_2(t, \alpha) \\
y_1(t, \alpha) \\
y_2(t, \alpha)
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
\alpha_1(\alpha)
\end{bmatrix}
\]

5. Modeling HIV Virus Infected Population in Interval environment

For these models we consider the three cases

Case 5.1: The infected populations i.e., initial condition is Interval number.
Case 5.2: The transference rate i.e., coefficients is Interval number.
Case 5.3: The infected populations and the transference rate both i.e., initial condition and coefficients are both interval number.

Solution of Case 5.1: The problem is

\[
\frac{dx(t; p)}{dt} = -atx(t; p)
\]

\[
\frac{dy(t; p)}{dt} = at(1 - y(t; p))
\]
With initial condition \( x(t_0; p) = (x_{0t})^{1-p}(x_{0u})^p \) and \( y(t_0) = 0 \)

**Solution:** The solutions are

\[
x(t; p) = (x_{0t})^{1-p}(x_{0u})^p e^{-\frac{(t^2-t_0^2)}{2}}
\]

\[
y(t; p) = 1 - e^{-\frac{(t^2-t_0^2)}{2}}
\]

**Solution of Case 5.2:** The problem is

\[
\frac{dx(t; p)}{dt} = -t(a_i)^p(a_u)^{1-p}x(t; p)
\]

\[
\frac{dy(t, p)}{dt} = t(a_i)^{1-p}(a_u)^p - t(a_i)^p(a_u)^{1-p}y(t; p)
\]

With initial condition \( x(t_0) = x_0 \) and \( y(t_0) = 0 \)

**Solution:** The solution is

\[
x(t; p) = x_0 e^{-\frac{(a_i)^p(a_u)^{1-p}(t^2-t_0^2)}{2}}
\]

\[
y(t, p) = (a_i)^{2p-1}(a_u)^{1-2p} \left\{ 1 - e^{-\frac{(a_i)^p(a_u)^{1-p}(t^2-t_0^2)}{2}} \right\}
\]

**Solution of Case 5.3:** The problem is

\[
\frac{dx(t; p)}{dt} = -t(a_i)^p(a_u)^{1-p}x(t; p)
\]

\[
\frac{dy(t, p)}{dt} = t(a_i)^{1-p}(a_u)^p - t(a_i)^p(a_u)^{1-p}y(t; p)
\]

With initial condition \( x(t_0; p) = (x_{0t})^{1-p}(x_{0u})^p \) and \( y(t_0) = 0 \)

**Solution:** The solution is

\[
x(t; p) = (x_{0t})^{1-p}(x_{0u})^p e^{-\frac{(a_i)^p(a_u)^{1-p}(t^2-t_0^2)}{2}}
\]

\[
y(t, p) = (a_i)^{2p-1}(a_u)^{1-2p} \left\{ 1 - e^{-\frac{(a_i)^p(a_u)^{1-p}(t^2-t_0^2)}{2}} \right\}
\]
6. Numerical Example

Example 1: HIV model in Fuzzy environment:

Case 1.1: Consider \( x(0) = (0.7, 1.4) \), \( y(0) = 0 \), \( \alpha = 0.237 \). Find solutions after \( t = 2 \)

Table 1: Value of \( x_1(t, \alpha), x_2(t, \alpha), y_1(t, \alpha) \) and \( y_2(t, \alpha) \) at \( t = 2 \) for different \( \alpha \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( x_1(t, \alpha) )</th>
<th>( x_2(t, \alpha) )</th>
<th>( y_1(t, \alpha) )</th>
<th>( y_2(t, \alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0914</td>
<td>1.2159</td>
<td>0.3775</td>
<td>0.3775</td>
</tr>
<tr>
<td>0.1</td>
<td>0.1445</td>
<td>1.1565</td>
<td>0.3775</td>
<td>0.3775</td>
</tr>
<tr>
<td>0.2</td>
<td>0.1976</td>
<td>1.0972</td>
<td>0.3775</td>
<td>0.3775</td>
</tr>
<tr>
<td>0.3</td>
<td>0.2507</td>
<td>1.0379</td>
<td>0.3775</td>
<td>0.3775</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3038</td>
<td>0.9785</td>
<td>0.3775</td>
<td>0.3775</td>
</tr>
<tr>
<td>0.5</td>
<td>0.3569</td>
<td>0.9192</td>
<td>0.3775</td>
<td>0.3775</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4101</td>
<td>0.8599</td>
<td>0.3775</td>
<td>0.3775</td>
</tr>
<tr>
<td>0.7</td>
<td>0.4632</td>
<td>0.8005</td>
<td>0.3775</td>
<td>0.3775</td>
</tr>
<tr>
<td>0.8</td>
<td>0.5163</td>
<td>0.7412</td>
<td>0.3775</td>
<td>0.3775</td>
</tr>
<tr>
<td>0.9</td>
<td>0.5694</td>
<td>0.6818</td>
<td>0.3775</td>
<td>0.3775</td>
</tr>
<tr>
<td>1</td>
<td>0.6225</td>
<td>0.6225</td>
<td>0.3775</td>
<td>0.3775</td>
</tr>
</tbody>
</table>

3D Plot of solution

Figure 1: \( x_1(t, \alpha), x_2(t, \alpha) \) vs. \( t \in [0, 2] \) vs. \( \alpha \in [0, 1] \)
Figure 2: $y_1(t, \alpha), y_2(t, \alpha)$ vs. $t \in [0, 2]$ vs. $\alpha \in [0, 1]$

Figure 3: $x_1(t, \alpha), x_2(t, \alpha), y_1(t, \alpha), y_2(t, \alpha)$ vs. $t \in [0, 2]$ vs. $\alpha \in [0, 1]$
**2D Plot of solution**

![2D Plot of solution](image)

**Figure 4:** Crisp and fuzzy solution for $\alpha = 0$

![Figure 5: Fuzzy solution for $\alpha = 0$](image)

**Figure 5:** Fuzzy solution for $\alpha = 0$
Remark: From above table and 2D figure we see that \( x_1(t, \alpha) \) is increasing and \( x_2(t, \alpha) \) is decreasing function hence the solution is strong solution. But \( y_1(t, \alpha) \) and \( y_2(t, \alpha) \) is crisp solution at point \( t = 2 \). From 3D figure we see the solution for \( x(t) \) form a triangular type shape fuzzy solution whereas \( y(t) \) gives crisp solution.

Here we also see that when \( \alpha \) increases the difference between \( x_1(t, \alpha) \) and \( x_2(t, \alpha) \) is decreases and at \( \alpha = 1 \) they becomes crisp solution. So from this figure we say that the case 1.1 is exist and the graphical solution of case 1.1 is biologically meaningful, furthermore the graphical solution is coherent with the crisp solution.
Case 1.2: Consider \( x(0) = 0, y(0) = 0, \alpha = (0.157, 0.237, 0.307) \). Find solutions after \( t = 2 \)

**Table 2:** Value of \( x_1(t, \alpha), x_2(t, \alpha), y_1(t, \alpha) \) and \( y_2(t, \alpha) \) at \( t = 2 \) for different \( \alpha \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( x_1(t, \alpha) )</th>
<th>( x_2(t, \alpha) )</th>
<th>( y_1(t, \alpha) )</th>
<th>( y_2(t, \alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.4640</td>
<td>1.0820</td>
<td>6.0701</td>
<td>0.5935</td>
</tr>
<tr>
<td>0.1</td>
<td>0.4806</td>
<td>1.0243</td>
<td>5.9292</td>
<td>0.4845</td>
</tr>
<tr>
<td>0.2</td>
<td>0.4971</td>
<td>0.9699</td>
<td>5.8046</td>
<td>0.3904</td>
</tr>
<tr>
<td>0.3</td>
<td>0.5134</td>
<td>0.9185</td>
<td>5.6942</td>
<td>0.3097</td>
</tr>
<tr>
<td>0.4</td>
<td>0.5295</td>
<td>0.8698</td>
<td>5.5963</td>
<td>0.2410</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5455</td>
<td>0.8235</td>
<td>5.5097</td>
<td>0.1834</td>
</tr>
<tr>
<td>0.6</td>
<td>0.5612</td>
<td>0.7795</td>
<td>5.4331</td>
<td>0.1361</td>
</tr>
<tr>
<td>0.7</td>
<td>0.5768</td>
<td>0.7376</td>
<td>5.3656</td>
<td>0.0984</td>
</tr>
<tr>
<td>0.8</td>
<td>0.5922</td>
<td>0.6975</td>
<td>5.3064</td>
<td>0.0698</td>
</tr>
<tr>
<td>0.9</td>
<td>0.6075</td>
<td>0.6592</td>
<td>5.2549</td>
<td>0.0500</td>
</tr>
<tr>
<td>1</td>
<td>0.6225</td>
<td>0.6225</td>
<td>5.2104</td>
<td>0.0387</td>
</tr>
</tbody>
</table>

**3D Plot of solution**

Figure 8: \( x_1(t, \alpha), x_2(t, \alpha) \) vs. \( t \in [0,2] \) vs. \( \alpha \in [0,1] \)
Figure 9: $y_1(t, \alpha), y_2(t, \alpha)$ vs. $t \in [0,2]$ vs. $\alpha \in [0,1]$

Figure 10: $x_1(t, \alpha), x_2(t, \alpha), y_1(t, \alpha), y_2(t, \alpha)$ vs. $t \in [0,2]$ vs. $\alpha \in [0,1]$
2D Plot of solution

Figure 11: Crisp and fuzzy solution for $\alpha = 0$

Figure 12: Fuzzy solution for $\alpha = 0$
Remark: From above table and 3D, 2D figure we see that \( x_1(t, \alpha) \) is increasing and \( x_2(t, \alpha) \) is decreasing function hence the solution is strong solution. But \( y_1(t, \alpha) \) and \( y_2(t, \alpha) \) is crisp solution at point \( t = 2 \) From 3D figure we see the solution for \( x(t) \) form triangular types shape fuzzy solution whereas \( y(t) \) also gives fuzzy solution.

Here we also see that when \( \alpha \) increases the difference between \( x_1(t, \alpha), x_2(t, \alpha) \) and \( y_1(t, \alpha), y_2(t, \alpha) \) is decreases and at \( \alpha = 1 \) they becomes crisp solution. So from this figure we say that the case 1.2 is exist and the graphical solution of case 1.2 is biologically meaningful, furthermore the graphical solution is coherent with the crisp solution.

Case 1.3: Consider \( x(0) = (0.7, 1.1, 1.4), y(0) = 0, \alpha = (0.157, 0.237, 0.307) \). Find solution after \( t = 0.5 \)
Table 3: Value of $x_1(t, \alpha), x_2(t, \alpha), y_1(t, \alpha)$ and $y_2(t, \alpha)$ at $t = 0.5$ for different $\alpha$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$x_1(t, \alpha)$</th>
<th>$x_2(t, \alpha)$</th>
<th>$y_1(t, \alpha)$</th>
<th>$y_2(t, \alpha)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.4889</td>
<td>1.3534</td>
<td>0.4892</td>
<td>-3.8559</td>
</tr>
<tr>
<td>0.1</td>
<td>0.5301</td>
<td>1.3081</td>
<td>0.4898</td>
<td>-3.7698</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5710</td>
<td>1.2625</td>
<td>0.4903</td>
<td>-3.6883</td>
</tr>
<tr>
<td>0.3</td>
<td>0.6117</td>
<td>1.2166</td>
<td>0.4908</td>
<td>-3.6107</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6521</td>
<td>1.1705</td>
<td>0.4914</td>
<td>-3.5362</td>
</tr>
<tr>
<td>0.5</td>
<td>0.6921</td>
<td>1.1241</td>
<td>0.4919</td>
<td>-3.4645</td>
</tr>
<tr>
<td>0.6</td>
<td>0.7319</td>
<td>1.0775</td>
<td>0.4923</td>
<td>-3.3950</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7714</td>
<td>1.0306</td>
<td>0.4928</td>
<td>-3.3273</td>
</tr>
<tr>
<td>0.8</td>
<td>0.8107</td>
<td>0.9834</td>
<td>0.4933</td>
<td>-3.2608</td>
</tr>
<tr>
<td>0.9</td>
<td>0.8496</td>
<td>0.9359</td>
<td>0.4937</td>
<td>-3.1953</td>
</tr>
<tr>
<td>1</td>
<td>0.8883</td>
<td>0.8883</td>
<td>0.4941</td>
<td>-3.1303</td>
</tr>
</tbody>
</table>

3D Plot of solution

![3D Plot of solution](image)

Figure 15: $x_1(t, \alpha), x_2(t, \alpha)$ vs. $t \in [0,0.5]$ vs. $\alpha \in [0,1]$
Figure 16: $y_1(t, \alpha), y_2(t, \alpha)$ vs. $t \in [0,0.5]$ vs. $\alpha \in [0,1]$

Figure 17: $x_1(t, \alpha), x_2(t, \alpha), y_1(t, \alpha), y_2(t, \alpha)$ vs. $t \in [0,0.5]$ vs. $\alpha \in [0,1]$
2D Plot of solution

Figure 18: Crisp and fuzzy solution for $\alpha = 0$

Figure 19: Fuzzy solution for $\alpha = 0$
Remark: From above table and 3D, 2D figure we see that $x_1(t, \alpha)$ is increasing and $x_2(t, \alpha)$ is decreasing function hence the solution is strong solution. But $y_1(t, \alpha)$ and $y_2(t, \alpha)$ is crisp solution at point $t = 0.5$. From 3D figure we see the solution for $x(t)$ form a triangular type shape fuzzy solution whereas $y(t)$ also gives fuzzy solution.

Here we also see that when $\alpha$ increases the difference between $x_1(t, \alpha), x_2(t, \alpha)$ and $y_1(t, \alpha), y_2(t, \alpha)$ is decreases and at $\alpha = 1$ they becomes crisp solution. So from this figure we say that the case 3 is exist and the graphical solution of case 3 is biologically meaningful, furthermore the graphical solution is coherent with the crisp solution.
Example 2: HIV model in Interval environment:

Case 2.1: Consider \( x(0) = (0.7)^{1-p} (1.4)^p, y(0) = 0, \alpha = 0.237 \). Find solutions after \( t = 2 \)

Table 4: Value of \( x(t;p) \) and \( y(t;p) \) at \( t = 2 \) for different \( p \)

<table>
<thead>
<tr>
<th>( p )</th>
<th>( x(t;p) )</th>
<th>( y(t;p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.4358</td>
<td>0.3775</td>
</tr>
<tr>
<td>0.1</td>
<td>0.4670</td>
<td>0.3775</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5006</td>
<td>0.3775</td>
</tr>
<tr>
<td>0.3</td>
<td>0.5365</td>
<td>0.3775</td>
</tr>
<tr>
<td>0.4</td>
<td>0.5750</td>
<td>0.3775</td>
</tr>
<tr>
<td>0.5</td>
<td>0.6163</td>
<td>0.3775</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6605</td>
<td>0.3775</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7079</td>
<td>0.3775</td>
</tr>
<tr>
<td>0.8</td>
<td>0.7587</td>
<td>0.3775</td>
</tr>
<tr>
<td>0.9</td>
<td>0.8131</td>
<td>0.3775</td>
</tr>
<tr>
<td>1</td>
<td>0.8715</td>
<td>0.3775</td>
</tr>
</tbody>
</table>

3D Plot of solution

Figure 22: \( x(t;p) \) vs. \( t \in [0,2] \) vs. \( p \in [0,1] \)
Figure 23: $y(t; p)$ vs. $t \in [0,2]$ vs. $p \in [0,1]$

Figure 24: $x(t; p), y(t; p)$ vs. $t \in [0,2]$ vs. $p \in [0,1]$
2D Plot of solution

Figure 25: Interval solution of problem (2) for $p = 0$

Figure 26: Interval solution of problem (2) for $p = 1$

Remark: From above table and 2D,3D figure we see that $x(t,p)$ is increasing and $y(t,p)$ is decreasing as $t$ and $p$ become larger. Because $x(t;p)$ become infected where $y(t;p)$ is uninfected.

Case 2.2: Consider $x(0) = 1$, $y(0) = 0$, $\alpha = (0.157)^{1-p}(0.307)^p$. Find solutions after $t = 2$
Table 5: Value of $x(t; p)$ and $y(t; p)$ at $t = 2$ for different $p$

<table>
<thead>
<tr>
<th>$p$</th>
<th>$x(t; p)$</th>
<th>$y(t; p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.5412</td>
<td>0.8972</td>
</tr>
<tr>
<td>0.1</td>
<td>0.5632</td>
<td>0.7470</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5845</td>
<td>0.6213</td>
</tr>
<tr>
<td>0.3</td>
<td>0.6053</td>
<td>0.5162</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6253</td>
<td>0.4285</td>
</tr>
<tr>
<td>0.5</td>
<td>0.6446</td>
<td>0.3554</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6632</td>
<td>0.2945</td>
</tr>
<tr>
<td>0.7</td>
<td>0.6811</td>
<td>0.2438</td>
</tr>
<tr>
<td>0.8</td>
<td>0.6983</td>
<td>0.2017</td>
</tr>
<tr>
<td>0.9</td>
<td>0.7148</td>
<td>0.1668</td>
</tr>
<tr>
<td>1</td>
<td>0.7305</td>
<td>0.1378</td>
</tr>
</tbody>
</table>

3D Plot of solution

Figure 27: $x(t; p)$ vs. $t \in [0,2]$ vs. $p \in [0,1]$
Figure 28: \( y(t;p) \) vs. \( t \in [0,2] \) vs. \( p \in [0,1] \)

Figure 29: \( x(t;p), y(t;p) \) vs. \( t \in [0,2] \) vs. \( p \in [0,1] \)
Figure 30: Interval solution of problem (2) for $p = 0$

Figure 31: Interval solution of problem (2) for $p = 1$

Remark: From above table and 2D, 3D figure we see that $x(t, p)$ is increasing and $y(t, p)$ is decreasing as $t$ and $p$ become larger. Because $x(t; p)$ become infected where $y(t; p)$ is uninfected.

Case 2.3: Consider $x(0) = (0.7)^{1-p}(1.4)^p$, $y(0) = 0$, $a = (0.157)^{1-p}(0.307)^p$. Find solutions after $t = 0.5$
Table 6: Value of $x(t;p)$ and $y(t;p)$ at $t = 0.5$ for different $p$

<table>
<thead>
<tr>
<th>$p$</th>
<th>$x(t;p)$</th>
<th>$y(t;p)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.6004</td>
<td>0.2783</td>
</tr>
<tr>
<td>0.1</td>
<td>0.6499</td>
<td>0.2287</td>
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<tr>
<td>0.2</td>
<td>0.7031</td>
<td>0.1878</td>
</tr>
<tr>
<td>0.3</td>
<td>0.7601</td>
<td>0.1543</td>
</tr>
<tr>
<td>0.4</td>
<td>0.8214</td>
<td>0.1267</td>
</tr>
<tr>
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<td>0.8870</td>
<td>0.1040</td>
</tr>
<tr>
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<td>0.9575</td>
<td>0.0853</td>
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<tr>
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<td>1.0331</td>
<td>0.0700</td>
</tr>
<tr>
<td>0.8</td>
<td>1.1141</td>
<td>0.0574</td>
</tr>
<tr>
<td>0.9</td>
<td>1.2011</td>
<td>0.0471</td>
</tr>
<tr>
<td>1</td>
<td>1.2943</td>
<td>0.0386</td>
</tr>
</tbody>
</table>

3D Plot of solution

Figure 32: $x(t;p)$ vs. $t \in [0,0.5]$ vs. $p \in [0,1]$
**Figure 33:** $y(t;p)$ vs. $t \in [0,0.5]$ vs. $p \in [0,1]$

**Figure 34:** $x(t;p), y(t;p)$ vs. $t \in [0,0.5]$ vs. $p \in [0,1]$
2D Plot of solution

Figure 35: Interval solution of problem (2) for $p = 0$

Figure 36: Interval solution of problem (2) for $p = 1$

**Remark:** From above table and 2D, 3D figure we see that $x(t; p)$ is increasing and $y(t; p)$ is decreasing as $t$ and $p$ become larger. Because $x(t; p)$ become infected where $y(t; p)$ is uninfected.
7. Stability Analysis for Imprecise model

The system (2) has only one equilibrium point which is (0, 1). At this point the variational matrix of system (2) is

\[
V = \begin{pmatrix} -at & 0 \\ 0 & -at \end{pmatrix}
\]

The eigenvalues of \( V \) are \( \lambda_1 = -at \) and \( \lambda_2 = -at \). So both are equal and negative and hence the equilibrium point (0, 1) of system (2) is exist and asymptotically stable.

7.1 Stability analysis in fuzzy environment

For Case 4.1 of system (2) has only one equilibrium point which is (0, 0, 1, 1). At this point the variational matrix of Case 4.1 of system (2) is

\[
V_1 = \begin{pmatrix} 0 & -at & 0 & 0 \\ -at & 0 & 0 & 0 \\ 0 & 0 & 0 & -at \\ 0 & 0 & -at & 0 \end{pmatrix}
\]

The eigenvalues of \( V \) are \( \lambda_1 = at, \lambda_2 = at, \lambda_3 = -at \) and \( \lambda_4 = -at \). Here \( \lambda_1, \lambda_2 \) are positive and hence the equilibrium point (0, 0, 1, 1) of case-1 of system (2) is unstable.

For Case 4.2 of system (2) has only one equilibrium point which is (0, 0, 1, 1). At this point the variational matrix of Case-2 of system (2) is

\[
V_2 = \begin{pmatrix} 0 & -a_1(\alpha)t & 0 & 0 \\ -a_1(\alpha)t & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_2(\alpha)t \\ 0 & 0 & -a_1(\alpha)t & 0 \end{pmatrix}
\]

The eigenvalues of \( V \) are
\( \lambda_1 = t\sqrt{a_1(\alpha)a_2(\alpha)}, \lambda_2 = t\sqrt{a_1(\alpha)a_2(\alpha)}, \lambda_3 = -t\sqrt{a_1(\alpha)a_2(\alpha)} \) and \( \lambda_4 = -t\sqrt{a_1(\alpha)a_2(\alpha)} \). Here \( \lambda_1, \lambda_2 \) are positive and hence the equilibrium point (0, 0, 1, 1) of case 4.2 of system (2) is unstable.

For Case 4.3 of system (2) has also only one equilibrium point which is (0, 0, 1, 1). At this point the variational matrix of Case-3 of system (2) is

\[
V_3 = \begin{pmatrix} 0 & -a_2(\alpha)t & 0 & 0 \\ -a_1(\alpha)t & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_2(\alpha)t \\ 0 & 0 & -a_1(\alpha)t & 0 \end{pmatrix}
\]
After analysis the above variational matrix $(V_3)$ and Case-2 of system (2), here we also say that the case 4.3 of system (2) is also unstable.

7.2 Stability analysis in interval environment

For Case 5.1 of system (2) has only one equilibrium point which is $(0, 1)$. At this point the variational matrix of case 1 of system (2) is

$$V_4 = \begin{pmatrix} -at & 0 \\ 0 & -at \end{pmatrix}$$

Therefore, the eigenvalues of $V_4$ are $\lambda_1 = -at$ and $\lambda_2 = -at$. Here we see that the variational matrix, $V$ and $V_4$ are similar. Hence the equilibrium point $(0, 1)$ of case 5.1 of system (2) is exist and asymptotically stable.

For Case 5.2 of system (2) has only one equilibrium point which is $(0, 1)$. At this point the variational matrix of case 5.2 of system (2) is

$$V_5 = \begin{pmatrix} -ta_l^p a_u^{1-p} & 0 \\ 0 & -ta_l^p a_u^{1-p} \end{pmatrix}$$

Therefore, the eigenvalues of $V$ are $\lambda_1 = -ta_l^p a_u^{1-p}$ and $\lambda_2 = -ta_l^p a_u^{1-p}$. So both are negative and equal and hence the equilibrium point $(0, 1)$ of case 2 of system (2) is exist and asymptotically stable.

For Case 5.3 of system (3) has only one equilibrium point which is $(0, 1)$. At this point the variational matrix of case 3 of system (2) is

$$V_6 = \begin{pmatrix} -ta_l^p a_u^{1-p} & 0 \\ 0 & -ta_l^p a_u^{1-p} \end{pmatrix}$$

Therefore, the eigenvalues of $V_6$ and $V_5$ are same. So here we also say that the equilibrium point $(0, 1)$ of case 5.3 of system (2) is exist and asymptotically stable. i.e. Stability analysis in interval environment of HIV model is exist and stable in all the cases.

8. Conclusion

For biological modeling it is not sure that the parameter present in it is known accurate or sharp then the only way to taken it as uncertain. In any real world of biological model, uncertainty plays an important role in recent time. The uncertainty can taken as fuzzy or interval environment. When we take the model with fuzzy or interval parameter the behavior of the model is changes. The crisp sense is not work here. For example we consider classical HIV model associated with differential equation. The differential equation is changes in both environments. The solution and stability is quite different from non uncertain case. The proposed solution procedure and uncertain
stability analysis can help various biological modeling in uncertain environments for researcher.

References


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