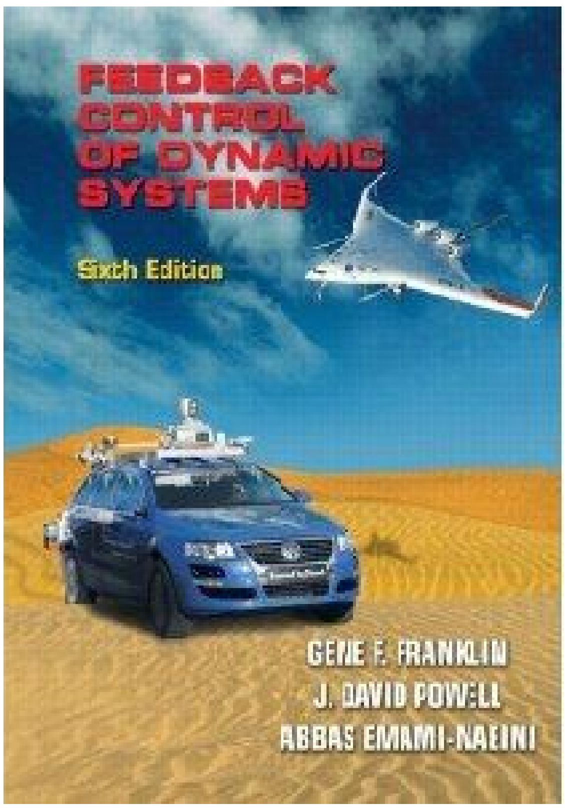
Solutions Manual

6th Edition

Feedback Control of Dynamic Systems



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Chapter 1

An Overview and Brief History of Feedback Control

1.1 Problems and Solutions

1. Draw a component block diagram for each of the following feedback control systems.

   (a) The manual steering system of an automobile

   (b) Drebbel's incubator

   (c) The water level controlled by a float and valve

   (d) Watt's steam engine with fly-ball governor

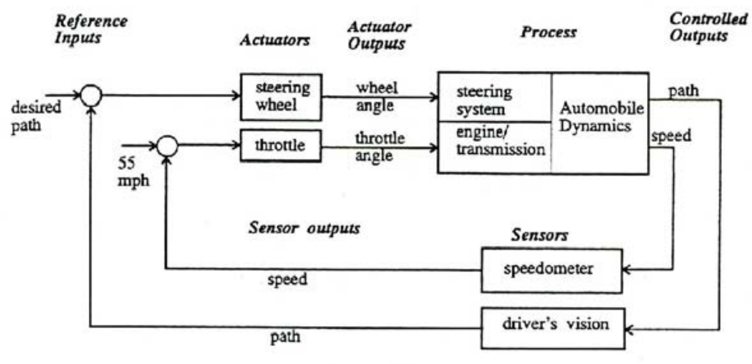
   In each case, indicate the location of the elements listed below and give the units associated with each signal.

   - the process
   - the process output signal
   - the sensor
   - the actuator
   - the actuator output signal
   - The reference signal

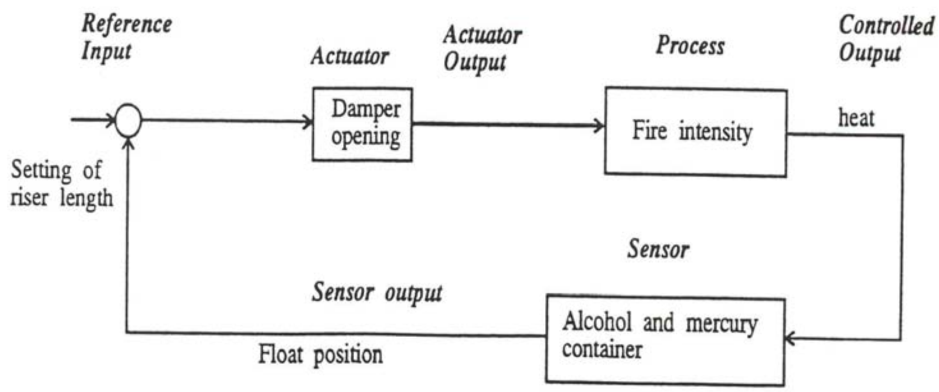
   Notice that in a number of cases the same physical device may perform more than one of these functions.

   Solution:

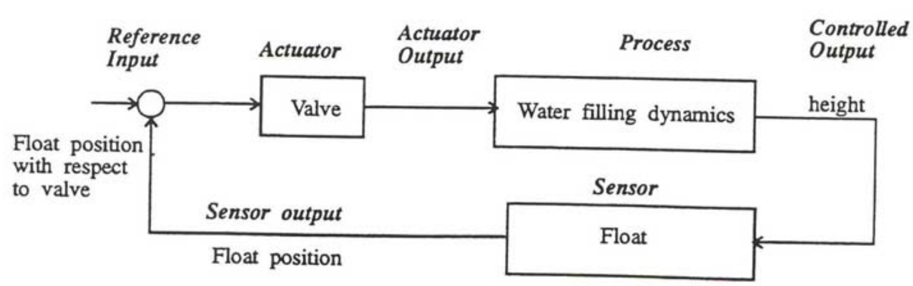
   (a) A manual steering system for an automobile:



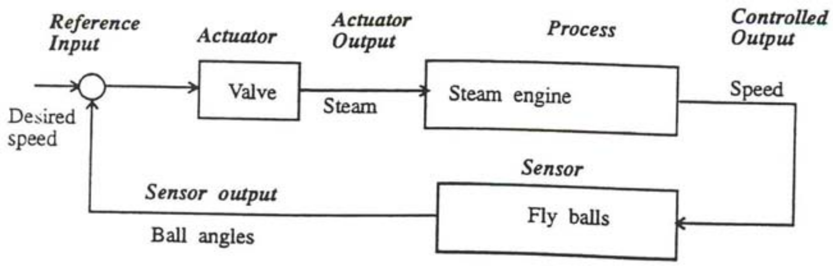
(b) Drebbel's incubator:



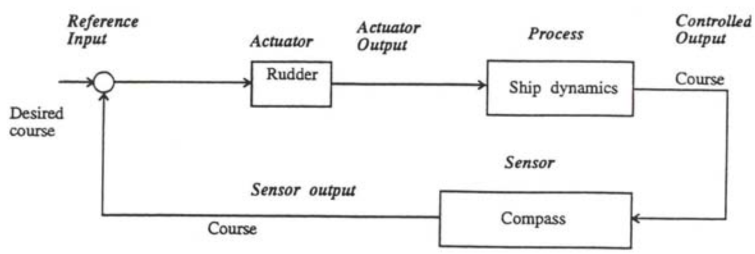
(c) Water level regulator:



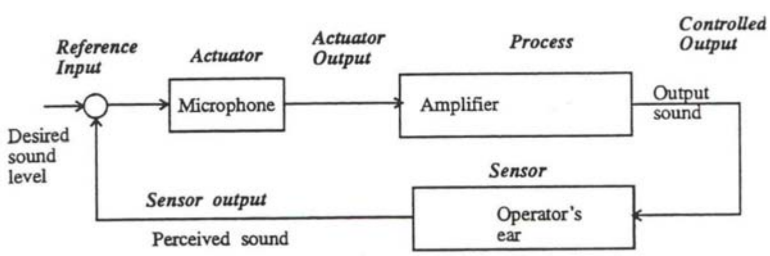
(d) Fly-ball governor:



(e) Automatic steering of a ship:



(f) A public address system:



2. Identify the physical principles and describe the operation of the thermostat in your home or office.

Solution:

A thermostat is a device for maintaining a temperature constant at a desired value. It is equipped with a temperature sensor which detects deviation from the desired value, determines whether the temperature setting is exceeded or not, and transmits the information to a furnace or air conditioner so that the termperature in the room is brought back

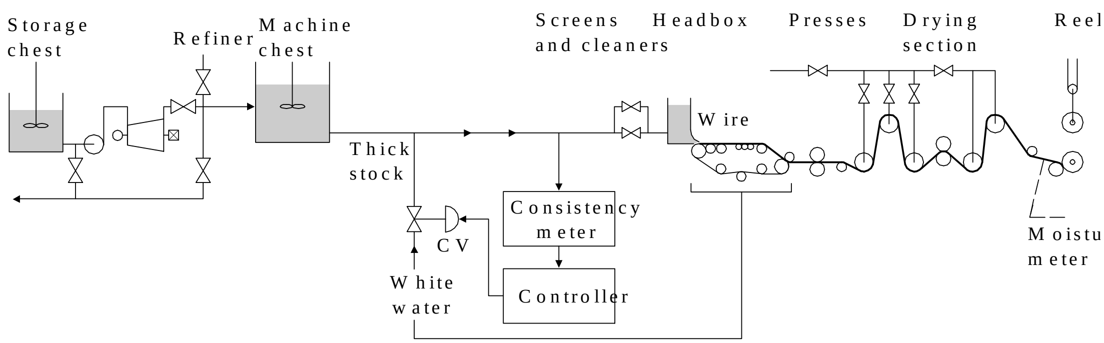


Figure 1.1: A paper making machine *From Karl Astrom, (1970, page 192) reprinted with permission.*

to the desired setting. Examples: Tubes filled with liquid mercury are attached to a bimetallic strip which tilt the tube and cause the mercury to slide over electrical contacts. A bimetallic strip consists of two strips of metal bonded together, each of a different expansion coefficient so that temperature changes bend the metal. In some cases, the bending of bimetallic strips simply cause electrical contacts to open or close directly. In most cases today, temperature is sensed electronically using,for example, a thermistor, a resistor whose resistance changes with temperature. Modern computer-based thermostats are programmable, sense the current from the thermistor and convert that to a digital signal.

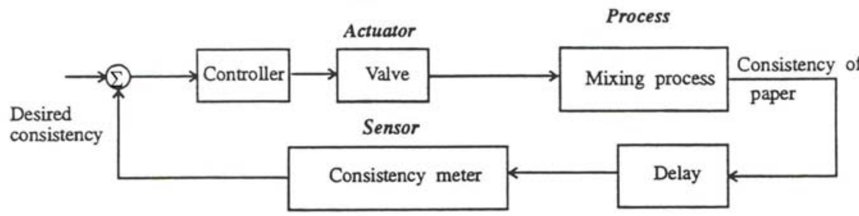
3. A machine for making paper is diagrammed in Fig. 1.12. There are two main parameters under feedback control: the density of fibers as controlled by the consistency of the thick stock that flows from the headbox onto the wire, and the moisture content of the final product that comes out of the dryers. Stock from the machine chest is diluted by white water returning from under the wire as controlled by a control valve (CV). A meter supplies a reading of the consistency. At the "dry end" of the machine, there is a moisture sensor. Draw a signal graph and identify the seven components listed in Problem 1 for

   (a) control of consistency

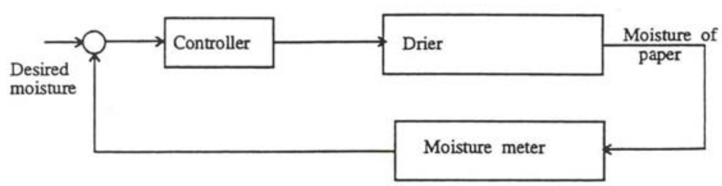
   (b) control of moisture

   Solution:

   (a) Control of paper machine consistency:



(b) Control of paper machine moisture:



4. Many variables in the human body are under feedback control. For each of the following controlled variables, draw a graph showing the process being controlled, the sensor that measures the variable, the actuator that causes it to increase and/or decrease, the information path that completes the feedback path, and the disturbances that upset the variable. You may need to consult an encyclopedia or textbook on human physiology for information on this problem.

(a) blood pressure

(b) blood sugar concentration

(c) heart rate

(d) eye-pointing angle

(e) eye-pupil diameter

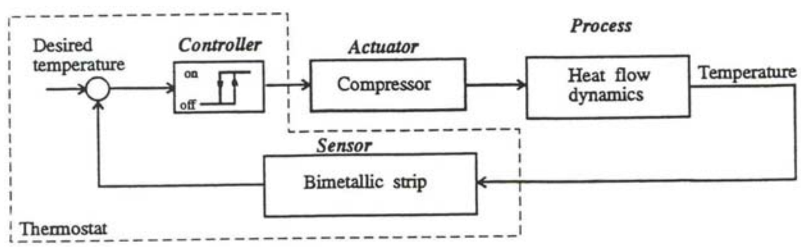
Solution:

Feedback control in human body:

Variable	Sensor	Actuator	Information path	Disturbances
a) Blood pressure	-Arterial baroreceptors	-Cardiac output -Arteriolar/venous dilation	-Afferent nerve fibers	-Bleeding -Drugs -Stress,Pain
b) Blood sugar concentration (Glucose)	-Pancreas	-Pancreas secreting insulin	-Blood flow to pancreas	-Diet -Exercise
c) Heart rate	-Diastolic volume sensors -Cardiac sympathetic nerves	-Electrical stimulation of sino-atrial node and cardiac muscle	-Mechanical draw of blood from heart -Circulating epinephrine	-Hormone releas -Exercise
d) Eye pointing angle	-Optic nerve -Image detection	-Extraocular muscles	-Cranial innervation	-Head movement -Muscle twitch
e) Pupil diameter	-Rods	-Pupillary sphincter muscles	-Autonomous system	-Ambient light -Drugs
f) Blood calcium level	-Parathyroid gland detectors	-Ca from bones to blood -Gastrointestinal absorption	- Parathormone hormone affecting effector sites	-Ca need in bone -Drugs

5. Draw a graph of the components for temperature control in a refrigerator or automobile air-conditioning system.

   Solution:

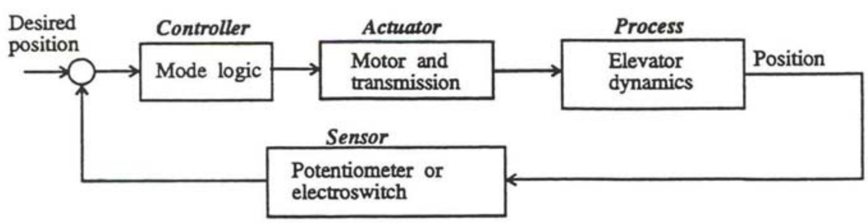


   This is the simplest possible system. Modern cases include computer control as described in later chapters.

6. Draw a graph of the components for an elevator-position control. Indicate how you would measure the position of the elevator car. Consider a combined coarse and fine measurement system. What accuracies do you suggest for each sensor? Your system should be able to correct for the fact that in elevators for tall buildings there is significant cable stretch as a function of cab load.

   Solution:

   A coarse measurement can be obtained by an electroswitch located before the desired floor level. When touched, the controller reduces the motor speed. A "fine" sensor can then be used to bring the elevator precisely to the floor level. With a sensor such as the one depicted in the figure, a linear control loop can be created (as opposed to the on-off type of the coarse control).Accuracy required for the course switch is around 5 cm; for the fine floor alignment, an accuracy of about 2 mm is desirable to eliminate any noticeable step for those entering or exiting the elevator.



7. Feedback control requires being able to sense the variable being controlled. Because electrical signals can be transmitted, amplified, and processed easily, often we want to have a sensor whose output is a voltage or current proportional to the variable being measured. Describe a sensor that would give an electrical output proportional to:

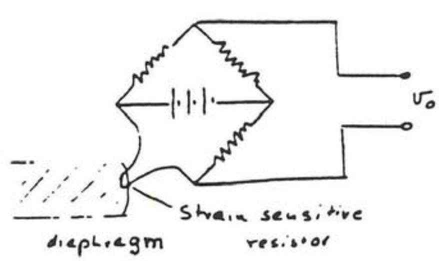
   (a) temperature
   (b) pressure
   (c) liquid level
   (d) flow of liquid along a pipe (or blood along an artery) force
   (e) linear position
   (f) rotational position
   (g) linear velocity
   (h) rotational speed
   (i) translational acceleration
   (j) torque

   Solution:

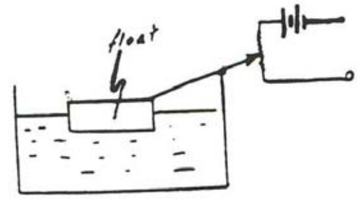
   Sensors for feedback control systems with electrical output. Examples

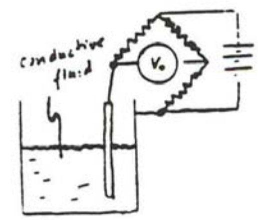
   (a) Temperature: Thermistor- temperature sensitive resistor with resistance change proportional to temperature; Thermocouple; Thyrister. Modern thermostats are computer controlled and programmable.

   (b) Pressure: Strain sensitive resistor mounted on a diaphragm which bends due to changing pressure

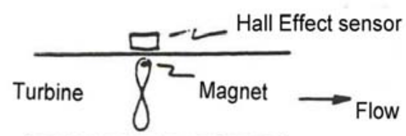


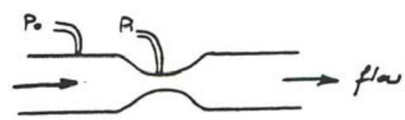
(c) Liquid level: Float connected to potentiometer. If liquid is conductive the impedance change of a rod immersed in the liquid may indicate the liquid level.





(d) Flow of liquid along a pipe: A turbine actuated by the flow with a magnet to trigger an external counting circuit. Hall effect produces an electronic output in response to magnetic field changes. Another way: Measure pressure difference from venturi into pressure sensor as in figure; Flowmeter. For blood flow, an ultrasound device like a SONAR can be used.





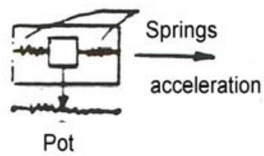
(e) Position.
When direct mechanical interaction is possible and for "small" displacements, the same ideas may be used. For example a potentiometer may be used to measure position of a mass in an accelerator (h). However in many cases such as the position of an aircraft, the task is much more complicated and measurement cannot be made directly. Calculation must be carried out based on other measurements, for example optical or electromagnetic direction measurements to several known references (stars,transmitting antennas ...); LVDT for linear, RVDT for rotational.

(f) Rotational position. The most common traditional device is a poteniometer. Also common are magnetic machines in shich a rotating magnet produces a variable output based on its angle.

(g) Linear velocity. For a vehicle, a RADAR can measure linear velocity. In other cases, a rack-and-pinion can be used to translate linear to rotational motion and an electric motor(tachometer) used to measure the speed.

(h) Speed: Any toothed wheel or gear on a rotating part may be used to trigger a magnetic field change which can be used to trigger an electrical counting circuit by use of a Hall effect (magnetic to electrical) sensor. The pulses can then be counted over a set time interval to produce angular velocity: Rate gyro; Tachometer

(i) Acceleration: A mass movement restrained by a spring measured by a potentiometer. A piezoelectric material may be used instead (a material that produces electrical current with intensity proportional to acceleration). In modern airbags, an integrated circuit chip contains a tiny lever and 'proof mass' whose motion is measured generating a voltage proportional to acceleration.



(j) Force, torque: A dynamometer based on spring or beam deflections, which may be measured by a potentiometer or a strain-gauge.

8. Each of the variables listed in Problem 7 can be brought under feedback control. Describe an actuator that could accept an electrical input and be used to control the variables listed. Give the units of the actuator output signal.

Solution:

(a) Resistor with voltage applied to it ormercury arc lamp to generate heat for small devices. a furnace for a building..

(b) Pump: Pumping air in or out of a chamberto generate pressure. Else, a 'torque motor' produces force..

(c) Valve and pump: forcing liquid in or out of the container.

(d) A valve is nromally used to control flow.

(e) Electric motor

(f) Electric motor

(g) Electric motor

(h) Electric motor

(i) Translational acceleration is usually controlled by a motor or engine to provide force on the vehicle or other object.

(j) Torque motor. In this motor the torque is directly proportional to the input (current).

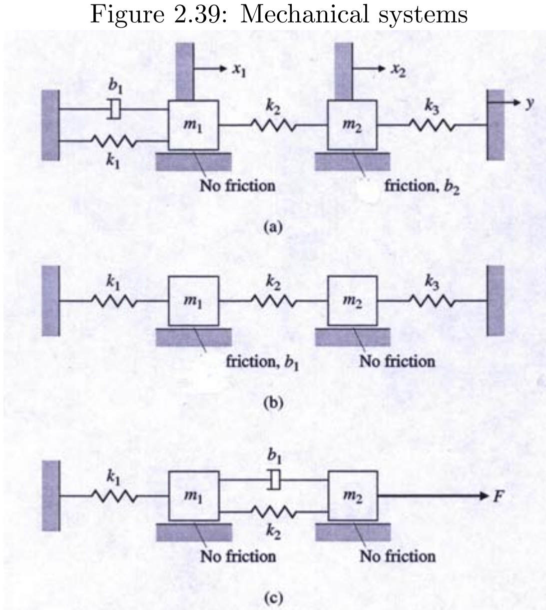
Chapter 2

Dynamic Models

Problems and Solutions for Section 2.1

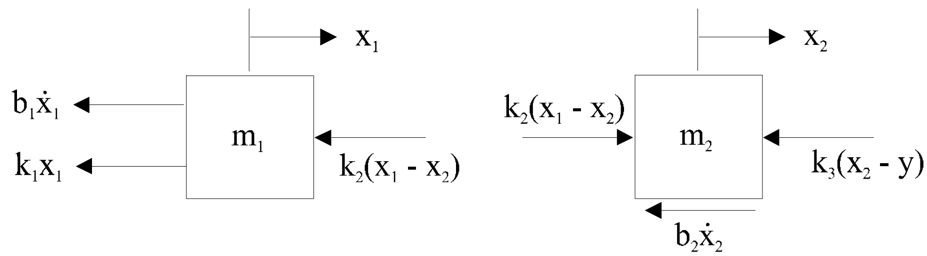
1. Write the differential equations for the mechanical systems shown in Fig. 2.39. For (a) and (b), state whether you think the system will eventually decay so that it has no motion at all, given that there are non-zero initial conditions for both masses, and give a reason for your answer.

Figure 2.39: Mechanical systems



Solution:

The key is to draw the Free Body Diagram (FBD) in order to keep the signs right. For (a), to identify the direction of the spring forces on the object, let $x_2 = 0$ and fixed and increase x_1 from 0. Then the k_1 spring will be stretched producing its spring force to the left and the k_2 spring will be compressed producing its spring force to the left also. You can use the same technique on the damper forces and the other mass.

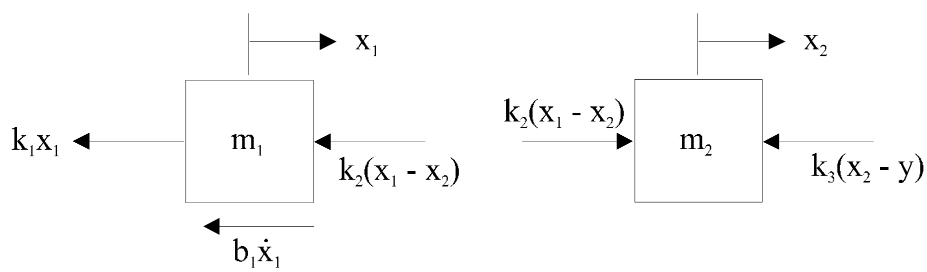


Free body diagram for Problem 2.1(a)

(a)

$$
\begin{aligned}
m_1\ddot{x}_1 &= -k_1x_1 - b_1\dot{x}_1 - k_2\left(x_1 - x_2\right) \\
m_2\ddot{x}_2 &= -k_2\left(x_2 - x_1\right) - k_3\left(x_2 - y\right) - b_2\dot{x}_2
\end{aligned}
$$

There is friction affecting the motion of both masses; therefore the system will decay to zero motion for both masses.



Free body diagram for Problem 2.1(b)

$$
\begin{aligned}
m_1\ddot{x}_1 &= -k_1x_1 - k_2(x_1 - x_2) - b_1\dot{x}_1 \\
m_2\ddot{x}_2 &= -k_2(x_2 - x_1) - k_3x_2
\end{aligned}
$$

Although friction only affects the motion of the left mass directly, continuing motion of the right mass will excite the left mass, and that interaction will continue until all motion damps out.

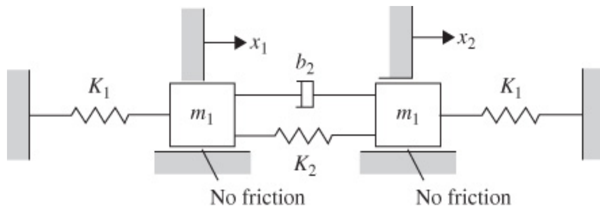
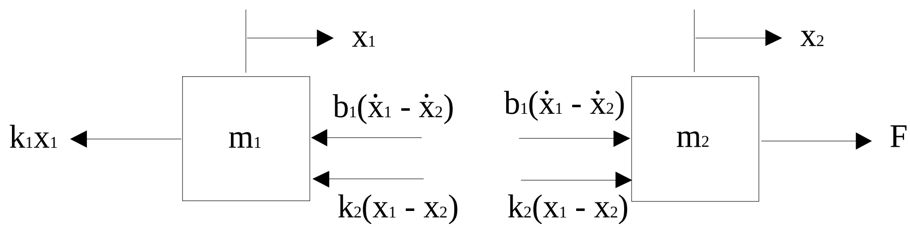


Figure 2.40: Mechanical system for Problem 2.2



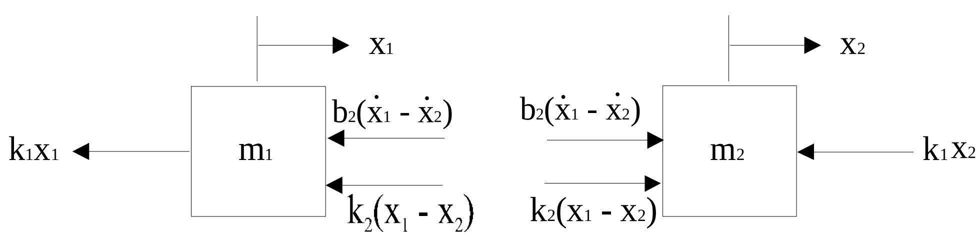
Free body diagram for Problem 2.1(c)

$$
\begin{aligned}
m_1\ddot{x}_1 &= -k_1x_1 - k_2(x_1 - x_2) - b_1(\dot{x}_1 - \dot{x}_2) \\
m_2\ddot{x}_2 &= F - k_2(x_2 - x_1) - b_1(\dot{x}_2 - \dot{x}_1)
\end{aligned}
$$

2. Write the differential equations for the mechanical systems shown in Fig. 2.40. State whether you think the system will eventually decay so that it has no motion at all, given that there are non-zero initial conditions for both masses, and give a reason for your answer.

   Solution:

   The key is to draw the Free Body Diagram (FBD) in order to keep the signs right. To identify the direction of the spring forces on the left side object, let $x_2 = 0$ and increase x_1 from 0. Then the k_1 spring on the left will be stretched producing its spring force to the left and the k_2 spring will be compressed producing its spring force to the left also. You can use the same technique on the damper forces and the other mass.



Free body diagram for Problem 2.2

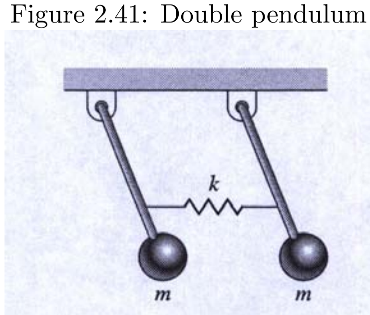
Then the forces are summed on each mass, resulting in

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1x_1 - k_2(x_1 - x_2) - b_1(\dot{x}_1 - \dot{x}_2) \\ m_2\ddot{x}_2 &= k_2(x_1 - x_2) - b_1(\dot{x}_1 - \dot{x}_2) - k_1x_2 \end{aligned}$$

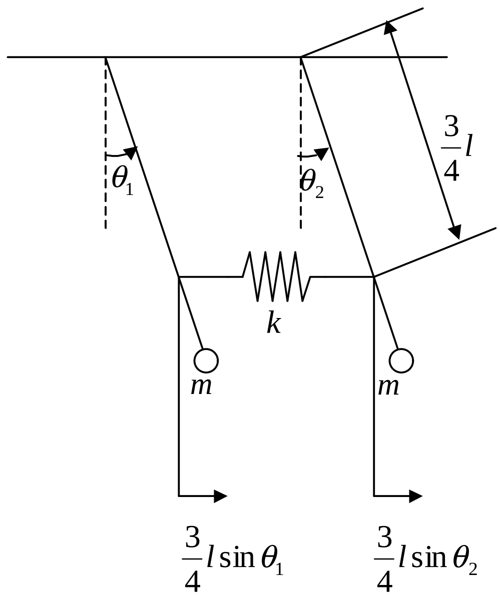
The *relative* motion between x_1 and x_2 will decay to zero due to the damper. However, the two masses will continue oscillating together without decay since there is no friction opposing that motion and no flexure of the end springs is all that is required to maintain the oscillation of the two masses.

3. Write the equations of motion for the double-pendulum system shown in Fig. 2.41. Assume the displacement angles of the pendulums are small enough to ensure that the spring is always horizontal. The pendulum rods are taken to be massless, of length l, and the springs are attached 3/4 of the way down.

Figure 2.41: Double pendulum



Solution:



Define coordinates

If we write the moment equilibrium about the pivot point of the left pendulem from the free body diagram,

$$M = -mgl\sin\theta_1 - k\frac{3}{4}l\left(\sin\theta_1 - \sin\theta_2\right)\cos\theta_1\frac{3}{4}l = ml^2\ddot{\theta}_1$$

$$ml^2\ddot{\theta}_1 + mgl\sin\theta_1 + \frac{9}{16}kl^2\cos\theta_1\left(\sin\theta_1 - \sin\theta_2\right) = 0$$

Similary we can write the equation of motion for the right pendulem

$$-mgl\sin\theta_2 + k\frac{3}{4}l\left(\sin\theta_1 - \sin\theta_2\right)\cos\theta_2\frac{3}{4}l = ml^2\ddot{\theta}_2$$

As we assumed the angles are small, we can approximate using $\sin\theta_1 \approx \theta_1, \sin\theta_2 \approx \theta_2$, $\cos\theta_1 \approx 1$, and $\cos\theta_2 \approx 1$. Finally the linearized equations of motion becomes,

$$\begin{aligned} ml\ddot{\theta}_1 + mg\theta_1 + \frac{9}{16}kl\left(\theta_1 - \theta_2\right) &= 0 \\ ml\ddot{\theta}_2 + mg\theta_2 + \frac{9}{16}kl\left(\theta_2 - \theta_1\right) &= 0 \end{aligned}$$

Or

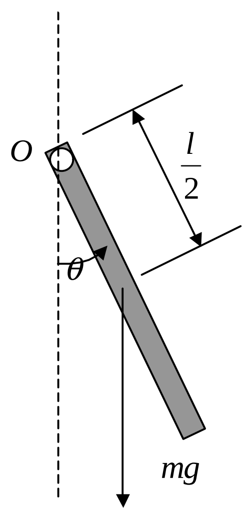
$$\begin{aligned}\ddot{\theta}_1 + \frac{g}{l}\theta_1 + \frac{9}{16}\frac{k}{m}(\theta_1 - \theta_2) &= 0 \\ \ddot{\theta}_2 + \frac{g}{l}\theta_2 + \frac{9}{16}\frac{k}{m}(\theta_2 - \theta_1) &= 0\end{aligned}$$

4. Write the equations of motion of a pendulum consisting of a thin, 4-kg stick of length l suspended from a pivot. How long should the rod be in order for the period to be exactly 2 secs? (The inertia I of a thin stick about an endpoint is $\frac{1}{3}ml^2$. Assume θ is small enough that $\sin\theta \cong \theta$.)

   Solution:

   Let's use Eq. (2.14)

$$M = I\alpha,$$



Define coordinates and forces

Moment about point O.

$$\begin{aligned}M_O &= -mg \times \frac{l}{2}\sin\theta = I_O\ddot{\theta} \\ &= \frac{1}{3}ml^2\ddot{\theta}\end{aligned}$$

$$\ddot{\theta} + \frac{3g}{2l}\sin\theta = 0$$

As we assumed θ is small,

$$\ddot{\theta} + \frac{3g}{2l}\theta = 0$$

The frequency only depends on the length of the rod

$$\omega^2 = \frac{3g}{2l}$$

$$\begin{aligned} T &= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2l}{3g}} = 2 \\ l &= \frac{3g}{2\pi^2} = 1.49\,\mathrm{m} \end{aligned}$$

<Notes>

(a) Compare the formula for the period, $T = 2\pi\sqrt{\frac{2l}{3g}}$ with the well known formula for the period of a point mass hanging with a string with length l. $T = 2\pi\sqrt{\frac{l}{g}}$.

(b) Important!
In general, Eq. (2.14) is valid only when the reference point for the moment and the moment of inertia is the mass center of the body. However, we also can use the formular with a reference point other than mass center when the point of reference is fixed or not accelerating, as was the case here for point O.

5. For the car suspension discussed in Example 2.2, plot the position of the car and the wheel after the car hits a "unit bump" (i.e., r is a unit step) using MATLAB. Assume that $m_1 = 10$ kg, $m_2 = 350$ kg, $k_w = 500,000$ N/m, $k_s = 10,000$ N/m. Find the value of b that you would prefer if you were a passenger in the car.

Solution:

The transfer function of the suspension was given in the example in Eq. (2.12) to be:

(a)

$$\frac{Y(s)}{R(s)} = \frac{\frac{k_w b}{m_1 m_2}(s + \frac{k_s}{b})}{s^4 + (\frac{b}{m_1} + \frac{b}{m_2})s^3 + (\frac{k_s}{m_1} + \frac{k_s}{m_2} + \frac{k_w}{m_1})s^2 + (\frac{k_w b}{m_1 m_2})s + \frac{k_w k_s}{m_1 m_2}}.$$

This transfer function can be put directly into MATLAB along with the numerical values as shown below. Note that b is not the damping ratio, but damping. We need to find the proper order of magnitude for b, which can be done by trial and error. What passengers feel is the position of the car. Some general requirements for the smooth ride will be, slow response with small overshoot and oscillation.

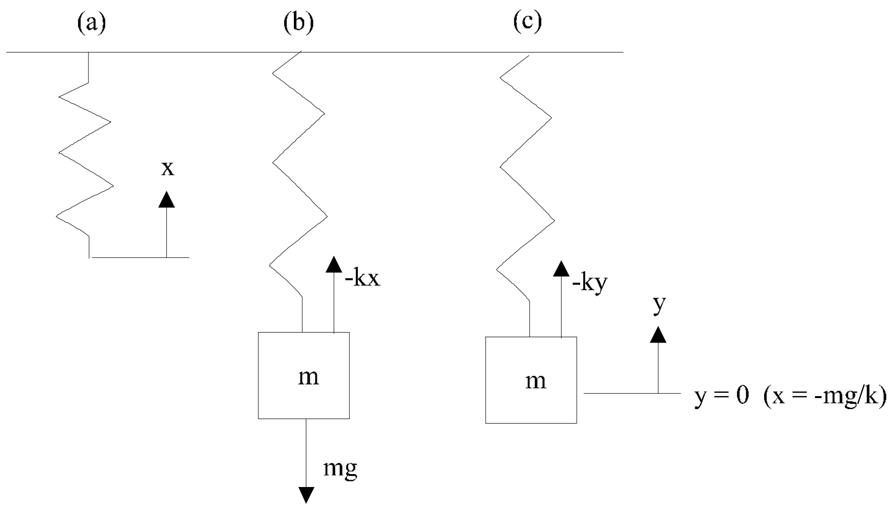
From the figures, $b \approx 3000$ would be acceptable. There is too much overshoot for lower values, and the system gets too fast (and harsh) for larger values.

```
% Problem 2.5
clear all, close all

m1 = 10;
m2 = 350;
kw = 500000;
ks = 10000;
B = [ 1000 2000 3000 4000 ];
t = 0:0.01:2;

for i = 1:4
    b = B(i);
    num = kw*b/(m1*m2)*[1 ks/b];
    den =[1  (b/m1+b/m2)    (ks/m1+ks/m2+kw/m1)
                (kw*b/(m1*m2)     kw*ks/(m1*m2)];
    sys=tf(num,den);
    y = step( sys, t );

    subplot(2,2,i);
    plot( t, y(:,1), ':', t, y(:,2), '-' );
    legend('Wheel','Car');
    ttl = sprintf('Response with b = %4.1f ',b );
    title(ttl);
end
```

6. Write the equations of motion for a body of mass M suspended from a fixed point by a spring with a constant k. Carefully define where the body's displacement is zero.

Solution:

Some care needs to be taken when the spring is suspended vertically in the presence of the gravity. We define $x = 0$ to be when the spring is unstretched with no mass attached as in (a). The static situation in (b) results from a balance between the gravity force and the spring.



From the free body diagram in (b), the dynamic equation results

$$m\ddot{x} = -kx - mg.$$

We can manipulate the equation

$$m\ddot{x} = -k\left(x + \frac{m}{k}g\right),$$

so if we replace x using $y = x + \frac{m}{k}g$,

$$\begin{aligned} m\ddot{y} &= -ky \\ m\ddot{y} + ky &= 0 \end{aligned}$$

The equilibrium value of x including the effect of gravity is at $x = -\frac{m}{k}g$ and y represents the motion of the mass about that equilibrium point.

An alternate solution method, which is applicable for any problem involving vertical spring motion, is to define the motion to be with respect to the static equilibrium point of the springs including the effect of gravity, and then to proceed as if no gravity was present. In this problem, we would define y to be the motion with respect to the equilibrium point, then the FBD in (c) would result directly in

$$m\ddot{y} = -ky.$$

7. Automobile manufacturers are contemplating building active suspension systems. The simplest change is to make shock absorbers with a changeable damping, $b(u_1)$. It is also possible to make a device to be placed in parallel with the springs that has the ability to supply an equal force, u_2, in opposite directions on the wheel axle and the car body.

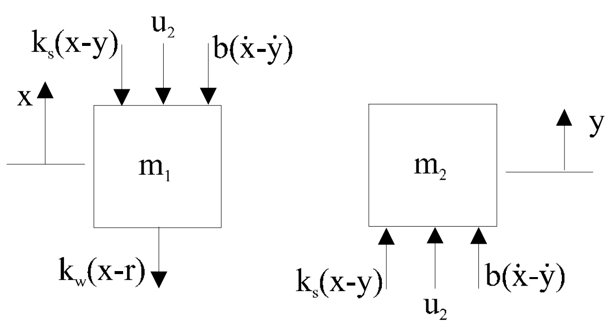
(a) Modify the equations of motion in Example 2.2 to include such control inputs.

(b) Is the resulting system linear?

(c) Is it possible to use the forcer, u_2, to completely replace the springs and shock absorber? Is this a good idea?

Solution:

(a) The FBD shows the addition of the variable force, u_2, and shows b as in the FBD of Fig. 2.5, however, here b is a function of the control variable, u_1. The forces below are drawn in the direction that would result from a positive displacement of x.



Free body diagram

$$\begin{aligned}
m_1\ddot{x} &= b(u_1)(\dot{y}-\dot{x}) + k_s(y-x) - k_w(x-r) - u_2 \\
m_2\ddot{y} &= -k_s(y-x) - b(u_1)(\dot{y}-\dot{x}) + u_2
\end{aligned}$$

(b) The system is linear with respect to u_2 because it is additive. But b is not constant so the system is non-linear with respect to u_1 because the control essentially multiplies a state element. So if we add controllable damping, the system becomes non-linear.

(c) It is technically possible. However, it would take very high forces and thus a lot of power and is therefore not done. It is a much better solution to modulate the damping coefficient by changing orifice sizes in the shock absorber and/or by changing the spring forces by increasing or decreasing the pressure in air springs. These features are now available on some cars... where the driver chooses between a soft or stiff ride.

8. Modify the equation of motion for the cruise control in Example 2.1, Eq(2.4), so that it has a control law; that is, let $u = K(v_r - v)$, where

$$\begin{aligned}
v_r &= \text{reference speed} \\
K &= \text{constant.}
\end{aligned}$$

This is a 'proportional' control law where the difference between v_r and the actual speed is used as a signal to speed the engine up or slow it down. Put the equations in the standard state-variable form with v_r as the input and v as the state. Assume that $m = 1000$ kg and $b = 50$ N · s/ m, and find the response for a unit step in v_r using MATLAB. Using trial and error, find a value of K that you think would result in a control system in which the actual speed converges as quickly as possible to the reference speed with no objectional behavior.

Solution:

$$\dot{v} + \frac{b}{m}v = \frac{1}{m}u$$

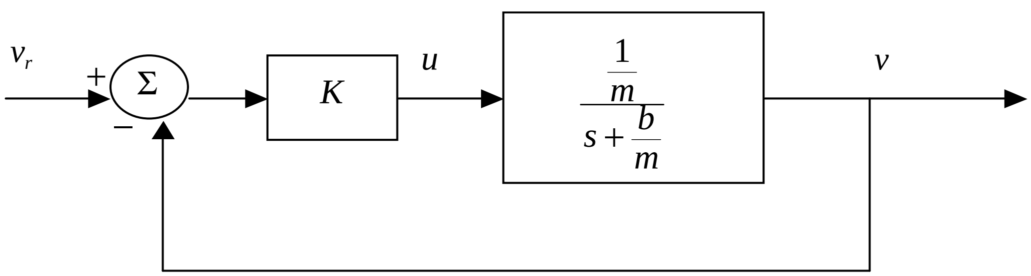
substitute in $u = K(v_r - v)$

$$\dot{v} + \frac{b}{m}v = \frac{1}{m}u = \frac{K}{m}(v_r - v)$$

Rearranging, yields the closed-loop system equations,

$$\dot{v} + \frac{b}{m}v + \frac{K}{m}v = \frac{K}{m}v_r$$

A block diagram of the scheme is shown below where the car dynamics are depicted by its transfer function from Eq. 2.7.



Block diagram

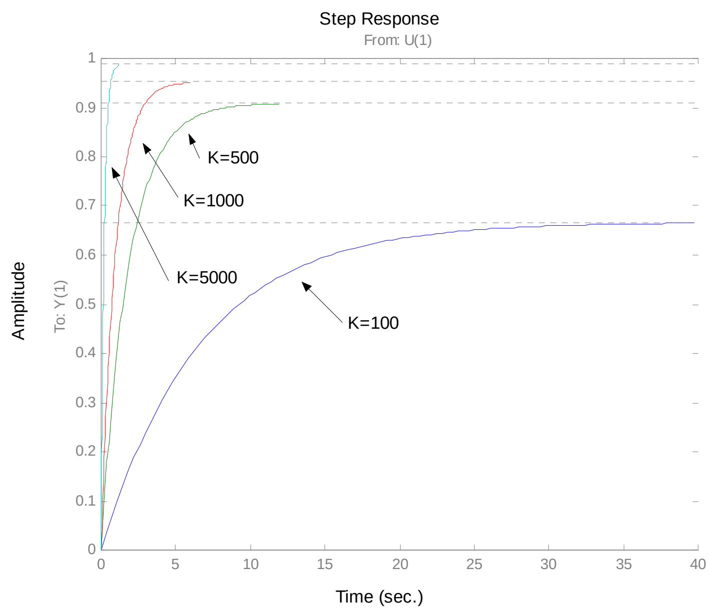
The transfer function of the closed-loop system is,

$$\frac{V(s)}{V_r(s)} = \frac{\frac{K}{m}}{s + \frac{b}{m} + \frac{K}{m}}$$

so that the inputs for Matlab are

$$num = \frac{K}{m}$$
$$den = [1 \quad \frac{b}{m} + \frac{K}{m}]$$

For $K = 100, 500, 1000, 5000$ We have,



Time responses

We can see that the larger the K is, the better the performance, with no objectionable behaviour for any of the cases. The fact that increasing K also results in the need for higher acceleration is less obvious from the plot but it will limit how fast K can be in the real situation because the engine has only so much poop. Note also that the error with this scheme gets quite large with the lower values of K. You will find out how to eliminate this error in chapter 4 using integral control, which is contained in all cruise control systems in use today. For this problem, a reasonable compromise between speed of response and steady state errors would be $K = 1000$, where it responds is 5 seconds and the steady state error is 5%.

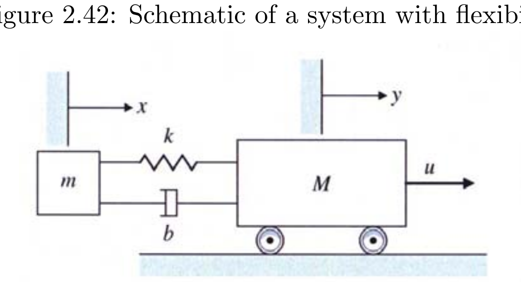
```
% Problem 2.8
clear all, close all

% data
m = 1000;
b = 50;
k = [ 100 500 1000 5000 ];

% Overlay the step response
hold on
for i=1:length(k)
    K=k(i);
    num =K/m;
    den = [1  b/m+K/m];
    step( num, den);
end
```

9. In many mechanical positioning systems there is flexibility between one part of the system and another. An example is shown in Figure 2.6 where there is flexibility of the solar panels. Figure 2.42 depicts such a situation, where a force u is applied to the mass M and another mass m is connected to it. The coupling between the objects is often modeled by a spring constant k with a damping coefficient b, although the actual situation is usually much more complicated than this.

   (a) Write the equations of motion governing this system.

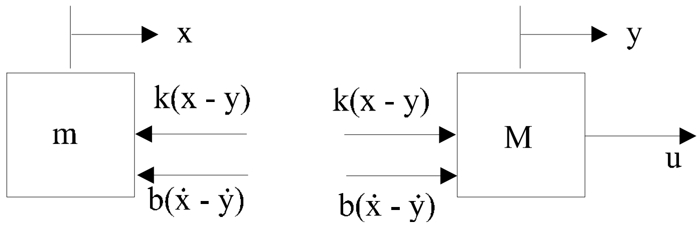
   (b) Find the transfer function between the control input, u, and the output, y.

Figure 2.42: Schematic of a system with flexibility



Solution:

(a) The FBD for the system is



Free body diagrams

which results in the equations

$$\begin{aligned} m\ddot{x} &= -k\,(x-y) - b\,(\dot{x}-\dot{y}) \\ M\ddot{y} &= u + k\,(x-y) + b\,(\dot{x}-\dot{y}) \end{aligned}$$

or

$$\begin{aligned} \ddot{x} + \frac{k}{m}x + \frac{b}{m}\dot{x} - \frac{k}{m}y - \frac{b}{m}\dot{y} &= 0 \\ -\frac{k}{M}x - \frac{b}{M}\dot{x} + \ddot{y} + \frac{k}{M}y + \frac{b}{M}\dot{y} &= \frac{1}{M}u \end{aligned}$$

(b) If we make Laplace Transform of the equations of motion

$$\begin{aligned} s^2X + \frac{k}{m}X + \frac{b}{m}sX - \frac{k}{m}Y - \frac{b}{m}sY &= 0 \\ -\frac{k}{M}X - \frac{b}{M}sX + s^2Y + \frac{k}{M}Y + \frac{b}{M}sY &= \frac{1}{M}U \end{aligned}$$

In matrix form,

$$\begin{bmatrix} ms^2+bs+k & -(bs+k) \\ -(bs+k) & Ms^2+bs+k \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ U \end{bmatrix}$$

From Cramer's Rule,

$$\begin{aligned} Y &= \frac{\det\begin{bmatrix} ms^2+bs+k & 0 \\ -(bs+k) & U \end{bmatrix}}{\det\begin{bmatrix} ms^2+bs+k & -(bs+k) \\ -(bs+k) & Ms^2+bs+k \end{bmatrix}} \\ &= \frac{ms^2+bs+k}{(ms^2+bs+k)(Ms^2+bs+k) - (bs+k)^2}U \end{aligned}$$

Finally,

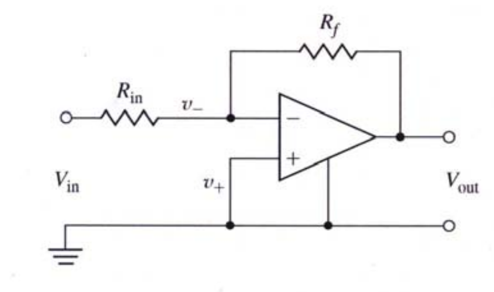
$$
\begin{aligned}
\frac{Y}{U} &= \frac{ms^2+bs+k}{\left(ms^2+bs+k\right)\left(Ms^2+bs+k\right)-\left(bs+k\right)^2} \\
&= \frac{ms^2+bs+k}{mMs^4+(m+M)bs^3+(M+m)ks^2}
\end{aligned}
$$

Problems and Solutions for Section 2.2

10. A first step toward a realistic model of an op amp is given by the equations below and shown in Fig. 2.43.

$$\begin{aligned} V_{out} &= \frac{10^7}{s+1}[V_+ - V_-] \\ i_+ &= i_- = 0 \end{aligned}$$

Figure 2.43: Circuit for Problem 10.



Find the transfer function of the simple amplification circuit shown using this model.

Solution:

As $i_- = 0$,

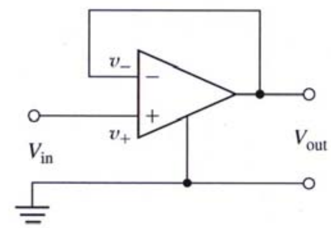
(a)

$$\frac{V_{in} - V_-}{R_{in}} = \frac{V_- - V_{out}}{R_f}$$

$$V_- = \frac{R_f}{R_{in} + R_f} V_{in} + \frac{R_{in}}{R_{in} + R_f} V_{out}$$

$$\begin{aligned} V_{out} &= \frac{10^7}{s+1}[V_+ - V_-] \\ &= \frac{10^7}{s+1}\left(V_+ - \frac{R_f}{R_{in} + R_f} V_{in} - \frac{R_{in}}{R_{in} + R_f} V_{out}\right) \\ &= -\frac{10^7}{s+1}\left(\frac{R_f}{R_{in} + R_f} V_{in} + \frac{R_{in}}{R_{in} + R_f} V_{out}\right) \end{aligned}$$

Figure 2.44: Circuit for Problem 11.



$$\frac{V_{out}}{V_{in}} = \frac{-10^7 \frac{R_f}{R_{in}+R_f}}{s+1+10^7 \frac{R_{in}}{R_{in}+R_f}}$$

11. Show that the op amp connection shown in Fig. 2.44 results in $V_o = V_{in}$ if the op amp is ideal. Give the transfer function if the op amp has the non-ideal transfer function of Problem 2.10.

 Solution:

 Ideal case:

$$\begin{aligned} V_{in} &= V_+ \\ V_+ &= V_- \\ V_- &= V_{out} \end{aligned}$$

 Non-ideal case:

$$V_{in} = V_+, V_- = V_{out}$$

 but,

$$V_+ \neq V_-$$

 instead,

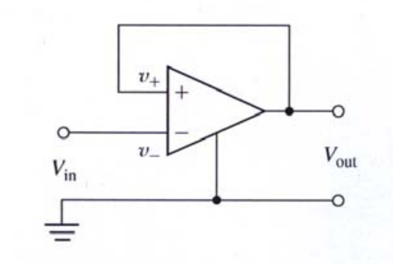
$$\begin{aligned} V_{out} &= \frac{10^7}{s+1}[V_+ - V_-] \\ &= \frac{10^7}{s+1}[V_{in} - V_{out}] \end{aligned}$$

so,

$$\frac{V_{out}}{V_{in}} = \frac{\frac{10^7}{s+1}}{1+\frac{10^7}{s+1}} = \frac{10^7}{s+1+10^7} \cong \frac{10^7}{s+10^7}$$

12. Show that, with the non-ideal transfer function of Problem 2.10, the op amp connection shown in Fig. 2.45 is *unstabl*e.

Figure 2.45: Circuit for Problem 12.



Solution:

$$V_{in} = V_-, V_+ = V_{out}$$

$$\begin{aligned} V_{out} &= \frac{10^7}{s+1}[V_+ - V_-] \\ &= \frac{10^7}{s+1}[V_{out} - V_{in}] \end{aligned}$$

$$\frac{V_{out}}{V_{in}} = \frac{\frac{10^7}{s+1}}{\frac{10^7}{s+1}-1} = \frac{10^7}{-s-1+10^7} \cong \frac{-10^7}{s-10^7}$$

The transfer function has a denominator with $s - 10^7$, and the minus sign means the exponential time function is increasing, which means that it has an unstable root.

13. A common connection for a motor power amplifier is shown in Fig. 2.46. The idea is to have the motor current follow the input voltage and the connection is called a current amplifier. Assume that the sense resistor, R_s is very small compared with the feedback resistor, R and find the transfer function from V_{in} to I_a. Also show the transfer function when $R_f = \infty$.

Solution:

At node A,

$$\frac{V_{in}-0}{R_{in}}+\frac{V_{out}-0}{R_f}+\frac{V_B-0}{R}=0 \tag{93}$$

At node B, with $R_s \ll R$

$$\begin{aligned} I_a+\frac{0-V_B}{R}+\frac{0-V_B}{R_s} &= 0 \qquad (94)\\ V_B &= \frac{RR_s}{R+R_s}I_a \\ V_B &\approx R_sI_a \end{aligned}$$

The dynamics of the motor is modeled with negligible inductance as

$$\begin{aligned} J_m\ddot{\theta}_m+b\dot{\theta}_m &= K_tI_a \qquad (95)\\ J_ms\Omega+b\Omega &= K_tI_a \end{aligned}$$

At the output, from Eq. 94. Eq. 95 and the motor equation $V_a = I_aR_a + K_es\Omega$

$$\begin{aligned} V_o &= I_aR_s+V_a \\ &= I_aR_s+I_aR_a+K_e\frac{K_tI_a}{J_ms+b} \end{aligned}$$

Substituting this into Eq.93

$$\frac{V_{in}}{R_{in}}+\frac{1}{R_f}\left[I_aR_s+I_aR_a+K_e\frac{K_tI_a}{J_ms+b}\right]+\frac{I_aR_s}{R}=0$$

This expression shows that, in the steady state when $s \to 0$, the current is proportional to the input voltage.

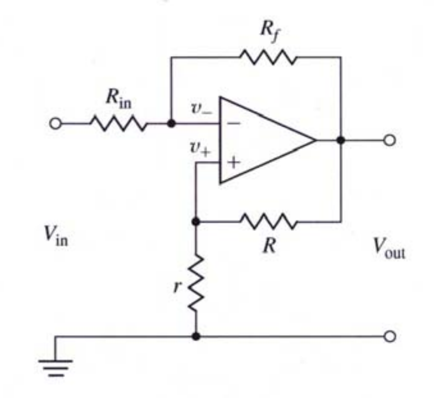
If fact, the current amplifier normally has no feedback from the output voltage, in which case $R_f \to \infty$ and we have simply

$$\frac{I_a}{V_{in}}=-\frac{R}{R_{in}R_s}$$

14. An op amp connection with feedback to both the negative and the positive terminals is shown in Fig 2.47. If the op amp has the non-ideal transfer function given in Problem 10, give the maximum value possible for the positive feedback ratio, $P = \dfrac{r}{r+R}$ in terms of the negative feedback ratio,$N = \dfrac{R_{in}}{R_{in}+R_f}$ for the circuit to remain stable.

Solution:

Figure 2.46: Op Amp circuit for Problem 14.



$$
\begin{aligned}
\frac{V_{in} - V_-}{R_{in}} + \frac{V_{out} - V_-}{R_f} &= 0 \\
\frac{V_{out} - V_+}{R} + \frac{0 - V_+}{r} &= 0
\end{aligned}
$$

$$
\begin{aligned}
V_- &= \frac{R_f}{R_{in} + R_f} V_{in} + \frac{R_{in}}{R_{in} + R_f} V_{out} \\
&= (1 - N) V_{in} + N V_{out} \\
V_+ &= \frac{r}{r + R} V_{out} = P V_{out}
\end{aligned}
$$

$$
\begin{aligned}
V_{out} &= \frac{10^7}{s+1} [V_+ - V_-] \\
&= \frac{10^7}{s+1} \left[P V_{out} - (1 - N) V_{in} - N V_{out} \right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{V_{out}}{V_{in}} &= \frac{\frac{10^7}{s+1} (1 - N)}{\frac{10^7}{s+1} P - \frac{10^7}{s+1} N - 1} \\
&= \frac{10^7 (1 - N)}{10^7 P - 10^7 N - (s + 1)} \\
&= \frac{-10^7 (1 - N)}{s + 1 - 10^7 P + 10^7 N}
\end{aligned}
$$

$$\begin{aligned} 0 &< 1 - 10^7 P + 10^7 N \\ P &< N + 10^{-7} \end{aligned}$$

15. Write the dynamic equations and find the transfer functions for the circuits shown in Fig. 2.48.

    (a) passive lead circuit
    (b) active lead circuit
    (c) active lag circuit.
    (d) passive notch circuit

    Solution:

    (a) Passive lead circuit
    With the node at y+, summing currents into that node, we get

    $$\frac{V_u - V_y}{R_1} + C\frac{d}{dt}(V_u - V_y) - \frac{V_y}{R_2} = 0 \tag{96}$$

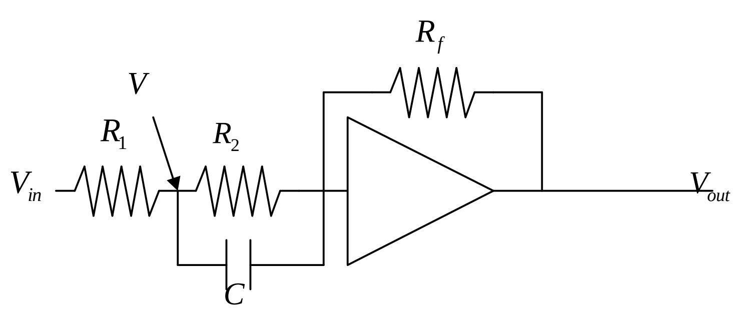
    rearranging a bit,

    $$C\dot{V}_y + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)V_y = C\dot{V}_u + \frac{1}{R_1}V_u$$

    and, taking the Laplace Transform, we get

    $$\frac{V_y(s)}{V_u(s)} = \frac{Cs + \frac{1}{R_1}}{Cs + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)}$$

    (b) Active lead circuit



Active lead circuit with node marked

$$\frac{V_{in} - V}{R_2} + \frac{0 - V}{R_1} + C\frac{d}{dt}(0 - V) = 0 \tag{97}$$

$$\frac{V_{in} - V}{R_2} = \frac{0 - V_{out}}{R_f} \tag{98}$$

We need to eliminate V. From Eq. 98,

$$V = V_{in} + \frac{R_2}{R_f} V_{out}$$

Substitute V's in Eq. 97.

$$\frac{1}{R_2}\left(V_{in} - V_{in} - \frac{R_2}{R_f}V_{out}\right) - \frac{1}{R_1}\left(V_{in} + \frac{R_2}{R_f}V_{out}\right) - C\left(\dot{V}_{in} + \frac{R_2}{R_f}\dot{V}_{out}\right) = 0$$

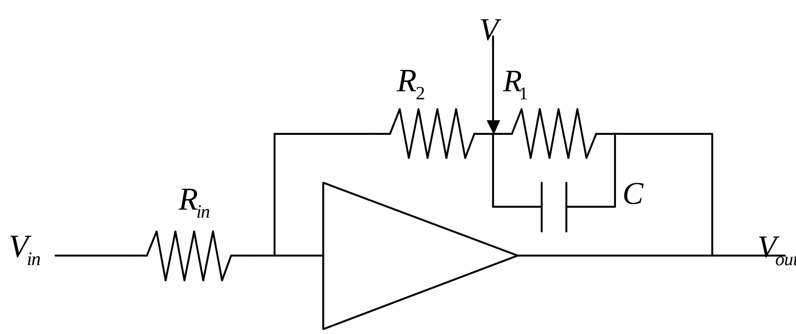
$$\frac{1}{R_1}V_{in} + C\dot{V}_{in} = -\frac{1}{R_f}\left[\left(1 + \frac{R_2}{R_1}\right)V_{out} + R_2 C\dot{V}_{out}\right]$$

Laplace Transform

$$\begin{aligned}\frac{V_{out}}{V_{in}} &= \frac{Cs + \frac{1}{R_1}}{-\frac{1}{R_f}\left(R_2 Cs + 1 + \frac{R_2}{R_1}\right)} \\ &= -\frac{R_f}{R_2}\frac{s + \frac{1}{R_1 C}}{s + \frac{1}{R_1 C} + \frac{1}{R_2 C}}\end{aligned}$$

We can see that the pole is at the left side of the zero, which means a lead compensator.

(c) active lag circuit



Active lag circuit with node marked

$$\frac{V_{in} - 0}{R_{in}} = \frac{0 - V}{R_2} = \frac{V - V_{out}}{R_1} + C\frac{d}{dt}\left(V - V_{out}\right)$$

$$V = -\frac{R_2}{R_{in}}V_{in}$$

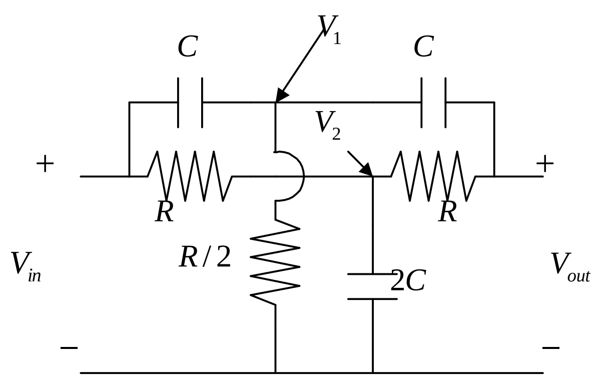
$$\begin{aligned}
\frac{V_{in}}{R_{in}} &= \frac{-\frac{R_2}{R_{in}}V_{in} - V_{out}}{R_1} + C\frac{d}{dt}\left(-\frac{R_2}{R_{in}}V_{in} - V_{out}\right) \\
&= \frac{1}{R_1}\left(-\frac{R_2}{R_{in}}V_{in} - V_{out}\right) + C\left(-\frac{R_2}{R_{in}}\dot{V}_{in} - \dot{V}_{out}\right)
\end{aligned}$$

$$\frac{1}{R_{in}}\left(1 + \frac{R_2}{R_1}\right)V_{in} + \frac{1}{R_{in}}R_2C\dot{V}_{in} = -\frac{1}{R_1}V_{out} - C\dot{V}_{out}$$

$$\begin{aligned}
\frac{V_{out}}{V_{in}} &= -\frac{R_1}{R_{in}}\frac{R_2Cs + 1 + \frac{R_2}{R_1}}{R_1Cs + 1} \\
&= -\frac{R_2}{R_{in}}\frac{s + \frac{1}{R_2C} + \frac{1}{R_1C}}{s + \frac{1}{R_1C}}
\end{aligned}$$

We can see that the pole is at the right side of the zero, which means a lag compensator.

(d) notch circuit



Passive notch filter with nodes marked

$$\begin{aligned}
C\frac{d}{dt}(V_{in} - V_1) + \frac{0 - V_1}{R/2} + C\frac{d}{dt}(V_{out} - V_1) &= 0 \\
\frac{V_{in} - V_2}{R} + 2C\frac{d}{dt}(0 - V_2) + \frac{V_{out} - V_2}{R} &= 0 \\
C\frac{d}{dt}(V_1 - V_{out}) + \frac{V_2 - V_{out}}{R} &= 0
\end{aligned}$$

We need to eliminat V_1, V_2 from three equations and find the relation between V_{in} and V_{out}

$$\begin{aligned}
V_1 &= \frac{Cs}{2\left(Cs + \frac{1}{R}\right)}(V_{in} + V_{out}) \\
V_2 &= \frac{\frac{1}{R}}{2\left(Cs + \frac{1}{R}\right)}(V_{in} + V_{out})
\end{aligned}$$

$$\begin{aligned}
& CsV_1 - CsV_{out} + \frac{1}{R}V_2 - \frac{1}{R}V_{out} \\
= \quad & Cs\frac{Cs}{2\left(Cs+\frac{1}{R}\right)}\left(V_{in}+V_{out}\right) + \frac{1}{R}\frac{\frac{1}{R}}{2\left(Cs+\frac{1}{R}\right)}\left(V_{in}+V_{out}\right) - \left(Cs+\frac{1}{R}\right)V_{out} \\
= \quad & 0
\end{aligned}$$

$$\begin{aligned}
\frac{C^2s^2+\frac{1}{R^2}}{2\left(Cs+\frac{1}{R}\right)}V_{in} &= \left[\left(Cs+\frac{1}{R}\right) - \frac{C^2s^2+\frac{1}{R^2}}{2\left(Cs+\frac{1}{R}\right)}\right]V_{out} \\
\frac{V_{out}}{V_{in}} &= \frac{\frac{C^2s^2+\frac{1}{R^2}}{2\left(Cs+\frac{1}{R}\right)}}{\left(Cs+\frac{1}{R}\right) - \frac{C^2s^2+\frac{1}{R^2}}{2\left(Cs+\frac{1}{R}\right)}} \\
&= \frac{\left(C^2s^2+\frac{1}{R^2}\right)}{2\left(Cs+\frac{1}{R}\right)^2 - \left(C^2s^2+\frac{1}{R^2}\right)} \\
&= \frac{C^2\left(s^2+\frac{1}{R^2C^2}\right)}{C^2s^2+4\frac{Cs}{R}+\frac{1}{R^2}} \\
&= \frac{s^2+\frac{1}{R^2C^2}}{s^2+\frac{4}{RC}s+\frac{1}{R^2C^2}}
\end{aligned}$$

16. The very flexible circuit shown in Fig. 2.49 is called a biquad because its transfer function can be made to be the ratio of two second-order or quadratic polynomials. By selecting different values for R_a, R_b, R_c, and R_d the circuit can realise a low-pass, band-pass, high-pass, or band-reject (notch) filter.

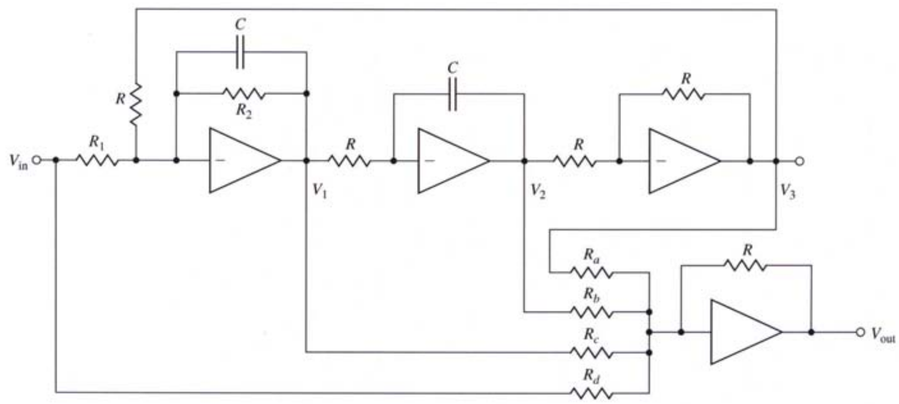
    (a) Show that if $R_a = R,$ and $R_b = R_c = R_d = \infty$, the transfer function from V_{in} to V_{out} can be written as the low-pass filter

$$\frac{V_{out}}{V_{in}} = \frac{A}{\frac{s^2}{\omega_n^2} + 2\zeta\frac{s}{\omega_n} + 1} \tag{99}$$

    where

$$\begin{aligned}
A &= \frac{R}{R_1} \\
\omega_n &= \frac{1}{RC} \\
\zeta &= \frac{R}{2R_2}
\end{aligned}$$

Figure 2.47: Op-amp biquad



(b) Using the MATLAB comand `step` compute and plot on the same graph the step responses for the biquad of Fig. 2.43 for $A = 1$, $\omega_n = 1$, and $\zeta = 0.1, \quad 0.5$, and 1.0.

Solution:

Before going in to the specific problem, let's find the general form of the transfer function for the circuit.

$$\begin{aligned}
\frac{V_{in}}{R_1} + \frac{V_3}{R} &= -\left(\frac{V_1}{R_2} + C\dot{V}_1\right) \\
\frac{V_1}{R} &= -C\dot{V}_2 \\
V_3 &= -V_2 \\
\frac{V_3}{R_a} + \frac{V_2}{R_b} + \frac{V_1}{R_c} + \frac{V_{in}}{R_d} &= -\frac{V_{out}}{R}
\end{aligned}$$

There are a couple of methods to find the transfer function from V_{in} to V_{out} with set of equations but for this problem, we will directly solve for the values we want along with the Laplace Transform.

From the first three equations, slove for V_1, V_2.

$$\begin{aligned}
\frac{V_{in}}{R_1} + \frac{V_3}{R} &= -\left(\frac{1}{R_2} + Cs\right) V_1 \\
\frac{V_1}{R} &= -CsV_2 \\
V_3 &= -V_2
\end{aligned}$$

$$\begin{aligned}\left(\frac{1}{R_2}+Cs\right)V_1-\frac{1}{R}V_2 &= -\frac{1}{R_1}V_{in}\\ \frac{1}{R}V_1+CsV_2 &= 0\end{aligned}$$

$$\begin{bmatrix}\frac{1}{R_2}+Cs & -\frac{1}{R}\\ \frac{1}{R} & Cs\end{bmatrix}\begin{bmatrix}V_1\\ V_2\end{bmatrix}=\begin{bmatrix}-\frac{1}{R_1}V_{in}\\ 0\end{bmatrix}$$

$$\begin{aligned}\begin{bmatrix}V_1\\ V_2\end{bmatrix} &= \frac{1}{\left(\frac{1}{R_2}+Cs\right)Cs+\frac{1}{R^2}}\begin{bmatrix}Cs & \frac{1}{R}\\ -\frac{1}{R} & \frac{1}{R_2}+Cs\end{bmatrix}\begin{bmatrix}-\frac{1}{R_1}V_{in}\\ 0\end{bmatrix}\\ &= \frac{1}{C^2s^2+\frac{C}{R_2}s+\frac{1}{R^2}}\begin{bmatrix}-\frac{C}{R_1}sV_{in}\\ \frac{1}{RR_1}V_{in}\end{bmatrix}\end{aligned}$$

Plug in V_1, V_2 and V_3 to the fourth equation.

$$\begin{aligned}&\frac{V_3}{R_a}+\frac{V_2}{R_b}+\frac{V_1}{R_c}+\frac{V_{in}}{R_d}\\ =\;& \left(-\frac{1}{R_a}+\frac{1}{R_b}\right)V_2+\frac{1}{R_c}V_1+\frac{1}{R_d}V_{in}\\ =\;& \left(-\frac{1}{R_a}+\frac{1}{R_b}\right)\frac{\frac{1}{RR_1}}{C^2s^2+\frac{C}{R_2}s+\frac{1}{R^2}}V_{in}+\frac{1}{R_c}\frac{-\frac{C}{R_1}s}{C^2s^2+\frac{C}{R_2}s+\frac{1}{R^2}}V_{in}+\frac{1}{R_d}V_{in}\\ =\;& \left[\left(-\frac{1}{R_a}+\frac{1}{R_b}\right)\frac{\frac{1}{RR_1}}{C^2s^2+\frac{C}{R_2}s+\frac{1}{R^2}}+\frac{1}{R_c}\frac{-\frac{C}{R_1}s}{C^2s^2+\frac{C}{R_2}s+\frac{1}{R^2}}+\frac{1}{R_d}\right]V_{in}\\ =\;& -\frac{V_{out}}{R}\end{aligned}$$

Finally,

$$\begin{aligned}\frac{V_{out}}{V_{in}} &= -R\left[\left(-\frac{1}{R_a}+\frac{1}{R_b}\right)\frac{\frac{1}{RR_1}}{C^2s^2+\frac{C}{R_2}s+\frac{1}{R^2}}+\frac{1}{R_c}\frac{-\frac{C}{R_1}s}{C^2s^2+\frac{C}{R_2}s+\frac{1}{R^2}}+\frac{1}{R_d}\right]\\ &= -R\frac{\left(-\frac{1}{R_a}+\frac{1}{R_b}\right)\frac{1}{RR_1}-\frac{1}{R_c}\frac{C}{R_1}s+\frac{1}{R_d}\left(C^2s^2+\frac{C}{R_2}s+\frac{1}{R^2}\right)}{C^2s^2+\frac{C}{R_2}s+\frac{1}{R^2}}\\ &= -\frac{R}{C^2}\frac{\frac{C^2}{R_d}s^2+\left(\frac{1}{R_d}\frac{C}{R_2}-\frac{1}{R_c}\frac{C}{R_1}\right)s+\left(\frac{1}{R_b}-\frac{1}{R_a}\right)\frac{1}{RR_1}+\frac{1}{R_d}\frac{1}{R^2}}{s^2+\frac{1}{R_2C}s+\frac{1}{(RC)^2}}\end{aligned}$$

(a) If $R_a = R,$ and $R_b = R_c = R_d = \infty,$

$$
\begin{aligned}
\frac{V_{out}}{V_{in}} &= -\frac{R}{C^2}\frac{\frac{C^2}{R_d}s^2 + \left(\frac{1}{R_d}\frac{C}{R_2} - \frac{1}{R_c}\frac{C}{R_1}\right)s + \left(\frac{1}{R_b} - \frac{1}{R_a}\right)\frac{1}{RR_1} + \frac{1}{R_d}\frac{1}{R^2}}{s^2 + \frac{1}{R_2 C}s + \frac{1}{(RC)^2}} \\
&= -\frac{R}{C^2}\frac{-\frac{1}{R}\frac{1}{RR_1}}{s^2 + \frac{1}{R_2 C}s + \frac{1}{(RC)^2}} = \frac{\frac{1}{RR_1C^2}}{s^2 + \frac{1}{R_2 C}s + \frac{1}{(RC)^2}} \\
&= \frac{\frac{R}{R_1}}{(RC)^2 s^2 + \frac{R^2 C}{R_2}s + 1}
\end{aligned}
$$

So,

$$
\begin{aligned}
\frac{R}{R_1} &= A \\
(RC)^2 &= \frac{1}{\omega_n^2} \\
2\frac{\zeta}{\omega_n} &= \frac{R^2 C}{R_2}
\end{aligned}
$$

$$
\begin{aligned}
\omega_n &= \frac{1}{RC} \\
\zeta &= \frac{\omega_n}{2}\frac{R^2 C}{R_2} = \frac{1}{2RC}\frac{R^2 C}{R_2} = \frac{R}{2R_2}
\end{aligned}
$$

(b) Step response using MatLab

```
% Problem 2.16
A = 1;
wn = 1;
z = [ 0.1 0.5 1.0 ];

hold on
for i = 1:3
    num = [ A ];
    den = [ 1/wn^2 2*z(i)/wn 1 ]
    step( num, den )
end
hold off
```

17. Find the equations and transfer function for the biquad circuit of Fig. 2.49 if $R_a = R$, $R_d = R_1$ and $R_b = R_c = \infty$.

Solution:

$$
\begin{aligned}
\frac{V_{out}}{V_{in}} &= -\frac{R}{C^2}\frac{\frac{C^2}{R_d}s^2+\left(\frac{1}{R_d}\frac{C}{R_2}-\frac{1}{R_c}\frac{C}{R_1}\right)s+\left(\frac{1}{R_b}-\frac{1}{R_a}\right)\frac{1}{RR_1}+\frac{1}{R_d}\frac{1}{R^2}}{s^2+\frac{1}{R_2C}s+\frac{1}{(RC)^2}} \\
&= -\frac{R}{C^2}\frac{\frac{C^2}{R_1}s^2+\left(\frac{1}{R_1}\frac{C}{R_2}\right)s+\left(-\frac{1}{R}\right)\frac{1}{RR_1}+\frac{1}{R_1}\frac{1}{R^2}}{s^2+\frac{1}{R_2C}s+\frac{1}{(RC)^2}} \\
&= -\frac{R}{R_1}\frac{s^2+\frac{1}{R_2C}s}{s^2+\frac{1}{R_2C}s+\frac{1}{(RC)^2}}
\end{aligned}
$$

Problems and Solutions for Section 2.3

18. The torque constant of a motor is the ratio of torque to current and is often given in ounce-inches per ampere. (ounce-inches have dimension force-distance where an ounce is 1/16 of a pound.) The electric constant of a motor is the ratio of back emf to speed and is often given in volts per 1000 rpm. In consistent units the two constants are the same for a given motor.

    (a) Show that the units ounce-inches per ampere are proportional to volts per 1000 rpm by reducing both to MKS (SI) units.

    (b) A certain motor has a back emf of 25 V at 1000 rpm. What is its torque constant in ounce-inches per ampere?

    (c) What is the torque constant of the motor of part (b) in newton-meters per ampere?

    Solution:

    Before going into the problem, let's review the units.

    - Some remarks on non SI units.
      - Ounce
        $$1\text{oz} = 2.835 \times 10^{-2}\,\text{kg}$$
        Originally ounce is a unit of mass, but like pounds, it is commonly used as a unit of force. If we translate it as force,
        $$1\text{oz(f)} = 2.835 \times 10^{-2}\ \text{kgf} = 2.835 \times 10^{-2} \times 9.81\,\text{N} = 0.2778\,\text{N}$$
      - Inch
        $$1\,\text{in} = 2.540 \times 10^{-2}\,\text{m}$$
      - RPM (Revolution per Minute)
        $$1\ \text{RPM} = \frac{2\pi\ \text{rad}}{60\ \text{s}} = \frac{\pi}{30}\ \text{rad/s}$$
    - Relation between SI units
      - Voltage and Current
        $$\begin{aligned} Volts \cdot Current(amps) &= Power = Energy(joules)/\sec \\ Volts &= \frac{Joules/\sec}{amps} = \frac{Newton - meters/\sec}{amps} \end{aligned}$$

(a) Relation between torque constant and electric constant.
Torque constant:

$$\frac{1 \text{ ounce} \times 1 \text{ inch}}{1 \text{ Ampere}} = \frac{0.2778\,\mathrm{N} \times 2.540 \times 10^{-2}\,\mathrm{m}}{1\,\mathrm{A}} = 7.056{\times}10^{-3}\,\mathrm{N\,m/\,A}$$

Electric constant:

$$\frac{1\,\mathrm{V}}{1000 \text{ RPM}} = \frac{1\,\mathrm{J/(A\ sec)}}{1000 \times \frac{\pi}{30} \text{ rad/\,s}} = 9.549 \times 10^{-3}\,\mathrm{N\,m/\,A}$$

So,

$$\begin{aligned} 1 \text{ oz in/\,A} &= \frac{7.056 \times 10^{-3}}{9.549 \times 10^{-3}} \text{ V/1000 RPM} \\ &= (0.739) \text{ V/1000 RPM} \end{aligned}$$

(b)

$$25\,\text{V/1000 RPM} = 25 \times \frac{1}{0.739} \text{ oz in/\,A} = 33.872 \text{ oz in/\,A}$$

(c)

$$25\,\text{V/1000 RPM} = 25 \times 9.549 \times 10^{-3}\,\mathrm{N\,m/\,A} = 0.239\,\mathrm{N\,m/\,A}$$

19. The electromechanical system shown in Fig. 2.50 represents a simplified model of a capacitor microphone. The system consists in part of a parallel plate capacitor connected into an electric circuit. Capacitor plate a is rigidly fastened to the microphone frame. Sound waves pass through the mouthpiece and exert a force $f_s(t)$ on plate b, which has mass M and is connected to the frame by a set of springs and dampers. The capacitance C is a function of the distance x between the plates, as follows:

$$C(x) = \frac{\varepsilon A}{x},$$

where

$$\begin{aligned} \varepsilon &= \text{dielectric constant of the material between the plates,} \\ A &= \text{surface area of the plates.} \end{aligned}$$

The charge q and the voltage e across the plates are related by

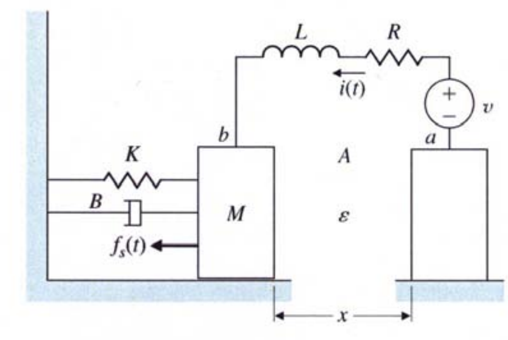
$$q = C(x)e.$$

The electric field in turn produces the following force f_e on the movable plate that opposes its motion:

$$f_e = \frac{q^2}{2\varepsilon A}$$

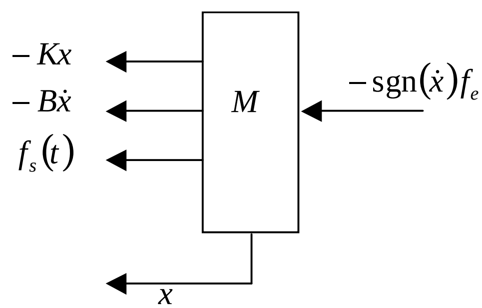
(a) Write differential equations that describe the operation of this system. (It is acceptable to leave in nonlinear form.)
(b) Can one get a linear model?
(c) What is the output of the system?

Figure 2.48: Simplified model for capacitor microphone



Solution:

(a) The free body diagram of the capacitor plate b



Free body diagram

So the equation of motion for the plate is

$$M\ddot{x} + B\dot{x} + Kx + f_e sgn\left(\dot{x}\right) = f_s\left(t\right).$$

The equation of motion for the circuit is

$$v = iR + L\frac{d}{dt}i + e$$

where e is the voltage across the capacitor,

$$e = \frac{1}{C}\int i(t)dt$$

and where $C = \varepsilon A/x$, a variable. Because $i = \frac{d}{dt}q$ and $e = q/C$, we can rewrite the circuit equation as

$$v = R\dot{q} + L\ddot{q} + \frac{qx}{\varepsilon A}$$

In summary, we have these two, couptled, non-linear differential equation.

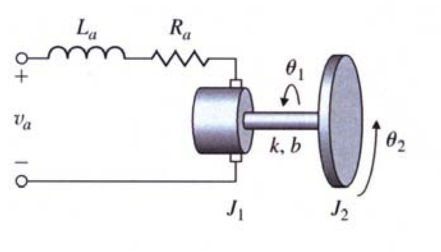
$$\begin{aligned} M\ddot{x} + b\dot{x} + kx + \operatorname{sgn}(\dot{x})\frac{q^2}{2\varepsilon A} &= f_s(t) \\ R\dot{q} + L\ddot{q} + \frac{qx}{\varepsilon A} &= v \end{aligned}$$

(b) The sgn function, q^2, and qx, terms make it impossible to determine a useful linearized version.

(c) The signal representing the voice input is the current, i, or $\dot{q}$.

20. A very typical problem of electromechanical position control is an electric motor driving a load that has one dominant vibration mode. The problem arises in computer-disk-head control, reel-to-reel tape drives, and many other applications. A schematic diagram is sketched in Fig. 2.51. The motor has an electrical constant K_e, a torque constant K_t, an armature inductance L_a, and a resistance R_a. The rotor has an inertia J_1 and a viscous friction B. The load has an inertia J_2. The two inertias are connected by a shaft with a spring constant k and an equivalent viscous damping b. Write the equations of motion.

Figure 2.49: Motor with a flexible load



(a)

Solution:

(a) Rotor:

$$J_1\ddot{\theta}_1 = -B\dot{\theta}_1 - b\left(\dot{\theta}_1 - \dot{\theta}_2\right) - k(\theta_1 - \theta_2) + T_m$$

Load:

$$J_2\ddot{\theta}_2 = -b\left(\dot{\theta}_2 - \dot{\theta}_1\right) - k\left(\theta_2 - \theta_1\right)$$

Circuit:

$$v_a - K_e\dot{\theta}_1 = L_a\frac{d}{dt}i_a + R_a i_a$$

Relation between the output torque and the armature current:

$$T_m = K_t i_a$$

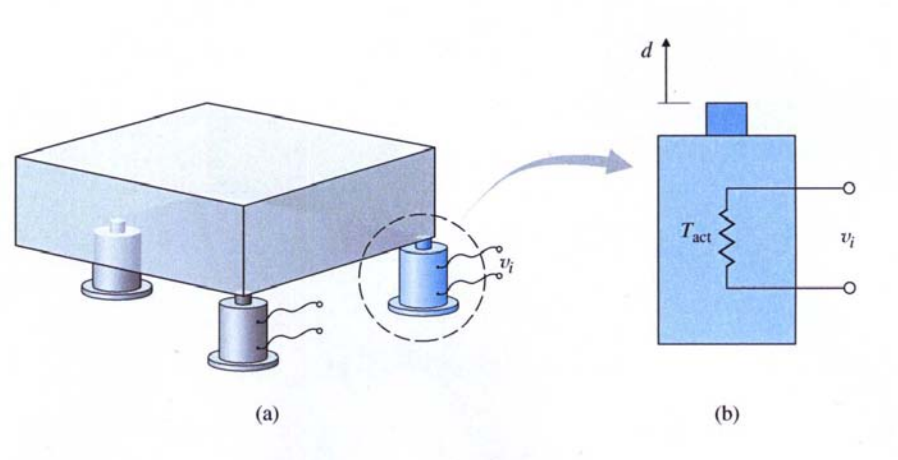
Problems and Solutions for Section 2.4

21. A precision-table leveling scheme shown in Fig. 2.52 relies on thermal expansion of actuators under two corners to level the table by raising or lowering their respective corners. The parameters are:

$$
\begin{aligned}
T_{\text{act}} &= \text{actuator temperature,} \\
T_{\text{amb}} &= \text{ambient air temperature,} \\
R_f &= \text{heat} - \text{flow coefficient between the actuator and the air,} \\
C &= \text{thermal capacity of the actuator,} \\
R &= \text{resistance of the heater.}
\end{aligned}
$$

Assume that (1) the actuator acts as a pure electric resistance, (2) the heat flow into the actuator is proportional to the electric power input, and (3) the motion d is proportional to the difference between T_{act} and T_{amb} due to thermal expansion. Find the differential equations relating the height of the actuator d versus the applied voltage v_i.

Figure 2.50: (a) Precision table kept level by actuators; (b) side view of one actuator



Solution:

Electric power in is proportional to the heat flow in

$$\dot{Q}_{in} = K_q \frac{v_i^2}{R}$$

and the heat flow out is from heat transfer to the ambient air

$$\dot{Q}_{out} = \frac{1}{R_f} \left(T_{act} - T_{amb} \right).$$

The temperature is governed by the difference in heat flows

$$\begin{aligned}\dot{T}_{act} &= \frac{1}{C}\left(\dot{Q}_{in} - \dot{Q}_{out}\right) \\ &= \frac{1}{C}\left(K_q\frac{v_i^2}{R} - \frac{1}{R_f}\left(T_{act} - T_{amb}\right)\right)\end{aligned}$$

and the actuator displacement is

$$d = K\left(T_{act} - T_{amb}\right).$$

where T_{amb} is a given function of time, most likely a constant for a table inside a room. The system input is v_i and the system output is d.

22. An air conditioner supplies cold air at the same temperature to each room on the fourth floor of the high-rise building shown in Fig. 2.53(a). The floor plan is shown in Fig. 2.53(b). The cold air flow produces an equal amount of heat flow q out of each room. Write a set of differential equations governing the temperature in each room, where

$$\begin{aligned}T_o &= \text{temperature outside the building,} \\ R_o &= \text{resistance to heat flow through the outer walls,} \\ R_i &= \text{resistance to heat flow through the inner walls.}\end{aligned}$$

Assume that (1) all rooms are perfect squares, (2) there is no heat flow through the floors or ceilings, and (3) the temperature in each room is uniform throughout the room. Take advantage of symmetry to reduce the number of differential equations to three.

Solution:

We can classify 9 rooms to 3 types by the number of outer walls they have.

Type 1	Type 2	Type 1
Type 2	Type 3	Type 2
Type 1	Type 2	Type 1

We can expect the hotest rooms on the outside and the corners hotest of all, but solving the equations would confirm this intuitive result. That is,

$$T_o > T_1 > T_2 > T_3$$

and, with a same cold air flow into every room, the ones with some sun load will be hotest.

Let's redefince the resistances

$$\begin{aligned}R_o &= \text{resistance to heat flow through one unit of outer wall} \\ R_i &= \text{resistance to heat flow through one unit of inner wall}\end{aligned}$$

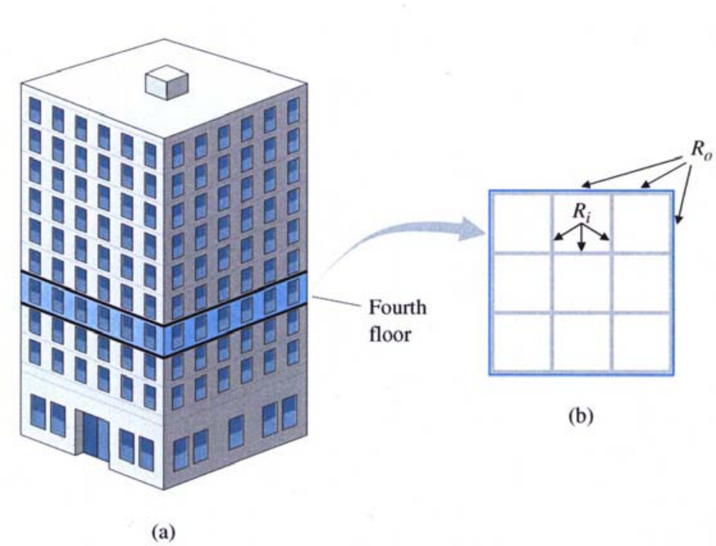


Figure 2.51: Building air conditioning: (a) high-rise building, (b) floor plan of the fourth floor

Room type 1:

$$
\begin{aligned}
q_{out} &= \frac{2}{R_i}(T_1 - T_2) + q \\
q_{in} &= \frac{2}{R_o}(T_o - T_1)
\end{aligned}
$$

$$
\begin{aligned}
\dot{T}_1 &= \frac{1}{C}(q_{in} - q_{out}) \\
&= \frac{1}{C}\left[\frac{2}{R_o}(T_o - T_1) - \frac{2}{R_i}(T_1 - T_2) - q\right]
\end{aligned}
$$

Room type 2:

$$
\begin{aligned}
q_{in} &= \frac{1}{R_o}(T_o - T_2) + \frac{2}{R_i}(T_1 - T_2) \\
q_{out} &= \frac{1}{R_i}(T_2 - T_3) + q
\end{aligned}
$$

$$
\dot{T}_2 = \frac{1}{C}\left[\frac{1}{R_o}(T_o - T_2) + \frac{2}{R_i}(T_1 - T_2) - \frac{1}{R_i}(T_2 - T_3) - q\right]
$$

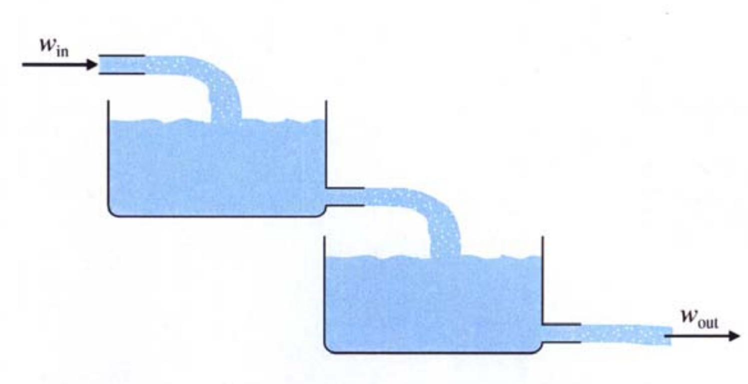
Room type 3:

$$\begin{aligned} q_{in} &= \frac{4}{R_i}(T_2 - T_3) \\ q_{out} &= q \end{aligned}$$

$$\dot{T}_3 = \frac{1}{C}\left[\frac{4}{R_i}(T_2 - T_3) - q\right]$$

23. For the two-tank fluid-flow system shown in Fig. 2.54, find the differential equations relating the flow into the first tank to the flow out of the second tank.

Figure 2.52: Two-tank fluid-flow system for Problem 23



Solution:

This is a variation on the problem solved in Example 2.18 and the definitions of terms is taken from that. From the relation between the height of the water and mass flow rate, the continuity equations are

$$\begin{aligned} \dot{m}_1 &= \rho A_1 \dot{h}_1 = w_{in} - w \\ \dot{m}_2 &= \rho A_2 \dot{h}_2 = w - w_{out} \end{aligned}$$

Also from the relation between the pressure and outgoing mass flow rate,

$$\begin{aligned} w &= \frac{1}{R_1}(\rho g h_1)^{\frac{1}{2}} \\ w_{out} &= \frac{1}{R_2}(\rho g h_2)^{\frac{1}{2}} \end{aligned}$$

Finally,

$$\begin{aligned}
\dot{h}_1 &= -\frac{1}{\rho A_1 R_1}(\rho g h_1)^{\frac{1}{2}} + \frac{1}{\rho A_1} w_{in} \\
\dot{h}_2 &= \frac{1}{\rho A_2 R_1}(\rho g h_1)^{\frac{1}{2}} - \frac{1}{\rho A_2 R_2}(\rho g h_2)^{\frac{1}{2}}.
\end{aligned}$$

24. A laboratory experiment in the flow of water through two tanks is sketched in Fig. 2.55. Assume that Eq. (2.74) describes flow through the equal-sized holes at points A, B, or C.

    (a) With holes at A and C but none at B, write the equations of motion for this system in terms of h_1 and h_2. Assume that $h_3 = 20$ cm, $h_1 > 20$ cm, and $h_2 < 20$ cm. When $h_2 = 10$ cm, the outflow is 200 g/min.

    (b) At $h_1 = 30$ cm and $h_2 = 10$ cm, compute a linearized model and the transfer function from pump flow (in cubic centimeters per minute) to h_2.

    (c) Repeat parts (a) and (b) assuming hole A is closed and hole B is open.

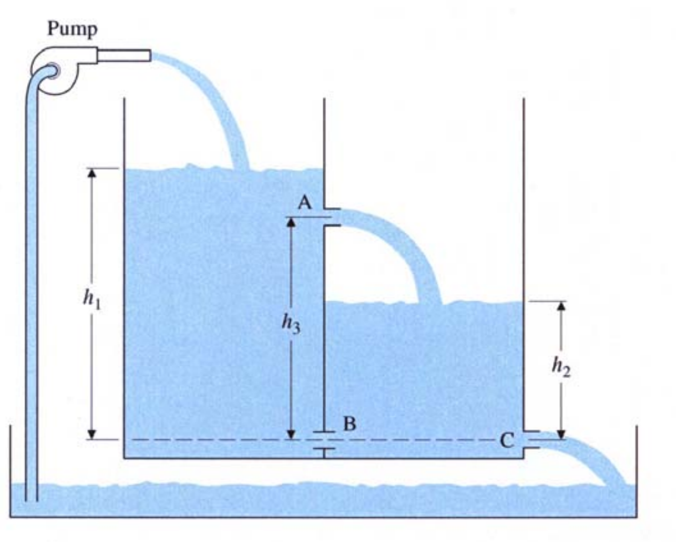


Figure 2.53: Two-tank fluid-flow system for Problem 24

Solution:

(a) Following the solution of Example 2.18, and assuming the area of both tanks is A, the values given for the heights ensure that the water will flow according to

$$\begin{aligned} W_A &= \frac{1}{R}\left[\rho g\left(h_1 - h_3\right)\right]^{\frac{1}{2}} \\ W_C &= \frac{1}{R}\left[\rho g h_2\right]^{\frac{1}{2}} \\ W_A - W_C &= \rho A \dot{h}_2 \\ W_{in} - W_A &= \rho A \dot{h}_1 \end{aligned}$$

From the outflow information given, we can compute the orifice resistance, R, noting that for water, $\rho = 1$ gram/cc and $g = 981$ cm/sec$^2 \simeq 1000$ cm/sec^2.

$$\begin{aligned} W_C &= 200\,\mathrm{g/\,mn} = \frac{1}{R}\sqrt{\rho g h_2} = \frac{1}{R}\sqrt{\rho g \times 10\,\mathrm{cm}} \\ R &= \frac{\sqrt{\rho g \times 10\,\mathrm{cm}}}{200\,\mathrm{g/\,mn}} = \frac{\sqrt{1\,\mathrm{g/\,cm^3} \times 1000\,\mathrm{cm/\,s^2} \times 10\,\mathrm{cm}}}{200\,\mathrm{g}/60\,\mathrm{s}} \\ &= \frac{100}{200}60\sqrt{\frac{\mathrm{g\,cm^2\,s^2}}{\mathrm{cm^3\,s^2\,g^2}}} = 30\,\mathrm{g}^{-\frac{1}{2}}\,\mathrm{cm}^{-\frac{1}{2}} \end{aligned}$$

(b) The nonlinear equations from above are

$$\begin{aligned} \dot{h}_1 &= -\frac{1}{\rho A R}\sqrt{\rho g\left(h_1 - h_3\right)} + \frac{1}{\rho A} W_{in} \\ \dot{h}_2 &= \frac{1}{\rho A R}\sqrt{\rho g\left(h_1 - h_3\right)} - \frac{1}{\rho A R}\sqrt{\rho g h_2} \end{aligned}$$

The square root functions need to be linearized about the nominal heights. In general the square root function can be linearized as below

$$\begin{aligned} \sqrt{x_0 + \delta x} &= \sqrt{x_0\left(1 + \frac{\delta x}{x_0}\right)} \\ &\cong \sqrt{x_0}\left(1 + \frac{1}{2}\frac{\delta x}{x_0}\right) \end{aligned}$$

So let's assume that $h_1 = h_{10} + \delta h_1$ and $h_2 = h_{20} + \delta h_2$ where $h_{10} = 30$ cm, $h_{20} = 10$ cm, and $h_3 = 20$ cm. And for round numbers, let's assume the area of each tank $A = 100$ cm^2. The equations above then reduce to

$$\begin{aligned} \delta\dot{h}_1 &= -\frac{1}{(1)(100)(30)}\sqrt{(1)(1000)\left(30 + \delta h_1 - 20\right)} + \frac{1}{(1)(100)} W_{in} \\ \delta\dot{h}_2 &= \frac{1}{(1)(100)(30)}\sqrt{(1)(1000)\left(30 + \delta h_1 - 20\right)} - \frac{1}{(1)(100)(30)}\sqrt{(1)(1000)(10 + \delta h_2)} \end{aligned}$$

which, with the square root approximations, is equivalent to,

$$\begin{aligned}
\delta\dot{h}_1 &= -\frac{1}{(30)}(1+\frac{1}{20}\delta h_1) + \frac{1}{(100)}W_{in} \\
\delta\dot{h}_2 &= \frac{1}{(30)}(1+\frac{1}{20}\delta h_1) - \frac{1}{(30)}(1+\frac{1}{20}\delta h_2)
\end{aligned}$$

The nominal inflow $W_{nom} = \frac{10}{3}$ cc/sec is required in order for the system to be in equilibrium, as can be seen from the first equation. So we will define the total inflow to be $W_{in} = W_{nom} + \delta W$, so the equations become

$$\begin{aligned}
\delta\dot{h}_1 &= -\frac{1}{(30)}(1+\frac{1}{20}\delta h_1) + \frac{1}{(100)}W_{nom} + \frac{1}{(100)}\delta W \\
\delta\dot{h}_2 &= \frac{1}{(30)}(1+\frac{1}{20}\delta h_1) - \frac{1}{(30)}(1+\frac{1}{20}\delta h_2)
\end{aligned}$$

or, with the nominal inflow included, the equations reduce to

$$\begin{aligned}
\delta\dot{h}_1 &= -\frac{1}{600}\delta h_1 + \frac{1}{100}\delta W \\
\delta\dot{h}_2 &= \frac{1}{600}\delta h_1 - \frac{1}{600}\delta h_2
\end{aligned}$$

Taking the Laplace transform of these two equations, and solving for the desired transfer function (in cc/sec) yields

$$\frac{\delta H_2(s)}{\delta W(s)} = \frac{1}{600}\frac{0.01}{(s+1/600)^2}.$$

which becomes, with the inflow in grams/min,

$$\frac{\delta H_2(s)}{\delta W(s)} = \frac{1}{600}\frac{(0.01)(60)}{(s+1/600)^2} = .\frac{0.001}{(s+1/600)^2}$$

(c) With hole B open and hole A closed, the relevant relations are

$$\begin{aligned}
W_{in} - W_B &= \rho A\dot{h}_1 \\
W_B &= \frac{1}{R}\sqrt{\rho g(h_1-h_2)} \\
W_B - W_C &= \rho A\dot{h}_2 \\
W_C &= \frac{1}{R}\sqrt{\rho g h_2}
\end{aligned}$$

$$\begin{aligned}
\dot{h}_1 &= -\frac{1}{\rho AR}\sqrt{\rho g(h_1-h_2)} + \frac{1}{\rho A}W_{in} \\
\dot{h}_2 &= \frac{1}{\rho AR}\sqrt{\rho g(h_1-h_2)} - \frac{1}{\rho AR}\sqrt{\rho g h_2}
\end{aligned}$$

With the same definitions for the perturbed quantities as for part (b), we obtain

$$\begin{aligned}
\delta\dot{h}_1 &= -\frac{1}{(1)(100)(30)}\sqrt{(1)(1000)(30+\delta h_1-10-\delta h_2)}+\frac{1}{(1)(100)}W_{in} \\
\delta\dot{h}_2 &= \frac{1}{(1)(100)(30)}\sqrt{(1)(1000)(30+\delta h_1-10-\delta h_2)} \\
&\quad -\frac{1}{(1)(100)(30)}\sqrt{(1)(1000)(10+\delta h_2)}
\end{aligned}$$

which, with the linearization carried out, reduces to

$$\begin{aligned}
\delta\dot{h}_1 &= -\frac{\sqrt{2}}{30}(1+\frac{1}{40}\delta h_1-\frac{1}{40}\delta h_2)+\frac{1}{100}W_{in} \\
\delta\dot{h}_2 &= \frac{\sqrt{2}}{30}(1+\frac{1}{40}\delta h_1-\frac{1}{40}\delta h_2)-\frac{1}{30}(1+\frac{1}{20}\delta h_2)
\end{aligned}$$

and with the nominal flow rate of $W_{in}=\frac{10\sqrt{2}}{3}$ removed

$$\begin{aligned}
\delta\dot{h}_1 &= -\frac{\sqrt{2}}{1200}(\delta h_1-\delta h_2)+\frac{1}{100}\delta W \\
\delta\dot{h}_2 &= \frac{\sqrt{2}}{1200}\delta h_1+(\frac{\sqrt{2}}{1200}-\frac{1}{600})\delta h_2+\frac{\sqrt{2}-1}{30}
\end{aligned}$$

However, unlike part (b), holding the nominal flow rate maintains h_1 at equilibrium, but h_2 will not stay at equilibrium. Instead, there will be a constant term increasing h_2. Thus the standard transfer function will not result.

25. The equations for heating a house are given by Eqs. (2.62) and (2.63) and, in a particular case can be written with time in *hours* as

$$C\frac{dT_h}{dt}=Ku-\frac{T_h-T_o}{R}$$

where

(a) C is the Thermal capacity of the house, $BTU/^oF$

(b) T_h is the temperature in the house, ^oF

(c) T_o is the temperature outside the house, ^oF

(d) K is the heat rating of the furnace, $= 90,000\ BTU/hour$

(e) R is the thermal resistance, ^oF per $BTU/hour$

(f) u is the furnace switch, =1 if the furnace is on and =0 if the furnace is off.

It is measured that, with the outside temperature at 32 ^oF and the house at 60 ^oF, the furnace raises the temperature 2 ^oF in 6 minutes (0.1 hour). With the furnace off, the house temperature falls 2 ^oF in 40 minutes. What are the values of C and R for the house?

Solution:

For the first case, the furnace is on which means $u = 1$.

$$\begin{aligned} C\frac{dT_h}{dt} &= K - \frac{1}{R}(T_h - T_o) \\ \dot{T}_h &= \frac{K}{C} - \frac{1}{RC}(T_h - T_o) \end{aligned}$$

and with the furnace off,

$$\dot{T}_h = -\frac{1}{RC}(T_h - T_o)$$

In both cases, it is a first order system and thus the solutions involve exponentials in time. The approximate answer can be obtained by simply looking at the slope of the exponential at the outset. This will be fairly accurate because the temperature is only changing by 2 degrees and this represents a small fraction of the 30 degree temperature difference. Let's solve the equation for the furnace off first

$$\frac{\Delta T_h}{\Delta t} = -\frac{1}{RC}(T_h - T_o)$$

plugging in the numbers available, the temperature falls 2 degrees in 2/3 hr, we have

$$\frac{2}{2/3} = -\frac{1}{RC}(60 - 32)$$

which means that

$$RC = 28/3$$

For the second case, the furnace is turned on which means

$$\frac{\Delta T_h}{\Delta t} = \frac{K}{C} - \frac{1}{RC}(T_h - T_o)$$

and plugging in the numbers yields

$$\frac{2}{0.1} = \frac{90,000}{C} - \frac{1}{28/3}(60 - 32)$$

and we have

$$\begin{aligned} C &= \frac{90,000}{23} = 3910 \\ R &= \frac{RC}{C} = \frac{28/3}{3910} = 0.00240 \end{aligned}$$

Chapter 3

Dynamic Response

Problems and Solutions for Section 3.1: Review of Laplace Transforms

1. Show that, in a partial-fraction expansion, complex conjugate poles have coefficients that are also complex conjugates. (The result of this relationship is that whenever complex conjugate pairs of poles are present, only one of the coefficients needs to be computed.)

   Solution:

   Consider the second-order system with poles at $-\alpha \pm j\beta$,

   $$H(s) = \frac{1}{(s+\alpha+j\beta)(s+\alpha-j\beta)}.$$

   Perform Partial Fraction Expansion:

   $$\begin{aligned} H(s) &= \frac{C_1}{s+\alpha+j\beta} + \frac{C_2}{s+\alpha-j\beta}. \\ C_1 &= \frac{1}{s+\alpha-j\beta}|_{s=-\alpha-j\beta} = \frac{1}{2\beta}j, \\ C_2 &= \frac{1}{s+\alpha+j\beta}|_{s=-\alpha+j\beta} = -\frac{1}{2\beta}j, \\ &\therefore \quad C_1 = C_2^*. \end{aligned}$$

2. Find the Laplace transform of the following time functions:

   (a) $f(t) = 1 + 2t$

   (b) $f(t) = 3 + 7t + t^2 + \delta(t)$

   (c) $f(t) = e^{-t} + 2e^{-2t} + te^{-3t}$

   (d) $f(t) = (t+1)^2$

(e) $f(t) = \sinh t$

Solution:

(a)

$$\begin{aligned} f(t) &= 1 + 2t. \\ \mathcal{L}\{f(t)\} &= \mathcal{L}\{1(t)\} + \mathcal{L}\{2t\}, \\ &= \frac{1}{s} + \frac{2}{s^2}, \\ &= \frac{s+2}{s^2}. \end{aligned}$$

We can verify the answer using MATLAB:

>> laplace(1+2*t)

ans =

(2+s)/s^2

(b)

$$\begin{aligned} f(t) &= 3 + 7t + t^2 + \delta(t), \\ \mathcal{L}\{f(t)\} &= \mathcal{L}\{3\} + \mathcal{L}\{7t\} + \mathcal{L}\{t^2\} + \mathcal{L}\{\delta(t)\}, \\ &= \frac{3}{s} + \frac{7}{s^2} + \frac{2!}{s^3} + 1, \\ &= \frac{s^3 + 3s^2 + 7s + 2}{s^3}. \end{aligned}$$

We can verify the answer using MATLAB:

>> laplace(3+7*t+t^2+dirac(t))

ans =

1+3/s+7/s^2+2/s^3

(c)

$$\begin{aligned} f(t) &= e^{-t} + 2e^{-2t} + te^{-3t}, \\ \mathcal{L}\{f(t)\} &= \mathcal{L}\{e^{-t}\} + \mathcal{L}\{2e^{-2t}\} + \mathcal{L}\{te^{-3t}\}, \\ &= \frac{1}{s+1} + \frac{2}{s+2} + \frac{1}{(s+3)^2}. \end{aligned}$$

We can verify the answer using MATLAB:

>> laplace(exp(-t)+2*exp(-2*t)+t*exp(-3*t))

ans =

1/(1+s)+2/(2+s)+1/(s+3)^2

(d)

$$\begin{aligned}
f(t) &= (t+1)^2, \\
&= t^2 + 2t + 1. \\
\mathcal{L}\{f(t)\} &= \mathcal{L}\{t^2\} + \mathcal{L}\{2t\} + \mathcal{L}\{1\}, \\
&= \frac{2!}{s^3} + \frac{2}{s^2} + \frac{1}{s}, \\
&= \frac{s^2 + 2s + 2}{s^3}.
\end{aligned}$$

We can verify the answer using MATLAB:

```
>> laplace((t+1)^2)
ans =
(2+2*s+s^2)/s^3
```

(e) Using the trigonometric identity,

$$\begin{aligned}
f(t) &= \sinh t, \\
&= \frac{e^t - e^{-t}}{2}, \\
\mathcal{L}\{f(t)\} &= \mathcal{L}\left\{\frac{e^t}{2}\right\} - \mathcal{L}\left\{\frac{e^{-t}}{2}\right\}, \\
&= \frac{1}{2}\left(\frac{1}{s-1}\right) - \frac{1}{2}\left(\frac{1}{s+1}\right), \\
&= \frac{1}{s^2 - 1}.
\end{aligned}$$

We can verify the answer using MATLAB:

```
>> laplace(sinh(t))
ans =
1/(s^2-1)
```

Remark: A useful reference for this problem and the next several problems is: K. R. Coombes, B. R. Hunt, R. L. Lipsman, J. E. Osborn, G. J. Stuck, *Differential Equations with* MATLAB, Wiley, 1998.

3. Find the Laplace transform of the following time functions:

(a) $f(t) = 3\cos 6t$

(b) $f(t) = \sin 2t + 2\cos 2t + e^{-t}\sin 2t$

(c) $f(t) = t^2 + e^{-2t}\sin 3t$

Solution:

(a)

$$\begin{aligned} f(t) &= 3\cos 6t \\ \mathcal{L}\{f(t)\} &= \mathcal{L}\{3\cos 6t\} \\ &= 3\frac{s}{s^2+36}. \end{aligned}$$

We can verify the answer using MATLAB:
>> laplace(3*cos(6*t))
ans =
3*s/(s^2+36)

(b)

$$\begin{aligned} f(t) &= \sin 2t + 2\cos 2t + e^{-t}\sin 2t \\ &= \mathcal{L}\{f(t)\} = \mathcal{L}\{\sin 2t\} + \mathcal{L}\{2\cos 2t\} + \mathcal{L}\{e^{-t}\sin 2t\}, \\ &= \frac{2}{s^2+4} + \frac{2s}{s^2+4} + \frac{2}{(s+1)^2+4}. \end{aligned}$$

We can verify the answer using MATLAB:
>> laplace(sin(2*t)+2*cos(2*t)+exp(-t)*sin(2*t))
ans =
2*(9+7*s+4*s^2+s^3)/(s^2+4)/(5+2*s+s^2)

(c)

$$\begin{aligned} f(t) &= t^2 + e^{-2t}\sin 3t, \\ &= \mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} + \mathcal{L}\{e^{-2t}\sin 3t\}, \\ &= \frac{2!}{s^3} + \frac{3}{(s+2)^2+9}, \\ &= \frac{2}{s^3} + \frac{3}{(s+2)^2+9}. \end{aligned}$$

We can verify the answer using MATLAB:
>> laplace(t^2+exp(-2*t)*sin(3*t))
ans =
2/s^3+3/(s^2+4*s+13)

4. Find the Laplace transform of the following time functions:

(a) $f(t) = t\sin t$
(b) $f(t) = t\cos 3t$
(c) $f(t) = te^{-t} + 2t\cos t$
(d) $f(t) = t\sin 3t - 2t\cos t$
(e) $f(t) = 1(t) + 2t\cos 2t$

Solution:

(a)

$$\begin{aligned} f(t) &= t\sin t \\ \mathcal{L}\{f(t)\} &= \mathcal{L}\{t\sin t\} \end{aligned}$$

Use multiplication by time Laplace transform property (Table A.1, entry #11),

$$\begin{aligned} \mathcal{L}\{tg(t)\} &= -\frac{d}{ds}G(s). \\ \text{Let } g(t) &= \sin t \quad \text{and use} \quad \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}. \\ \mathcal{L}\{t\sin t\} &= -\frac{d}{ds}\left(\frac{1}{s^2+1^2}\right), \\ &= \frac{2s}{(s^2+1)^2}, \\ &= \frac{2s}{s^4+2s^2+1}. \end{aligned}$$

We can verify the answer using MATLAB:

```
>> laplace(t*sin(t))
ans =
2*s/(s^2+1)^2
```

(b)

$$f(t) = t\cos 3t$$

Use multiplication by time Laplace transform property (Table A.1, entry #11),

$$\begin{aligned} \mathcal{L}\{tg(t)\} &= -\frac{d}{ds}G(s). \\ \text{Let } g(t) &= \cos 3t \quad \text{and use} \quad \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}. \\ \mathcal{L}\{t\cos 3t\} &= -\frac{d}{ds}\left(\frac{s}{s^2+9}\right), \\ &= \frac{-[(s^2+9)-(2s)s]}{(s^2+9)^2}, \\ &= \frac{s^2-9}{s^4+18s^2+81}. \end{aligned}$$

We can verify the answer using MATLAB:

```
>> laplace(t*cos(3*t))
ans =
(s^2-9)/(s^2+9)^2
```

(c)

$$f(t) = te^{-t} + 2t\cos t$$

Use the following Laplace transforms and properties (Table A.1, entries 4,11, and 3),

$$\begin{aligned}
\mathcal{L}\{te^{-at}\} &= \frac{1}{(s+a)^2}, \\
\mathcal{L}\{tg(t)\} &= -\frac{d}{ds}G(s), \\
\mathcal{L}\{\cos at\} &= \frac{s}{s^2+a^2}, \\
\mathcal{L}\{f(t)\} &= \mathcal{L}\{te^{-t}\} + 2\mathcal{L}\{t\cos t\}, \\
&= \frac{1}{(s+1)^2} + 2\left(-\frac{d}{ds}\frac{s}{s^2+1}\right), \\
&= \frac{1}{(s+1)^2} - 2\left[\frac{(s^2+1)-(2s)s}{(s^2+1)^2}\right], \\
&= \frac{1}{(s+1)^2} + \frac{2(s^2-1)}{(s^2+1)^2}.
\end{aligned}$$

We can verify the answer using MATLAB:

```
>> laplace(t*exp(-t)+2*t*cos(t))
ans =
1/(1+s)^2+2*(s^2-1)/(s^2+1)^2
```

(d)

$$f(t) = t\sin 3t - 2t\cos t.$$

Use the following Laplace transforms and properties (Table A.1, entries 11, 3),

$$\begin{aligned}
\mathcal{L}\{tg(t)\} &= -\frac{d}{ds}G(s), \\
\mathcal{L}\{\sin at\} &= \frac{a}{s^2+a^2}, \\
\mathcal{L}\{\cos at\} &= \frac{s}{s^2+a^2}, \\
\mathcal{L}\{f(t)\} &= \mathcal{L}\{t\sin 3t\} - 2\mathcal{L}\{t\cos t\}, \\
&= -\frac{d}{ds}\left[\frac{3}{s^2+9}\right] - 2\left(-\frac{d}{ds}\left[\frac{s}{s^2+1}\right]\right), \\
&= \frac{(2s\times 3)}{(s^2+9)^2} + 2\frac{[(s^2+1)-(2s)s]}{(s^2+1)^2}, \\
&= \frac{6s}{(s^2+9)^2} - \frac{2(s^2-1)}{(s^2+1)^2}.
\end{aligned}$$

We can verify the answer using MATLAB:

```
>> laplace(t*sin(3*t)-2*t*cos(t))
ans =
6*s/(s^2+9)^2-2*(s^2-1)/(s^2+1)^2
```

(e)

$$\begin{aligned} f(t) &= 1(t) + 2t\cos 2t, \\ \mathcal{L}\{1(t)\} &= \frac{1}{s}, \\ \mathcal{L}\{tg(t)\} &= -\frac{d}{ds}G(s), \\ \mathcal{L}\{\cos at\} &= \frac{s}{s^2+a^2}, \\ \mathcal{L}\{f(t)\} &= \mathcal{L}\{1(t)\} + 2\mathcal{L}\{t\cos 2t\}, \\ &= \frac{1}{s} + 2\left(-\frac{d}{ds}\frac{s}{s^2+4}\right), \\ &= \frac{1}{s} - 2\left[\frac{(s^2+4)-(2s)s}{(s^2+4)^2}\right], \\ &= \frac{1}{s} - 2\frac{(-s^2+4)}{(s^2+4)^2}. \end{aligned}$$

We can verify the answer using MATLAB:

```
>> laplace(1+2*t*cos(2*t))
ans =
1/s+2*(s^2-4)/(s^2+4)^2
```

5. Find the Laplace transform of the following time functions (* denotes convolution):

   (a) $f(t) = \sin t \sin 3t$
   (b) $f(t) = \sin^2 t + 3\cos^2 t$
   (c) $f(t) = (\sin t)/t$
   (d) $f(t) = \sin t * \sin t$
   (e) $f(t) = \int_0^t \cos(t-\tau)\sin\tau d\tau$

Solution:

(a)

$$f(t) = \sin t \sin 3t.$$

Use the trigonometric relation,

$$
\begin{aligned}
\sin \alpha t \sin \beta t &= \frac{1}{2}\cos(|\alpha-\beta|t) - \frac{1}{2}\cos(|\alpha+\beta|t), \\
\alpha &= 1 \quad \text{and} \quad \beta = 3. \\
f(t) &= \frac{1}{2}\cos(|1-3|t) - \frac{1}{2}\cos(|1+3|t), \\
&= \frac{1}{2}\cos 2t - \frac{1}{2}\sin 4t. \\
\mathcal{L}\{f(t)\} &= \frac{1}{2}\mathcal{L}\{\cos 2t\} - \frac{1}{2}\mathcal{L}\{\cos 4t\}, \\
&= \frac{1}{2}\left[\frac{s}{s^2+4} - \frac{s}{s^2+16}\right]. \\
&= \frac{6s}{(s^2+4)(s^2+16)}.
\end{aligned}
$$

We can verify the answer using MATLAB:

>> laplace(sin(t)*sin(3*t))

ans =

6*s/(s^2+16)/(s^2+4)

(b)

$$f(t) = \sin^2 t + 3\cos^2 t.$$

Use the trigonometric formulas,

$$
\begin{aligned}
\sin^2 t &= \frac{1-\cos 2t}{2}, \\
\cos^2 t &= \frac{1+\cos 2t}{2}, \\
f(t) &= \frac{1-\cos 2t}{2} + 3\left(\frac{1+\cos 2t}{2}\right), \\
&= 2 + \cos 2t. \\
\mathcal{L}\{f(t)\} &= \mathcal{L}\{2\} + \mathcal{L}\{\cos 2t\} \\
&= \frac{2}{s} + \frac{s}{s^2+4}, \\
&= \frac{3s^2+8}{s(s^2+4)}.
\end{aligned}
$$

We can verify the answer using MATLAB:

>> laplace(sin(t)^2+3*cos(t)^2)

ans =

(8+3*s^2)/s/(s^2+4)

(c) We first show the result that division by time is equivalent to integration in the frequency domain. This can be done as follows,

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st} f(t)dt, \\ \int_s^\infty F(s)ds &= \int_s^\infty \left[\int_0^\infty e^{-st} f(t)dt\right] ds, \end{aligned}$$

Interchanging the order of integration,

$$\begin{aligned} \int_s^\infty F(s)ds &= \int_0^\infty \left[\int_s^\infty e^{-st} ds\right] f(t)dt, \\ \int_s^\infty F(s)ds &= \int_0^\infty \left[-\frac{1}{t}e^{-st}\right]_s^\infty f(t)dt, \\ &= \int_0^\infty \frac{f(t)}{t} e^{-st} dt. \end{aligned}$$

Using this result then,

$$\begin{aligned} \mathcal{L}\{\sin t\} &= \frac{1}{s^2+1}, \\ \mathcal{L}\left\{\frac{\sin t}{t}\right\} &= \int_s^\infty \frac{1}{\xi^2+1} d\xi, \\ &= \tan^{-1}(\infty) - \tan^{-1}(s), \\ &= \frac{\pi}{2} - \tan^{-1}(s), \\ &= \tan^{-1}\left(\frac{1}{s}\right). \end{aligned}$$

where a table of integrals was used and the last simplification follows from the related trigonometric identity.

(d)

$$f(t) = \sin t * \sin t.$$

Use the convolution Laplace transform property (Table A.1, entry 7),

$$\begin{aligned} \mathcal{L}\{\sin t * \sin t\} &= \left(\frac{1}{s^2+1}\right)\left(\frac{1}{s^2+1}\right), \\ &= \frac{1}{s^4+2s^2+1}. \end{aligned}$$

(e)

$$\begin{aligned} f(t) &= \int_0^t \cos(t-\tau)\sin\tau d\tau. \\ \mathcal{L}\{f(t)\} &= \mathcal{L}\left\{\int_0^t \cos(t-\tau)\sin\tau d\tau\right\} = \mathcal{L}\{\cos(t) * \sin(t)\}. \end{aligned}$$

This is just the definition of the convolution theorem,

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \frac{s}{s^2+1}\frac{1}{s^2+1}, \\ &= \frac{s}{s^4+2s^2+1}.\end{aligned}$$

6. Given that the Laplace transform of *f(t)* is *F(s)*, find the Laplace transform of the following:

   (a) $g(t) = f(t)\cos t$

   (b) $g(t) = \int_0^t \int_0^{t_1} f(\tau)d\tau dt_1$

   Solution:

   (a) First write $\cos t$ in terms of the related Euler identity (Eq. B.33),

   $$g(t) = f(t)\cos t = f(t)\frac{e^{jt}+e^{-jt}}{2} = \frac{1}{2}f(t)e^{jt} + \frac{1}{2}f(t)e^{-jt}.$$

   Then using entry 4 of Table A.1 we have,

   $$G(s) = \frac{1}{2}F(s-j) + \frac{1}{2}F(s+j) = \frac{1}{2}\left[F(s-j)+F(s+j)\right].$$

   (b) Let us define

   $$\widetilde{f}(t_1) = \int_0^{t_1} f(\tau)d\tau,$$

   then

   $$g(t) = \int_0^t \widetilde{f}(t_1)dt_1,$$

   and from entry 6 of Table A.1 we have

   $$L\{\widetilde{f}(t)\} = \widetilde{F}(s) = \frac{1}{s}F(s)$$

   and using the same result again, we have

   $$G(s) = \frac{1}{s}\widetilde{F}(s) = \frac{1}{s}\left[\frac{1}{s}F(s)\right] = \frac{1}{s^2}F(s).$$

7. Find the time function corresponding to each of the following Laplace transforms using partial fraction expansions:

   (a) $F(s) = \frac{2}{s(s+2)}$

   (b) $F(s) = \frac{10}{s(s+1)(s+10)}$

   (c) $F(s) = \frac{3s+2}{s^2+4s+20}$

(d) $F(s) = \frac{3s^2+9s+12}{(s+2)(s^2+5s+11)}$

(e) $F(s) = \frac{1}{s^2+4}$

(f) $F(s) = \frac{2(s+2)}{(s+1)(s^2+4)}$

(g) $F(s) = \frac{s+1}{s^2}$

(h) $F(s) = \frac{1}{s^6}$

(i) $F(s) = \frac{4}{s^4+4}$

(j) $F(s) = \frac{e^{-s}}{s^2}$

Solution:

(a) Perform partial fraction expansion,

$$\begin{aligned}
F(s) &= \frac{2}{s(s+2)}, \\
&= \frac{C_1}{s} + \frac{C_2}{s+2}. \\
C_1 &= \frac{2}{s+2}|_{s=0} = 1, \\
C_2 &= \frac{2}{s}|_{s=-2} = -1, \\
F(s) &= \frac{1}{s} - \frac{1}{s+2}. \\
\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}, \\
f(t) &= 1(t) - e^{-2t}1(t).
\end{aligned}$$

We can verify the answer using MATLAB:

```
>> ilaplace(2/(s*(s+2)))
ans =
1-exp(-2*t)
```

(b) Perform partial fraction expansion,

$$\begin{aligned}
F(s) &= \frac{10}{s(s+1)(s+10)}, \\
&= \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3}{s+10}. \\
C_1 &= \frac{10}{(s+1)(s+10)}|_{s=0} = 1, \\
C_2 &= \frac{10}{s(s+10)}|_{s=-1} = -\frac{10}{9}, \\
C_3 &= \frac{10}{s(s+1)}|_{s=-10} = \frac{1}{9}, \\
F(s) &= \frac{1}{s} - \frac{\frac{10}{9}}{s+1} + \frac{\frac{1}{9}}{s+10}, \\
f(t) &= \mathcal{L}^{-1}\{F(s)\} = 1(t) - \frac{10}{9}e^{-t}1(t) + \frac{1}{9}e^{-10t}1(t).
\end{aligned}$$

We can verify the answer using MATLAB:
>> ilaplace(10/(s*(s+1)*(s+10)))
ans =
-10/9*exp(-t)+1+1/9*exp(-10*t)

(c) Re-write and carry out partial fraction expansion,

$$\begin{aligned}
F(s) &= \frac{3s+2}{s^2+4s+20}, \\
&= 3\frac{(s+2) - \frac{4}{3}}{(s+2)^2+4^2}, \\
&= \frac{3(s+2)}{(s+2)^2+4^2} - \frac{4}{(s+2)^2+4^2}, \\
f(t) &= \mathcal{L}^{-1}\{F(s)\} = (3e^{-2t}\cos 4t - e^{-2t}\sin 4t)1(t).
\end{aligned}$$

We can verify the answer using MATLAB:
>> ilaplace((3*s+2)/(s^2+4*s+20))
ans =
exp(-2*t)*(3*cos(4*t)-sin(4*t))

(d) Perform partial fraction expansion,

$$\begin{aligned}
F(s) &= \frac{3s^2+9s+12}{(s+2)(s^2+5s+11)} \\
&= \frac{C_1}{s+2} + \frac{C_2 s + C_3}{s^2+5s+11} \\
C_1 &= \frac{3(s^2+3s+4)}{(s^2+5s+11)}|_{s=-2} = \frac{6}{5}.
\end{aligned}$$

Equate numerators:

$$\frac{\frac{6}{5}}{(s+2)} + \frac{C_2 s + C_3}{(s^2+5s+11)} = \frac{3s^2+9s+12}{(s+2)(s^2+5s+11)},$$
$$(C_2 + \frac{6}{5})s^2 + (6 + C_3 + 2C_2)s + (2C_3 + \frac{66}{5}) = 3s^2 + 9s + 12.$$

Equate like powers of s to find C_2 and C_3:

$$\begin{aligned}
C_2 + \frac{6}{5} &= 3 \Rightarrow C_2 = \frac{9}{5}, \\
2C_3 + \frac{66}{5} &= 12 \Rightarrow C_3 = -\frac{3}{5}, \\
F(s) &= \frac{\frac{6}{5}}{(s+2)} + \frac{\frac{9}{5}s - \frac{3}{5}}{(s^2+5s+11)}, \\
&= \frac{\frac{6}{5}}{(s+2)} + \frac{9}{5}\frac{s+\frac{5}{2}}{\left(s+\frac{5}{2}\right)^2 + \frac{19}{4}} - \frac{9}{5}\frac{17\sqrt{19}}{57}\frac{\frac{\sqrt{19}}{2}}{\left(s+\frac{5}{2}\right)^2 + \left(\frac{\sqrt{19}}{2}\right)^2}. \\
f(t) &= \mathcal{L}^{-1}\{F(s)\} = \left(\frac{6}{5}e^{-2t} + \frac{9}{5}e^{-\frac{5}{2}t}\cos\frac{\sqrt{19}}{2}t - \frac{153\sqrt{19}}{285}e^{-\frac{5}{2}t}\sin\frac{\sqrt{19}}{2}t\right)1(t).
\end{aligned}$$

We can verify the answer using MATLAB:

```
>> ilaplace((3*s^2+9*s+12)/((s+2)*(s^2+5*s+11)))
ans =
6/5*exp(-2*t)+3/95*exp(-5/2*t)*(57*cos(1/2*19^(1/2)*t)-17*19^(1/2)*sin(1/2*19^(1/2)*t))
```

(e) Re-write and use entry #17 of Table A.2,

$$\begin{aligned}
F(s) &= \frac{1}{s^2+4}. \\
&= \frac{1}{2}\frac{2}{(s^2+2^2)}. \\
f(t) &= \frac{1}{2}\sin 2t.
\end{aligned}$$

We can verify the answer using MATLAB:

```
>> ilaplace(1/(s^2+4))
ans =
1/2*sin(2*t)
```

(f)

$$\begin{aligned}
F(s) &= \frac{2(s+2)}{(s+1)(s^2+4)}. \\
&= \frac{C_1}{(s+1)} + \frac{C_2 s + C_3}{(s^2+4)}. \\
C_1 &= \frac{2(s+2)}{(s^2+4)}|_{s=-1} = \frac{2}{5}.
\end{aligned}$$

Equate numerators and like powers of s terms:

$$\begin{aligned}
\left(\frac{2}{5} + C_2\right)s^2 + (C_2 + C_3)s + \left(\frac{8}{5} + C_3\right) &= 2s + 4, \\
\frac{8}{5} + C_3 &= 4 \qquad \Rightarrow C_3 = \frac{12}{5}, \\
C_2 + C_3 &= 2 \qquad \Rightarrow C_2 = -\frac{2}{5}, \\
\frac{2}{5} + C_2 &= 0.
\end{aligned}$$

$$\begin{aligned}
F(s) &= \frac{\frac{2}{5}}{(s+1)} + \frac{-\frac{2}{5}s + \frac{12}{5}}{(s^2+4)}, \\
&= \frac{\frac{2}{5}}{(s+1)} + \frac{-\frac{2}{5}s}{(s^2+2^2)} + \frac{6}{5}\frac{2}{(s^2+2^2)}. \\
f(t) &= \frac{2}{5}e^{-t} - \frac{2}{5}\cos 2t + \frac{6}{5}\sin 2t.
\end{aligned}$$

We can verify the answer using MATLAB:

```
>> ilaplace(2*(s+2)/((s+1)*(s^2+4)))
ans =
-4/5*cos(t)^2+12/5*sin(t)*cos(t)+2/5+2/5*exp(-t)
```

(g) Perform partial fraction expansion,

$$\begin{aligned}
F(s) &= \frac{s+1}{s^2}, \\
&= \frac{1}{s} + \frac{1}{s^2}. \\
f(t) &= (1+t)1(t).
\end{aligned}$$

We can verify the answer using MATLAB:

```
>> ilaplace((s+1)/(s^2))
ans =
t+1
```

(h) Use entry #6 of Table A.2,

$$\begin{aligned} F(s) &= \frac{1}{s^6}, \\ f(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^6}\right\} = \frac{t^5}{5!} = \frac{t^5}{120}. \end{aligned}$$

We can verify the answer using MATLAB:
```
>> ilaplace(1/s^6)
ans =
1/120*t^5
```

(i) Re-write as,

$$\begin{aligned} F(s) &= \frac{4}{s^4+4}, \\ &= \frac{\frac{1}{2}s+1}{s^2+2s+2} + \frac{-\frac{1}{2}s+1}{s^2-2s+2}, \\ &= \frac{(s+1)-\frac{1}{2}s}{(s+1)^2+1} - \frac{(s-1)-\frac{1}{2}s}{(s-1)^2+1}. \end{aligned}$$

Use Table A.2 entry #19 and Table A.1 entry #5,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} = e^{-t}\cos(t) - \frac{1}{2}\frac{d}{dt}\left\{e^{-t}\sin(t)\right\} - e^{t}\cos(t), \\ & -\frac{1}{2}\frac{d}{dt}\{e^{t}\sin(t)\}, \\ &= e^{-t}\cos(t) - \frac{1}{2}\{-e^{-t}\sin(t) + \cos(t)e^{-t}\} \\ & -e^{t}\cos(t) + \frac{1}{2}\{e^{t}\sin(t) + \cos(t)e^{t}\}, \\ &= -\cos(t)\left\{\frac{-e^{-t}+e^{t}}{2}\right\} + \sin(t)\left\{\frac{-e^{-t}+e^{t}}{2}\right\}, \\ f(t) &= -\cos(t)\sinh(t) + \sin(t)\cosh(t). \end{aligned}$$

We can verify the answer using MATLAB:
```
>> ilaplace(4/(s^4+4))
ans =
cosh(t)*sin(t)-sinh(t)*cos(t)
```

(j) Using entry #2 of Table A.1,

$$\begin{aligned} F(s) &= \frac{e^{-s}}{s^2}. \\ f(t) &= \mathcal{L}^{-1}\{F(s)\} = (t-1)1(t-1). \end{aligned}$$

We can verify the answer using MATLAB:
>> ilaplace(exp(-s)/(s^2))
ans =
heaviside(t-1)*(t-1)

8. Find the time function corresponding to each of the following Laplace transforms:

(a) $F(s) = \frac{1}{s(s+2)^2}$

(b) $F(s) = \frac{2s^2+s+1}{s^3-1}$

(c) $F(s) = \frac{2(s^2+s+1)}{s(s+1)^2}$

(d) $F(s) = \frac{s^3+2s+4}{s^4-16}$

(e) $F(s) = \frac{2(s+2)(s+5)^2}{(s+1)(s^2+4)^2}$

(f) $F(s) = \frac{(s^2-1)}{(s^2+1)^2}$

(g) $F(s) = \tan^{-1}(\frac{1}{s})$

Solution:

(a) Perform partial fraction expansion,

$$\begin{aligned}
F(s) &= \frac{1}{s(s+2)^2}, \\
&= \frac{C_1}{s} + \frac{C_2}{(s+2)} + \frac{C_3}{(s+2)^2}. \\
C_1 &= sF(s)|_{s=0} = \frac{1}{(s+2)^2}|_{s=0} = \frac{1}{4}, \\
C_3 &= (s+2)^2F(s)|_{s=-2} = \frac{1}{s}|_{s=-2} = -\frac{1}{2}, \\
C_2 &= \frac{d}{ds}[(s+2)^2F(s)]_{s=-2}, \\
&= \frac{d}{ds}[s^{-1}]_{s=-2}, \\
&= -\frac{1}{s^2}|_{s=-2}, \\
&= -\frac{1}{4}, \\
F(s) &= \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}}{(s+2)} + \frac{-\frac{1}{2}}{(s+2)^2}. \\
f(t) &= \mathcal{L}^{-1}\{F(s)\} = \left(\frac{1}{4} - \frac{1}{4}e^{-2t} - \frac{1}{2}te^{-2t}\right)1(t).
\end{aligned}$$

We can verify the answer using MATLAB:
>> ilaplace(1/(s*(s+2)^2))
ans =
1/4-1/4*exp(-2*t)*(1+2*t)

(b) Perform partial fraction expansion,

$$\begin{aligned}
F(s) &= \frac{2s^2+s+1}{s^3-1},\\
&= \frac{2s^2+s+1}{(s-1)(s^2+s+1)},\\
&= \frac{C_1}{s-1}+\frac{C_2 s+C_3}{s^2+s+1}.\\
C_1 &= (s-1)F(s)|_{s=1} = \frac{2s^2+s+1}{s^2+s+1}|_{s=1} = \frac{4}{3}.
\end{aligned}$$

Equate numerators and match the coefficients of like powers of s:

$$\begin{aligned}
\frac{\frac{4}{3}}{s-1}+\frac{C_2 s+C_3}{s^2+s+1} &= \frac{2s^2+s+1}{(s-1)(s^2+s+1)},\\
s^2(\frac{4}{3}+C_2)+s(\frac{4}{3}-C_2+C_3)+(\frac{4}{3}-C_3) &= 2s^2+s+1,\\
\frac{4}{3}+C_2 &= 2 \qquad \Rightarrow C_2=\frac{2}{3},\\
\frac{4}{3}-C_3 &= 1 \qquad \Rightarrow C_3=\frac{1}{3}.
\end{aligned}$$

$$\begin{aligned}
F(s) &= \frac{\frac{4}{3}}{s-1}+\frac{\frac{2}{3}s+\frac{1}{3}}{s^2+s+1},\\
&= \frac{\frac{4}{3}}{s-1}+\frac{2}{3}\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2},\\
f(t) &= \mathcal{L}^{-1}\{F(s)\} = \frac{4}{3}e^t+\frac{2}{3}e^{-\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t,\\
&= \frac{2}{3}\left\{2e^t+e^{-\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t\right\}1(t).
\end{aligned}$$

We can verify the answer using MATLAB:
>> ilaplace((2*s^2+s+1)/(s^3-1))
ans =
4/3*exp(t)+2/3*exp(-1/2*t)*cos(1/2*3^(1/2)*t)

(c) Carry out partial fraction expansion,

$$
\begin{aligned}
F(s) &= \frac{2(s^2+s+1)}{s\,(s+1)^2}, \\
&= \frac{C_1}{s} + \frac{C_2}{(s+1)} + \frac{C_3}{(s+1)^2}. \\
C_1 &= sF(s)|_{s=0} = \frac{2(s^2+s+1)}{(s+1)^2}|_{s=0} = 2, \\
C_3 &= (s+1)^2F(s)|_{s=-1} = \frac{2(s^2+s+1)}{s}|_{s=-1} = -2, \\
C_2 &= \frac{d}{ds}[(s+1)^2F(s)]_{s=-1}, \\
&= \frac{d}{ds}[\frac{2(s^2+s+1)}{s}]_{s=-1}, \\
&= \frac{2(2s+1)s-2(s^2+s+1)}{s^2}|_{s=-1}, \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
F(s) &= \frac{2}{s} + \frac{0}{(s+1)} + \frac{-2}{(s+1)^2}. \\
f(t) &= \mathcal{L}^{-1}\{F(s)\} = 2\{1-te^{-t}\}1(t).
\end{aligned}
$$

We can verify the answer using MATLAB:

```
>> ilaplace((2*s^2+2*s+2)/(s*(s+1)^2))
ans =
2-2*t*exp(-t)
```

(d) Carry out partial fraction expansion,

$$
\begin{aligned}
F(s) &= \frac{s^3+2s+4}{s^4-16} = \frac{As+B}{s^2-4} + \frac{Cs+D}{s^2+4} = \frac{\frac{3}{4}s+\frac{1}{2}}{s^2-4} + \frac{\frac{1}{4}s-\frac{1}{2}}{s^2+4}, \\
&= \frac{1}{4}\sinh(2t) + \frac{3}{4}\frac{d}{dt}\left\{\frac{1}{2}\sinh(2t)\right\} - \frac{1}{4}\sin(2t) - \frac{1}{4}\frac{d}{dt}\left\{\frac{1}{2}\sin(2t)\right\}, \\
&= \frac{1}{4}\sinh(2t) + \frac{3}{4}\cosh(2t) - \frac{1}{4}\sin(2t) + \frac{1}{4}\cos(2t).
\end{aligned}
$$

We can verify the answer using MATLAB:

```
>> ilaplace((s^3+2*s+4)/(s^4-16))
ans =
-1/4*sin(2*t)+1/2*exp(2*t)+1/4*exp(-2*t)+1/4*cos(2*t)
```

(e) Expand in partial fraction expansion and compute the residues using the results from Appendix A,

$$\begin{aligned}
F(s) &= \frac{2(s+2)(s+5)^2}{(s+1)(s^2+4)^2}, \\
&= \frac{C_1}{s+1} + \frac{C_2}{s-2j} + \frac{C_3}{s+2j} + \frac{C_4}{(s-2j)^2} + \frac{C_5}{(s+2j)^2}. \\
C_1 &= (s+1)F(s)|_{s=-1} = \frac{32}{25} = 1.280, \\
C_4 &= (s-2j)^2F(s)|_{s=2j} = \frac{-83-39j}{20} = -4.150 - j1.950, \\
C_5 &= C_4^* = -4.150 + j1.950, \\
C_2 &= \frac{d}{ds}\left[(s-2j)^2F(s)\right]_{s=2j} = \frac{-128-579j}{200}, \\
&= -0.64 - j2.895, \\
C_3 &= C_2^* = -0.64 + j2.895.
\end{aligned}$$

These results can also be verified with the MATLAB residue command,

```
a =[1 1 8 8 16 16];
b =[2 24 90 100];
[r,p,k]=residue(b,a)
r =
-0.64000000000000 - 2.89500000000002i
-4.15000000000002 - 1.95000000000000i
-0.64000000000000 + 2.89500000000002i
-4.15000000000002 + 1.95000000000000i
1.28000000000001
p =
0.00000000000000 + 2.00000000000000i
0.00000000000000 + 2.00000000000000i
0.00000000000000 - 2.00000000000000i
0.00000000000000 - 2.00000000000000i
-1.00000000000000
k =
[]
```

We then have,

$$\begin{aligned}
f(t) &= 1.28e^{-t} + 2|C_2|\cos(2t + \arg C_2) + 2|C_4|t\cos(2t + \arg C_4), \\
&= 1.28e^{-t} + 5.92979\cos(2t - 1.788) + 9.1706t\cos(2t - 2.702).
\end{aligned}$$

where

$$|C_2| = 2.96489, \quad |C_4| = 4.5853, \quad \arg C_2 = \tan^{-1}\left(\frac{-2.895}{-0.64}\right) = -1.788,$$

using the atan2 command in MATLAB, and

$$\arg C_4 = \tan^{-1}\left(\frac{-1.950}{-4.150}\right) = -2.702,$$

also using the atan2 command in MATLAB.

(f)

$$F(s) = \frac{(s^2-1)}{(s^2+1)^2}.$$

Using the multiplication by time Laplace transform property (Table A.1 entry #11):

$$-\frac{d}{ds}G(s) = \mathcal{L}\{tg(t)\}.$$

We can see that

$$-\frac{d}{ds}\left[\frac{s}{(s^2+1)}\right] = \frac{s^2-1}{(s^2+1)^2}.$$

So the inverse Laplace transform of $F(s)$ is:

$$\mathcal{L}^{-1}\{F(s)\} = t\cos t.$$

We can verify the answer using MATLAB:
>> ilaplace((s^2-1)/(s^2+1)^2)
ans =
t*cos(t)

(g) Follows from Problem 5 (c), or expand in series,

$$\tan^{-1}(\frac{1}{s}) = \frac{1}{s} - \frac{1}{3s^3} + \frac{1}{5s^5} - ...$$

Then,

$$\mathcal{L}^{-1}\left\{\tan^{-1}(\frac{1}{s})\right\} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - ..... = \frac{\sin(t)}{t}.$$

Alternatively, let us assume

$$\mathcal{L}^{-1}\left\{\tan^{-1}(\frac{1}{s})\right\} = f(t).$$

We use the identity

$$\frac{d}{ds}\left[\tan^{-1}s\right] = \frac{1}{1+s^2},$$

which means that

$$\mathcal{L}^{-1}\left\{-\frac{1}{s^2+1}\right\} = -tf(t) = -\sin(t).$$

Therefore,

$$f(t) = \frac{\sin(t)}{t}.$$

We can verify the answer using MATLAB:
>> ilaplace(atan(1/s))
ans =
1/t*sin(t)

9. Solve the following ordinary differential equations using Laplace transforms:

   (a) $\ddot{y}(t) + \dot{y}(t) + 3y(t) = 0; y(0) = 1,\ \dot{y}(0) = 2$
   (b) $\ddot{y}(t) - 2\dot{y}(t) + 4y(t) = 0; y(0) = 1,\ \dot{y}(0) = 2$
   (c) $\ddot{y}(t) + \dot{y}(t) = \sin t; y(0) = 1,\ \dot{y}(0) = 2$
   (d) $\ddot{y}(t) + 3y(t) = \sin t; y(0) = 1,\ \dot{y}(0) = 2$
   (e) $\ddot{y}(t) + 2\dot{y}(t) = e^t; y(0) = 1,\ \dot{y}(0) = 2$
   (f) $\ddot{y}(t) + y(t) = t; y(0) = 1,\ \dot{y}(0) = -1$

Solution:

(a)

$$\ddot{y}(t) + \dot{y}(t) + 3y(t) = 0; \qquad y\,(0) = 1, \qquad \dot{y}\,(0) = 2$$

Using Table A.1 entry #5, the differentiation Laplace transform property,

$$s^2 Y\,(s) - sy\,(0) - \dot{y}\,(0) + sY\,(s) - y\,(0) + 3Y\,(s) = 0$$

$$\begin{aligned} Y\,(s) &= \frac{s+3}{s^2+s+3}, \\ &= \frac{\left(s+\frac{1}{2}\right)+\frac{5}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{11}{4}}, \\ &= \frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\frac{11}{4}} + \frac{5\sqrt{11}}{11}\frac{\sqrt{\frac{11}{4}}}{\left(s+\frac{1}{2}\right)^2+\frac{11}{4}}. \end{aligned}$$

Using Table A.2 entries #19 and #20,

$$y(t) = e^{-\frac{1}{2}t}\cos\frac{\sqrt{11}}{2}t + \frac{5\sqrt{11}}{11}e^{-\frac{1}{2}t}\sin\frac{\sqrt{11}}{2}t.$$

We can verify the answer using MATLAB:
>> dsolve('D2y+Dy+3*y=0','y(0)=1','Dy(0)=2','t')
ans =
5/11*11^(1/2)*exp(-1/2*t)*sin(1/2*11^(1/2)*t)+exp(-1/2*t)*cos(1/2*11^(1/2)*t)

(b)

$$\ddot{y}(t) - 2\dot{y}(t) + 4y(t) = 0; y(0) = 1, \dot{y}(0) = 2.$$
$$s^2Y(s) - sy(0) - \dot{y}(0) - 2sY(s) + 2y(0) + 4Y(s) = 0.$$

$$\begin{aligned} Y(s) &= \frac{s}{s^2 - 2s + 4}, \\ &= \frac{s}{(s-1)^2 + 3}, \end{aligned}$$

Using Table A.1 entry #5 and Table A.2 entry #20,

$$\begin{aligned} y(t) &= \frac{d}{dt}\left[e^t \sin\sqrt{3}t\right] \\ y(t) &= \frac{1}{\sqrt{3}}e^t \sin\sqrt{3}t + e^t \cos\sqrt{3}t \end{aligned}$$

We can verify the answer using MATLAB:

```
>> dsolve('D2y-2*Dy+4*y=0','y(0)=1','Dy(0)=2','t')
ans =
1/3*3^(1/2)*exp(t)*sin(3^(1/2)*t)+exp(t)*cos(3^(1/2)*t)
```

(c)

$$\ddot{y}(t) + \dot{y}(t) = \sin t; \quad y(0) = 1, \quad \dot{y}(0) = 2$$
$$s^2Y(s) - sy(0) - \dot{y}(0) + sY(s) - y(0) = \frac{1}{s^2+1}$$

$$\begin{aligned} Y(s) &= \frac{s^3 + 3s^2 + s + 4}{s(s+1)(s^2+1)}, \\ &= \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3 s + C_4}{s^2+1}. \end{aligned}$$

$$\begin{aligned} C_1 &= \frac{s^3 + 3s^2 + s + 4}{(s+1)(s^2+1)}\Big|_{s=0} = 4, \\ C_2 &= \frac{s^3 + 3s^2 + s + 4}{s(s^2+1)}\Big|_{s=-1} = -\frac{5}{2}. \end{aligned}$$

$$\begin{aligned} \frac{4}{s} + \frac{-\frac{5}{2}}{s+1} + \frac{C_3 s + C_4}{s^2+1} &= \frac{s^3 + 3s^2 + s + 4}{s(s+1)(s^2+1)} \\ s^3\left(\frac{3}{2} + C_3\right) + s^2(4 + C_3 + C_4) + s\left(\frac{3}{2} + C_4\right) + 4 &= s^3 + 3s^2 + s + 4. \end{aligned}$$

Match coefficients of like powers of s

$$\begin{aligned} C_4 + \frac{3}{2} &= 1 \qquad \Longrightarrow C_4 = -\frac{1}{2}, \\ C_3 + \frac{3}{2} &= 1 \qquad \Longrightarrow C_3 = -\frac{1}{2}. \end{aligned}$$

$$\frac{4}{s}+\frac{-\frac{5}{2}}{s+1}+\frac{-\frac{1}{2}s-\frac{1}{2}}{s^2+1}=\frac{4}{s}+\frac{-\frac{5}{2}}{s+1}-\frac{1}{2}\frac{s}{s^2+1}-\frac{1}{2}\frac{1}{s^2+1}.$$

Using Table A.2 entries #2, #7, #17, and #18

$$y(t)=4-\frac{5}{2}e^{-t}-\frac{1}{2}\cos t-\frac{1}{2}\sin t.$$

We can verify the answer using MATLAB:

```
>> dsolve('D2y+Dy-sin(t)=0','y(0)=1','Dy(0)=2','t')
ans =
-1/2*sin(t)-1/2*cos(t)-5/2*exp(-t)+4
```

(d)

$$\ddot{y}(t)+3y(t)=\sin t;\quad y(0)=1,\quad \dot{y}(0)=2,$$

$$s^2Y(s)-sy(0)-\dot{y}(0)+3Y(s)=\frac{1}{s^2+1},$$

$$\begin{aligned} Y(s) &= \frac{s^3+2s^2+s+3}{(s^2+3)(s^2+1)},\\ &= \frac{C_1s+C_2}{s^2+3}+\frac{C_3s+C_4}{s^2+1}. \end{aligned}$$

$$\frac{(C_1s+C_2)\left(s^2+1\right)+(C_3s+C_4)\left(s^2+3\right)}{(s^2+3)(s^2+1)}=\frac{s^3+2s^2+s+3}{(s^2+3)(s^2+1)}.$$

Match coefficients of like powers of s:

$$s^3(C_1+C_3)+s^2(C_2+C_4)+s(C_1+3C_3)+(C_2+3C_4)=s^3+2s^2+s+3,$$

$$\begin{aligned} C_1+C_3 &= 1 && \Longrightarrow C_1=-C_3+1,\\ C_2+C_4 &= 2 && \Longrightarrow C_2=2-C_4,\\ C_1+3C_3 &= 1 && \Longrightarrow -C_3+1+3C_3=1 \qquad \Longrightarrow C_3=0,\\ &\Longrightarrow C_1=1,\\ C_2+3C_4 &= 3 && \Longrightarrow (2-C_4)+3C_4=3 \qquad \Longrightarrow C_4=\frac{1}{2},\\ &\Longrightarrow C_2=\frac{3}{2}, \end{aligned}$$

$$\begin{aligned} Y(s) &= \frac{\frac{1}{2}s+\frac{3}{2}}{s^2+3}+\frac{\frac{1}{2}}{s^2+1},\\ &= \frac{\frac{1}{2}}{s^2+3}+\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{s^2+3}+\frac{1}{2}\frac{1}{s^2+1}. \end{aligned}$$

$$y(t)=\frac{1}{2}\cos\sqrt{3}t+\frac{\sqrt{3}}{2}\sin\sqrt{3}t+\frac{1}{2}\sin t.$$

We can verify the answer using MATLAB:

```
>> dsolve('D2y+3*y-sin(t)=0','y(0)=1','Dy(0)=2','t')
ans =
1/2*sin(3^(1/2)*t)*3^(1/2)+cos(3^(1/2)*t)+1/2*sin(t)
```

(e)

$$\ddot{y}(t) + 2\dot{y}(t) = e^t; \quad y(0) = 1, \quad \dot{y}(0) = 2$$

$$s^2Y(s) - sy(0) - \dot{y}(0) + 2sY(s) - 2y(0) = \frac{1}{s-1}$$

$$\begin{aligned} Y(s) &= \frac{s^2+3s-3}{s(s-1)(s+2)}, \\ &= \frac{C_1}{s} + \frac{C_2}{s-1} + \frac{C_3}{s+2}. \end{aligned}$$

$$\begin{aligned} C_1 &= \frac{s^2+3s-3}{(s-1)(s+2)}|_{s=0} = \frac{3}{2}, \\ C_2 &= \frac{s^2+3s-3}{s(s+2)}|_{s=1} = \frac{1}{3}, \\ C_3 &= \frac{s^2+3s-3}{s(s-1)}|_{s=-2} = -\frac{5}{6}, \end{aligned}$$

$$Y(s) = \frac{\frac{3}{2}}{s} + \frac{1}{3}\frac{1}{s-1} - \frac{5}{6}\frac{1}{s+2}.$$

$$y(t) = \frac{3}{2} + \frac{1}{3}e^t - \frac{5}{6}e^{-2t}.$$

We can verify the answer using MATLAB:

```
>> dsolve('D2y+2*Dy-exp(t)=0','y(0)=1','Dy(0)=2','t')
ans =
1/3*exp(t)-5/6*exp(-2*t)+3/2
```

(f) Using the results from Appendix A,

$$\ddot{y}(t) + y(t) = t; \quad y(0) = 1, \quad \dot{y}(0) = -1,$$

$$s^2Y(s) - sy(0) - \dot{y}(0) + Y(s) = \frac{1}{s^2},$$

$$\begin{aligned} Y(s) &= \frac{s^3-s^2+1}{s^2(s^2+1)}, \\ &= \frac{C_1}{s} + \frac{C_2}{s^2} + \frac{C_3 s + C_4}{s^2+1}. \end{aligned}$$

$$C_1 = \frac{d}{ds}\frac{\left(s^3 - s^2 + 1\right)}{(s^2+1)}|_{s=0} = 0,$$
$$C_2 = \frac{\left(s^3 - s^2 + 1\right)}{(s^2+1)}|_{s=0} = 1.$$

$$\frac{1}{s^2} + \frac{C_3 s + C_4}{s^2+1} = \frac{s^3 - s^2 + 1}{s^2\left(s^2+1\right)},$$
$$\frac{\left(s^2+1\right) + \left(C_3 s + C_4\right)s^2}{s^2\left(s^2+1\right)} = \frac{s^3 - s^2 + 1}{s^2\left(s^2+1\right)}.$$

Match coefficients of like powers of s:

$$C_3 = 1$$
$$C_4 + 1 = -1 \qquad \Longrightarrow C_4 = -2$$

$$Y(s) = \frac{1}{s^2} + \frac{s}{s^2+1} - 2\frac{1}{s^2+1}$$
$$y(t) = t + \cos t - 2\sin t.$$

We can verify the answer using MATLAB:

```
>> dsolve('D2y+y-t=0','y(0)=1','Dy(0)=-1','t')
ans =
-2*sin(t)+cos(t)+t
```

10. Using the convolution integral, find the step response of the system whose impulse response is given below and shown in Figure 3.47:

$$h(t) = \begin{cases} te^{-t} & t \geq 0 \\ 0 & t < 0. \end{cases}$$

Solution: There are only two cases to consider.

Case (a): For the case $t \leq 0$, the situation is illustrated in the following Figure part (c). There is no overlap between the two functions ($u(t-\tau)$ and $h(\tau)$) and the output is zero

$$y_1(t) = 0.$$

Case (b): For the case $t \geq 0$, the situation is displayed in the following Figure part (d). The output of the system is given by

$$y_2(t) = \int_0^t h(\tau)u(t-\tau)d\tau = \int_0^t (\tau e^{-\tau})(1)d\tau = 1 - (t+1)e^{-t}.$$

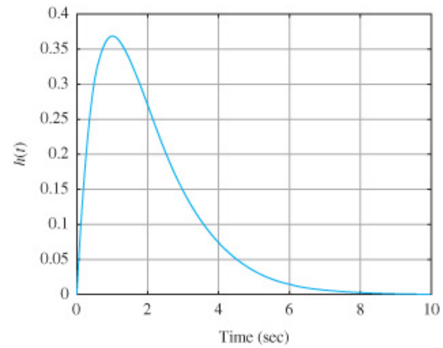


Figure 3.47: Impulse response for Problem 3.10.

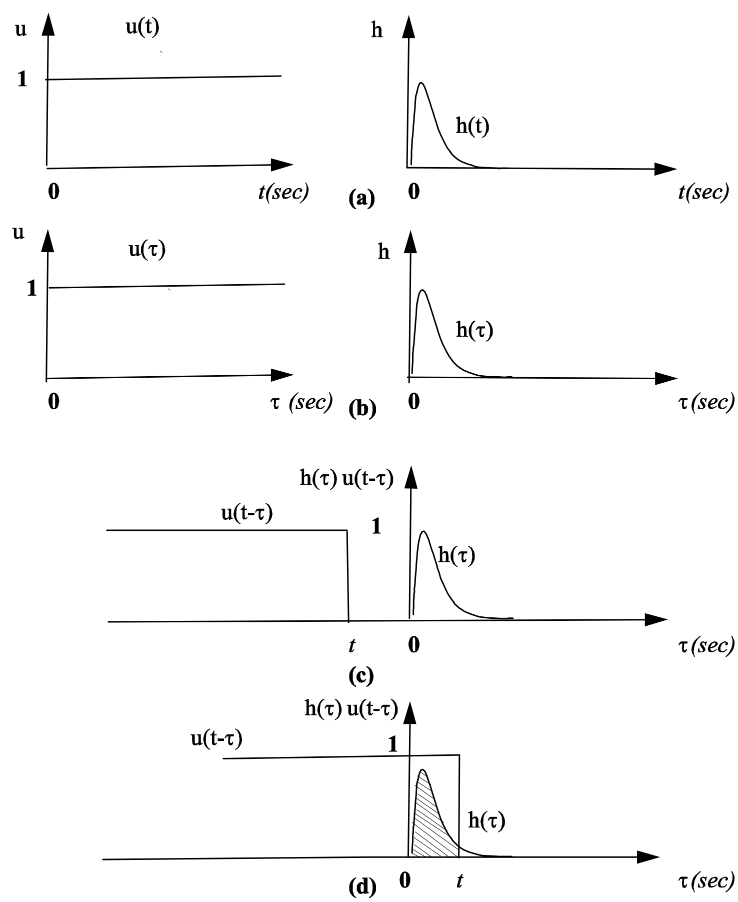


Illustration of convolution.

The output of the system is the composite of the two segments computed

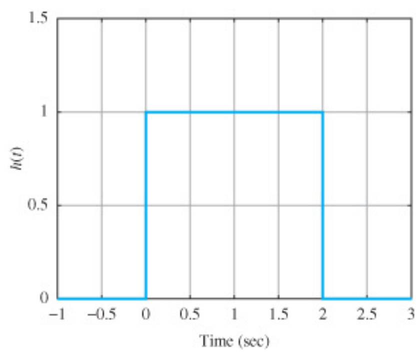
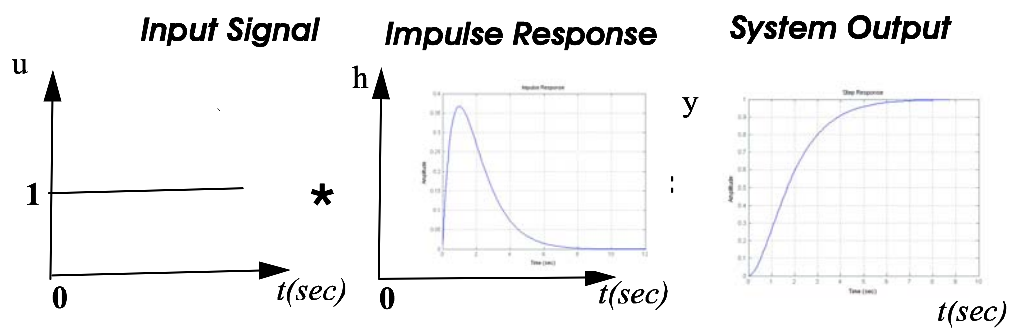


Figure 3.48: Impulse response for Problem 3.11.

above as shown in the following Figure.



System output response.

11. Using the convolution integral, find the step response of the system whose impulse response is given below and shown in Figure 3.48:

$$h(t) = \begin{cases} 1 & 0 \leq t \leq 2 \\ 0 & t < 0 \text{ and } t > 2 \end{cases}$$

Solution: There are three cases to consider as shown in the following figure.

Case (a): For the case $t \leq 0$, the situation is illustrated in the following Figure part (c). There is no overlap between the two functions ($u(t-\tau)$ and $h(\tau)$) and the output is zero

$$y_1(t) = 0$$

Case (b): For the case $0 \geqslant t \geq 2$, the situation is displayed in the following Figure part (d) and shows partial overlap. The output of the system is given by

$$y_2(t) = \int_0^t h(\tau)u(t-\tau)d\tau = \int_0^t (1)(1)d\tau = t.$$

Case (c): For the case $t \geq 2$, the situation is displayed in the following Figure part (e) and shows total overlap. The output of the system is given by

$$y_3(t) = \int_0^t h(\tau)u(t-\tau)d\tau = \int_0^2 (1)(1)d\tau = 2.$$

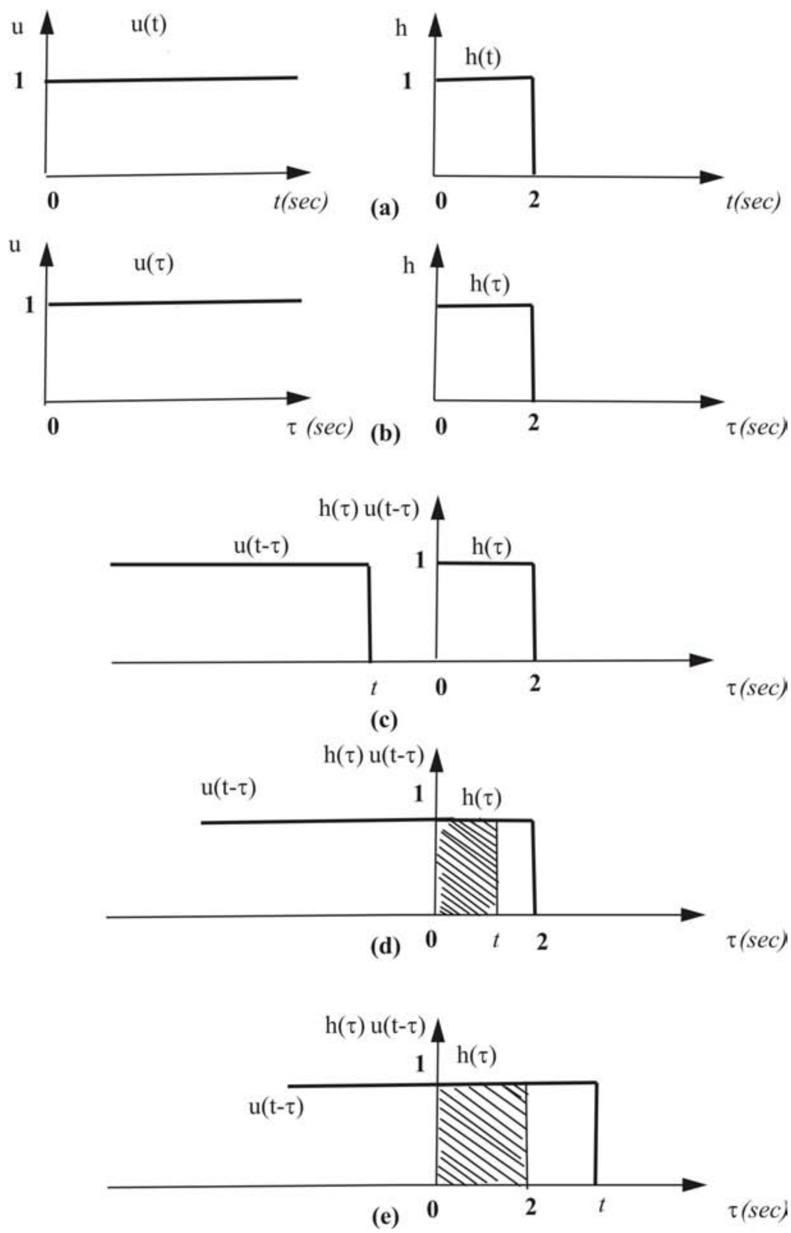
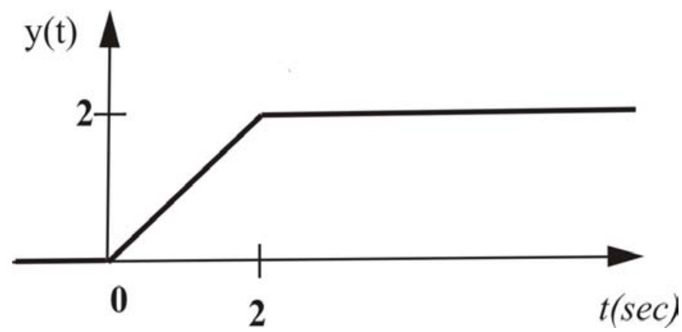


Illustration of convolution.

The output of the system is the composite of the three segments computed above as shown in the following figure.



System output response.

12. Consider the standard second-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

a) Write the Laplace transform of the signal in Fig. 3.49. b). What is the transform of the output if this signal is applied to $G(s)$. c) Find the output of the system for the input shown in Fig. 3.49.

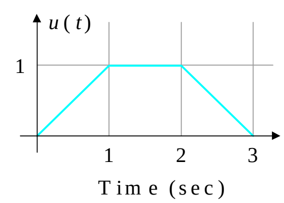


Figure 3.49: Plot of input for Problem 3.12

Solution:

(a) The input signal in Figure 3.40 may be written as:

$$u(t) = t - t[1(t-1)] - t[1((t-2)] + t[1(t-3)],$$

where $1(t-\tau)$ denotes a delayed unit step.
The Laplace transform of the input signal is:

$$U(s) = \frac{1}{s^2}\left(1 - e^{-s} - e^{-2s} - e^{-3s}\right).$$

(b) The Laplace transform of the output if this signal is applied is:

$$Y(s) = G(s)U(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\left(\frac{1}{s^2}\right)\left(1 - e^{-s} - e^{-2s} - e^{-3s}\right).$$

(c) However to make the mathematical manipulation easier, consider only the response of the system to a ramp input:

$$Y_1(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\left(\frac{1}{s^2}\right).$$

Partial fractions yields the following:

$$Y_1(s) = \frac{1}{s^2} - \frac{\frac{2\zeta}{\omega_n}}{s} + \frac{\frac{2\zeta}{\omega_n}\left(s + 2\zeta\omega_n - \frac{\omega_n}{2\zeta}\right)}{(s+\omega_n\zeta)^2 + \left(\omega_n\sqrt{1-\zeta^2}\right)^2}.$$

Use the following Laplace transform pairs for the case $0 \le \zeta < 1$:

$$\mathcal{L}^{-1}\left\{\frac{s+z_1}{(s+a)^2+\omega^2}\right\} = \sqrt{\frac{(z_1-a)^2+\omega^2}{\omega^2}}e^{-at}\sin(\omega t + \phi),$$

where

$$\phi \equiv \tan^{-1}\left(\frac{\omega}{z_1 - a}\right).$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t \qquad \text{ramp}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1(t) \qquad \text{unit step}$$

and the following Laplace transform pairs for the case $\zeta = 1$:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+a)^2}\right\} = te^{-at}.$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+a)^2}\right\} = (1-at)e^{-at}.$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t \qquad \text{ramp,}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1(t) \qquad \text{unit step,}$$

the following is derived:

$$y_1(t) = \begin{cases} t - \frac{2\zeta}{\omega_n} + \frac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t + \tan^{-1}\frac{2\zeta\sqrt{1-\zeta^2}}{2\zeta^2-1}\right) & \begin{array}{c} 0 \le \zeta < 1 \\ t \ge 0 \end{array} \\ t - \frac{2}{\omega_n} + \frac{2}{\omega_n}e^{-\omega_n t}\left(\frac{\omega_n}{2}t + 1\right) & \begin{array}{c} \zeta = 1 \\ t \ge 0 \end{array} \end{cases}$$

Since $u(t)$ consists of a ramp and three delayed ramp signals, using superposition (the system is linear), then:

$$y(t) = y_1(t) - y_1(t-1) - y_1(t-2) + y_1(t-3) \qquad t \geq 0.$$

13. A rotating load is connected to a field-controlled DC motor with negligible field inductance. A test results in the output load reaching a speed of 1 rad/sec within 1/2 sec when a constant input of 100 V is applied to the motor terminals. The output steady-state speed from the same test is found to be 2 rad/sec. Determine the transfer function $\theta(s)/V_f(s)$ of the motor.

Solution:

Equations of motion for a DC motor:

$$J_m\ddot{\theta}_m + b\dot{\theta}_m = K_m i_a,$$

$$K_e\dot{\theta}_m + L_a\frac{di_a}{dt} + R_a i_a = v_a,$$

but since there's negligible field inductance $L_a = 0$.

Combining the above equations yields:

$$R_a J_m\ddot{\theta}_m + R_a b\dot{\theta}_m = K_t v_a - K_t K_e\dot{\theta}_m.$$

Applying Laplace transforms yields the following transfer function:

$$\frac{\theta(s)}{V_f(s)} = \frac{\frac{K_t}{J_m R_a}}{s(s + \frac{K_t K_e}{R_a J_m} + \frac{b}{J_m})} = \frac{K}{s(s+a)},$$

where $K = \frac{K_t}{J_m R_a}$ and $a = \frac{K_t K_e}{R_a J_m} + \frac{b}{J_m}$.

K and a are found using the given information:

$$\begin{aligned} V_f(s) &= \frac{100}{s} \text{ since } V_f(t) = 100V, \\ \dot{\theta}\left(\frac{1}{2}\right) &= 2 \text{ rad/sec.} \end{aligned}$$

For the given information we need to utilize $\dot{\theta}_m(t)$ instead of $\theta_m(t)$:

$$s\theta(s) = \frac{100K}{s(s+a)}.$$

Using the Final Value Theorem and assuming that the system is stable:

$$\lim_{s\to 0}\frac{100K}{s+a}=\lim_{s\to 0}\dot{\theta}\left(\frac{1}{2}\right)=2=\frac{100K}{a}.$$

Take the inverse Laplace transform:

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{100K}{a}\frac{a}{s(s+a)}\right\} &= \frac{100K}{a}(1-e^{-at})=2(1-e^{-at})=1,\\
0.5 &= e^{-\frac{a}{2}} \quad \text{yields} \quad a=1.39,\\
K &= \frac{2}{100}a \quad \text{yields} \quad K=0.0278,\\
\frac{\theta(s)}{V_f(s)} &= \frac{0.0278}{s(s+1.39)}.
\end{aligned}$$

14. A simplified sketch of a computer tape drive is given in Fig. 3.50

    (a) Write the equations of motion in terms of the parameters listed below. K and B represent the spring constant and the damping of tape stretch, respectively, and ω_1 and ω_2 are angular velocities. A positive current applied to the DC motor will provide a torque on the capstan in the clockwise direction as shown by the arrow. Find the value of current that just cancels the force, F, then eliminate the constant current and its balancing force, F, from your equations. Assume positive angular velocities of the two wheels are in the directions shown by the arrows.

$$\begin{aligned}
J_1 &= 5\times 10^{-5}\ \text{kg}\cdot\text{m}^2,\ \text{motor and capstan inertia}\\
B_1 &= 1\times 10^{-2}\ \text{N}\cdot\text{m}\cdot\text{sec},\ \text{motor damping}\\
r_1 &= 2\times 10^{-2}\ \text{m}\\
K_t &= 3\times 10^{-2}\ \text{N}\cdot\text{m/A},\ \text{motor}-\text{torque constant}\\
K &= 2\times 10^{4}\ \text{N/m}\\
B &= 20\ \text{N/m}\cdot\text{sec}\\
r_2 &= 2\times 10^{-2}\ \text{m}\\
J_2 &= 2\times 10^{-5}\ \text{kg}\cdot\text{m}^2\\
B_2 &= 2\times 10^{-2}\ \text{N}\cdot\text{m}\cdot\text{sec},\ \text{viscous damping, idler}\\
F &= 6\ \text{N},\ \text{constant force}\\
\dot{x}_1 &= \text{tape velocity m/sec} \quad (\text{variable to be controlled})
\end{aligned}$$

    (b) Find the transfer function from the motor current to the tape position;

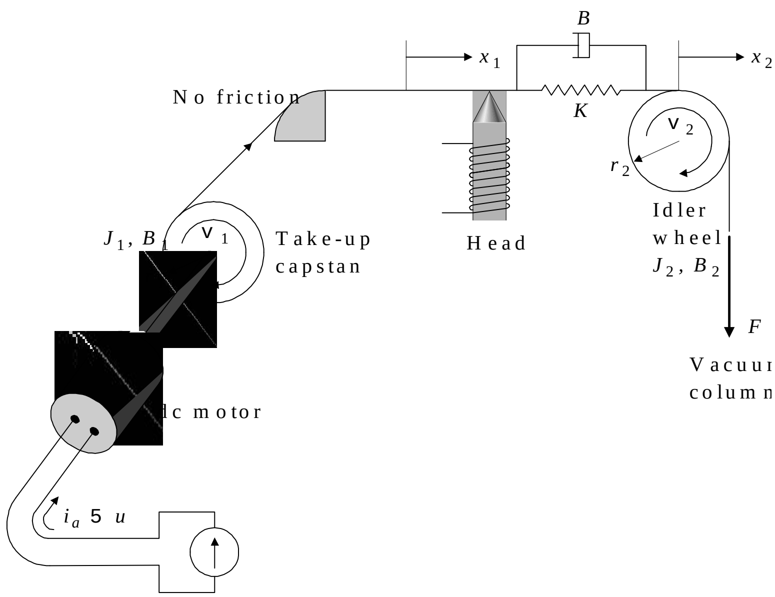


Figure 3.50: Tape drive schematic

(c) Find the poles and zeros for the transfer function in part (a).

(d) Use MATLAB to find the response of x_1 to a step input in i_a.

Solution:

(a) Because of the force F from the vacuum column, the spring will be stretched in the steady-state by and the motor torque will have a constant component

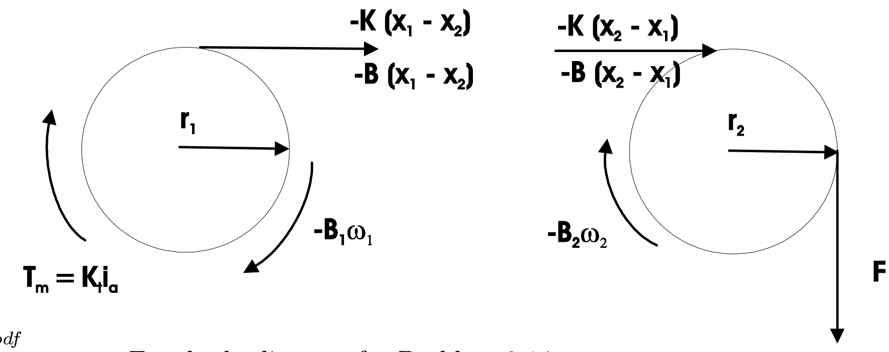
$$T_{m_{ss}} = -Fr_1,$$

and thus the steady-state current to provide the torque will be

$$i_{a_{ss}} = \frac{T_{m_{ss}}}{K_t}.$$

We can then assume $F = 0$ in the equations from now on.

Fig



9.pdf

Free body diagram for Problem 3.14.

On the capstan side:

$$\underbrace{K_t i_a}_{\text{Motor Torque}} = \underbrace{J_1\dot{\omega}_1}_{\text{Wheel Inertia}} + \underbrace{B_1\omega_1}_{\text{Wheel Damping}} + r_1\underbrace{[B(\dot{x}_1 - \dot{x}_2) + r_1(x_1 - x_2)]}_{\text{Spring and damper}},$$

On the idler side:

$$\underbrace{Fr_2}_{\text{Vac uum Col}} = \underbrace{J_2\dot{\omega}_2}_{\text{Wheel Inertia}} + \underbrace{B_2\omega_2}_{\text{Wheel Damping}} + r_2\underbrace{[-B(\dot{x}_1 - \dot{x}_2) - K(x_1 - x_2)]}_{\text{Spring and damper}}.$$

We also have:

$$\begin{aligned}
\dot{x}_1 &= r_1\omega_1, \\
\dot{x}_2 &= r_2\omega_2, \\
x_1 &= r_1\theta_1.
\end{aligned}$$

(b) From part (a):

$$\begin{aligned}
J_1\dot{\omega}_1 &= -B_1\omega_1 + K_t i_a + Br_1(\dot{x}_2 - \dot{x}_1) + Kr_1(x_2 - x_1), \\
J_2\dot{\omega}_2 &= -B_2\omega_2 + Br_2(\dot{x}_1 - \dot{x}_2) + Kr_2(x_1 - x_2), \\
\dot{x}_1 &= r_1\omega_1, \\
\dot{x}_2 &= r_2\omega_2.
\end{aligned}$$

$$\begin{bmatrix} J_1 s + B_1 & 0 & (Bs+K)r_1 & -(Bs+K)r_1 \\ 0 & J_2 s + B_2 & -(Bs+K)r_2 & (Bs+K)r_2 \\ -r_1 & 0 & s & 0 \\ 0 & -r_2 & 0 & s \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} K_t I_a \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

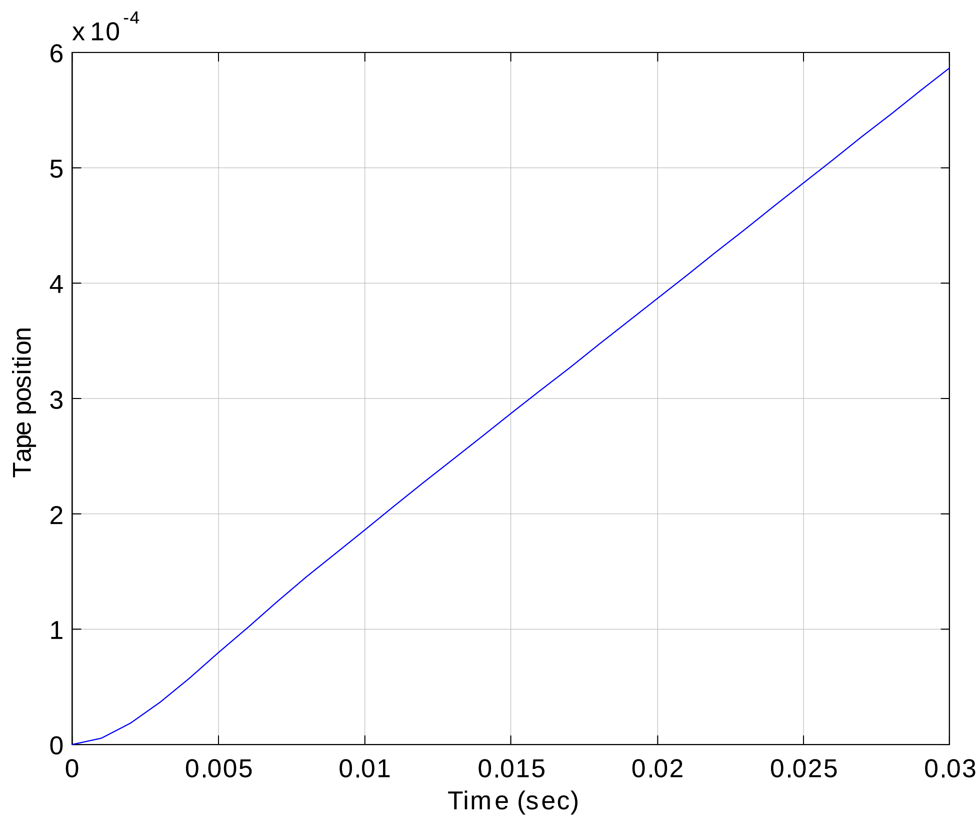
$$\frac{X_1(s)}{I_a(s)} = \frac{K_t r_1[J_2 s^2 + (B_2 + r_2^2 B)s + r_2^2 K]}{s\left[\begin{array}{c} J_1 J_2 s^3 + (J_1 B_2 + B_1 J_2 + r_2^2 J_1 B + r_1^2 J_2 B)s^2 + \\ (B_1 B_2 + r_2^2 J_1 K + r_2^2 B_1 B + r_1^2 J_2 K + r_1^2 B_2 B)s \\ + r_2^2 B_1 K + r_1^2 B_2 K \end{array}\right]},$$

$$X_1(s) = \frac{12(s+400)(s+1000)}{s(s+380 \pm j309)(s+1000)} I_a(s) = \frac{num(s)}{den(s)} I_a(s).$$

(c)

$$\begin{aligned} \text{Poles are at} &: \quad 0,\ -380 \pm j309,\ -1000, \\ \text{Zeros are at} &: \quad -400,\ -1000. \end{aligned}$$

(d) The step response [step(num,den)] is shown in the figure below.



Tape position response to a step in current.

15. For the system in Fig. 2.51, compute the transfer function from the motor voltage to position θ_2.

Solution:

From Problem 2.19:

$$
\begin{aligned}
L\frac{di_a}{dt} + R_a i_a + k_e\dot{\theta}_1 &= v_a \\
k_t i_a = J_1\ddot{\theta}_1 + b(\dot{\theta}_1 - \dot{\theta}_2) &+ k(\theta_1 - \theta_2) + B\dot{\theta}_1 \\
J_2\ddot{\theta}_2 + b(\dot{\theta}_2 - \dot{\theta}_1) &+ k(\theta_2 - \theta_1) = 0
\end{aligned}
$$

So we have:

$$
\begin{aligned}
LsI_a(s) + R_a I_a(s) + sk_e\Theta_1(s) &= V_a(s), \\
k_t I_a(s) = s^2 J_1\Theta_1(s) + b[\Theta_1(s) - \Theta_2(s)]s &+ k[\Theta_1(s) - \Theta_2(s)] + Bs\Theta_1(s), \\
s^2 J_2\Theta_2(s) + b[\Theta_2(s) - \Theta_1(s)]s &+ k[\Theta_2(s) - \Theta_1(s)] = 0,
\end{aligned}
$$

we have:

$$
\begin{aligned}
\frac{\Theta_2(s)}{V_a(s)} &= \frac{k_t(bs+k)}{\det\begin{bmatrix} sk_e & 0 & Ls + R_a \\ J_1 s^2 + Bs + bs + k & -bs - k & -k_t \\ -bs - k & J_2 s^2 + bs + k & 0 \end{bmatrix}}, \\
&= \frac{k_t(bs+k)}{\begin{array}{c}(Ls + Ra)[J_1J_2s^4 + (J_1b + BJ_2 + bJ_2)s^3 + (J_1k + Bb + KJ_2)s^2 + Bks] \\ +k_ek_tJ_2s^3 + k_ek_tbs^2 + kk_ek_ts\end{array}}, \\
&= \frac{k_t(bs+k)}{\begin{array}{c}J_1J_2s^5 + J_2[J_1R_a + L(b+B)]s^4 \\ +[J_2k_ek_tJ_1L(b+k) + LJ_2k + R_a(b+B)J_2 - Lb^2]s^3 \\ +[L(b+B)(b+k) - 2bkL + J_1R_a(b+k) + R_aJ_2k - b^2R_a]s^2 \\ +[k_ek_t(b+k) + kL(b+k) - bk^2 + R_a(b+B)(b+k) - 2bkR_a]s + kR_ab\end{array}}.
\end{aligned}
$$

16. Compute the transfer function for the two-tank system in Fig. 2.55 with holes at A and C.

Solution:

From Problem 2.23 but with $s = a$ tank area we have:

$$
\begin{bmatrix} \Delta\dot{h}_1 \\ \Delta\dot{h}_2 \end{bmatrix} = \frac{1}{6a}\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \Delta h_1 \\ \Delta h_2 \end{bmatrix} + \frac{\omega_{in}}{a}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{-10}{3a} \\ 0 \end{bmatrix},
$$

$$\begin{aligned}
\Delta\dot{h}_1 &= \frac{-\Delta h_1 + 6\omega_{in} - 20}{6a}, \\
\Delta\dot{h}_2 &= \frac{1}{6a}(\Delta h_1 - \Delta h_2), \\
s\Delta h_1(s) &= \frac{-\Delta h_1(s) + 6\omega_{in}(s)}{6a}, \\
s\Delta h_2(s) &= \frac{1}{6a}[\Delta h_1(s) - \Delta h_2(s)], \\
\Delta h_2(s) &= \frac{\omega_{in}(s)}{6a[a(\frac{1}{6a} + s)]^2}, \\
\frac{\Delta h_2(s)}{\omega_{in}(s)} &= \frac{1}{6[a(\frac{1}{6a} + s)]^2}.
\end{aligned}$$

17. For a second-order system with transfer function

$$G(s) = \frac{3}{s^2 + 2s - 3},$$

determine the following:

(a) The DC gain;

(b) The final value to a step input.

Solution:

(a) DC gain $G(0) = \frac{3}{-3} = -1$

(b) $\lim_{t\to\infty} y(t) = ?$

$s^2 + 2s + 3 = 0 \Longrightarrow s = 1, -3$

Since the system has an unstable pole, the Final Value Theorem is not applicable. The output is unbounded.

18. Consider the continuous rolling mill depicted in Fig. 3.51. Suppose that the motion of the adjustable roller has a damping coefficient b, and that the force exerted by the rolled material on the adjustable roller is proportional to the material's change in thickness: $F_s = c(T - x)$. Suppose further that the DC motor has a torque constant K_t and a back-emf constant K_e, and that the rack-and-pinion has effective radius of R.

(a) What are the inputs to this system? The output?

(b) Without neglecting the effects of gravity on the adjustable roller, draw a block diagram of the system that explicitly shows the following quantities: $V_s(s)$, $I_0(s), F(s)$ (the force the motor exerts on the adjustable roller), and $X(s)$.

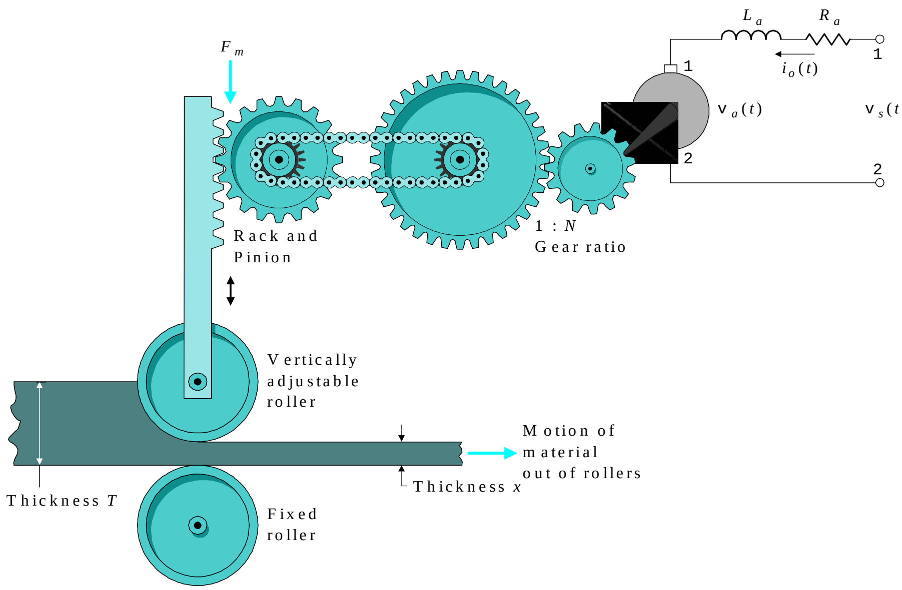


Figure 3.51: Continuous rolling mill

(c) Simplify your block diagram as much as possible while still identifying output and each input separately.

Solution:

(a)

Inputs:	input voltage $\longrightarrow v_s(t)$
	thickness $\longrightarrow$ T
	gravity $\longrightarrow$ mg
Output:	thickness $\longrightarrow x$

(b) Dynamic analysis of adjustable roller:

$$m\ddot{x} = c(T - x) - mg - b\dot{x} - F_m,$$

$$\Longrightarrow (s^2m + sb + c)X(s) + F_m(s) + \frac{mg - cT}{s} = 0. \qquad (1)$$

Torque in rack and pinion:

$$T_{RP} = RF_m = NT_{motor},$$

$$\text{but} \quad T_{motor} = K_t I_f i_o,$$

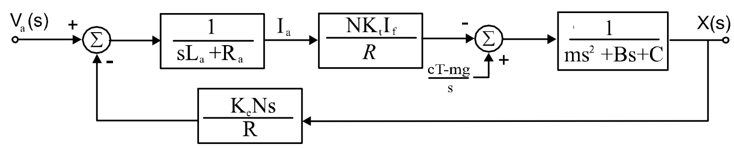
$$F_m = \frac{NK_tI_f}{R} i_o \qquad (2)$$

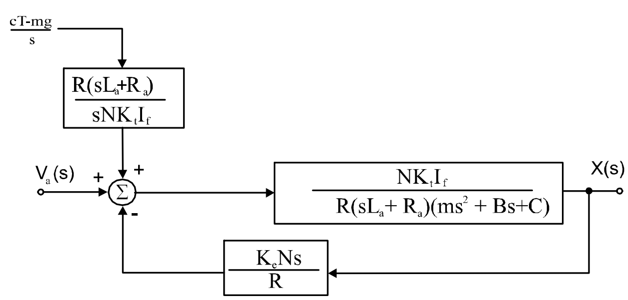
DC motor circuit analysis:

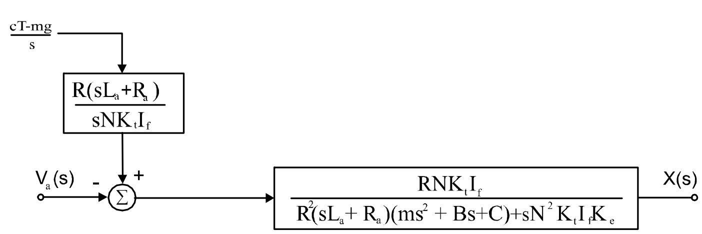
$$v_s(t) = R_a i_o + L_a \frac{di_o}{dt} + v_a(t),$$
$$v_a(t) = u_e \dot{\theta},$$
$$\frac{\theta R}{N} = x,$$
$$I_o(s) = \frac{V_s(s) - \frac{K_e N}{R} sX(s)}{R_a + sL_a} \qquad (3)$$

Combining (1), (2), and (3):

$$0 = (s^2 m + sb + c)X(s) + \frac{mg - cT}{s} + \frac{NK_t I_f}{R}\left[\frac{V_s(s) - \frac{K_e N}{R} sX(s)}{sL_a + R_a}\right].$$







Block diagrams for rolling mill.

Problems and Solutions for Section 3.2: System Modeling Diagrams

19. Consider the block diagram shown in Fig. 3.52. Note that a_i and b_i are constants. Compute the transfer function for this system. This special structure is called the "control canonical form" and will be discussed further in Chapter 7.)

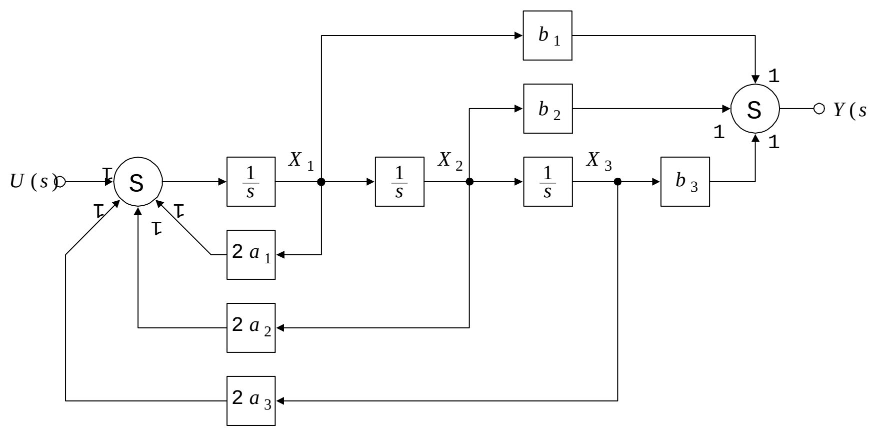
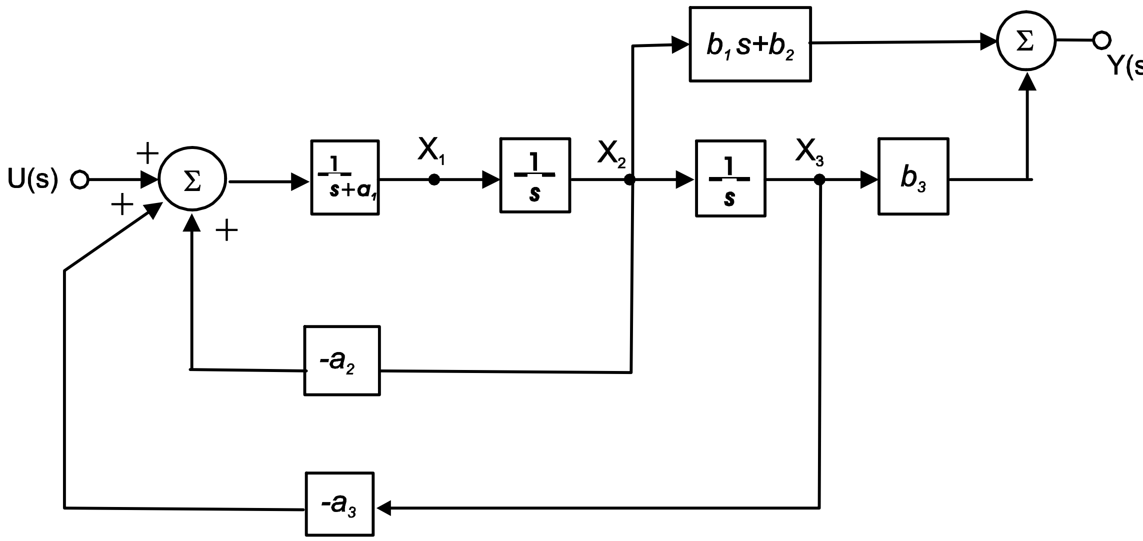


Figure 3.52: Block diagram for Problem 3.19

Solution:

We move the pickoff point at X_1 to the right past the second integrator to get $b_1 s + b_2$ as shown in the figure on the next page.



Block diagram reduction for Problem 3.19.

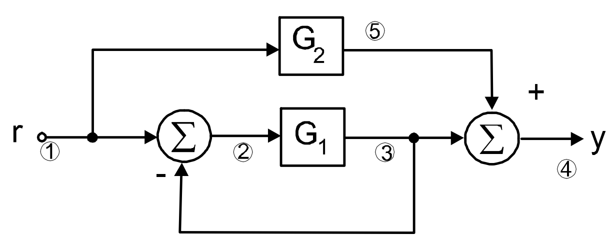
We then move the pickoff point at X_2 past the third integrator to get $s(b_1s+b_2)+b_3$. We have a block with the transfer function $b_1s^2+b_2s+b_3$ at the output. Meanwhile we apply the feedback rule to the first inner loop to get $\frac{1}{s+a_1}$ as shown in the figure and repeat for the second and third loops. We finally have:

$$\frac{Y}{U} = \frac{b_1s^2+b_2s+b_3}{s^3+a_1s^2+a_2s+a_3}.$$

Example on the web in Chapter 3 shows that we can obtain the same answer using Mason's rule.

20. Find the transfer functions for the block diagrams in Fig. 3.53.

Solution:



(a)

Block diagram for Fig. 3.53 (a)

$$\frac{Y}{R} = \frac{G_1}{1+G_1} + G_2.$$

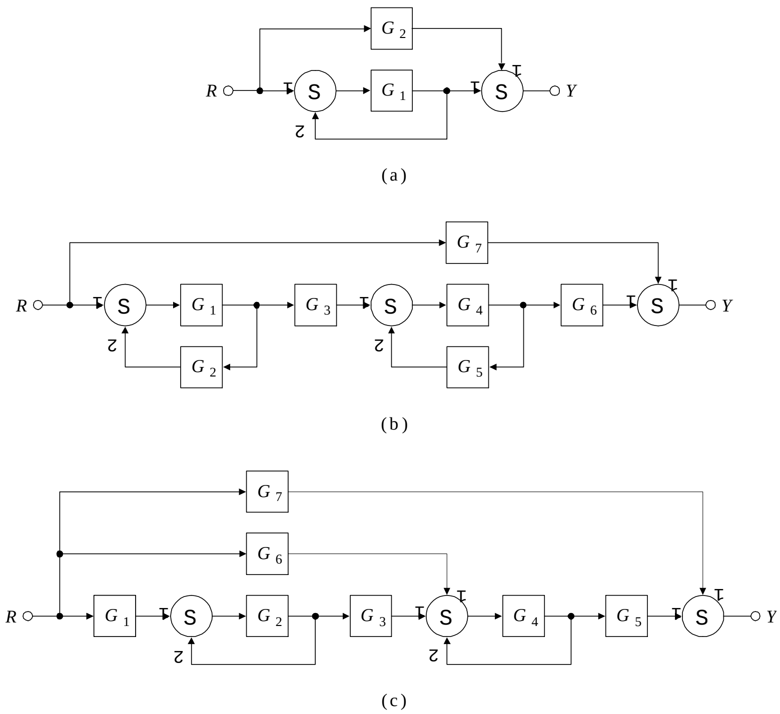
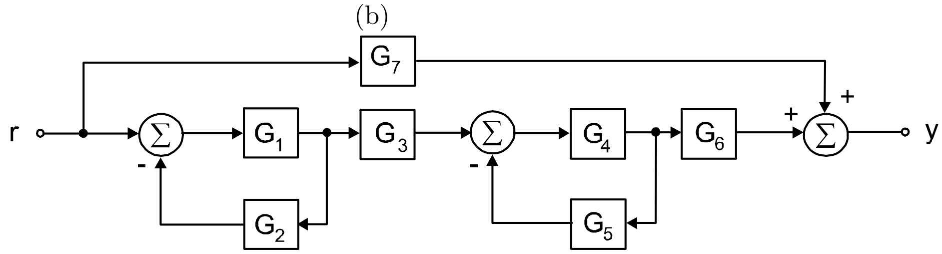
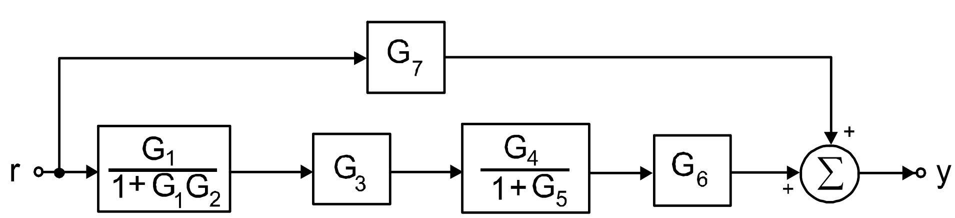


Figure 3.53: Block diagrams for Problem 3.20

(b)



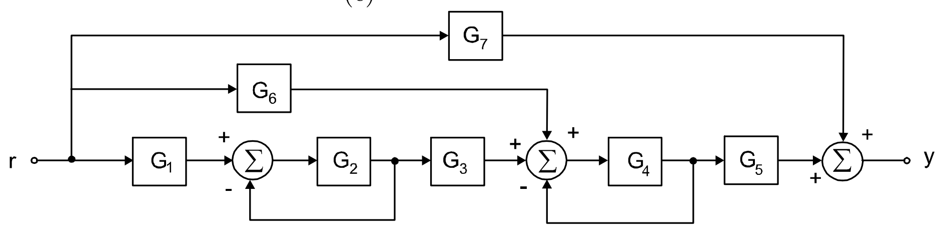
Block diagram for Fig. 3.53 (b)

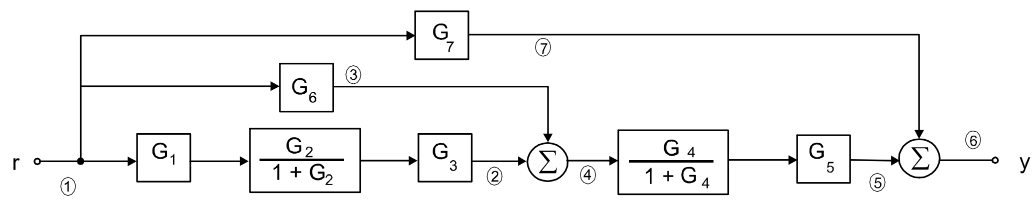


Block diagram for Fig. 3.53 (b): reduced

$$\frac{Y}{R} = G_7 + \frac{G_1 G_3 G_4 G_6}{(1 + G_1 G_2)(1 + G_4 G_5)}.$$

(c)





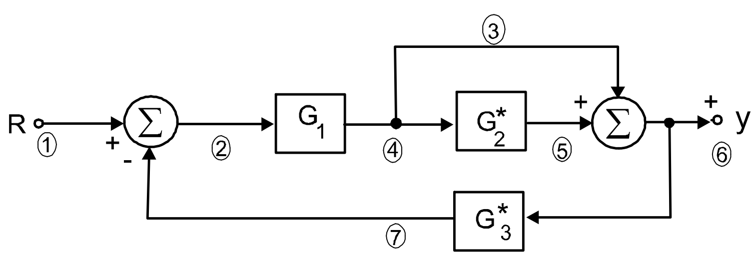
Top: Block diagram for Fig. 3.53 (c) ; Bottom: Block diagram for Fig 3.43 (c) reduced.

$$\frac{Y}{R} = G_7 + \frac{G_6 G_4 G_5}{1 + G_4} + \frac{G_1 G_2 G_3}{1 + G_2} \times \frac{G_4 G_5}{1 + G_4}.$$

21. Find the transfer functions for the block diagrams in Fig. 3.54, using the ideas of block diagram simplification. The special; structure in Fig. 3.54 (b) is called the "observer canonical form" and will be discussed in Chapter 7.

Solution:

Part (a): Transfer functions found using the ideas of Figs. 3.8 and 3.9**:**



(a)

(a) Block diagram for Fig. 3.54 (a).

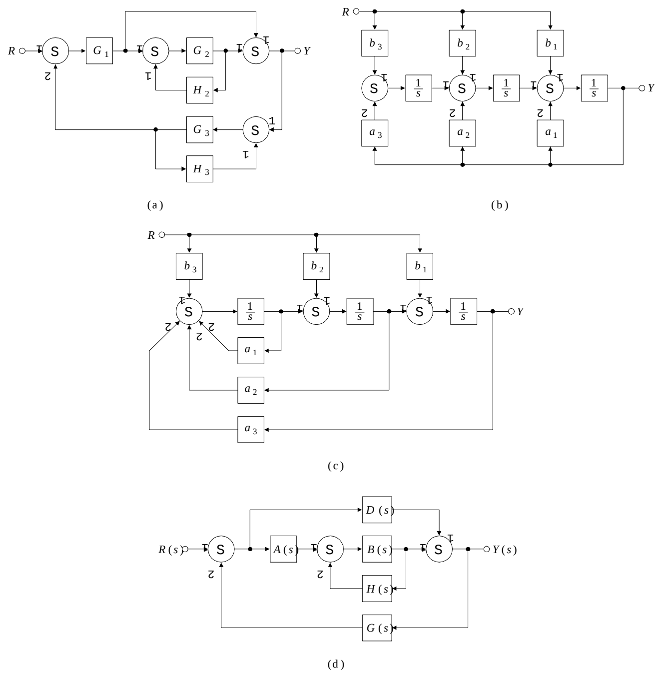
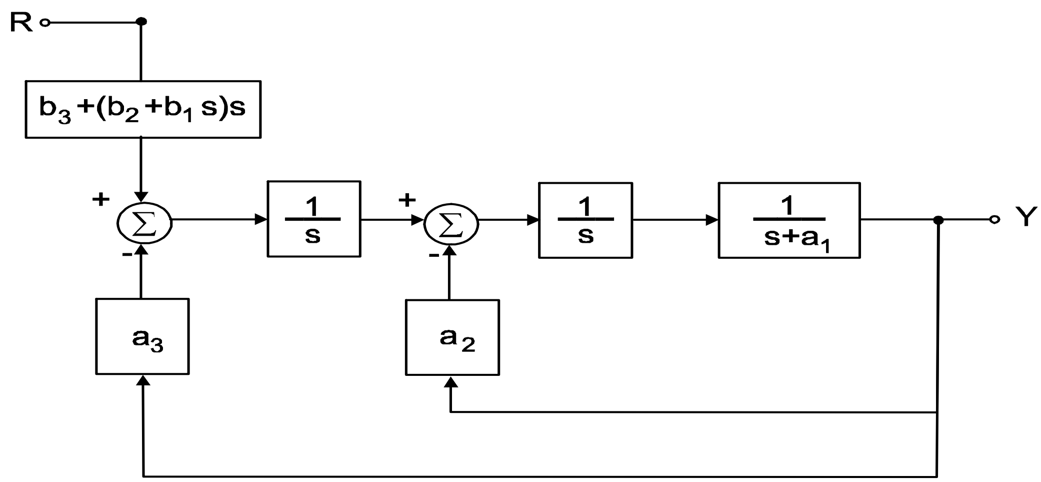


Figure 3.54: Block diagrams for Problem 3.21

$$
\begin{aligned}
G_2^* &= \frac{G_2}{1 - G_2 H_2}, \\
G_3^* &= \frac{G_3}{1 - G_3 H_3},
\end{aligned}
$$

$$
\frac{Y}{R} = \frac{G_1(1 + G_2^*)}{1 + G_1(1 + G_2^*)G_3^*} = \frac{G_1(1 - G_2H_2)(1 - G_3H_3) + G_1G_2(1 - G_3H_3)}{1 + (1 - G_2H_2)(1 - G_3H_3) + G_1G_3(1 - G_2H_2) + G_1G_2G_3}.
$$

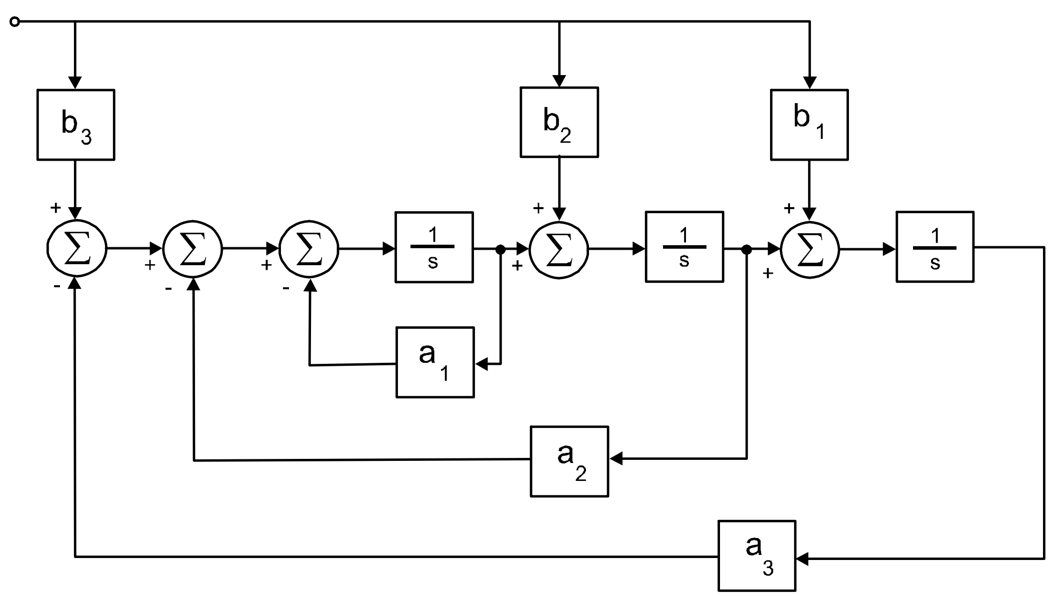
(b) We move the summer on the right past the integrator to get $b_1 s$ and repeat to get $(b_2 + b_1 s)s$. Meanwhile we apply the feedback rule to the first inner loop to get $\frac{1}{s+a_1}$ as shown in the figure and repeat for the second and third loops to get:



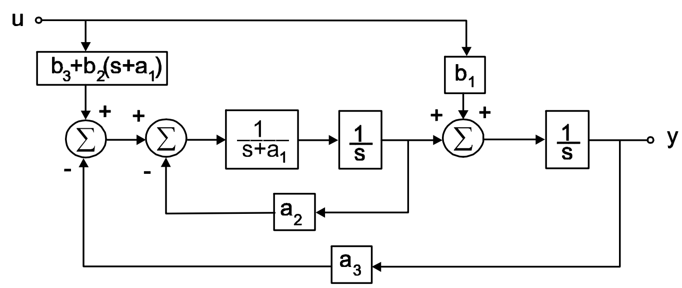
(b) Block diagram for Fig. 3.54(b).

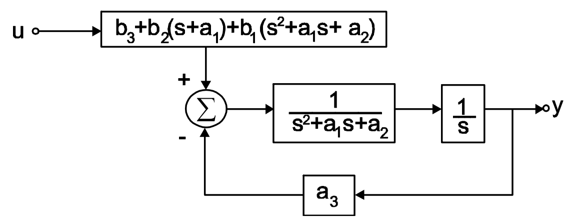
$$\frac{Y}{R} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}.$$

(c) Applying block diagram reduction: reduce innermost loop, shift b_2 to the b_3 node, reduce next innermost loop and continue systematically to obtain:



(c) Block diagram for Fig. 3.54(c).

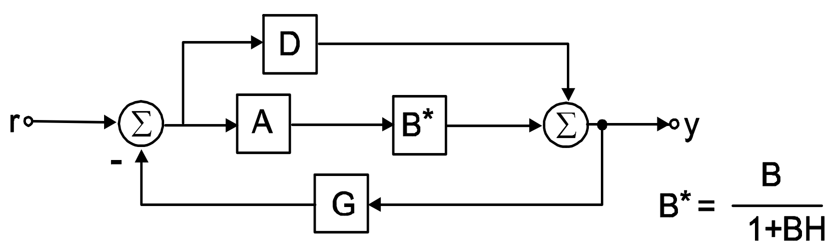




(c) Block diagram for Fig. 3.54(c).

$$\frac{Y}{R} = \frac{b_1 s^2 + (a_1 b_1 + b_2)s + a_1 b_2 + a_2 b_1 + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}.$$

(d)



(d) Block diagram for Fig. 3.54(d).

$$\frac{Y}{R} = \frac{D + AB^*}{1 + G(D + AB^*)} = \frac{D + DBH + AB}{1 + BH + GD + GBDH + GAB}.$$

22. Use block-diagram algebra to determine the transfer function between $R(s)$ and $Y(s)$ in Fig. 3.55.

Solution:

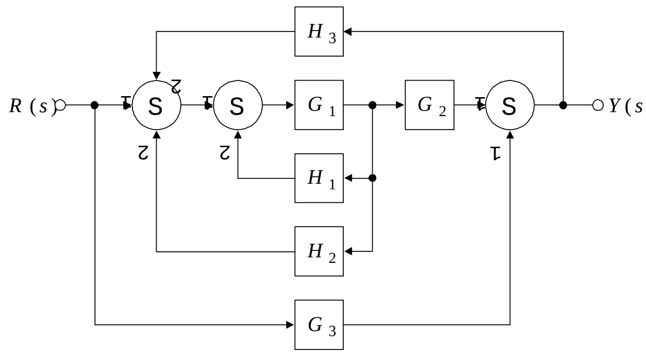
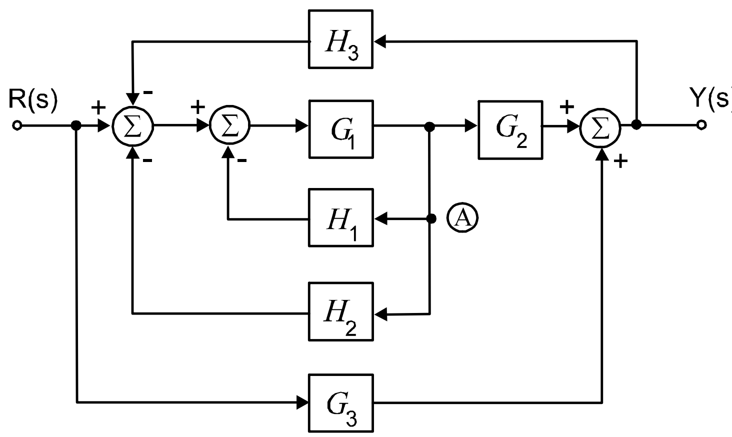
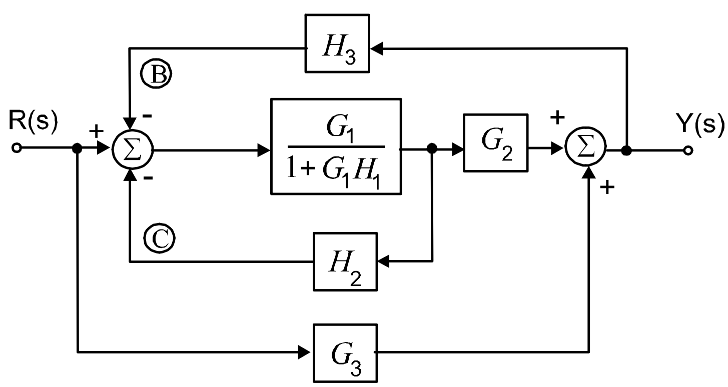


Figure 3.55: Block diagram for Problem 3.22



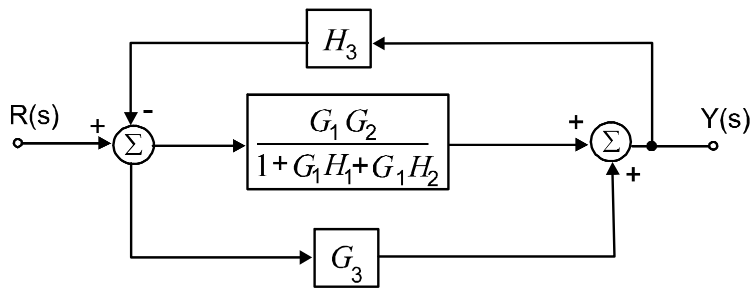
Block diagram for Fig. 3.55.

Move node A and close the loop:



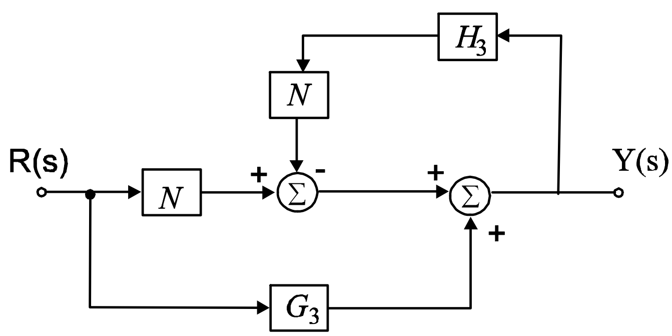
Block diagram for Fig. 3.55: reduced.

Add signal B, close loop and multiply before signal C:



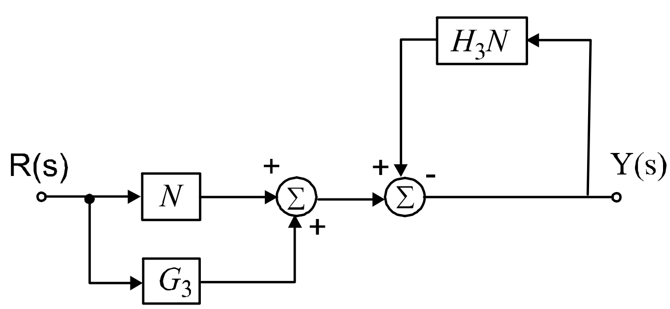
Block diagram for Fig. 3.55: reduced.

Move middle block N past summer:



Block diagram for Fig. 3.55: reduced.

Now reverse order of summers and close each block separately:



Block diagram for Fig. 3.55: reduced.

$$\frac{Y}{R} = \overbrace{(N + G_3)}^{feedforward}\underbrace{(\frac{1}{1 + NH_3})}_{feedback}.$$

$$\frac{Y}{R} = \frac{G_1G_2 + G_3(1 + G_1H_1 + G_1H_2)}{1 + G_1H_1 + G_1H_2 + G_1G_2H_3}.$$

Problems and Solutions for Section 3.3: Effect of Pole Locations

23. For the electric circuit shown in Fig. 3.56, find the following:

    (a) The time-domain equation relating *i(t)* and $v_1(t)$;

    (b) The time-domain equation relating *i(t)* and $v_2(t)$;

    (c) Assuming all initial conditions are zero, the transfer function $V_2(s)/V_1(s)$ and the damping ratio ζ and undamped natural frequency ω_n of the system;

    (d) The values of R that will result in $v_2(t)$ having an overshoot of no more than 25%, assuming $v_1(t)$ is a unit step, L = 10 mH, and $C = 4\ \mu$F.

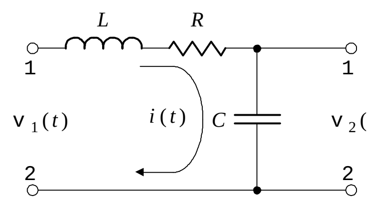


Figure 3.56: Circuit for Problem 3.23

Solution:

(a)

$$v_1(t) = L\frac{di}{dt} + Ri + \frac{1}{C}\int i(t)dt.$$

(b)

$$v_2(t) = \frac{1}{C}\int i(t)dt.$$

(c)

$$\frac{v_2(s)}{v_1(s)} = \frac{\frac{1}{sC}}{sL + R + \frac{1}{sC}} = \frac{1}{s^2LC + sRC + 1}.$$

(d) For 25% overshoot $\zeta \approx 0.4$,

$$0.4 \approx \zeta = \frac{R}{2\sqrt{\frac{L}{C}}}$$

$$R = 2\zeta\sqrt{\frac{L}{C}} = (2)(0.4)\sqrt{\frac{10\times10^{-3}}{4\times10^{-6}}} = 40\ \Omega.$$

24. For the unity feedback system shown in Fig. 3.57, specify the gain K of the proportional controller so that the output $y(t)$ has an overshoot of no more than 10% in response to a unit step.

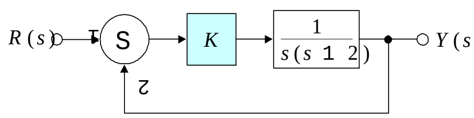


Figure 3.57: Unity feedback system for Problem 3.24

Solution:

$$\frac{Y(s)}{R(s)} = \frac{K}{s^2+2s+K} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2},$$
$$\omega_n = \sqrt{K},$$
$$\zeta = \frac{2}{2\omega_n} = \frac{1}{\sqrt{K}}. \qquad (1)$$

In order to have an overshoot of no more than 10%:

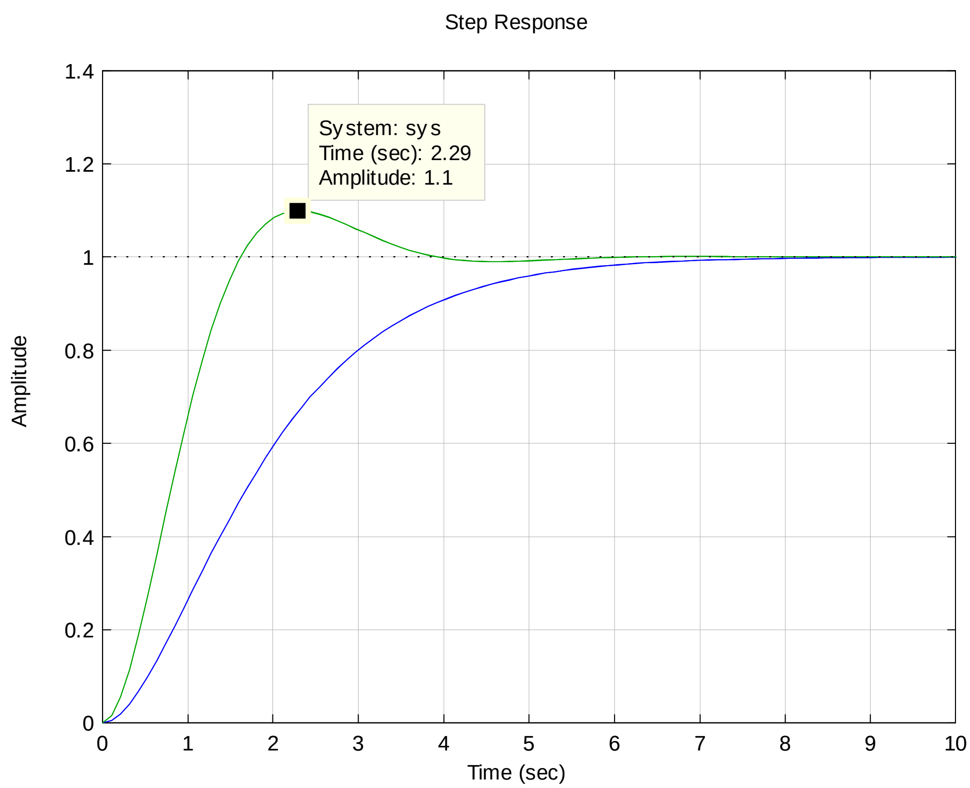
$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \le 0.10.$$

Solving for ζ :

$$\zeta = \sqrt{\frac{(\ln M_p)^2}{\pi^2+(\ln M_p)^2}} \ge 0.591.$$

Using (1) and the solution for ζ:

$$K = \frac{1}{\zeta^2} \le 2.86,$$
$$\therefore 0 < K \le 2.86.$$



Step responses for $K = 1$ (blue) and $K = 2.86$ (green).

25. For the unity feedback system shown in Fig. 3.58, specify the gain and pole location of the compensator so that the overall closed-loop response to a unit-step input has an overshoot of no more than 25%, and a 1% settling time of no more than 0.1 sec. Verify your design using MATLAB.

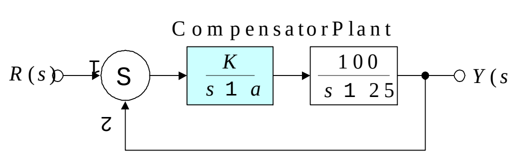


Figure 3.58: Unity feedback system for Problem 3.25

Solution:

$$\frac{Y(s)}{R(s)} = \frac{100K}{s^2 + (25 + a)s + 25a + 100K} = \frac{100K}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Using the given information:

$$\begin{aligned} R(s) &= \frac{1}{s} \qquad \text{unit step}, \\ M_p &\leq 25\%, \\ t_s &\leq 0.1\,\text{sec}. \end{aligned}$$

Solve for ζ :

$$\begin{aligned} M_p &= e^{-\pi\zeta/\sqrt{1-\zeta^2}}, \\ \zeta &= \sqrt{\frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}} \geq 0.4037. \end{aligned}$$

Solve for ω_n:

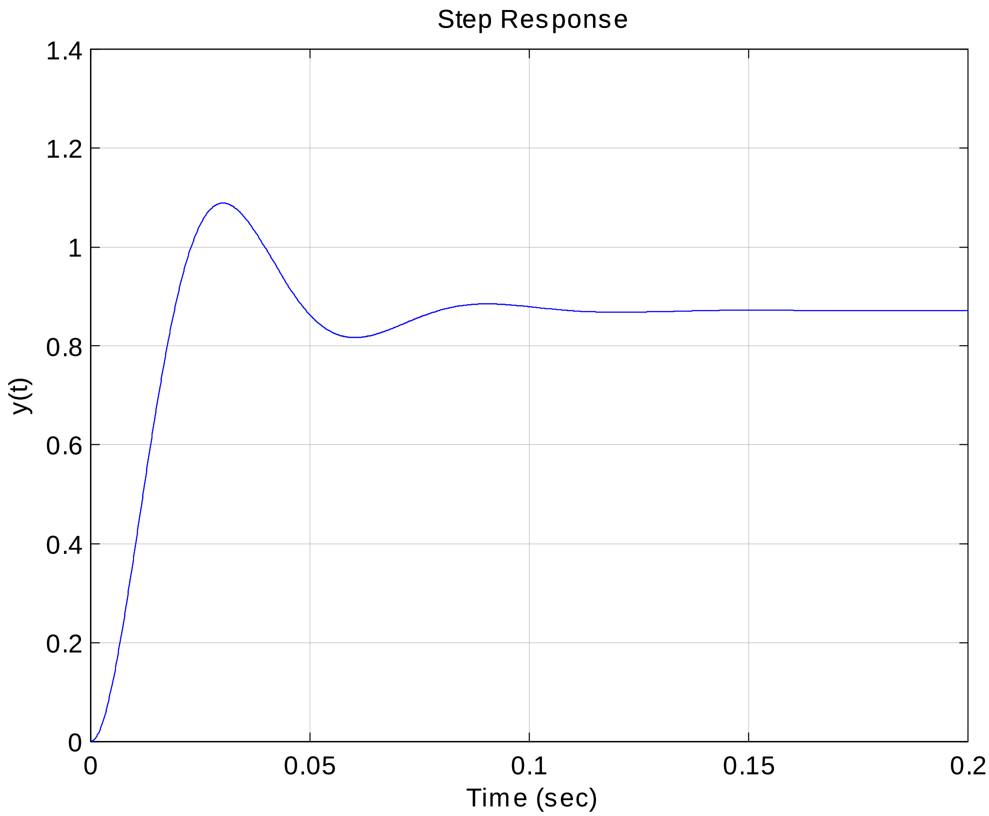
$$e^{-\zeta\omega_n t_s} = 0.01 \qquad \text{For a 1\% settling time.}$$

$$\begin{gathered} t_s \leq \frac{4.605}{\zeta\omega_n} = 0.1, \\ \Longrightarrow \omega_n \approx 114.07. \end{gathered}$$

Now find a and K :

$$\begin{gathered} 2\zeta\omega_n = (25 + a), \\ a = 2\zeta\omega_n - 25 = 92.10 - 25 = 67.10, \\ \omega_n^2 = (25a + 100K), \\ K = \frac{\omega_n^2 - 25a}{100} \approx 113.34. \end{gathered}$$

The step response of the system using Matlab is shown below.

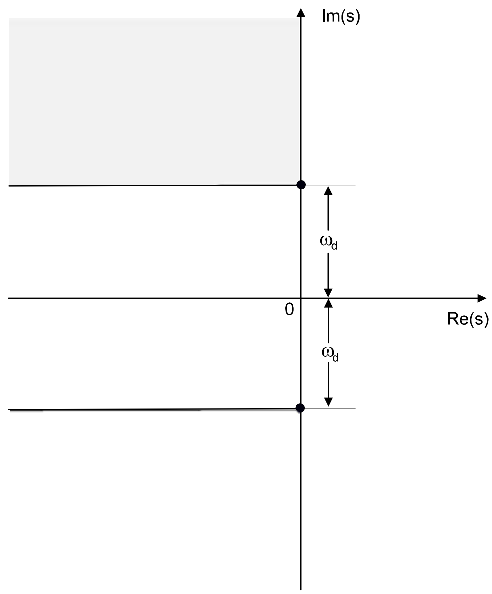


Step response for Problem 3.25.

Problems and Solutions for Section 3.4: Time-Domain Specifications

26. Suppose you desire the peak time of a given second-order system to be less than t_p'. Draw the region in the s-plane that corresponds to values of the poles that meet the specification $t_p < t_p'$.

Solution:



s-plane region to meet peak time constraint: shaded.

$$\omega_d t_p = \pi \Longrightarrow t_p = \frac{\pi}{\omega_d} < t_p^{'},$$
$$\frac{\pi}{t_p'} < \omega_d.$$

27. A certain servomechanism system has dynamics dominated by a pair of complex poles and no finite zeros. The time-domain specifications on the rise time (t_r), percent overshoot

(M_p), and settling time (t_s) are given by,

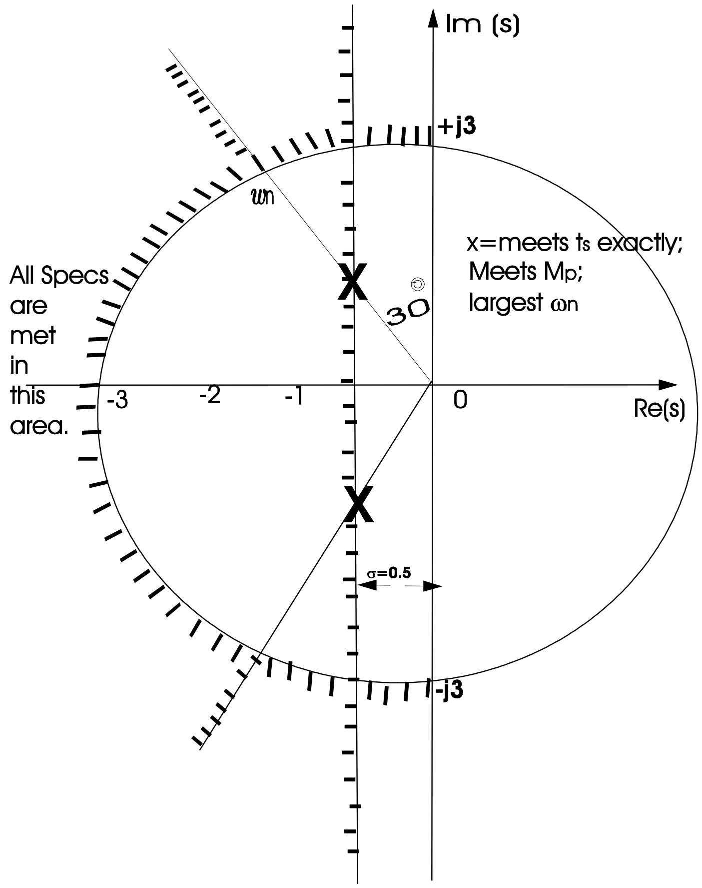
$$\begin{aligned} t_r &\leq 0.6\,\mathrm{sec}\,, \\ M_p &\leq 17\%, \\ t_s &\leq 9.2\,\mathrm{sec}\,. \end{aligned}$$

(a) Sketch the region in the s-plane where the poles could be placed so that the system will meet *all* three specifications.

(b) Indicate on your sketch the specific locations (denoted by $\times$) that will have the smallest rise-time and also meet the settling time specification *exactly*.

Solution:

(a)-(b)



s-plane region to meet the specifications.

28. Suppose you are to design a unity feedback controller for a first-order plant depicted in Fig. 3.59. (As you will learn in Chapter 4, the configuration shown is referred to as a proportional-integral controller.) You are to design the controller so that the closed-loop poles lie within the shaded regions shown in Fig. 3.60.

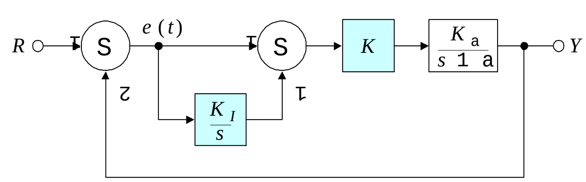


Figure 3.59: Unity feedback system for Problem 3.28

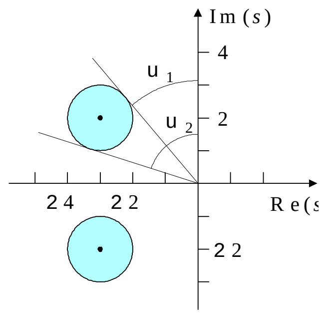


Figure 3.60: Desired closed-loop pole locations for Problem 3.28

(a) What values of ω_n and ζ correspond to the shaded regions in Fig. 3.60? (A simple estimate from the figure is sufficient.)

(b) Let $K_\alpha = \alpha = 2$. Find values for K and K_I so that the poles of the closed-loop system lie within the shaded regions.

(c) Prove that no matter what the values of K_α and α are, the controller provides enough flexibility to place the poles anywhere in the complex (left-half) plane.

Solution:

(a) The values could be worked out mathematically but working from the diagram:

$$\begin{aligned}
\sqrt{3^2+2^2} &= 3.6 \Longrightarrow 2.6 \leq \omega_n \leq 4.6, \\
\theta &= \sin^{-1}\zeta, \\
\zeta &= \sin\theta.
\end{aligned}$$

From the figure:

$$\begin{aligned}
\theta &\approx 34^\circ \qquad \zeta_1 = 0.554, \\
\theta &\approx 70^\circ \qquad \zeta_2 = 0.939,
\end{aligned}$$

$$\Longrightarrow 0.6 \leq \zeta \leq 0.9 \qquad \text{(approximately)}$$

(b) Closed-loop pole positions:

$$\begin{aligned}
s(s+\alpha) + (Ks + KK_I)K_\alpha &= 0, \\
s^2 + (\alpha + KK_\alpha)s + KK_IK_\alpha &= 0.
\end{aligned}$$

For this case:

$$s^2 + (2+2K)s + 2KK_I = 0 \qquad (*)$$

Choose roots that lie in the center of the shaded region,

$$\begin{aligned}
(s+(3+j2))(s+(3-j2)) &= s^2+6s+13=0,\\
s^2+(2+2K)s+2KK_I &= s^2+6s+13,\\
2+2K=6 &\Longrightarrow K=2,\\
13=4K_I &\Longrightarrow K_I=\frac{13}{4}.
\end{aligned}$$

(c) For the closed-loop pole positions found in part (b), in the (*) equation the value of K can be chosen to make the coefficient of s take on any value. For this value of K a value of K_I can be chosen so that the quantity KK_IK_α takes on any value desired. This implies that the poles can be placed anywhere in the complex plane.

29. The open-loop transfer function of a unity feedback system is

$$G(s) = \frac{K}{s(s+2)}.$$

The desired system response to a step input is specified as peak time $t_p = 1$ sec and overshoot $M_p = 5\%$.

(a) Determine whether both specifications can be met simultaneously by selecting the right value of K.

(b) Sketch the associated region in the s-plane where both specifications are met, and indicate what root locations are possible for some likely values of K.

(c) Pick a suitable value for K, and use MATLAB to verify that the specifications are satisfied.

Solution:

(a)

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{K}{s^2+2s+K} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}.$$

Equate the coefficients of like powers of s:

$$\begin{aligned}2 &= 2\zeta\omega_n\\ K &= \omega_n^2\end{aligned} \quad (*)$$

$$\Longrightarrow \quad \omega_n = \sqrt{K} \qquad \zeta = \frac{1}{\sqrt{K}}.$$

We would need:

$$\begin{aligned}
\frac{M_p\%}{100} = 0.05 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \qquad & \Longrightarrow \quad \zeta = 0.69, \\
t_p = 1\,\mathrm{sec} = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \qquad & \Longrightarrow \quad \omega_n = 4.34.
\end{aligned}$$

But the combination ($\zeta = 0.69$, $\omega_n = 4.34$) that we need is <u>not</u> possible by varying K alone. Observe that from equations (*) $\zeta\omega_n = 1 \neq 0.69 \times 4.34$.

(b) Now we wish to have:

$$\begin{matrix} M_p^* = r \times 0.05 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \\ t_p^* = r \times 1\,\mathrm{sec} = \frac{\pi}{\omega_d} \end{matrix} \quad (**)$$

where $r \equiv$ relaxation factor.

Recall the conditions of our system:

$$\begin{aligned}
\omega_n &= \sqrt{K}, \\
\zeta &= \frac{1}{\sqrt{K}},
\end{aligned}$$

replace ω_n and ζ in the system (**):

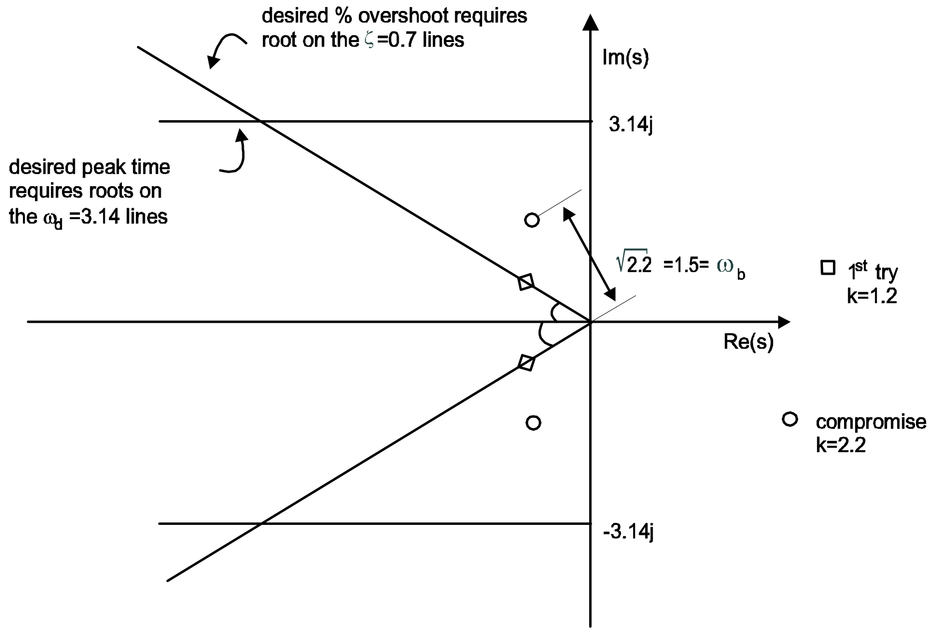
$$\Longrightarrow \quad \begin{matrix} -\frac{\pi}{\sqrt{K-1}} = r \times 0.05 \\ 1\,\mathrm{sec} = \frac{\pi}{\sqrt{K-1}} \end{matrix}$$

$$\begin{aligned}
\Longrightarrow r \times 0.05 = e^{-r} \qquad & \Longrightarrow \quad r \cong 2.21. \\
K = 1 + \frac{\pi^2}{r^2} \qquad & \Longrightarrow \quad K = 3.02.
\end{aligned}$$

then with $K = 3.02$ we will have:

$$\begin{aligned}
M_p^* &= rM_p = 2.21 \times 0.05 = 0.11. \\
t_p^* &= rt_p = 2.21 \times 1\,\mathrm{sec} = 2.21\,\mathrm{sec}\,.
\end{aligned}$$

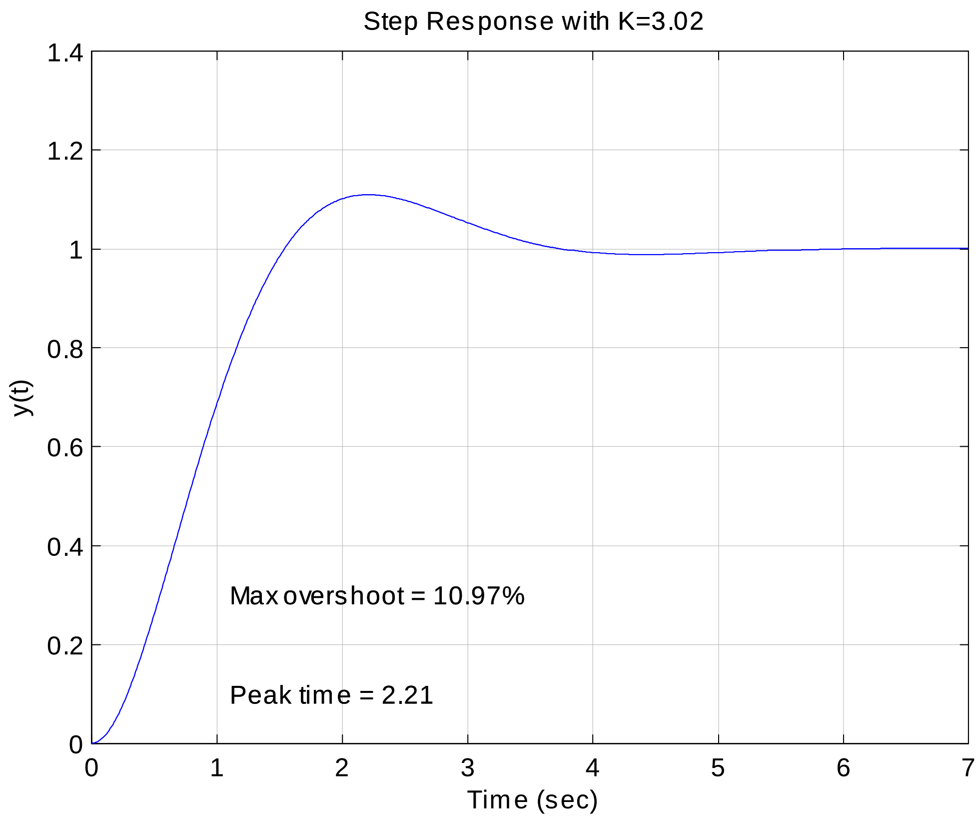
Note: * denotes actual location of closed-loop roots.



s-plane regions.

```
% Problem 3.29 FPE6e
K=3.02;
num=[K];
den=[1, 2, K];
sys=tf(num,den);
t=0:.01:7;
y=step(sys,t);
plot(t,y);
yss = dcgain(sys);
Mp = (max(y) - yss)*100;
% Finding maximum overshoot
msg_overshoot = sprintf('Max overshoot = %3.2f%%', Mp);
% Finding peak time
idx = max(find(y==(max(y))));
tp = t(idx);
msg_peaktime = sprintf('Peak time = %3.2f', tp);
xlabel('Time (sec)');
ylabel('y(t)');
msg_title = sprintf('Step Response with K=%3.2f',K);
title(msg_title);
text(1.1, 0.3, msg_overshoot);
```

```
text(1.1, 0.1, msg_peaktime);
grid on;
```



Problem 3.29: Closed-loop step response.

30. ▲The equations of motion for the DC motor shown in Fig. 2.32 were given in Eqs. (2.52-53) as

$$J_m \ddot{\theta}_m + \left(b + \frac{K_t K_e}{R_a} \right) \dot{\theta}_m = \frac{K_t}{R_a} v_a.$$

Assume that

$$\begin{aligned} J_m &= 0.01 \ \mathrm{kg} \cdot \mathrm{m}^2, \\ b &= 0.001 \ \mathrm{N} \cdot \mathrm{m} \cdot \mathrm{sec}, \\ K_e &= 0.02 \ \mathrm{V} \cdot \mathrm{sec}, \\ K_t &= 0.02 \ \mathrm{N} \cdot \mathrm{m/A}, \\ R_a &= 10 \ \Omega. \end{aligned}$$

(a) Find the transfer function between the applied voltage v_a and the motor speed $\dot{\theta}_m$.

(b) What is the steady-state speed of the motor after a voltage $v_a = 10$ V has been applied?

(c) Find the transfer function between the applied voltage v_a and the shaft angle θ_m.

(d) Suppose feedback is added to the system in part (c) so that it becomes a position servo device such that the applied voltage is given by

$$v_a = K(\theta_r - \theta_m),$$

where K is the feedback gain. Find the transfer function between θ_r and θ_m.

(e) What is the maximum value of K that can be used if an overshoot $M_p < 20\%$ is desired?

(f) What values of K will provide a rise time of less than 4 sec? (Ignore the M_p constraint.)

(g) Use MATLAB to plot the step response of the position servo system for values of the gain $K = 0.5$, 1, and 2. Find the overshoot and rise time for each of the three step responses by examining your plots. Are the plots consistent with your calculations in parts (e) and (f)?

Solution:

$$J_m \ddot{\theta}_m + \left(b + \frac{K_t K_e}{R_a}\right)\dot{\theta}_m = \frac{K_t}{R_a} v_a.$$

(a)

$$J_m \Theta_m s^2 + \left(b + \frac{K_t K_e}{R_a}\right)\Theta_m s = \frac{K_t}{R_a} V_a(s)$$

$$\frac{s\Theta_m(s)}{V_a(s)} = \frac{\frac{K_t}{R_a J_m}}{s + \frac{b}{J_m} + \frac{K_t K_e}{R_a J_m}}.$$

$$\begin{aligned}
J_m &= 0.01 \text{ kg} \cdot \text{m}^2, \\
b &= 0.001 \text{ N} \cdot \text{m} \cdot \text{sec}, \\
K_e &= 0.02 \text{ V} \cdot \text{sec}, \\
K_t &= 0.02 \text{ N} \cdot \text{m/A}, \\
R_a &= 10\ \Omega.
\end{aligned}$$

$$\frac{s\Theta_m(s)}{V_a(s)} = \frac{0.2}{s + 0.104}.$$

(b) Final Value Theorem

$$\dot{\theta}(\infty) = \frac{s(10)(0.2)}{s(s+0.104)}|_{s=0} = \frac{2}{0.104} = 19.23.$$

(c)

$$\frac{\Theta_m(s)}{V_a(s)} = \frac{0.2}{s(s+0.104)}.$$

(d)

$$\begin{aligned}
\Theta_m(s) &= \frac{0.2K(\Theta_r - \Theta_m)}{s(s+0.104)}. \\
\frac{\Theta_m(s)}{\Theta_r(s)} &= \frac{0.2K}{s^2 + 0.104s + 0.2K}.
\end{aligned}$$

(e)

$$\begin{aligned}
M_p &= e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 0.2 \qquad (20\%), \\
\zeta &= 0.4559. \\
Y(s) &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \\
2\zeta\omega_n &= 0.104, \\
\omega_n &= \frac{0.104}{2(0.4559)} = 0.114 \text{ rad/sec}, \\
\omega_n^2 &= 0.2K, \\
K &< 6.50 \times 10^{-2}.
\end{aligned}$$

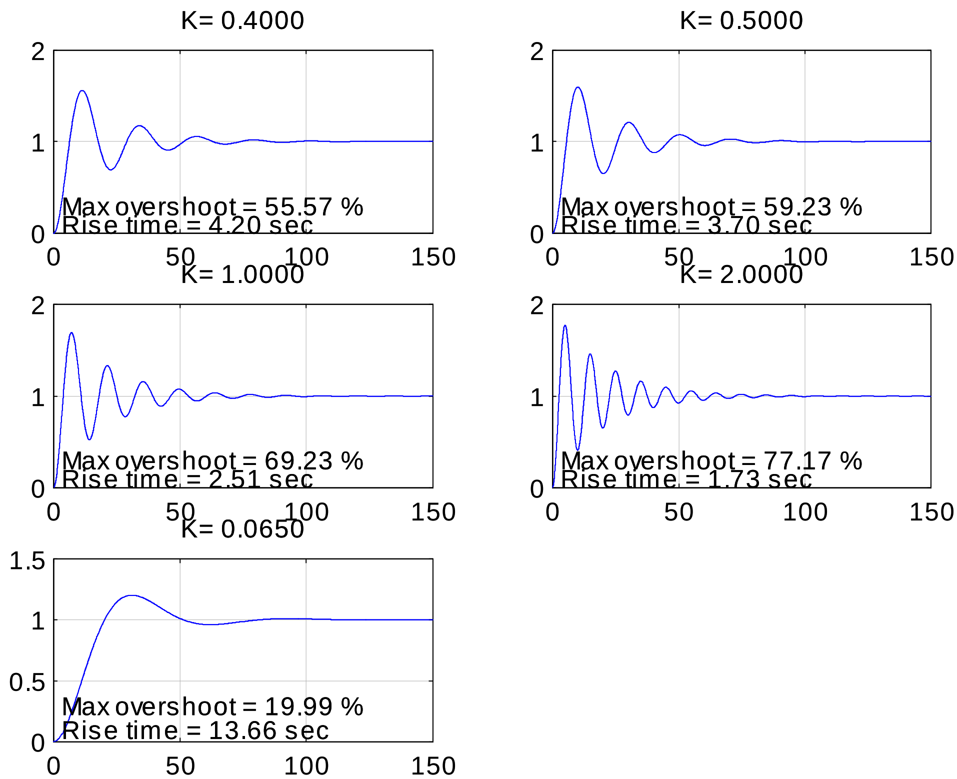
(f)

$$\begin{aligned}
\omega_n &\geq \frac{1.8}{t_r} \\
\omega_n^2 &= 0.2K \\
K &\geq 1.01.
\end{aligned}$$

(g) Matlab

```
% Problem 3.30 FPE6e
clear all
close all
K1=[0.5 1.0 2.0 6.5e-2];
t=0:0.01:150;
for i=1:1:length(K1)
K = K1(i);
titleText = sprintf(' K= %1.4f ', K);
```

```
wn = sqrt(0.2*K);
num=wn^2;
den=[1 0.104 wn^2];
zeta=0.104/2/wn;
sys = tf(num, den);
y= step(sys, t);
% Finding maximum overshoot
if zeta < 1
Mp = (max(y) - 1)*100;
overshootText = sprintf(' Max overshoot = %3.2f %', Mp);
else
overshootText = sprintf(' No overshoot');
end
% Finding rise time
idx_01 = max(find(y<0.1));
idx_09 = min(find(y>0.9));
t_r = t(idx_09) - t(idx_01);
risetimeText = sprintf(' Rise time = %3.2f sec', t_r);
% Plotting
subplot(3,2,i);
plot(t,y);
grid on;
title(titleText);
text( 0.5, 0.3, overshootText);
text( 0.5, 0.1, risetimeText);
end
%%%%%%%%%%%%%%%
% Function for computing rise time
function tr = risetime(t,y)
% A. Emami 2006
% normalize y to 1:
y = y/y(length(y));
idx1 = min(find(y>=0.1))
idx2 = min(find(y>=0.9))
if ~isempty(idx1) & ~isempty(idx2)
tr = t(idx2) - t(idx1);
else
tr = 0
end
```



Problem 3.30: Closed-loop step responses for several values of K.

For part (e) we concluded that $K < 6.50 \times 10^{-2}$ in order for $M_p < 20\%$. This is consistent with the above plots. For part (f) we found that $K \geq 1.01$ in order to have a rise time of less than 4 seconds. We actually see that our calculations is slightly off and that K can be $K \geq 0.5$, but since $K \geq 1.01$ is included in $K \geq 0.5$, our answer in part (f) is consistent with the above plots.

31. You wish to control the elevation of the satellite-tracking antenna shown in Figs. 3.61 and 3.62. The antenna and drive parts have a moment of inertia J and a damping B; these arise to some extent from bearing and aerodynamic friction, but mostly from the back emf of the DC drive motor. The equations of motion are

$$J\ddot{\theta} + B\dot{\theta} = T_c,$$

where T_c is the torque from the drive motor. Assume that

$$J = 600{,}000 \text{ kg}\cdot\text{m}^2 \quad B = 20{,}000 \text{ N}\cdot\text{m}\cdot\text{sec}.$$

Figure 3.61: Satellite Antenna (Courtesy Space Systems/Loral)

(a) Find the transfer function between the applied torque T_c and the antenna angle θ.

(b) Suppose the applied torque is computed so that θ tracks a reference command θ_r according to the feedback law

$$T_c = K(\theta_r - \theta),$$

where K is the feedback gain. Find the transfer function between θ_r and θ.

(c) What is the maximum value of K that can be used if you wish to have an overshoot $M_p < 10\%$?

(d) What values of K will provide a rise time of less than 80 sec? (Ignore the M_p constraint.)

(e) ▲Use MATLAB to plot the step response of the antenna system for K = 200, 400, 1000, and 2000. Find the overshoot and rise time of the four step responses by examining your plots. Do the plots confirm your calculations in parts (c) and (d)?

Solution:

$$J\ddot{\theta} + B\dot{\theta} = T_c$$

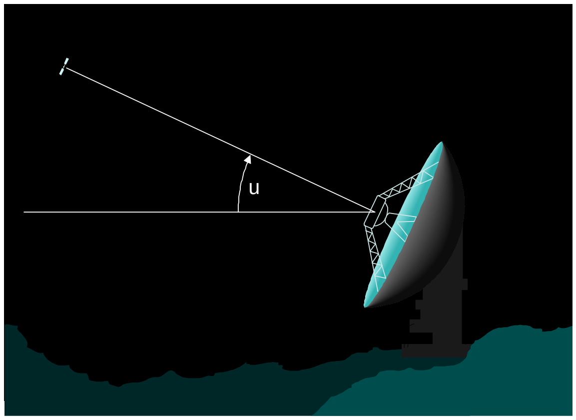


Figure 3.62: Schematic of antenna for Problem 3.31

(a)

$$\begin{aligned}
J\Theta s^2 + B\Theta s &= T_c(s), \\
\frac{\Theta(s)}{T_c(s)} &= \frac{1}{s(Js+B)}, \\
J &= 600,000 \text{ kg} \cdot \text{m}^2, \\
B &= 20,000 \text{ N} \cdot \text{m} \cdot \text{sec}, \\
\frac{\Theta(s)}{T_c(s)} &= \frac{1.667 \times 10^{-6}}{s(s+\frac{1}{30})}.
\end{aligned}$$

(b)

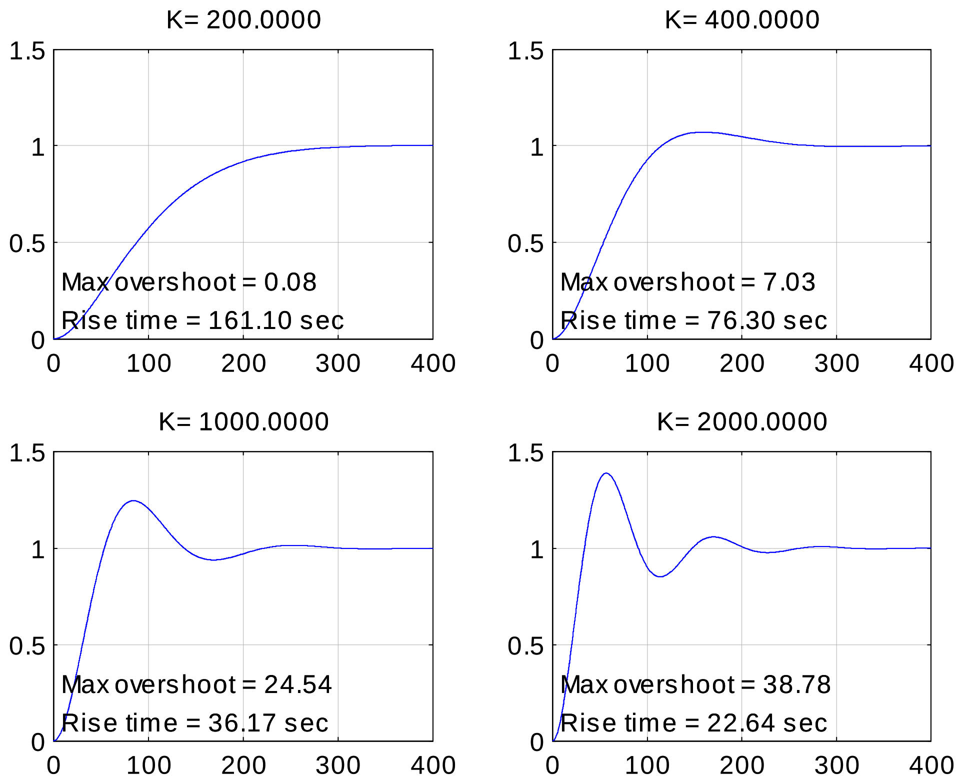
$$\begin{aligned}
\Theta(s) &= \frac{1.667 \times 10^{-6} K(\Theta_r - \Theta)}{s(s+\frac{1}{30})}, \\
\frac{\Theta(s)}{\Theta_r(s)} &= \frac{1.667K \times 10^{-6}}{s^2 + \frac{1}{30}s + 1.667K \times 10^{-6}}.
\end{aligned}$$

(c)

$$
\begin{aligned}
M_p &= e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 0.1 \qquad (10\%), \\
\zeta &= 0.591. \\
Y(s) &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}, \\
2\zeta\omega_n &= \frac{1}{30}, \\
\omega_n &= \frac{\frac{1}{30}}{2(0.591)} = 0.0282 \text{ rad/sec}, \\
\omega_n^2 &= 1.667K \times 10^{-6}, \\
K &< 477.
\end{aligned}
$$

(d)

$$
\begin{aligned}
\omega_n &\geq \frac{1.8}{t_r}, \\
\omega_n^2 &= 1.667K \times 10^{-6}, \\
K &\geq 304.
\end{aligned}
$$



(e) Problem 3.31: Step responses for several values of K.

(e) The results compare favorably with the predictions made in parts (c) and (d). For $K < 477$ the overshoot was less than 10, the rise-time was less than 80 seconds.

32. (a) Show that the second-order system

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0, \quad y(0) = y_o, \quad \dot{y}(0) = 0,$$

has the response

$$y(t) = y_o \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \cos^{-1}\zeta).$$

(b) Prove that, for the underdamped case ($\zeta < 1$), the response oscillations decay at a predictable rate (see Fig. 3.63) called the **logarith-**

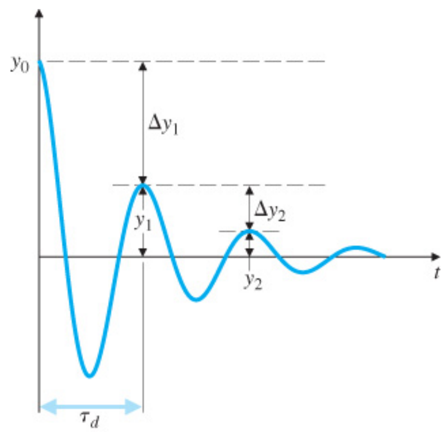


Figure 3.63: Definition of logarithmic decrement

mic decrement

$$\begin{aligned}\delta &= \ln \frac{y_o}{y_1} = \sigma\tau_d \\ &= \ln \frac{\Delta y_1}{y_1} \cong \ln \frac{\Delta y_i}{y_i},\end{aligned}$$

where

$$\tau_d = \frac{2\pi}{\omega_d}$$

is the damped natural period of vibration.

Solution:

(a) The system is second order $\Longrightarrow Q(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$. The initial condition response can be obtained by plugging a dirac delta at the input at the time 0 (this "charges" the system immediately to its initial condition and after that the system evolves by itself).

$$\begin{aligned}\text{Input}_{effective} &= y_0\delta(t) \\ \mathcal{L}\left[\text{Input}_{effective}\right] &= y_0\end{aligned}$$

We do not know whether the transfer function has finite zeros or not, but further thought will reveal the presence of at least one finite zero in the $H(s)$.

$$\lim_{s\to\infty} sH(s)y_0 = y(t)|_{0+}$$

where

$$H(s) = \frac{P(s)}{Q(s)} = \frac{P(s)}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

If $P(s)$ were a constant (no zeros in the $H(s)$), then the limit in the initial value theorem would give always zero (which is wrong because we know that the initial value must be y_0.) So we need a zero. We suggest using the following $H(s)$:

$$\begin{aligned} H(s) &= \frac{-s}{s^2+2\zeta\omega_n s+\omega_n^2}, \\ Y(s) &= H(s)y_0 = \frac{-sy_0}{s^2+2\zeta\omega_n s+\omega_n^2}, \\ &= \frac{R_+}{s-P_+}+\frac{R_-}{s-P_-}. \end{aligned}$$

where

$$\begin{aligned} P_+ &= -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}, \\ P_- &= -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}, \\ R_+ &= \frac{-\omega_n e^{j(\pi=\cos^{-1}\zeta)}}{2\omega_n\sqrt{1-\zeta^2}e^{j\pi/s}}, \\ R_- &= R_+^*. \end{aligned}$$

Note: The residues can be calculated graphically.

$$R_+ = \lim_{s\to P_+}[(s-P_+)Y(s)],$$

$$\Longrightarrow \qquad y(t) = R_+e^{P_+t}+R_-e^{P_-t}$$

$$y(t) = \frac{-e^{-\zeta\omega_n t}}{2\sqrt{1-\zeta^2}}\left[e^{+j(\omega_n\sqrt{1-\zeta^2}t+\pi/2-\cos^{-1}\zeta)}+e^{-j(\omega_n\sqrt{1-\zeta^2}t+\pi/2-\cos^{-1}\zeta)}\right],$$

$$\Longrightarrow y(t) = y_0\frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t-\cos^{-1}\zeta).$$

(b)

$$\begin{aligned} \frac{dy(t)}{dt} &= 0 \qquad \Longrightarrow t = \frac{n\pi}{\omega_d} \qquad (n \text{ is any integer}) \\ t_{Max} &= \frac{2\pi}{\omega_d}n \\ y(t)|_{t_{Max}} &\equiv y_n = y_0\frac{e^{-\sigma n\tau_d}}{\sqrt{1-\zeta^2}}\sin(\cos^{-1}\zeta). \end{aligned}$$

Note:

$$\sin(-\cos^{-1}\zeta) = \sqrt{1-\zeta^2}$$

$$y_n = \frac{y_0\sqrt{1-\zeta^2}}{\sqrt{1-\zeta^2}}e^{-\sigma n \tau_d} \qquad (*)$$

(Proof of the first line)

$$\sigma = \ln\frac{y_0}{y_n} = \sigma\tau_d,$$

From (*)

$$y_1 = y_0 e^{-\sigma\tau_d} \Longrightarrow \ln\frac{y_0}{y_n} = \sigma\tau_d.$$

(Proof of the second line)

$$\begin{aligned}
\Delta y_n &= y_{n-1} - y_n, \\
\Delta y_n &= y_0 e^{-\sigma n\tau_d} - y_0 e^{-(n-1)\sigma\tau_d} = y_0 e^{-\sigma n\tau_d}(1 - e^{\sigma\tau_d}),
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \quad \frac{\Delta y_n}{y_n} &= \frac{y_0 e^{-\sigma n\tau_d}}{y_0 e^{-\sigma n\tau_d}}(1 - e^{\sigma\tau_d}), \\
\Longrightarrow \quad \frac{\Delta y_n}{y_n} &= \frac{\Delta y_i}{yi} \qquad \text{for all } i, n.
\end{aligned}$$

Problems and Solutions for Section 3.5: Effects of Zeros and Additional Poles

33. In aircraft control systems, an ideal pitch response (q_o) versus a pitch command (q_c) is described by the transfer function

$$\frac{Q_o(s)}{Q_c(s)} = \frac{\tau\omega_n^2(s + 1/\tau)}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

The actual aircraft response is more complicated than this ideal transfer function; nevertheless, the ideal model is used as a guide for autopilot design. Assume that t_r is the desired rise time and that

$$\begin{aligned}
\omega_n &= \frac{1.789}{t_r}, \\
\frac{1}{\tau} &= \frac{1.6}{t_r}, \\
\zeta &= 0.89.
\end{aligned}$$

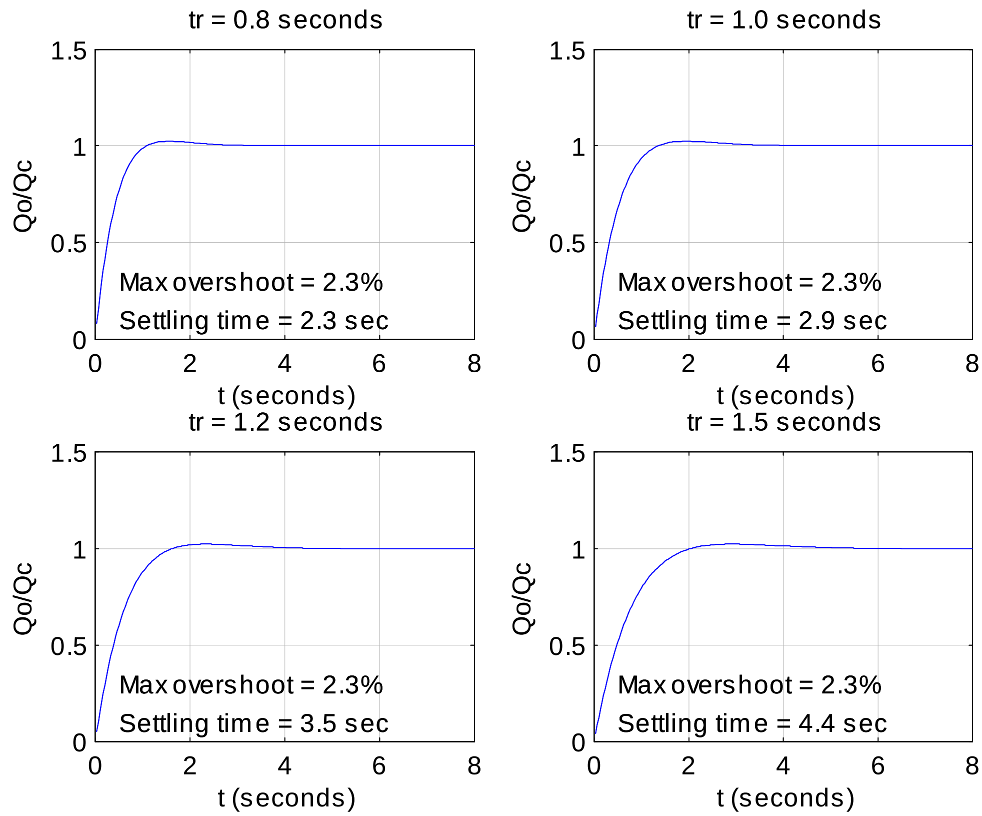
Show that this ideal response possesses a fast settling time and minimal overshoot by plotting the step response for $t_r = 0.8$, 1.0, 1.2, and 1.5 sec.

Solution:

The following program statements in MATLAB produce the following plots:

```
% Problem 3.33 FPE6e
```

```
tr = [0.8 1.0 1.2 1.5];
t=[1:240]/30;
tback=fliplr(t);
clf;
for I=1:4,
    wn=(1.789)/tr(I); %Rads/second
    tau=tr(I)/(1.6); %tau
    zeta=0.89; %
    b=tau*(wn^2)*[1 1/tau];
    a=[1 2*zeta*wn (wn^2)];
    y=step(b,a,t);
    subplot(2,2,I);
    plot(t,y);
    titletext=sprintf('tr=%3.1f seconds',tr(I));
    title(titletext);
    xlabel('t (seconds)');
    ylabel('Qo/Qc');
    ymax=(max(y)-1)*100;
    msg=sprintf('Max overshoot=%3.1f%%',ymax);
    text(.50,.30,msg);
    yback=flipud(y);
    yind=find(abs(yback-1)>0.01);
    ts=tback(min(yind));
     msg=sprintf('Settling time =%3.1f sec',ts);
    text(.50,.10,msg);
     grid;
end
```



Problem 3.33: Ideal pitch response.

34. Consider the system shown in Fig. 3.64, where

$$G(s) = \frac{1}{s(s+3)} \quad \text{and} \quad D(s) = \frac{K(s+z)}{s+p}. \tag{1}$$

Find K, z, and p so that the closed-loop system has a 10% overshoot to a step input and a settling time of 1.5 sec (1% criterion).

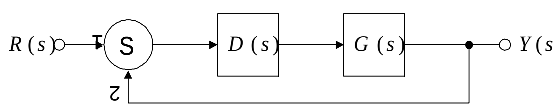


Figure 3.64: Unity feedback system for Problem 3.34

Solution:

For the 10% overshoot:

$$\begin{aligned} M_p &= e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 10\% \\ &\Longrightarrow \zeta = \sqrt{\frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}} = 0. \end{aligned}$$

For the 1.5sec (1% criterion):

$$\omega_n = \frac{4.6}{\zeta t_s} = \frac{4.6}{(0.6)(1.5)} = 5.11.$$

The closed-loop transfer function is:

$$\frac{Y(s)}{R(s)} = \frac{K\frac{s+z}{s+p} \times \frac{1}{s(s+3)}}{1 + K\frac{s+z}{s+p} \times \frac{1}{s(s+3)}} = \frac{K(s+z)}{s(s+3)(s+p) + K(s+z)}.$$

Method I.

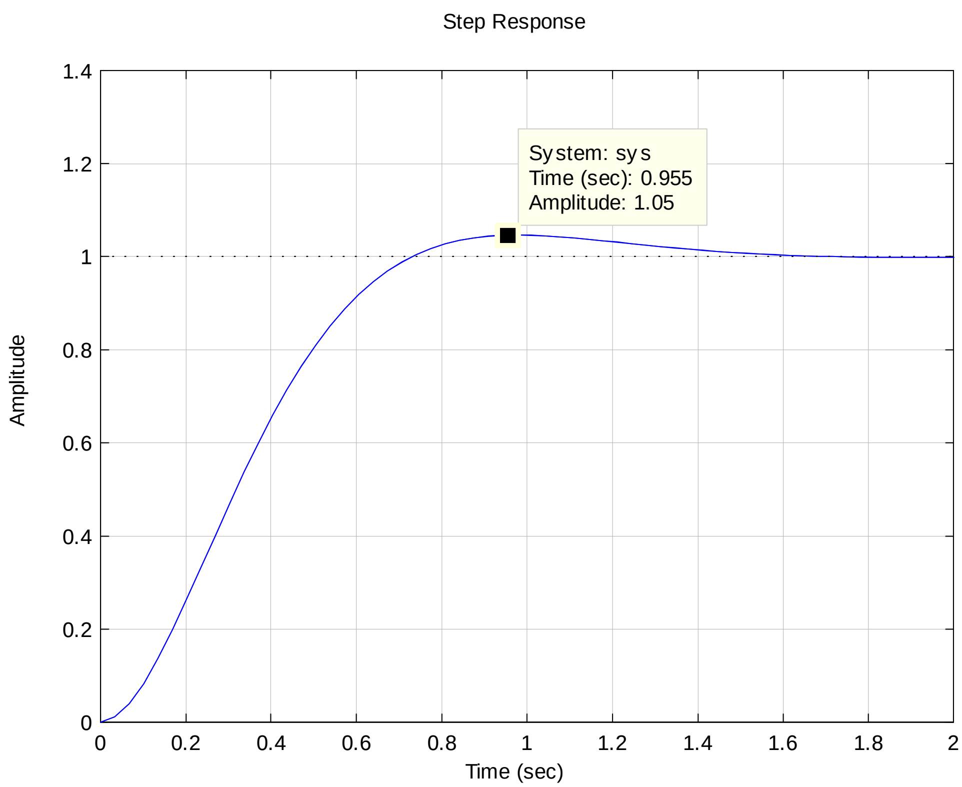
From inspection, if $z = 3$, $(s+3)$ will cancel out and we will have a standard form transfer function. As perfect cancellation is impossible, assign z a value that is very close to 3, say 3.1. But in determining the K and p, assume that $(s+3)$ and$(s+3.1)$ cancelled out each other. Then:

$$\frac{Y(s)}{R(s)} = \frac{K}{s^2 + ps + K}$$

As the additional pole and zero will affect the system response, pick some larger damping ratio.

Let $\zeta = 0.7$

$$\begin{aligned} \omega_n &= \frac{4.6}{\zeta t_s} = \frac{4.6}{(0.7)(1.5)} = 4.38, \text{ so let } \omega_n = 4.5, \\ p &= 2\zeta\omega_n = 2 \times 0.7 \times 4.5 = 6.3, \\ K &= \omega_n^2 = 20.25. \end{aligned}$$



Step response: Method I.

Method II.

There are 3 unknowns $(z,\ p,\ K)$ and only 2 specified conditions. We can arbitrarily choose p large such that complex poles will dominate in the system response.

$$\text{Try} \quad p = 10z$$

Choose a damping ratio corresponding to an overshoot of 5% (instead of 10%, just to be safe).

$$\zeta = 0.707.$$

From the formula for settling time (with a 1% criterion)

$$\omega_n = \frac{4.6}{\zeta t_s} = \frac{4.6}{0.707 \times 1.5} = 4.34,$$

adding some margin, let $\omega_n = 4.88$. The characteristic equation is

$$Q(s) = s^3 + (3+p)\,s^2 + (3p+K)s + Kz = (s+a)(s^2 + 2\zeta\omega_n s + \omega_n^2).$$

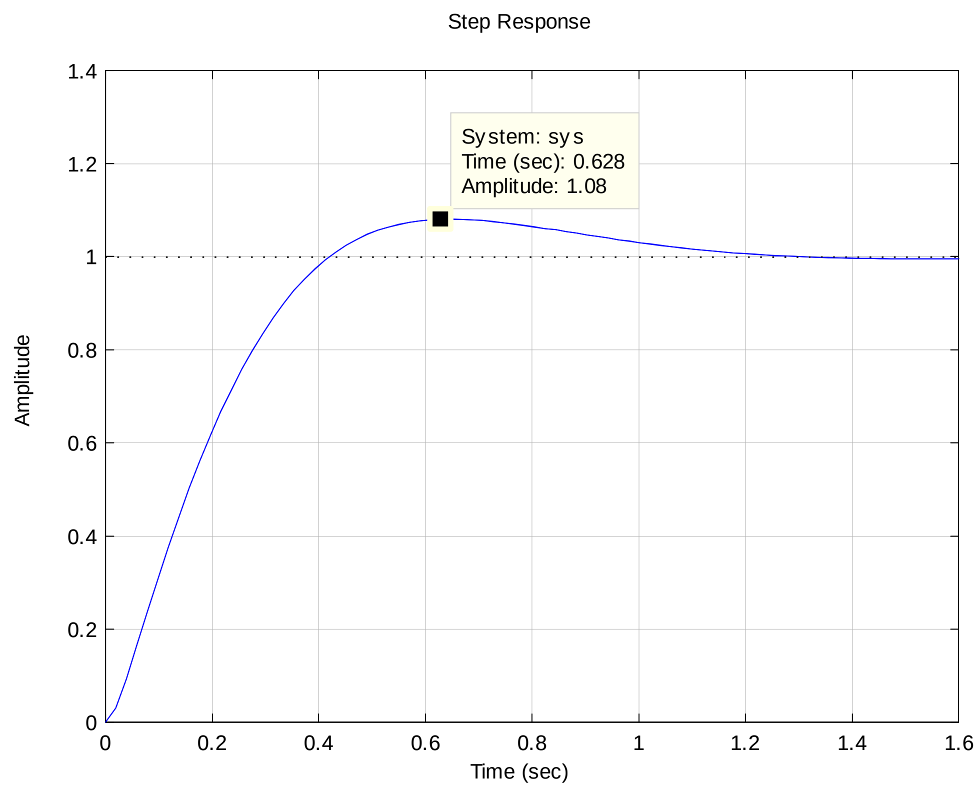
We want the characteristic equation to be the product of two factors, a couple of conjugated poles (dominant) and a non-dominant real pole far form the dominant poles.

Equate the coefficients of like powers of s in the expressions of the characteristic equation.

$$\begin{aligned} \omega_n^2 a &= Kz, \\ 2\zeta\omega_n a + \omega_n^2 &= 30z + K, \\ 2\zeta\omega_n + a &= 3 + 10z. \end{aligned}$$

Solving the three equations we get

$$\begin{aligned} z &= 5.77, \\ p &= 57.7, \\ K &= 222.45, \\ a &= 53.79. \end{aligned}$$



Step response: Method II.

35. ▲Sketch the step response of a system with the transfer function

$$G(s) = \frac{s/2+1}{(s/40+1)[(s/4)^2+s/4+1]}.$$

Justify your answer on the basis of the locations of the poles and zeros. (Do not find inverse Laplace transform.) Then compare your answer with the step response computed using MATLAB.

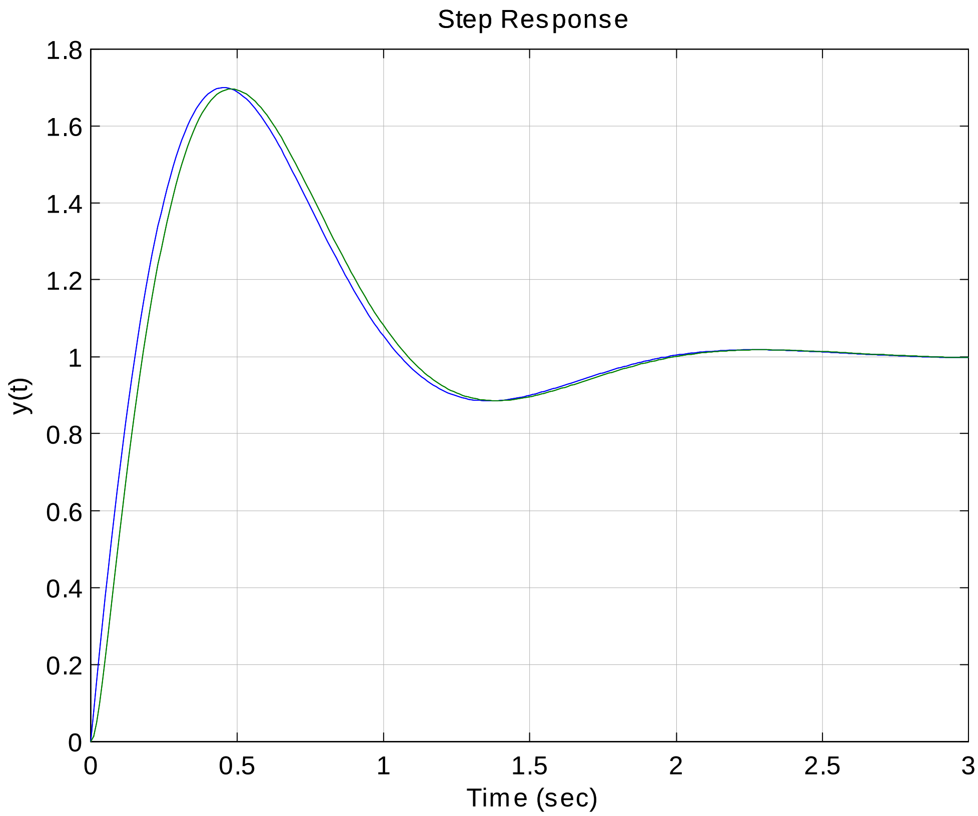
Solution:

From the location of the poles, we notice that the real pole is a factor of 20 away from the complex pair of poles. Therefore, the response of the system is *dominated* by the complex pair of poles.

$$G(s) \approx \frac{(s/2+1)}{[(s/4)^2+s/4+1]}.$$

This is now in the same form as equation (3.72) where $\alpha = 1$, $\zeta = 0.5$ and $\omega_n = 4$. Therefore, Fig. 3.27 suggests an overshoot of over 70%. The step response is the same as shown in Fig. 3.26, for $\alpha = 1$, with more than 70% overshoot and settling time of 3 seconds. The MATLAB plots below confirm this.

```
% Problem 3.35 FPE 6e

num=[1/2, 1];

den1=[1/16, 1/4, 1];

sys1=tf(num,den1);

t=0:.01:3;

y1=step(sys1,t);

den=conv([1/40, 1],den1);

sys=tf(num,den);

y=step(sys,t);

plot(t,y1,t,y);

xlabel('Time (sec)');

ylabel('y(t)');

title('Step Response');

grid on;
```



Problem 3.35: Comparison of step responses: third-order system (green), second-order approximation (blue).

36. Consider the two nonminimum phase systems,

$$G_1(s) = -\frac{2(s-1)}{(s+1)(s+2)}, \tag{2}$$

$$G_2(s) = \frac{3(s-1)(s-2)}{(s+1)(s+2)(s+3)}. \tag{3}$$

(a) Sketch the unit step responses for $G_1(s)$ and $G_2(s)$, paying close attention to the transient part of the response.

(b) Explain the difference in the behavior of the two responses as it relates to the zero locations.

(c) Consider a stable, strictly proper system (that is, m zeros and n poles, where $m < n$). Let $y(t)$ denote the step response of the system. The step response is said to have an undershoot if it initially starts off in the "wrong" direction. Prove that a stable, strictly proper system has an undershoot if and only if its transfer function has an *odd* number of *real* RHP zeros.

Solution:

(a) For $G_1(s)$:

$$
\begin{aligned}
Y_1(s) &= \frac{1}{s}G_1(s) = \frac{-2(s-1)}{s(s+1)(s+2)}, \\
H(s) &= k\frac{\prod^j (s-z_j)}{\prod^l (s-p_l)}, \\
R_{p_i} &= \lim_{s\to p_i}[(s-p_i)H(s)] = \lim_{s\to p_i} k\frac{\prod^j (s-z_j)}{\prod^l_{l\neq i}(s-p_l)} = k\frac{\prod^j (p_i-z_j)}{\prod^l_{l\neq i}(p_i-p_l)}.
\end{aligned}
$$

Each factor $(p_i - z_j)$ or $(p_i - p_l)$ can be thought of as a complex number (a magnitude and a phase) whose pictorial representation is a vector pointing to p_i and coming from z_j or p_l respectively.

The method for calculating the residue at a pole p_i is:

(1) Draw vectors from the rest of the poles and from all the zeros to the pole p_i.

(2) Measure magnitude and phase of these vectors.

(3) The residue will be equal to the gain, multiplied by the product of the vectors coming from the zeros and divided by the product of the vectors coming from the poles.

In our problem:

$$
\begin{aligned}
Y_1(s) &= \frac{-2(s-1)}{s(s+1)(s+2)} = \frac{R_0}{s} + \frac{R_{-1}}{(s+1)} + \frac{R_{-2}}{(s+2)} = \frac{1}{s} - \frac{4}{s+1} + \frac{3}{s+2}, \\
y_1(t) &= 1 - 4e^{-t} + 3e^{-2t}.
\end{aligned}
$$

For $G_2(s)$:

$$
\begin{aligned}
Y_2(s) &= \frac{3(s-1)(s-2)}{s(s+1)(s+2)(s+3)} = \frac{1}{s} + \frac{-9}{(s+1)} + \frac{18}{(s+2)} + \frac{-10}{(s+3)}, \\
y_2(t) &= 1 - 9e^{-t} + 18e^{-2t} - 10e^{-3t}.
\end{aligned}
$$

(b) The first system presents an “undershoot”. The second system, on the other hand, starts off in the right direction.

The reasons for this initial behavior of the step response will be analyzed in part c.

In $y_1(t)$: dominant at $t = 0$ the term $-4e^{-t}$

In $y_2(t)$: dominant at $t = 0$ the term $18e^{-2t}$

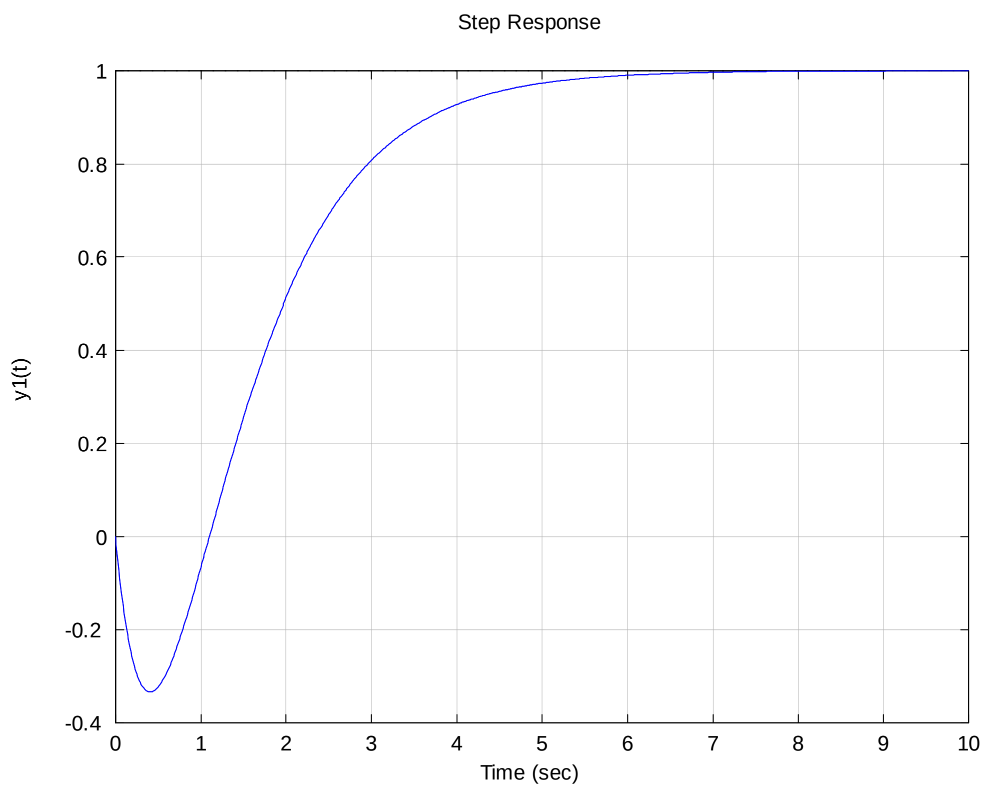


Figure 3.65: Problem 3.36: Step response for a non-minimum phase system with one *real* RHP zero.

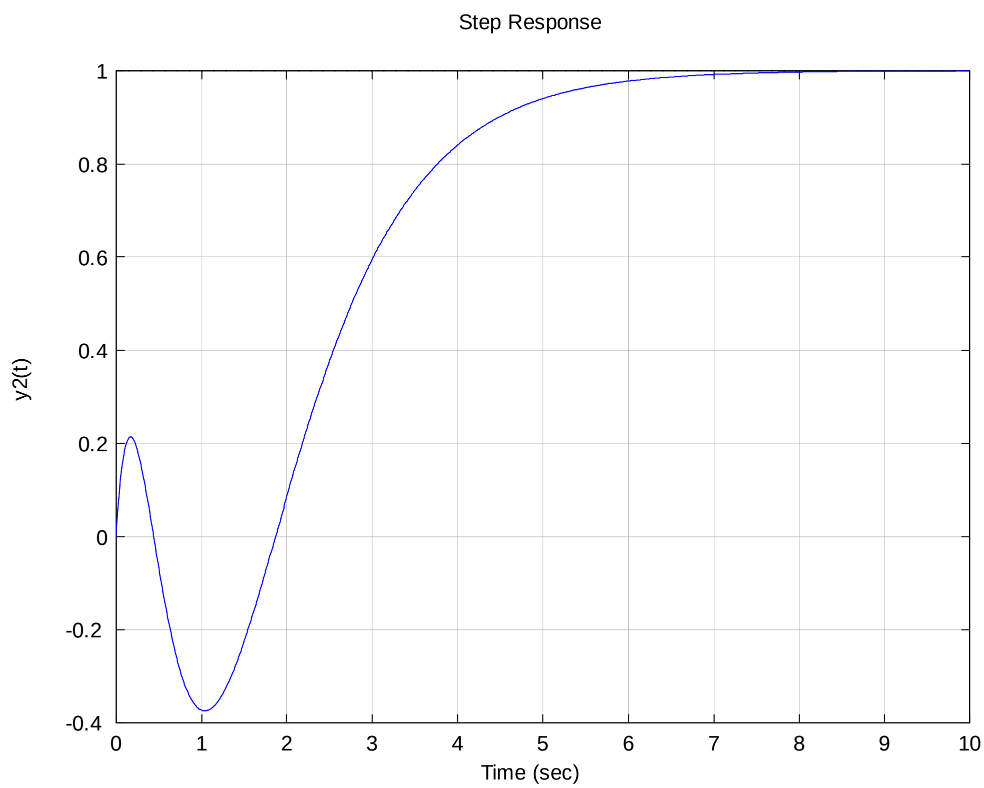


Figure 3.66: Problem 3.36: Step response of a non-minimum phase system with two *real* zeros in the RHP.

(c) The following concise proof is from Reference [1] (see also References [2]-[3]).

Without loss of generality assume the system has unity DC gain $(G(0) = 1)$ Since the system is stable, $y(\infty) = G(0) = 1$, and it is reasonable to assume $y(\infty) \neq 0$. Let us denote the pole-zero excess as $r = n - m$. Then, $y(t)$ and its $r - 1$ derivatives are zero at $t = 0$, and $y^r(0)$ is the first non-zero derivative. The system has an undershoot if $y^r(0)y(\infty) < 0$. The transfer function may be re-written as

$$G(s) = \frac{\prod_{i=1}^{m}(1 - \frac{s}{z_i})}{\prod_{i=1}^{m+r}(1 - \frac{s}{p_i})}$$

The *numerator* terms can be classified into three types of terms:

(1). The first group of terms are of the form $(1 - \alpha_i s)$ with $\alpha_i > 0$.

(2). The second group of terms are of the form $(1+\alpha_i s)$ with $\alpha_i > 0$.

(3). Finally, the third group of terms are of the form, $(1+\beta_i s+\alpha_i s^2)$ with $\alpha_i > 0$, and β_i could be negative.

However, $\beta_i^2 < 4\alpha_i$, so that the corresponding zeros are complex.

All the *denominator* terms are of the form (2), (3), above. Since,

$$y^r(0) = \lim_{s\to\infty} s^r G(s)$$

it is seen that the *sign* of $y^r(0)$ is determined entirely by the number of terms of group 3 above. In particular, if the number is *odd*, then $y^r(0)$ is *negative* and if it is even, then $y^r(0)$ is positive. Since $y(\infty) = G(0) = 1$, then we have the desired result.

References

[1] Vidyasagar, M., "On Undershoot and Nonminimum Phase Zeros," *IEEE Trans. Automat. Contr.*, Vol. AC-31, p. 440, May 1986.

[2] Clark, R., N., *Introduction to Automatic Control Systems*, John Wiley, 1962.

[3] Mita, T. and H. Yoshida, "Undershooting phenomenon and its control in linear multivariable servomechanisms, " *IEEE Trans. Automat. Contr.*, Vol. AC-26, pp. 402-407, 1981.

37. Find the relationships for the impulse response and the step response corresponding to Equation (3.57) for the cases where,

(a) the roots are repeated.

(b) the roots are both real. Express your answers in terms of hyperbolic functions (sinh, cosh) to best show the properties of the system response.

(c) the value of the damping coefficient, ζ, is negative.

Solution:

(a) In this case we have $\zeta = 1$

$$H(s) = \frac{Y(s)}{U(s)} = \frac{\omega_n^2}{(s+\omega_n)^2}.$$

For the impulse response,$U(s) = 1$, and using Item #8 from Table A.2 we find

$$h(t) = \omega_n^2 t e^{-\omega_n t}.$$

We can then integrate the impulse response to obtain the step response. Alternatively, for a unit step input, $U(s) = \frac{1}{s}$ and

$$Y(s) = \frac{\omega_n^2}{(s+\omega_n)^2}\frac{1}{s}.$$

Using Item #15 from Table A.2 we find

$$y(t) = 1 - e^{-\omega_n t}\left(1 + \omega_n t\right).$$

(b) We re-write $H(s)$ as follows

$$H(s) = \frac{Y(s)}{U(s)} = \frac{\omega_n^2}{(s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})(s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})},$$

where $|\zeta| > 1$. For the impulse response,$U(s) = 1$ and using Item #13 from Table A.2,

$$\begin{aligned} h(t) &= -\frac{\omega_n^2}{2\omega_n\sqrt{\zeta^2-1}}\left(e^{-(\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})t} - e^{-(\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})t}\right), \\ &= \frac{\omega_n}{2\sqrt{\zeta^2-1}}e^{-\zeta\omega_n t}\left(e^{+(\omega_n\sqrt{\zeta^2-1})t} - e^{-(\omega_n\sqrt{\zeta^2-1})t}\right), \\ &= \frac{\omega_n}{\sqrt{\zeta^2-1}}e^{-\zeta\omega_n t}\sinh(\omega_n\sqrt{\zeta^2-1t}). \end{aligned}$$

We can then integrate the impulse response to obtain the unit step response. Alternatively, for a unit step input, $U(s) = \frac{1}{s}$ and using partial fraction expansion

$$Y(s) = \frac{1}{s} + \frac{\frac{1}{2\sqrt{\zeta^2-1}\left(\zeta+\sqrt{\zeta^2-1}\right)}}{s+\zeta\omega_n+\omega_n\sqrt{\zeta^2-1}} - \frac{\frac{1}{2\sqrt{\zeta^2-1}\left(\zeta-\sqrt{\zeta^2-1}\right)}}{s+\zeta\omega_n-\omega_n\sqrt{\zeta^2-1}},$$

and using Item #7 from Table A.2

$$\begin{aligned}
y(t) &= 1 + \frac{1}{2\sqrt{\zeta^2-1}\left(\zeta+\sqrt{\zeta^2-1}\right)} e^{-(\zeta\omega_n+\omega_n\sqrt{\zeta^2-1})t} \\
&\quad - \frac{1}{2\sqrt{\zeta^2-1}\left(\zeta-\sqrt{\zeta^2-1}\right)} e^{-(\zeta\omega_n-\omega_n\sqrt{\zeta^2-1})t}, \\
&= 1 + \frac{1}{2\sqrt{\zeta^2-1}} e^{-\zeta\omega_n t}\left(\frac{1}{\left(\zeta+\sqrt{\zeta^2-1}\right)} e^{-\omega_n\sqrt{\zeta^2-1}t} - \frac{1}{\left(\zeta-\sqrt{\zeta^2-1}\right)} e^{+\omega_n\sqrt{\zeta^2-1}t}\right), \\
&= 1 + \frac{1}{2\sqrt{\zeta^2-1}} e^{-\zeta\omega_n t}\left(\left(\zeta-\sqrt{\zeta^2-1}\right) e^{-\omega_n\sqrt{\zeta^2-1}t} - \left(\zeta+\sqrt{\zeta^2-1}\right) e^{+\omega_n\sqrt{\zeta^2-1}t}\right), \\
y(t) &= 1 - e^{-\zeta\omega_n t}\left(\cosh\left(-\omega_n\sqrt{\zeta^2-1t}\right) + \frac{\zeta}{\sqrt{\zeta^2-1}}\sinh\left(\omega_n\sqrt{\zeta^2-1t}\right)\right).
\end{aligned}$$

Notice that unlike the expression for the impulse response on FPE 6e page 112 (Eq. 3.58) and the step response on FPE 6e page 117 (Eq. 3.61), these responses *do not oscillate* due to the behavior of the cosh and sinh functions.

(c) Now we have the remaining case where ζ is negative and $|\zeta| < 1$, since we already dealt with the case of $|\zeta| > 1$ in the previous part (b). The impulse response and the step responses are exactly the same as given in on pages 112 (Eq. 3.58) and 117 (Eq. 3.61)

$$\begin{aligned}
h(t) &= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t}\sin(\omega_d t) \\
y(t) &= 1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)
\end{aligned}$$

except now ζ is negative and the exponential terms become unbounded and the system is unstable.

38. Consider the following second-order system with an extra pole:

$$H(s) = \frac{\omega_n^2 p}{(s+p)(s^2+2\zeta\omega_n s+\omega_n^2)}.$$

Show that the unit step response is

$$y(t) = 1 + Ae^{-pt} + Be^{-\sigma t}\sin(\omega_d t - \theta),$$

where

$$A = \frac{-\omega_n^2}{\omega_n^2 - 2\zeta\omega_n p + p^2},$$
$$B = \frac{p}{\sqrt{(p^2 - 2\zeta\omega_n p + \omega_n^2)(1-\zeta^2)}},$$
$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{-\zeta} + \tan^{-1}\frac{\omega_n\sqrt{1-\zeta^2}}{p - \zeta\omega_n}.$$

(a) Which term dominates $y(t)$ as p gets large?

(b) Give approximate values for A and B for small values of p.

(c) Which term dominates as p gets small? (Small with respect to what?)

(d) Using the preceding explicit expression for $y(t)$ or the step command in MATLAB, and assuming that $\omega_n = 1$ and $\zeta = 0.7$, plot the step response of the preceding system for several values of p ranging from very small to very large. At what point does the extra pole cease to have much effect on the system response?

Solution:

Second-order system:

$$H(s) = \frac{\omega_n^2 p}{(s+p)(s^2 + 2\zeta\omega_n s + \omega_n^2)}.$$

Unit step response:

$$Y(s) = \frac{1}{s}H(s), \quad y(t) = \mathcal{L}^{-1}\{Y(s)\},$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = (s + \sigma + j\omega_d)(s + \sigma - j\omega_d),$$

where

$$\sigma = \zeta\omega_n, \quad \omega_d = \omega_n\sqrt{1-\zeta^2}.$$

Thus from partial fraction expansion:

$$Y(s) = \frac{k_1}{s} + \frac{k_2}{s+p} + \frac{k_3}{s+\sigma+j\omega_d} + \frac{k_4}{s+\sigma-j\omega_d},$$

solving for k_1, k_2, k_3, and k_4:

$$
\begin{aligned}
k_1 &= H(0) \Longrightarrow k_1 = 1, \\
k_2 &= \frac{\omega_n^2 p}{s(s+\sigma+j\omega_d)(s+\sigma-j\omega_d)}|_{s=-p} \Longrightarrow k_2 = \frac{-\omega_n^2}{\omega_n^2 - 2p\zeta\omega_n + p^2}, \\
k_3 &= (s+\sigma+j\omega_d)Y(s)|_{s=-\sigma-j\omega_d} \\
\Longrightarrow & \; k_3 = \frac{p}{2\sqrt{(1-\zeta^2)\left(p^2 - 2p\zeta\omega_n + \omega_n^2\right)}} e^{-i\theta} = |k_3|\, e^{-i\theta} \\
k_4 &= k_3^*
\end{aligned}
$$

where

$$\theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{-\zeta}\right) + \tan^{-1}\left(\frac{\omega_n\sqrt{1-\zeta^2}}{p-\zeta\omega_n}\right).$$

Thus

$$Y(s) = \frac{1}{s} + \frac{k_2}{s+p} + |k_3|\left(\frac{e^{-i\theta}}{s+\sigma+j\omega_d} + \frac{e^{+i\theta}}{s+\sigma-j\omega_d}\right).$$

Inverse Laplace:

$$y(t) = 1 + k_2 e^{-pt} + |k_3|\left(e^{-i\theta}e^{-(\sigma+j\omega_d)t} + e^{+i\theta}e^{-(\sigma-j\omega_d)t}\right),$$

or

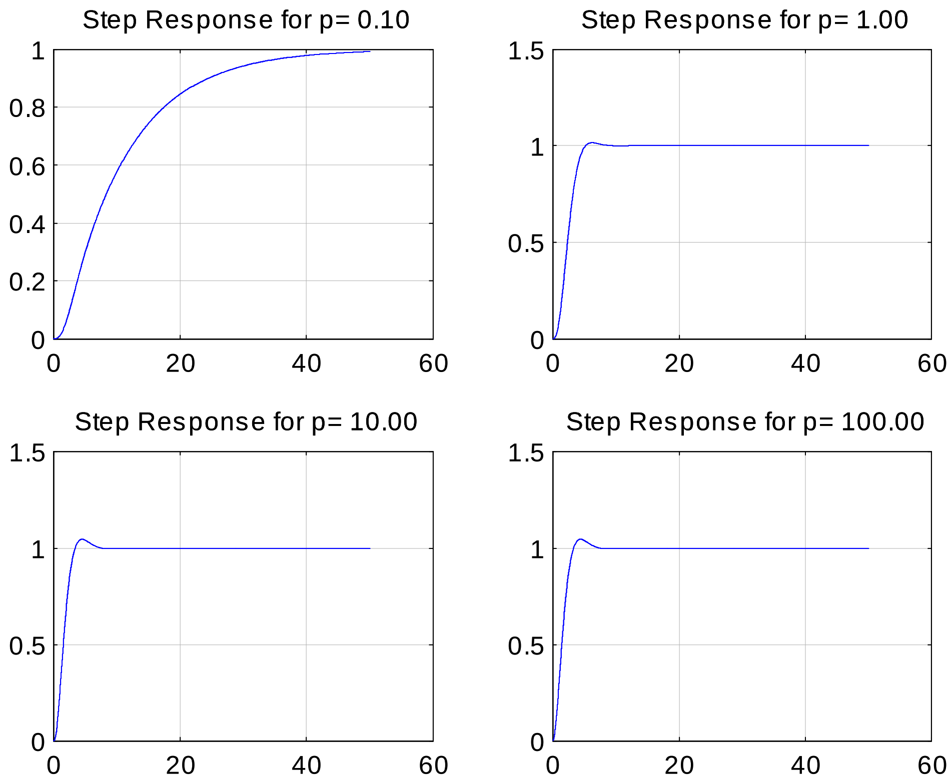
$$y(t) = 1 + \underbrace{\frac{-\omega_n^2}{\omega_n^2 - 2p\zeta\omega_n + p^2}}_{A} e^{-pt} + \underbrace{\frac{p}{\sqrt{(1-\zeta^2)\left(p^2 - 2p\zeta\omega_n + \omega_n^2\right)}}}_{B} e^{-\sigma t}\cos(\omega_d t + \theta).$$

(a) As p gets large the B term dominates.

(b) For small p: $A \approx -1, B \approx 0$.

(c) As p gets small A dominates.

(d) The effect of a change in p is not noticeable above $p \approx 10$.



Problem 3.38: Step responses for several values of p.

39. Consider the second order unity DC gain system with one finite zero,

$$H(s) = \frac{\omega_n^2(s+z)}{z(s^2 + 2\zeta\omega_n s + \omega_n^2)}.$$

(a) Show that the unit-step response is

$$y(t) = 1 + \frac{1}{z}\frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\sqrt{\omega_n^2 + z^2 - 2\zeta\omega_n}\cos(\omega_d t - \beta_1),$$

where

$$\beta_1 = \tan^{-1}\frac{\zeta z - \omega_n}{\sqrt{1-\zeta^2}z}.$$

(b) Derive an expression for the overshoot, M_p, for this system.

(c) For a given value of overshoot, M_p, how do we solve for ζ and ω_n?

Solution:

(a). We write the transfer function in partial fraction form,

$$H(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} + \frac{1}{z}\frac{\omega_n^2 s}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}.$$

The step response of the first term is as given in Chapter 3, and that of the second term is simply the derivative of that (i.e., the impulse response) scaled by $1/z$:

$$\begin{aligned}
y(t) &= y_1 + \frac{1}{z}\frac{dy_1}{dt}, \\
y(t) &= 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \beta) + \frac{1}{z}\left[\frac{\sigma e^{-\sigma t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \beta) + \frac{\omega_d e^{-\sigma t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t - \beta)\right] \\
y(t) &= 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\left(-1 + \frac{\sigma}{z}\right)\cos(\omega_d t - \beta) + \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\frac{\omega_d}{z}\sin(\omega_d t - \beta) \\
\text{where } \beta &= \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}.
\end{aligned}$$

Now as in Chapter 3 we combine the last two terms to yield,

$$\begin{aligned}
y(t) &= 1 + \frac{1}{z}\frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\left(\sqrt{\omega_n^2 + z^2 - 2\zeta\omega_n}\right)\cos(\omega_d t - (\beta + \beta_2)), \\
\text{where } \beta_2 &= \tan^{-1}\frac{\omega_n\sqrt{1-\zeta^2}}{\zeta\omega_n - z}.
\end{aligned}$$

Using the trigonometric identity,

$$\tan^{-1} A + \tan^{-1} B = \tan^{-1}\frac{A+B}{1-AB},$$

we combine the last two terms in the argument of the cosine term,

$$\beta_1 = \beta + \beta_2 = \tan^{-1}\left(\frac{\frac{\zeta}{\sqrt{1-\zeta^2}} + \frac{\omega_n\sqrt{1-\zeta^2}}{\zeta\omega_n - z}}{1 - \frac{\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_n\sqrt{1-\zeta^2}}{\zeta\omega_n - z}}\right) = \tan^{-1}\frac{\zeta z - \omega_n}{\sqrt{1-\zeta^2}\,z}.$$

Hence we have the final desired result,

$$y(t) = 1 + \frac{1}{z}\frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\left(\sqrt{\omega_n^2 + z^2 - 2\zeta\omega_n}\right)\cos(\omega_d t - \beta_1).$$

Alternative Solution for (a):

$$
\begin{aligned}
y(t) &= y_1 + \frac{1}{z}\frac{dy_1}{dt}, \\
&= 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\cos(\omega_d t - \beta) + \frac{1}{z}\left[\frac{\omega_n e^{-\sigma t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right], \\
&= 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\left[(\frac{-\omega_n}{z} + \sin(\beta))\sin(\omega_d t) + \cos(\omega_d t)\cos(\beta)\right], \\
\text{where } \beta &= \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}.
\end{aligned}
$$

$$
\begin{aligned}
\cos(\beta) &= \cos\left(\tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}\right) = \frac{1}{\sqrt{1+\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)^2}} = \sqrt{1-\zeta^2}, \\
\sin(\beta) &= \sin\left(\tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}\right) = \frac{\frac{\zeta}{\sqrt{1-\zeta^2}}}{\sqrt{1+\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)^2}} = \zeta,
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\left[(\frac{-\omega_n}{z} + \zeta)\sin(\omega_d t) + \sqrt{1-\zeta^2}\cos(\omega_d t)\right], \\
y(t) &= 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\left[\sqrt{(\frac{-\omega_n}{z} + \zeta)^2 + \left(\sqrt{1-\zeta^2}\right)^2}\cos(\omega_d t - \beta_1)\right], \\
y(t) &= 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\left[\sqrt{(\frac{-\omega_n}{z} + \zeta)^2 + 1 - \zeta^2}\cos(\omega_d t - \beta_1)\right], \\
y(t) &= 1 + \frac{1}{z}\frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\sqrt{\omega_n^2 + z^2 - 2\zeta\omega_n}\cos(\omega_d t - \beta_1), \\
\text{where } \beta_1 &= \tan^{-1}\frac{\zeta z - \omega_n}{\sqrt{1-\zeta^2}z}.
\end{aligned}
$$

(b) At peak time t_p, we have that

$$
\frac{dy(t_p)}{dt} = 0,
$$

$$
-\frac{\sigma}{z}\frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\left(\sqrt{\omega_n^2 + z^2 - 2\zeta\omega_n}\right)\cos(\omega_d t - \beta_1) -
$$

$$
\frac{\omega_d}{z}\frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\left(\sqrt{\omega_n^2 + z^2 - 2\zeta\omega_n}\right)\sin(\omega_d t - \beta_1) = 0,
$$

$$\begin{aligned}
\cos(\omega_d t - \beta_1 - \beta_3) &= 0, \\
\beta_3 &= \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}, \\
\beta_1 + \beta_3 &= \tan^{-1}\frac{\frac{\zeta z - \omega_n}{\sqrt{1-\zeta^2}z} + \frac{\sqrt{1-\zeta^2}}{\zeta}}{1 - \frac{\zeta z-\omega_n}{\sqrt{1-\zeta^2}z}\frac{\sqrt{1-\zeta^2}}{\zeta}} = \tan^{-1}\left(\frac{z-\omega_n}{\omega_n}\right), \\
t_p &= \frac{1}{\omega_d}\left[\tan^{-1}\left(\frac{z-\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\right) + \frac{3}{2}\pi\right], \\
M_p &= y(t_p) - 1, \\
M_p &= \frac{1}{z}\sqrt{z^2 - z\zeta\omega_n + \omega_n^2}e^{-\sigma t_p}.
\end{aligned}$$

(c) For a given overshoot M_p, the values of ω_n and ζ have to be found by trial and error. In general, they will be different than the standard second order system values unless z is large that is the zero is far away.

40. The block diagram of an autopilot designed to maintain the pitch attitude θ of an aircraft is shown in Fig. 3.67. The transfer function relating the elevator angle δ_e and the pitch attitude θ is

$$\frac{\theta(s)}{\delta_e(s)} = G(s) = \frac{50(s+1)(s+2)}{(s^2+5s+40)(s^2+0.03s+0.06)},$$

where θ is the pitch attitude in degrees and δ_e is the elevator angle in degrees. The autopilot controller uses the pitch attitude error e to adjust the elevator according to the transfer function

$$\frac{\delta_e(s)}{e(s)} = D(s) = \frac{K(s+3)}{s+10}.$$

Using MATLAB, find a value of K that will provide an overshoot of less than 10% and a rise time faster than 0.5 sec for a unit-step change in θ_r. After examining the step response of the system for various values of K, comment on the difficulty associated with making rise-time and overshoot measurements for complicated systems.

Solution:

$$\begin{aligned}
G(s) &= \frac{\Theta(s)}{\delta_e(s)} = \frac{50\,(s+1)\,(s+2)}{(s^2+5s+40)\,(s^2+0.03s+0.06)}, \\
D(s) &= \frac{\delta_e(s)}{e(s)} = \frac{K(s+3)}{(s+10)},
\end{aligned}$$

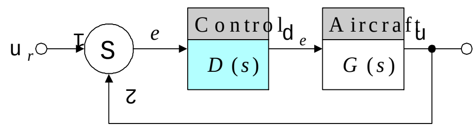
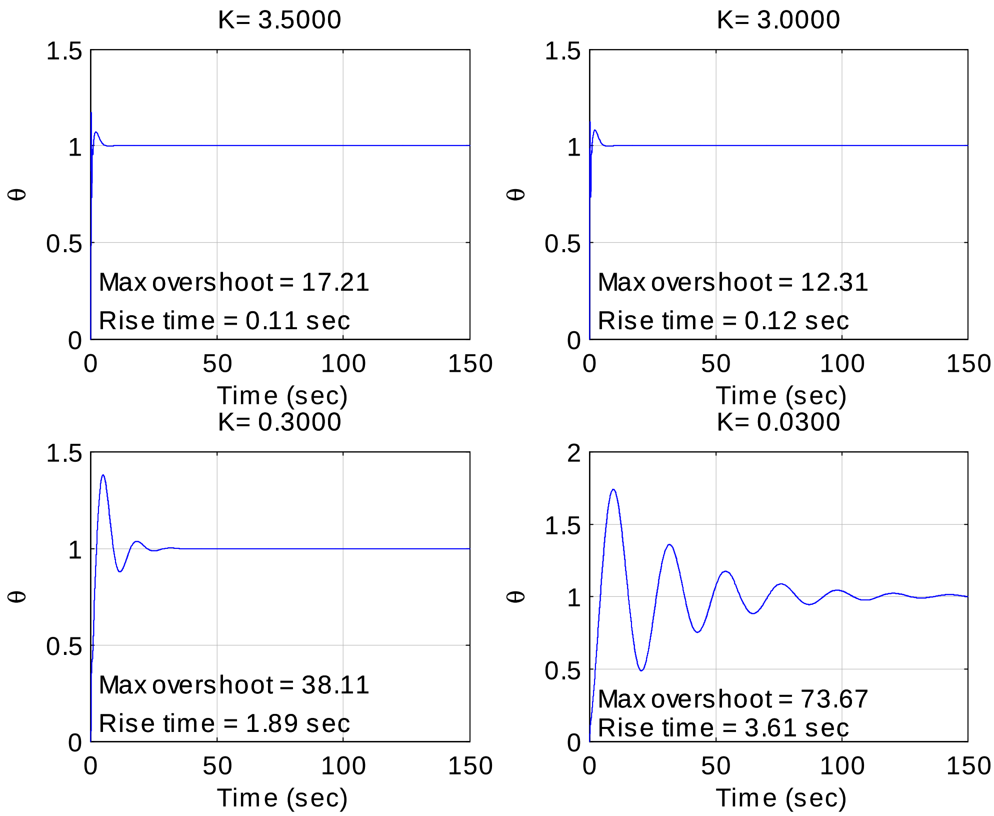


Figure 3.67: Block diagram of autopilot

where

$$
\begin{aligned}
e(s) &= \Theta_r - \Theta, \\
\frac{\Theta(s)}{\Theta_r(s)} &= \frac{G(s)D(s)}{1+G(s)D(s)}, \\
&= \frac{50K\,(s+1)\,(s+2)\,(s+3)}{(s^2+5s+40)\,(s^2+0.03s+0.06)\,(s+10)+K\,(s+3)}, \\
&= \frac{50K\left(s^3+6s^2+11s+6\right)}{s^5+15.03s^4+(50K+90.51)s^3+(300K+403.6)s^2+(17.4+550K)\,s+(24+300K)}.
\end{aligned}
$$

Output must be normalized to the final value of $\frac{\Theta(s)}{\Theta_r(s)}$ for easy computation of the overshoot and rise-time. In this case the design criterion for overshoot cannot be met easily which is indicated in the sample plots.



Problem 3.40: Step responses for an autopilot for various values of K.

Problems and Solutions for Section 3.7: Stability

41. A measure of the degree of instability in an unstable aircraft response is the amount of time it takes for the *amplitude* of the time response to double (see Fig. 3.68), given some nonzero initial condition.

    (a) For a first-order system, show that the **time to double** is

    $$\tau_2 = \frac{\ln 2}{p},$$

    where p is the pole location in the RHP.

    (b) For a second-order system (with two complex poles in the RHP), show that

    $$\tau_2 = \frac{\ln 2}{-\zeta\omega_n}.$$

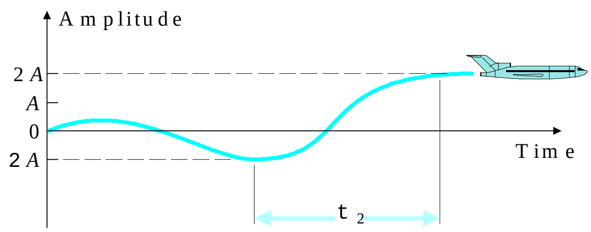


Figure 3.68: Time to double

Solution:

(a) First-order system, $H(s)$ could be:

$$\begin{aligned}
H(s) &= \frac{k}{(s-p)}, \\
h(t) &= \mathcal{L}^{-1}\left[H(s)\right] = ke^{pt}, \\
h(\tau_0) &= ke^{p\tau_0}, \\
h\left(\tau_0+\tau_2\right) &= 2h\left(\tau_0\right) = ke^{p(\tau_0+\tau_2)},
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \qquad & 2ke^{p\tau_0} = ke^{p\tau_0}e^{p\tau_2}, \\
\Longrightarrow \qquad & \tau_2 = \frac{\ln 2}{p}.
\end{aligned}$$

(b) Second-order system:

$$y\left(t\right) = y_0 \frac{e^{\omega_n|\zeta|t}}{\sqrt{1-\left|\zeta^2\right|}} \sin\left(\omega_n\sqrt{1-\left|\zeta^2\right|}t + \cos^{-1}\zeta\right),$$

where

$$\cos^{-1}\zeta = \cos^{-1}|\zeta| + \pi$$

$$\Longrightarrow y\left(t\right) = y_0 \frac{e^{\omega_n|\zeta|t}}{\sqrt{1-\left|\zeta^2\right|}} (-1)\sin\left(\omega_n\sqrt{1-\left|\zeta^2\right|}t + \cos^{-1}|\zeta|\right)$$

Note: Instead of working with a negative ζ, everything is changed to $|\zeta|$.

$$
\begin{aligned}
|t_0| &= -y_0 \frac{e^{\omega_n |\zeta| t}}{\sqrt{1 - |\zeta^2|}}, \\
|\tau_0| &= -y_0 \frac{e^{\omega_n |\zeta| \tau_0}}{\sqrt{1 - |\zeta^2|}}, \\
|\tau_0 + \tau_2| &= -y_0 \frac{e^{\omega_n |\zeta| (\tau_0 + \tau_2)}}{\sqrt{1 - |\zeta^2|}} = 2 |\tau_0|
\end{aligned}
$$

$$
\begin{aligned}
&\Longrightarrow \qquad e^{\omega_n |\zeta| \tau_2} = 2 \\
&\Longrightarrow \qquad \tau_2 = \frac{\ln 2}{\omega_n |\zeta|} = \frac{\ln 2}{-\omega_n \zeta} \qquad (\zeta \le 0)
\end{aligned}
$$

Note: This problem shows that $\sigma = \omega_n |\zeta|$ (the real part of the poles) is inversely proportional to the time to double.

The further away from the imaginary axis the poles lie, the faster the response is (either increasing faster for RHP poles or decreasing faster for LHP poles).

42. Suppose that unity feedback is to be applied around the listed open-loop systems. Use Routh's stability criterion to determine whether the resulting closed-loop systems will be stable.

(a) $KG(s) = \frac{4(s+2)}{s(s^3+2s^2+3s+4)}$

(b) $KG(s) = \frac{2(s+4)}{s^2(s+1)}$

(c) $KG(s) = \frac{4(s^3+2s^2+s+1)}{s^2(s^3+2s^2-s-1)}$

Solution:

(a)

$$1 + KG = s^4 + 2s^3 + 3s^2 + 8s + 8 = 0.$$

The Rouh array is,

$$
\begin{array}{lllll}
s^4 & : & 1 & 3 & 8 \\
s^3 & : & 2 & 8 & \\
s^2 & : & a & b & \\
s^1 & : & c & & \\
s^0 & : & d & &
\end{array}
$$

where

$$\begin{aligned} a &= \frac{2\times 3 - 8\times 1}{2} = -1 \qquad b = \frac{2\times 8 - 1\times 0}{2} = 8, \\ c &= \frac{3a-2b}{a} = \frac{-8-16}{-1} = 24, \\ d &= b = 8. \end{aligned}$$

2 sign changes in the first column $\Longrightarrow$ 2 roots not in the LHP$\Longrightarrow$ unstable.

(b)

$$1 + KG = s^3 + s^2 + 2s + 8 = 0.$$

The Routh's array is,

$$\begin{array}{ccll} s^3 & : & 1 & 2 \\ s^2 & : & 1 & 8 \\ s^1 & : & -6 & \\ s^0 & : & 8 & \end{array}$$

There are two sign changes in the first column of the Routh array. Therefore, there are two roots not in the LHP.

(c)

$$1 + KG = s^5 + 2s^4 + 3s^3 + 7s^2 + 4s + 4 = 0.$$

The Routh array is,

$$\begin{array}{cclll} s^5 & : & 1 & 3 & 4 \\ s^4 & : & 2 & 7 & 4 \\ s^3 & : & a_1 & a_2 & \\ s^2 & : & b_1 & b_2 & \\ s^1 & : & c_1 & & \\ s^0 & : & d_1 & & \end{array}$$

where

$$\begin{aligned} a_1 &= \frac{6-7}{2} = \frac{-1}{2} \qquad a_2 = \frac{8-4}{2} = 2 \\ b_1 &= \frac{-7/2-4}{-1/2} = 15 \qquad b_2 = \frac{-4/2-0}{-1/2} = 4 \\ c_1 &= \frac{30+2}{15} = \frac{32}{15} \\ d_1 &= 4 \end{aligned}$$

2 sign changes in the first column $\Longrightarrow$ 2 roots not in the LHP $\Longrightarrow$ unstable.

43. Use Routh's stability criterion to determine how many roots with positive real parts the following equations have:

(a) $s^4 + 8s^3 + 32s^2 + 80s + 100 = 0.$

(b) $s^5 + 10s^4 + 30s^3 + 80s^2 + 344s + 480 = 0.$

(c) $s^4 + 2s^3 + 7s^2 - 2s + 8 = 0.$

(d) $s^3 + s^2 + 20s + 78 = 0.$

(e) $s^4 + 6s^2 + 25 = 0.$

Solution:

(a)
$$s^4 + 8s^3 + 32s^2 + 80s + 100 = 0$$

The Routh array is,

$$\begin{array}{llll}
s^4 & : & 1 & 32 \quad 100 \\
s^3 & : & 8 & 80 \\
s^2 & : & 22 & 100 \\
s^1 & : & 80 - \dfrac{800}{22} = 43.6 & \\
s^0 & : & 100 &
\end{array}$$

$\Longrightarrow$ No roots not in the LHP

(b)
$$s^5 + 10s^4 + 30s^3 + 80s^2 + 344s + 480 = 0$$

$$\begin{array}{lllll}
s^5 & : & 1 & 30 & 344 \\
s^4 & : & 10 & 80 & 480 \\
s^3 & : & 22 & 296 & \\
s^2 & : & -545 & 480 & \\
s^1 & : & 490 & & \\
s^0 & : & 480 & &
\end{array}$$

$\Longrightarrow$ 2 roots not in the LHP.

(c)
$$s^4 + 2s^3 + 7s^2 - 2s + 8 = 0$$

There are roots in the RHP (not all coefficients are >0). The Routh array is,

$$
\begin{array}{lcccc}
s^4 & : & 1 & 7 & 8 \\
s^3 & : & 2 & -2 & \\
s^2 & : & 8 & 8 & \\
s^1 & : & -4 & & \\
s^0 & : & 8 & &
\end{array}
$$

$\Longrightarrow$ 2 roots not in the LHP.

(d) The Routh array is,

$$
\begin{array}{lccc}
s^3 & : & 1 & 20 \\
s^2 & : & 1 & 78 \\
s^1 & : & -58 & \\
s^0 & : & 78 &
\end{array}
$$

There are two sign changes in the first column of the Routh array. Therefore, there are two roots not in the LHP.

(e)

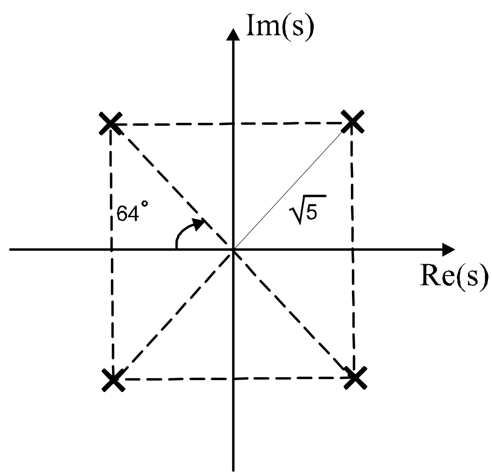
$$a(s) = s^4 + 6s^2 + 25 = 0$$

Two coefficients are missing so there are roots outside the LHP.
Create a new row by $\frac{da(s)}{ds}$.
The Routh array with the new row is,

$$
\begin{array}{lcccl}
s^4 & : & 1 & 6 & 25 \\
s^3 & : & 4 & 12 & \longleftarrow \textbf{new row} \\
s^2 & : & 3 & 25 & \\
s^1 & : & 12 - \dfrac{100}{3} = -21.3 & & \\
s^0 & : & 25 & &
\end{array}
$$

$\Longrightarrow$ 2 roots not in the LHP
check:

$$
\begin{aligned}
a(s) &= 0 \Longrightarrow s^2 = -3 \pm 4j = 5e^{j(\pi \pm 0.92)} \\
s &= \sqrt{5}e^{j\left(\frac{\pi}{2} \pm 0.46\right) + n\pi j} \qquad n = 0, 1
\end{aligned}
$$



Problem 3.43: s-plane pole locations.

44. Find the range of K for which all the roots of the following polynomial are in the LHP:

$$s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + K = 0.$$

Use MATLAB to verify your answer by plotting the roots of the polynomial in the s-plane for various values of K.

Solution:

$$s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + K = 0.$$

The Routh array is,

$$\begin{array}{llccc} s^5 & : & 1 & 10 & 5 \\ s^4 & : & 5 & 10 & K \\ s^3 & : & a_1 & a_2 & \\ s^2 & : & b_1 & K & \\ s^1 & : & c_1 & & \\ s^0 & : & K & & \end{array}$$

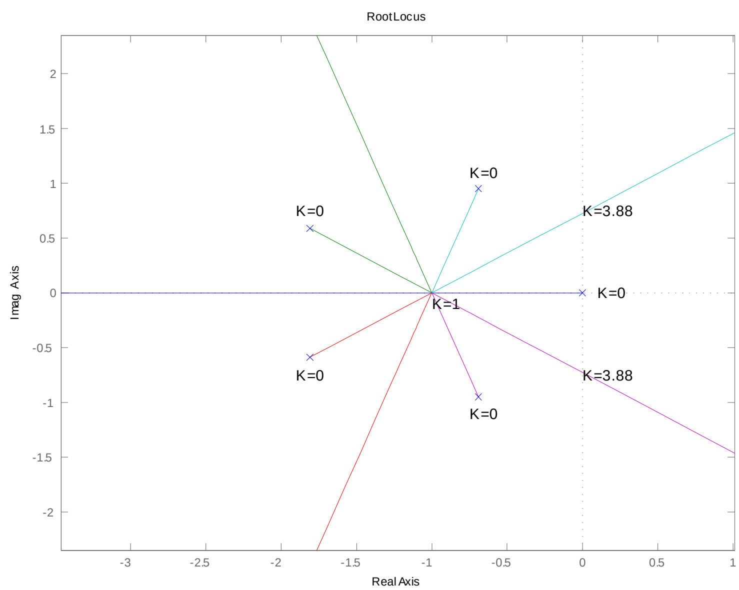
where

$$\begin{aligned} a_1 &= \frac{5\,(10) - 1\,(10)}{5} = 8 \qquad a_2 = \frac{5\,(5) - 1\,(K)}{5} = \frac{25 - K}{8} \\ b_1 &= \frac{(a_1)\,(10) - (5)\,(a_2)}{a_1} = \frac{55 + K}{8} \\ c_1 &= \frac{(b_1)\,(a_2) - (a_1)\,(K)}{b_1} = \frac{-\left(K^2 + 350K - 1375\right)}{5\,(55 + K)} \end{aligned}$$

For stability: all elements in first the first column must be positive that results in the following set of constraints:

$$
\begin{aligned}
(1)\ b_1 &= \frac{55+K}{8} > 0 \Longrightarrow K > -55 \\
(2)\ c_1 &= \frac{-\left(K^2+350K-1375\right)}{5\left(55+K\right)} > 0, \frac{-\left(K-3.88\right)\left(K+354\right)}{5\left(55+K\right)} > 0 \Longrightarrow -55 < K < 3.88 \\
(3)\ d_1 &= K > 0
\end{aligned}
$$

Combining (1), (2), and (3) $\Longrightarrow 0 < K < 3.88$. If we plot the roots of the polynomial for various values of K we obtain the following root locus plot (see Chapter 5),



Problem 3.44: Roots of the polynomial in the s-plane for various values of K.

45. The transfer function of a typical tape-drive system is given by

$$G(s) = \frac{K(s+4)}{s[(s+0.5)(s+1)(s^2+0.4s+4)]},$$

where time is measured in milliseconds. Using Routh's stability criterion, determine the range of K for which this system is stable when the characteristic equation is $1+G(s)=0$.

Solution:

$$1+G\left(s\right) = s^5 + 1.9s^4 + 5.1s^3 + 6.2s^2 + \left(2+K\right)s + 4K = 0.$$

The Routh array is,

$$\begin{array}{lllll} s^5 & : & 1.0 & 5.1 & 2+K \\ s^4 & : & 1.9 & 6.2 & 4K \\ s^3 & : & a_1 & a_2 & \\ s^2 & : & b_1 & 4K & \\ s^1 & : & c_1 & & \\ s^0 & : & 4K & & \end{array}$$

where

$$\begin{aligned} a_1 &= \frac{(1.9)(5.1)-(1)(6.2)}{1.9} = 1.837 \qquad a_2 = \frac{(1.9)(2+K)-(1)(4K)}{1.9} = 2-1.1K \\ b_1 &= \frac{(a_1)(6.2)-(a_2)(1.9)}{a_1} = 1.138\,(K+3.63) \\ c_1 &= \frac{(b_1)(a_2)-(4K)(a_1)}{b_1} = \frac{-\left(1.25K^2+9.61K-8.26\right)}{1.138\,(K+363)} = \frac{-(K+8.47)(K-0.78)}{0.91\,(K+3.63)} \end{aligned}$$

For stability we must have all the elements in the first column to be positive and that results in the following set of constraints:

$$\begin{aligned} (1)\ b_1 &= K+3.63>0 \Longrightarrow K>-3.63, \\ (2)\ c_1 &> 0 \Longrightarrow -8.43<K<0.78, \\ (3)\ d_1 &> 0 \Longrightarrow K>0. \end{aligned}$$

Intersection of (1), (2), and (3) $\Longrightarrow 0<K<0.78$.

46. Consider the closed-loop magnetic levitation system shown in Figure 3.69.

(a) Compute the transfer function from the input (R) to the output (Y).

(b) Assume $K_o = 1$. Determine the conditions on the system parameters (a, K, z, p), to guarantee closed-loop system stability.

Solution:

(a) The transfer function is

$$\frac{Y}{R} = \frac{\frac{K(s+z)}{s+p}\frac{K_\circ}{s^2-a^2}}{1+\frac{K(s+z)}{s+p}\frac{K_\circ}{s^2-a^2}} = \frac{KK_\circ(s+z)}{s^3+ps^2+(KK_\circ-a^2)s+KK_\circ z-pa^2}$$

(b) With $K_\circ = 1$ we have *Denominator* $(s) = s^3+ps^2+(K-a^2)s+Kz-pa^2$;constructing the Routh array we obtain

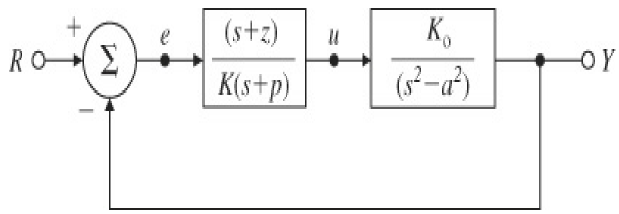


Figure 3.69: Magnetic levitation system

$$
\begin{array}{lcc}
s^3: & 1 & K-a^2 \\
s^2: & p & Kz-pa^2 \\
s^1: & \frac{-Kz+pa^2+Kp-pa^2}{p}=\frac{-Kz+Kp}{p} & \\
s^0: & Kz-pa^2 &
\end{array}
$$

Therefore, for stability, all the elements in the first column to be positive and we obtain the following set of constraints:

$$
\begin{aligned}
p &> 0, \\
Kp-Kz &> 0 \quad \text{if} \quad K>0 \Rightarrow p>z, \\
Kz-pa^2 &> 0 \quad \text{if} \quad K>0 \Rightarrow z>\frac{pa^2}{K}.
\end{aligned}
$$

47. Consider the system shown in Fig. 3.70.

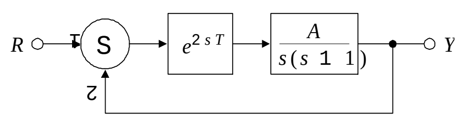


Figure 3.70: Control system for Problem 3.47

(a) Compute the closed-loop characteristic equation.

(b) For what values of (T, A) is the system stable? *Hint*: An approximate answer may be found using

$$e^{-Ts} \cong 1 - Ts$$

or

$$e^{-Ts} \cong \frac{1-\frac{T}{2}s}{1+\frac{T}{2}s}$$

for the pure delay. As an alternative, you could use the computer MATLAB (Simulink) to simulate the system or to find the roots of the system's characteristic equation for various values of T and A.

Solution:

(a) The characteristic equation is,

$$s(s+1) + Ae^{-Ts} = 0$$

(b) Using $e^{-Ts} \cong 1 - Ts$, the characteristic equation is,

$$s^2 + (1 - TA)s + A = 0$$

The Routh's array is,

$$\begin{array}{lcll} s^2 & : & 1 & A \\ s^1 & : & 1 - TA & 0 \\ s^0 & : & A & \end{array}$$

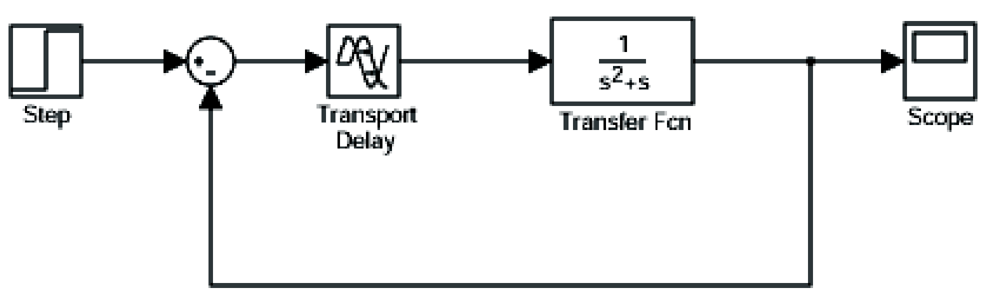
For stability we must have $A > 0$ and $TA < 1$.
Using $e^{-Ts} \cong \frac{(1-\frac{T}{2}s)}{(1+\frac{T}{2}s)}$, the characteristic equation is,

$$s^3 + \left(1 + \frac{2}{T}\right)s^2 + \left(\frac{2}{T} - A\right)s + \frac{2}{T}A = 0$$

The Routh's array is,

$$\begin{array}{lcll} s^3 & : & 1 & \left(\frac{2}{T} - A\right) \\ s^2 & : & \left(1 + \frac{2}{T}\right) & \frac{2A}{T} \\ s^1 & : & \dfrac{\left(1+\frac{2}{T}\right)\left(\frac{2}{T} - A\right) - \frac{2A}{T}}{\left(1+\frac{2}{T}\right)} & 0 \\ s^0 & : & \dfrac{2A}{T} & \end{array}$$

For stability we must have all the coefficients in the first column be positive. The following Simulink diagram simulates the closed-loop system.



Problem 3.47: Simulink simulation diagram.

48. Modify the Routh criterion so that it applies to the case in which all the poles are to be to the left of $-\alpha$ when $\alpha > 0$. Apply the modified test to the polynomial

$$s^3 + (6+K)s^2 + (5+6K)s + 5K = 0,$$

finding those values of K for which all poles have a real part less than -1.

Solution:

Let $p = s+\alpha$ and substitute $s = p-\alpha$ to obtain a polynomial in terms of p. Apply the standard Routh test to the polynomial in p.

For the example $p = s+1$ or $s = p-1$. Substitute this in the polynomial,

$$(p-1)^3 + (6+K)(p-1)^2 + (5+6K)(p-1) + 5K = 0$$

or

$$p^3 + (3+K)p^2 + (4K-4)p + 1 = 0.$$

The Routh's array is,

$$\begin{array}{lllll} p^3 & : & 1 & 4K-4 & \\ p^2 & : & 3+K & 1 & \\ p^1 & : & \dfrac{(3+K)(4K-4)-1}{3+K} & & 0 \\ p^0 & : & 1 & & \end{array}$$

For stability, all the elements in the first column must be positive We must have $K > -3$ and $4K^2 + 8K - 13 > 0$. The roots of the second-order polynomial in K are $K = 1.06$ and $K = -3.061$. The second-order polynomial remains positive if $K > 1.06$ or $K < -3.061$. Therefore, we must have $K > 1.06$.

49. Suppose the characteristic polynomial of a given closed-loop system is computed to be

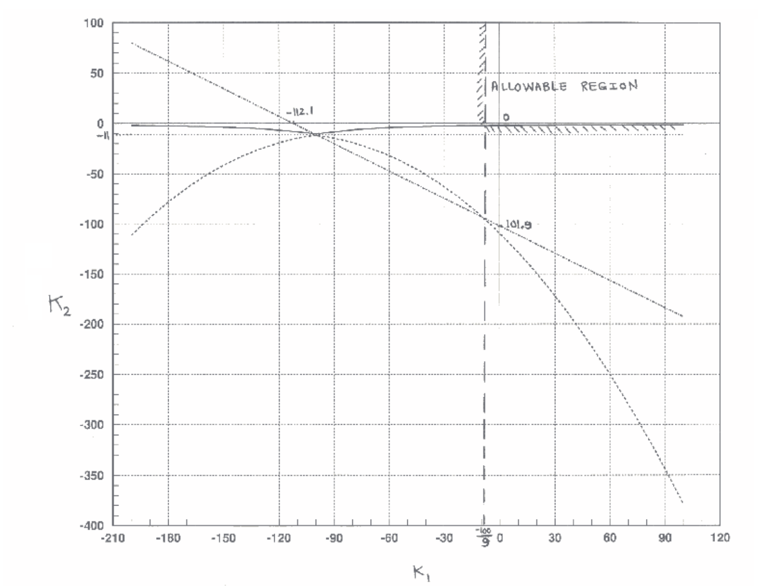
$$s^4+(11+K_2)s^3+(121+K_1)s^2+(K_1+K_1K_2+110K_2+210)s+11K_1+100 = 0.$$

Find constraints on the two gains K_1 and K_2 that guarantee a stable closed-loop system, and plot the allowable region(s) in the (K_1, K_2) plane. You may wish to use the computer to help solve this problem.

Solution: The Routh array is,

$$
\begin{array}{lllll}
s^4 & : & 1 & 121+K_1 & 11K_1+100 \\
s^3 & : & 11+K_2 & K_1+K_1K_2+110K_2+210 & 0 \\
s^2 & : & \dfrac{(11K_2+10K_1+1121)}{K_2+11} & 11K_1+100 & \\
s^1 & : & \multicolumn{3}{l}{\dfrac{10\left(111K_2^2+K_1^2K_2+199K_1K_2+12342K_2+K_1^2+189K_1+22331\right)}{(11K_2+10K_1+1121)}} \\
s^0 & : & 11K_1+100 & &
\end{array}
$$

For stability, the elements in the first column must all be positive. This means that $K_2 > -11$ and $K_1 > -\frac{100}{11}$. The region of stability is shown in the following figure.



Problem 3.49: s-plane region for stability.

50. Overhead electric power lines sometimes experience a low-frequency, high-amplitude vertical oscillation, or **gallop**, during winter storms when the line conductors become covered with ice. In the presence of wind, this ice can assume aerodynamic lift and drag forces that result in a gallop up to several meters in amplitude. Large-amplitude gallop can cause clashing conductors and structural damage to the line support structures caused by the large dynamic loads. These effects in turn can lead to power outages. Assume that the line conductor is a rigid rod, constrained to vertical motion only, and suspended by springs and dampers as shown in Fig. 3.71.

A simple model of this conductor galloping is

$$m\ddot{y} + \frac{D(\alpha)\dot{y} - L(\alpha)v}{(\dot{y}^2 + v^2)^{1/2}} + T\left(\frac{n\pi}{\ell}\right) y = 0,$$

where

$$\begin{aligned}
m &= \text{mass of conductor,}\\
y &= \text{conductor}'\text{s vertical displacement,}\\
D &= \text{aerodynamic drag force,}\\
L &= \text{aerodynamic lift force,}\\
v &= \text{wind velocity,}\\
\alpha &= \text{aerodynamic angle of attack} = -\tan^{-1}(\dot{y}/v),\\
T &= \text{conductor tension,}\\
n &= \text{number of harmonic frequencies,}\\
\ell &= \text{length of conductor.}
\end{aligned}$$

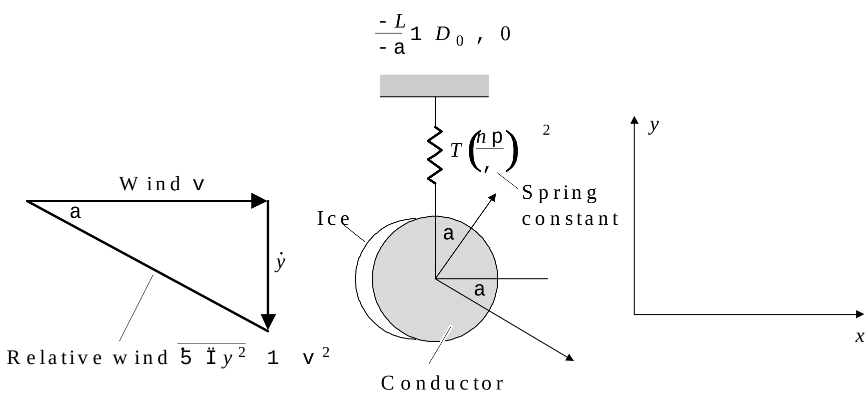


Figure 3.71: Electric power-line conductor

Assume that $L(0) = 0$ and $D(0) = D_0$ (a constant), and linearize the equation around the value $y = \dot{y} = 0$. Use Routh's stability criterion to show that galloping can occur whenever

$$\frac{\partial L}{\partial \alpha} + D_0 < 0.$$

Solution:

$$m\ddot{y} + \left[\frac{D\,(\alpha)\,\dot{y} - L\,(\alpha)\,v}{\sqrt{\dot{y}^2 + v^2}}\right] + T\left(\frac{n\pi}{l}\right)^2 y = 0,$$

Let $x_1 = y$ and $x_2 = \dot{y} = \dot{x}_1$

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{1}{m}\left[\frac{D(\alpha)\,x_2 - L(\alpha)\,v}{\sqrt{x_2^2+v^2}}\right] - \frac{T}{m}\left(\frac{n\pi}{l}\right)^2 x_1 = 0 \\
\alpha &= -\tan^{-1}\left(\frac{x_2}{v}\right) \\
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2)
\end{aligned}$$

$$\begin{aligned}
\dot{x}_1 &= \dot{x}_2 = 0 \quad \text{implies} \quad x_2 = 0 \\
x_2 &= 0 \quad \text{implies} \quad \alpha = 0 \\
\alpha &= 0 \quad \text{implies} \ -\frac{T}{m}\left(\frac{n\pi}{l}\right)^2 x_1 = 0 \quad \text{implies } x_1 = 0.
\end{aligned}$$

$$\frac{\partial f_1}{\partial x_1} = 0, \qquad \frac{\partial f_2}{\partial x_2} = 1, \qquad \frac{\partial f_2}{\partial x_1} = -\frac{T}{m}\left(\frac{n\pi}{l}\right)^2$$

$$\begin{aligned}
\frac{\partial f_2}{\partial x} &= \frac{\partial}{\partial x_2}\left\{-\frac{1}{m}\left[\frac{D(\alpha)\,x_2 - L(\alpha)\,v}{\sqrt{x_2^2+v^2}}\right]\right\} \\
&= -\frac{1}{m}\{\frac{1}{\sqrt{x_2^2+v^2}}\left[\frac{\partial D}{\partial\alpha}\frac{\partial\alpha}{\partial x_2}x_2 + D(\alpha) - \frac{\partial L}{\partial\alpha}\frac{\partial\alpha}{\partial x_2}\right] - \\
&\qquad - \left[\frac{D(\alpha)\,x_2 - L(\alpha)\,v}{\sqrt{x_2^2+v^2}}\right]\left[\frac{-x_2}{(x_2^2+v^2)^{\frac{3}{2}}}\right]\}
\end{aligned}$$

Now,

$$\frac{\partial\alpha}{x_2} = \frac{\partial}{\partial x_2}\left(-\tan^{-1}\left(\frac{x_2}{v}\right)\right) = \frac{-1}{1+\frac{x_2^2}{v^2}}\left(\frac{1}{v}\right)$$

so,

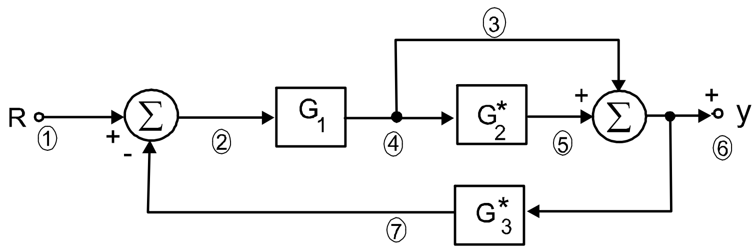
$$\begin{aligned}
\frac{\partial f_2}{\partial x_2} &= \frac{-1}{m}\{\frac{1}{\sqrt{x_2^2+v^2}}\left[\frac{-\frac{\partial D}{\alpha}x_2}{v\left(1+\frac{x_2^2}{v^2}\right)} + D(\alpha) + \frac{\frac{\partial L}{\partial\alpha}v}{v\left(1+\frac{x_2^2}{v^2}\right)}\right] \\
&\quad - \left[\frac{D(\alpha)\,x_2 - L(\alpha)\,v}{\sqrt{x_2^2+v^2}}\right]\left[\frac{-x_2}{(x_2^2+v^2)^{\frac{3}{2}}}\right]\} \\
\frac{\partial f_2}{\partial x_2}|_{x_2=0} &= -\frac{1}{m}\left\{\frac{1}{v}\left[D_0 + \frac{\partial L}{\partial\alpha}\right]\right\} = -\frac{1}{mv}\left(D_0 + \frac{\partial L}{\partial\alpha}\right)
\end{aligned}$$

For no damping (or negative damping) δx_2 term must be ≤ 0 so this implies $D_0 + \frac{\partial L}{\partial \alpha} < 0$.

Problems and Solutions for .Mason's rule and the Signal-Flow Graph On the Web

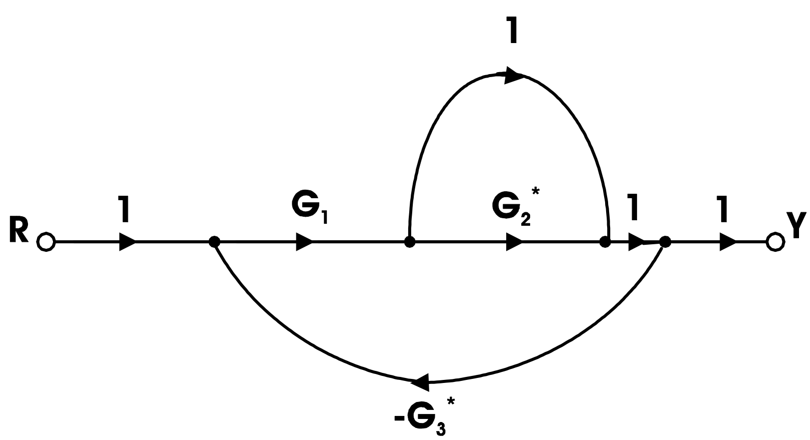
51. ▲Find the transfer functions for the block diagrams in Fig. 3.54, using Mason's rule.

Solution: Transfer functions are found using Mason's rule,



Block diagram for Fig. 3.54 (a).

$$G_2^* = \frac{G_2}{1 - G_2 H_2}$$
$$G_3^* = \frac{G_3}{1 - G_3 H_3}$$



Flow graph for Fig. 3.54(a).

(a) Mason's rule for Fig. 3.54(a):

Forward Path Gains

1 2 4 5 6	$p_1 = G_1G_2^*$
1 2 4 3 6	$p_2 = G_1$

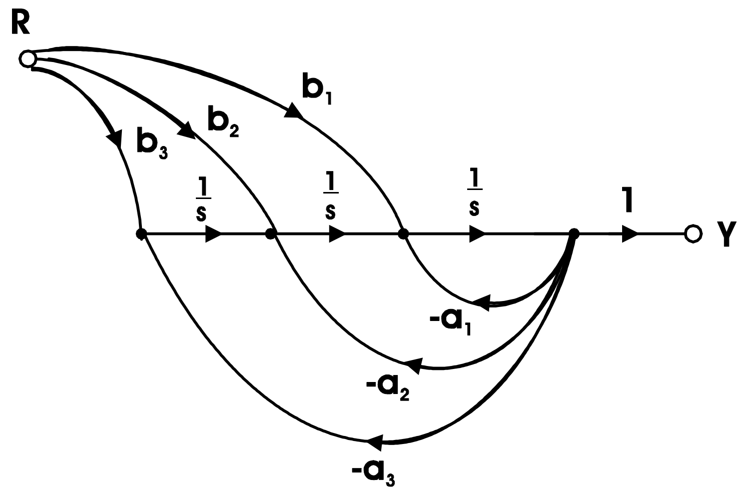
Loop Path Gains

2 4 3 7 2	$\ell_1 = -G_1G_3^*$
2 4 5 7 2	$\ell_2 = -G_1G_2^*G_3^*$

$$\begin{aligned}\frac{Y}{R} &= \frac{p_1+p_2}{1-\ell_1-\ell_2} = \frac{G_1(1+G_2^*)}{1+G_1G_3^*(1+G_2^*)} \\ &= \frac{G_1(1-G_2H_2)(1-G_3H_3)+G_1G_2(1-G_3H_3)}{1+(1-G_2H_2)(1-G_3H_3)+G_1G_3(1-G_2H_2)+G_1G_2G_3}.\end{aligned}$$

This is the same answer as in Problem 3.21(a).

(b) Mason's rule for Fig. 3.54(b):



Flow graph for Fig. 3.54(b).

Forward path gains:

$$p_1 = \frac{b_3}{s^3}, \quad p_2 = \frac{b_2}{s^2}, \quad p_3 = \frac{b_1}{s}$$

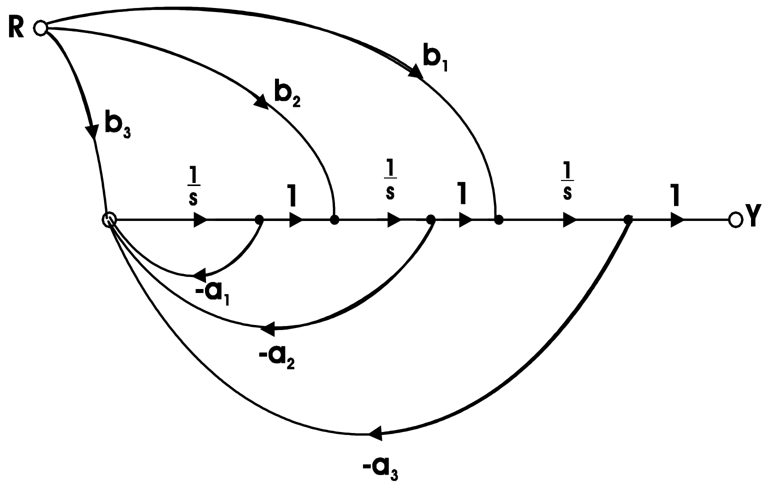
Loop path gains:

$$\ell_1 = -\frac{a_3}{s^3}, \quad \ell_2 = -\frac{a_2}{s^2}, \quad \ell_3 = -\frac{a_1}{s}$$

$$\frac{Y}{R} = \frac{p_1+p_2+p_3}{1-\ell_1-\ell_2-\ell_3} = \frac{\frac{b_3}{s^3}+\frac{b_2}{s^2}+\frac{b_1}{s}}{1+\frac{a_3}{s^3}+\frac{a_2}{s^2}+\frac{a_1}{s}} = \frac{b_1s^2+b_2s+b_3}{s^3+a_1s^2+a_2s+a_3}$$

This is the same answer as in Problem 3.21(b).

(c) Mason's Rule for Fig. 3.54(c):



Flow graph for Fig. 3.54(c).

Forward path gains:

$$p_1 = \frac{b_3}{s^3}, \quad p_2 = \frac{b_2}{s^2}[1 + \frac{a_1}{s}], \quad p_3 = \frac{b_1}{s}[1 + \frac{a_1}{s} + \frac{a_2}{s^2}]$$

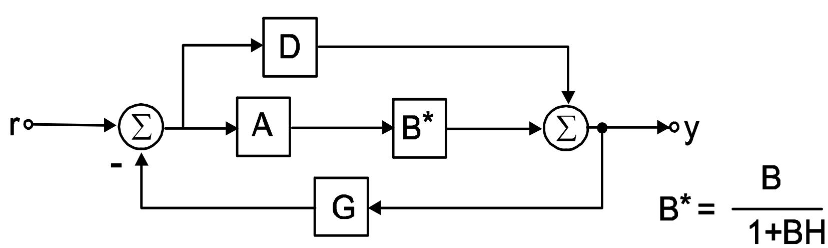
Loop path gains:

$$\ell_1 = -\frac{a_3}{s^3}, \quad \ell_2 = -\frac{a_2}{s^2}, \quad \ell_3 = -\frac{a_1}{s}$$

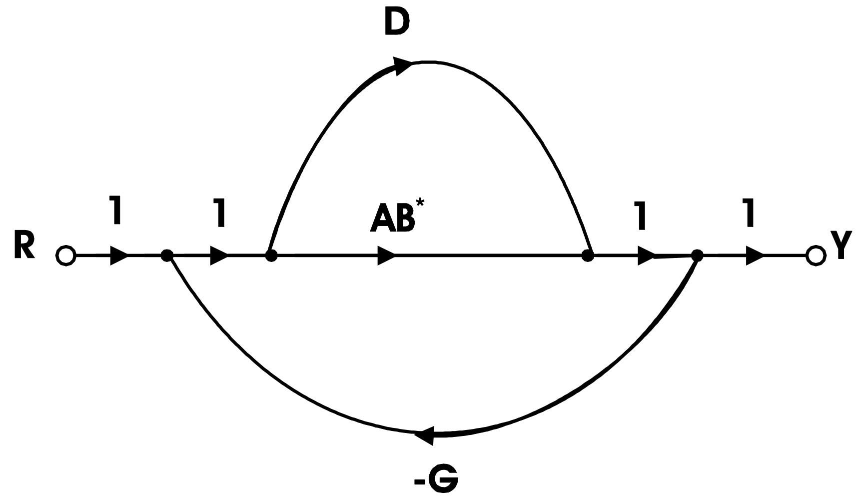
$$\frac{Y}{R} = \frac{p_1 + p_2 + p_3}{1 - \ell_1 - \ell_2 - \ell_3} = \frac{b_1 s^2 + (a_1 b_1 + b_2)s + a_1 b_2 + a_2 b_1 + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

This is the same answer as in Problem 3.21(c).

(d) Mason's rule for Fig. 3.54(d): The system is tightly connected, easy to apply Mason's rule.



Block diagram for Fig. 3.54(d).



Flow graph for Fig. 3.54(d).

Forward path gains:

$$p_1 = D, \quad p_2 = AB^*$$

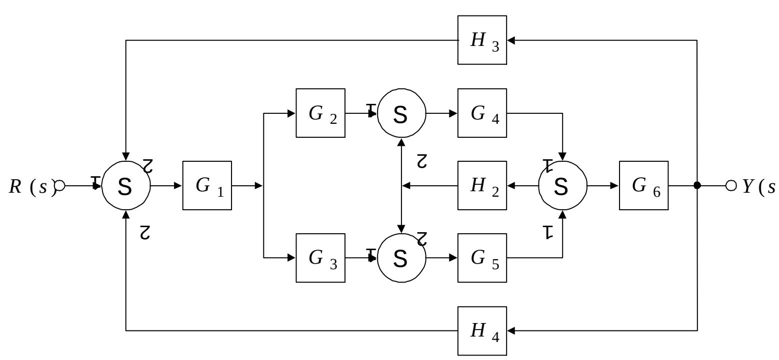
Loop path gains:

$$\ell_1 = -GD, \quad \ell_2 = -AB^*G$$

$$\frac{Y}{R} = \frac{p_1 + p_2}{1 - \ell_1 - \ell_2} = \frac{D + AB^*}{1 + G(D + AB^*)} = \frac{D + DBH + AB}{1 + BH + GD + GBDH + GAB}$$

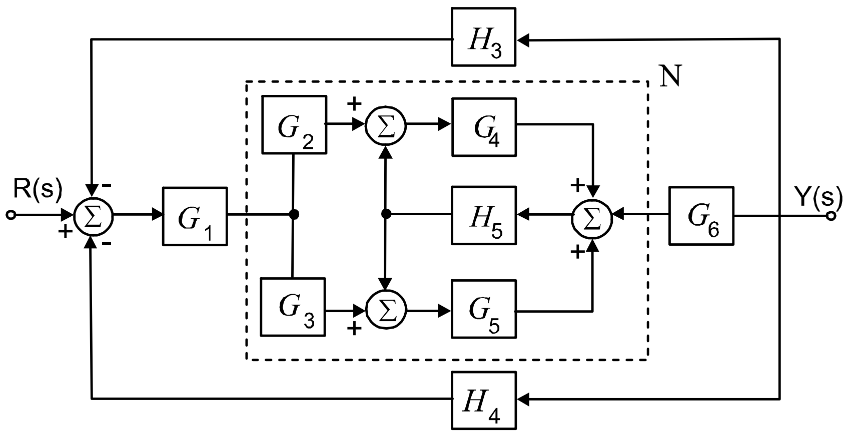
This is the same answer as in Problem 3.21(d).

52. ▲Use block-diagram algebra or Mason's rule to determine the transfer function between $R(s)$ and $Y(s)$ in Fig. 3.55.



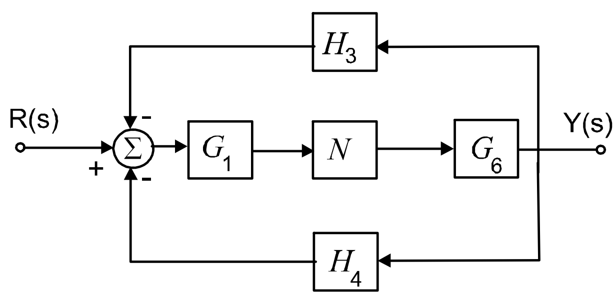
Text Fig. 3.55 Block diagram for Problem 3.52

Solution:



Block diagram for Fig. 3.55.

Block diagram algebra:



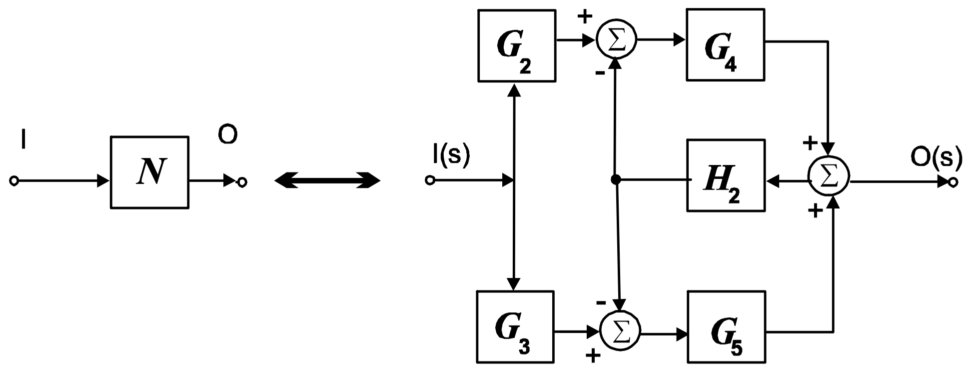
Block diagram for Fig. 3.55: reduced.

$$
\begin{aligned}
Q &= R - PH_3 - PH_4 \\
&= R - P(H_3 - H_4) \\
P &= G_1 N G_6 = Y
\end{aligned}
$$

So:

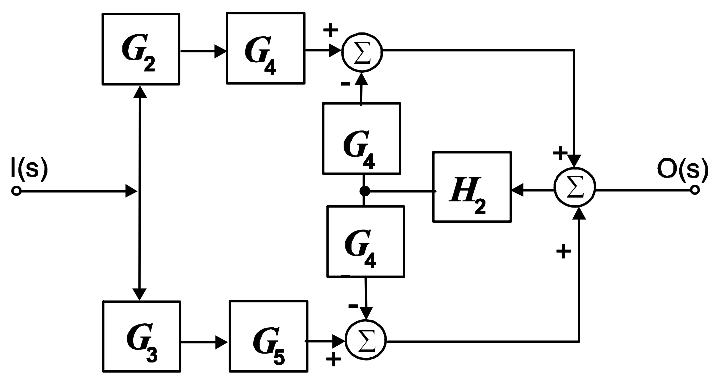
$$
\frac{Y}{R} = \frac{G_1 N G_6}{1 + (H_3 + H_4) G_1 N G_6}
$$

Now, what is N?



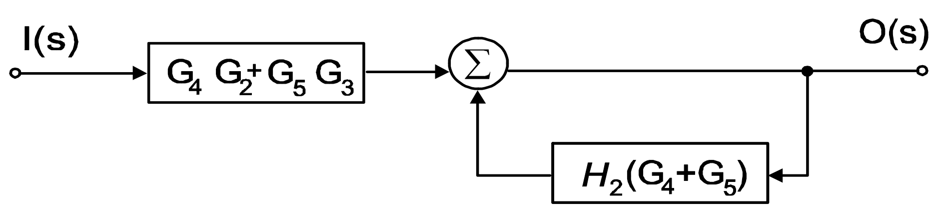
Block diagrams for Fig. 3.55.

Move G_4 and G_3 :



Block diagram for Fig. 3.55: reduced.

Combine symmetric loops as in the first step:



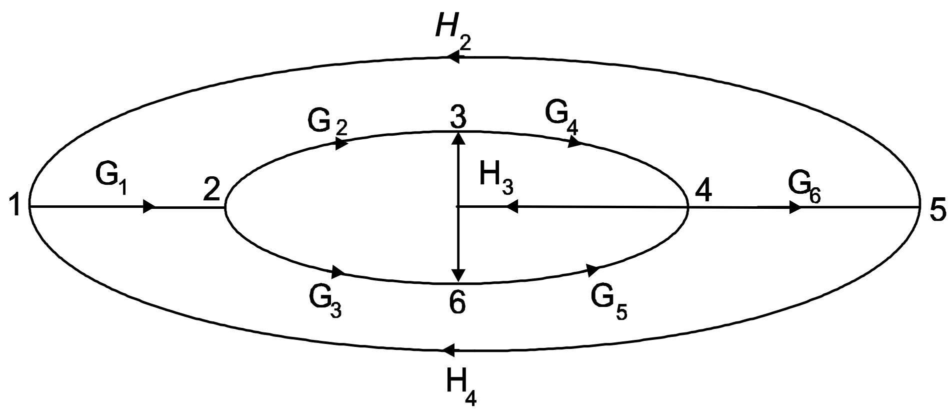
Block diagram for Fig. 3.55: reduced.

Which is:

$$
\begin{aligned}
N &= \frac{O}{I} = \frac{G_4G_2 + G_5G_3}{1 + H_2(G_4 + G_5)} \\
\frac{Y(s)}{R(s)} &= \frac{G_1(G_4G_2 + G_5G_3)G_6}{1 + H_2(G_4 + G_5) + (H_3 + H_4)G_1(G_4G_2 + G_5G_3)G_6}
\end{aligned}
$$

Mason's Rule:

The signal flow graph is:



Flow graph for Fig. 3.55.

Forward Path	**Gain**
1 2 3 4 5	$p_1 = G_1G_2G_4G_6$
1 2 6 4 5	$p_2 = G_1G_3G_5G_6$
Loop Path	**Gain**
1 2 3 4 5 1	$\ell_1 = -G_1G_2G_4G_6H_3$
1 2 3 4 5 1	$\ell_2 = -G_1G_2G_4G_6H_4$
1 2 6 4 5 1	$\ell_3 = -G_1G_3G_5G_6H_3$
1 2 6 4 5 1	$\ell 4 = -G_1G_3G_5G_6H_4$
3 4 3	$\ell_5 = -G_4H_2$
3 4 3	$\ell_6 = -G_5H_2$

and the determinants are

$$\begin{aligned}
\Delta &= 1 + [(H_3 + H_4)G_1(G_2G_4 + G_3G_5)G_6 + H_2(G_4 + G_5)] \\
\Delta_1 &= 1 - (0) \\
\Delta_2 &= 1 - (0) \\
\Delta_3 &= 1 - (0) \\
\Delta_4 &= 1 - (0).
\end{aligned}$$

Applying the rule, the transfer function is

$$\begin{aligned}
\frac{Y(s)}{R(s)} &= \frac{1}{\Delta}\sum G_i\Delta_i = \frac{p_1 + p_2}{1 - \ell_1 - \ell_2 - \ell_3 - \ell_4 - \ell_5 - \ell_6} \\
&= \frac{G_1(G_4G_2 + G_5G_3)G_6}{1 + H_2(G_4 + G_5) + (H_3 + H_4)G_1(G_4G_2 + G_5G_3)G_6}.
\end{aligned}$$

Chapter 4: A First Analysis of Feedback

Problems and Solutions for Section 4.1: The Basic Equations of Control

1. If S is the sensitivity of the unity feedback system to changes in the plant transfer function and T is the transfer function from reference to output, show that $S + T = 1$.

   Solution:

$$\begin{aligned} S+T &= \frac{1}{1+DG} + \frac{DG}{1+DG} \\ &= 1 \end{aligned}$$

   OK, this one was free!

2. We define the sensitivity of a transfer function G to one of its parameters k as the ratio of percent change in G to percent change in k.

$$S_K^G = \frac{dG/G}{dK/K} = \frac{d\ln G}{d\ln K} = \frac{K}{G}\frac{dG}{dK}.$$

   The purpose of this problem is to examine the effect of feedback on sensitivity. In particular, we would like to compare the topologies shown in Fig. 4.23 for connecting three amplifier stages with a gain of $-K$ into a single amplifier with a gain of -10.

   (a) For each topology in Fig. 4.23, compute β_i so that if $K = 10$, $Y = -10R$.

   (b) For each topology, compute S_k^G when $G = Y/R$. [Use the respective β_i values found in part (a).] Which case is the *least* sensitive?

   (c) Compute the sensitivities of the systems in Fig. 4.23(b, c) to β_2 and β_3. Using your results, comment on the relative need for precision in sensors and actuators.

   Solution:

   (a) For $K = 10$ and $y = -10r$, we have:

   Case a:

$$\frac{Y}{R} = -\beta_1 K^3 \implies \beta_1 = 0.01$$

   Case b:

$$\frac{Y}{R} = (\frac{-K}{1+\beta_2 K})^3 \implies \beta_2 = 0.364$$

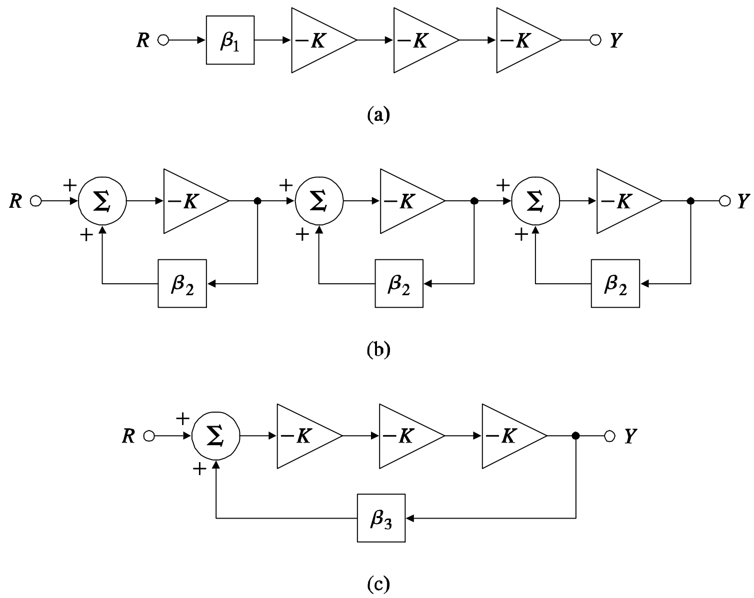


Figure 4.23: Three-amplifier topologies for Problem 2

Case c:

$$\frac{Y}{R} = \frac{-K^3}{1+\beta_3 K^3} \quad \Longrightarrow \quad \beta_3 = 0.099$$

(b) Sensitivity S_K^G, $G = \dfrac{Y}{R}$

Case a:

$$\frac{dG}{dK} = -3\beta_1 K^2$$

$$\mathcal{S}_K^G = \frac{K}{G}\frac{dG}{dK} = \frac{K}{-\beta_1 K^3}(-3\beta_1 K^2) = 3$$

Similarly:
Case b: $S_K^G = 0.646$
Case c: $S_K^G = 0.03$
Case c is the least sensitive.

(c) Sensitivities w.r.t. feedback gains:

Case b:

$$\mathcal{S}_{\beta_2}^G = -2.354$$

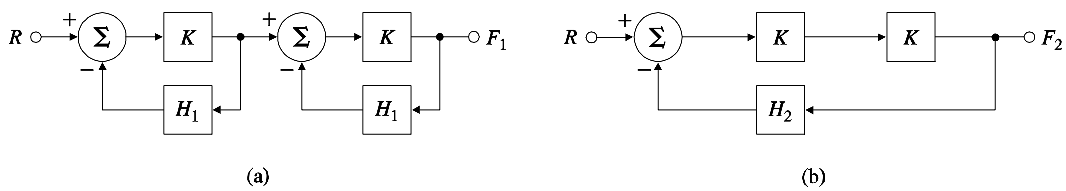


Figure 4.24: Block diagrams for Problem 3

Case c:

$$\mathcal{S}_{\beta_3}^{G} = -0.99$$

The results indicate that the closed-loop system is much more sensitive to errors in the feedback path than in the forward path. It is 33 times as sensitive in case c. We conclude that sensors need to have much higher precision than actuators.

3. Compare the two structures shown in Fig. 4.24 with respect to sensitivity to changes in the overall gain due to changes in the amplifier gain. Use the relation

$$S = \frac{d\ln F}{d\ln K} = \frac{K}{F}\frac{dF}{dK}.$$

as the measure. Select H_1 and H_2 so that the nominal system outputs satisfy $F_1 = F_2$, and assume $KH_1 > 0$.

Solution:

$$F_1 = (\frac{K}{1+KH_1})^2; \ F_2 = \frac{K^2}{1+K^2H_2}$$

$$\mathcal{S}_K^{F_1} = \frac{2}{1+KH_1}; \ \mathcal{S}_K^{F_2} = \frac{2}{1+K^2H_2}$$

$$F_1 = F_2 \implies H_2 = H_1^2 + \frac{2H_1}{K}$$

$$\mathcal{S}_K^{F_2} = \frac{2}{(1+KH_1)^2} = \frac{\mathcal{S}_K^{F_1}}{1+KH_1}$$

System 2 is less sensitive. The conclusion is to put as much gain in the feedback loop as you can.

4. A unity feedback control system has the open-loop transfer function

$$G(s) = \frac{A}{s(s+a)}.$$

(a) Compute the sensitivity of the closed-loop transfer function to changes in the parameter A.

(b) Compute the sensitivity of the closed-loop transfer function to changes in the parameter a.

(c) If the unity gain in the feedback changes to a value of $\beta \neq 1$, compute the sensitivity of the closed-loop transfer function with respect to β.

Solution:

(a)

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{\frac{A}{s(s+a)}}{1+\frac{A}{s(s+a)}} = \frac{A}{s^2+as+A}$$

$$\frac{dT}{dA} = \frac{(s^2+as+A)-A}{(s^2+as+A)^2}$$

$$\mathcal{S}_A^T = \frac{A}{T}\frac{dT}{dA} = \frac{A(s^2+as+A)}{A}\frac{s^2+as}{(s^2+as+A)^2} = \frac{s(s+a)}{s(s+a)+A}$$

(b)

$$\frac{dT}{da} = \frac{-sA}{(s^2+as+A)^2}$$

$$\frac{a}{T}\frac{dT}{da} = \frac{a(s^2+as+A)}{A}\frac{-sA}{(s^2+as+A)^2}$$

$$\mathcal{S}_a^T = \frac{-as}{s(s+a)+A}$$

(c) In this case,

$$T(s) = \frac{G(s)}{1+\beta G(s)}$$

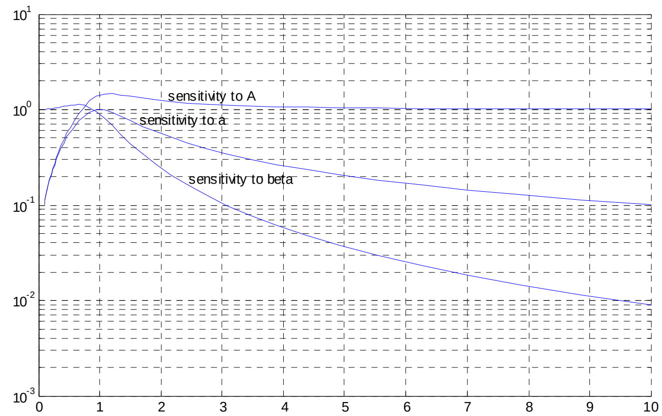
$$\frac{dT}{d\beta} = \frac{-G(s)^2}{(1+\beta G(s))^2}$$

$$\frac{\beta}{T}\frac{dT}{d\beta} = \frac{\beta(1+\beta G)}{G}\frac{-G^2}{(1+\beta G)^2} = \frac{-\beta G}{1+\beta G}$$

$$\mathcal{S}_\beta^T = \frac{\frac{-\beta A}{s(s+A)}}{1+\frac{\beta A}{s(s+a)}} = \frac{-\beta A}{s(s+a)+\beta A}$$

- If $a = A = 1$,the transfer function is most sensitive to variations in a and A near $\omega = 1$ rad/sec .
- The steady-state response is not affected by variations in A and a ($\mathcal{S}_A^T(0)$ and $\mathcal{S}_a^T(0)$ are both zero).
- The steady-state response *is* heavily dependent on β since $|\mathcal{S}_\beta^T(0)|$ $= 1.0$

See attached plots of sensitivities versus radian frequency for $a = A = 1.0$.



5. Compute the equation for the system error for the filtered feedback system shown in Fig. 4.4.

   Solution:

   For this figure, the equation for the output is:

$$Y = \frac{FDG}{1+DGH}R + \frac{G}{1+DGH}W - \frac{DGH}{1+DGH}V$$

   And the resulting equation for the error is:

$$\begin{aligned} E &= R - Y \\ &= \frac{1+DG(H-F)}{1+DGH}R - \frac{G}{1+DGH}W + \frac{DGH}{1+DGH}V \end{aligned}$$

   Therefore, as we have seen, increasing the loop gain does not necessarily reduce the error as the result depends on the structure of the system..

6. If S is the sensitivity of the filtered feedback system to changes in the plant transfer function and T is the transfer function from reference to output, compute the sum of $S + T$. Show that $S + T = 1$ if $F = H$.

   (a) Compute the sensitivity of the filtered feedback system shown in Fig 4.4 with respect to changes in the plant transfer function, G.

   (b) Compute the sensitivity of the filtered feedback system shown in Fig 4.4 with respect to changes in the controller transfer function, D_{cl}.

   (c) Compute the sensitivity of the filtered feedback system shown in Fig 4.4 with respect to changes in the filter transfer function, F.

(d) Compute the sensitivity of the filtered feedback system shown in Fig 4.4 with respect to changes in the sensor transfer function, H.

Solution:

To answer the first question, we need the answer to part a) so let's start there.

a.

Applying the formula for sensitivity of T to changes in G:

$$\begin{aligned} T &= \frac{FDG}{1+DGH} \\ S &= G\frac{1+DGH}{FDG}\frac{(1+DGH)FD-(FDG)(DH}{(1+DGH)^2} \\ &= \frac{1}{1+DGH} \end{aligned}$$

Now we can do

$$\begin{aligned} S+T &= \frac{1}{1+DGH}+\frac{FDG}{1+DGH} \\ &= \frac{1+FDG}{1+DGH} \\ &= 1 \qquad if\ F=H \end{aligned} \tag{4.1}$$

b. Applying the formula for sensitivity of T to changes in D:

$$\begin{aligned} S_T^D &= D\frac{1+DGH}{FDG}\frac{(1+DGH)FG-FDG(GH)}{(1+DGH)^2} \\ &= \frac{1}{1+DGH} \end{aligned}$$

This is not surprising as D and G are in series.

c. Applying the formula for sensitivity of T to changes in F.

$$\begin{aligned} S_T^F &= F\frac{1+DGH}{FDG}\frac{(1+DGH)(DG)}{(1+DGH)^2} \\ & \frac{1+DGH}{1+DGH} \\ &= 1 \end{aligned}$$

In this case, the F term is in the open loop so it has sensitivity of unity.

d. Applying the formula for sensitivity of T to changes in H.

$$\begin{aligned} S_T^H &= H\frac{1+DGH}{FDG}\frac{(1+DGH)0-FDG(DG)}{(1+DGH)^2} \\ &= -\frac{DGH}{(1+DGH)} \end{aligned}$$

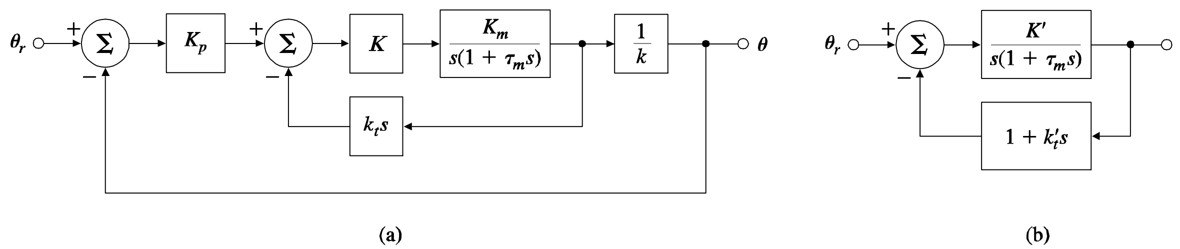


Figure 4.25: Control system for Problem 7

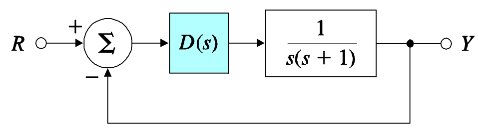


Figure 4.26: Control system for Problem 8

Problems and Solutions for Section 4.2: Control of Steady-State Error

7. Consider the DC-motor control system with rate (tachometer) feedback shown in Fig. 4.25(a).

   (a) Find values for K' and k_t' so that the system of Fig. 4.25(b) has the same transfer function as the system of Fig. 4.25(a).

   (b) Determine the system type with respect to tracking θ_r and compute the system K_v in terms of parameters K' and k_t'.

   (c) Does the addition of tachometer feedback with positive k_t increase or decrease K_v?

   Solution:

   (a) Using block diagram reduction techniques:

   - Move the pickoff point from the input of the $\frac{1}{k}$ to its output.

   - Eliminate the second summer by absorbing K_p.

   This will result in Figure 4.25(b) where

$$K' = \frac{K_p K K_m}{k}$$

$$k_t' = \frac{k k_t}{K_p}.$$

(b) The inner-loop in Fig. 4.25(a) may be reduced to

$$\frac{kK_m}{s(1+\tau_m s + kK_m k_t)}$$

which means that the unity feedback system has a pure integrator in the forward loop and hence it is Type 1 with respect to reference input (θ_r) and $K_v = \dfrac{kK_m}{(1+kK_m k_t)}$

(c) *We conclude that the introduction of k_t reduces the velocity constant and therefore makes the error to a ramp larger*

8. Consider the system shown in Fig. 4.26, where

$$D(s) = K\frac{(s+\alpha)^2}{s^2+\omega_o^2}.$$

(a) Prove that if the system is stable, it is capable of tracking a sinusoidal reference input $r = \sin\omega_o t$ with zero steady-state error. (Look at the transfer function from R to E and consider the gain at ω_o.)

(b) Use Routh's criterion to find the range of K such that the closed-loop system remains stable if $\omega_o = 1$ and $\alpha = 0.25$.

Solution:

(a)

$$\begin{aligned}
D(s)G(s) &= \frac{K(s+\alpha)^2}{(s^2+\omega_o^2)s(s+1)} \\
\frac{E(s)}{R(s)} &= \frac{1}{1+DG} \\
&= \frac{s(s+1)(s^2+\omega_o^2)}{(s^2+\omega_o^2)s(s+1)+K(s+\alpha)^2}
\end{aligned}$$

The gain of this transfer function is zero at $s = \pm j\omega_o$ and we expect the error to be zero if R is a sinusoid at that frequency. More formally, let $R(s) = \dfrac{\omega_n}{s^2+\omega_n^2}$ then

$$E(s) = \frac{s(s+1)(s^2+\omega_o^2)}{(s^2+\omega_o^2)s(s+1)+K(s+\alpha)^2}\ \frac{\omega_n}{s^2+\omega_n^2}$$

Assuming the (closed-loop) system is stable, then if $\omega_n \neq \omega_o$ Then $E(s)$ has a pole on the imaginary axis and the FVT does not apply. The final error will NOT be zero in this case. However, if $\omega_n = \omega_o$ we *can* use the FVT and

$$e_{ss} = \lim_{s\to 0} sE(s) = 0$$

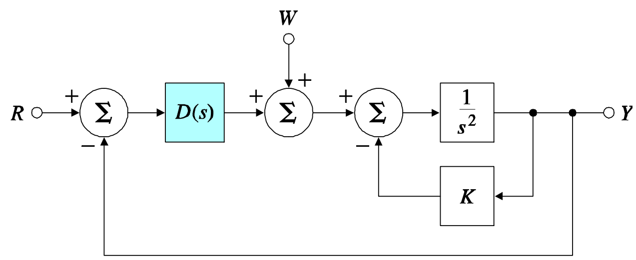


Figure 4.27: Control system for problem 9

(b) To test for stability, the characteristic equation is,

$$s^4 + (K + \omega_o^2)s^2 + s^3 + (\omega_o^2 + 2\alpha K)s + K\alpha^2 = 0$$

Using the Routh array

$$\begin{array}{llll}
s^4: & 1 & \omega_o^2 + K & K\alpha^2 \\
s^3: & 1 & (\omega_o^2 + 2\alpha K) & \\
s^2: & K(1-2\alpha) & K\alpha^2 & \\
s^1: & \omega_o^2 + 2\alpha K + \dfrac{\alpha^2}{(1-2\alpha)} & & \\
s^0: & K\alpha^2 & &
\end{array}$$

If $\alpha = 0.25$, we must have $K > 0$, and $K > 0.25 - 2\omega_o^2$.

9. Consider the system shown in Fig. 4.27 which represents control of the angle of a pendulum that has no damping.

   (a) What condition must $D(s)$ satisfy so that the system can track a ramp reference input with constant steady-state error?

   (b) For a transfer function $D(s)$ that stabilizes the system and satisfies the condition in part (a), find the class of disturbances $w(t)$ that the system can reject with zero steady-state error.

   Solution:

   (a) For a unity feedback system to be Type 1 the open loop transfer function must have a pole at $s = 0$. Thus in this case, since G has no such pole, it is necessary for D to have a pole at $s = 0$.

   (b)

$$Y(s) = \frac{1}{s^2 + D(s) + K} W(s)$$

$$\lim_{s \to 0} s\left(\frac{1}{s^2 + D(s) + K}\right)\frac{1}{s^\ell} = 0$$

iff

$$\lim_{s\to 0} s^{\ell-1}D(s) = \infty$$

iff $\ell = 1$ since $D(s)$ has a pole at the origin. Therefore system will reject step disturbanceswith zero error.

10. A unity feedback system has the overall transfer function

$$\frac{Y(s)}{R(s)} = \mathcal{T}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

Give the system type and corresponding error constant for tracking polynomial reference inputs in terms of ζ and ω_n.

Solution:

$$\frac{E(s)}{R(s)} = \frac{s^2 + 2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Therefore the system is Type 1 and the velocity constant is $K_v = \dfrac{\omega_n}{2\zeta}$

11. Consider the second-order system

$$G(s) = \frac{1}{s^2 + 2\zeta s + 1}.$$

We would like to add a transfer function of the form $D(s) = K(s+a)/(s+b)$ in series with $G(s)$ in a unity-feedback structure.

(a) Ignoring stability for the moment, what are the constraints on K, a, and b so that the system is Type 1?

(b) What are the constraints placed on K, a, and b so that the system is both stable and Type 1?

(c) What are the constraints on a and b so that the system is both Type 1 and remains stable for every positive value for K?

Solution:

(a) In a unity feedback structure, the error is $1/(1+GD)$ and, as we saw, to be Type 1, there needs to be a pole at $s = 0$ in the product GD. Since there is no such pole in G, it must be supplied by D, thus, the answer is

$$b = 0$$

(b) To assure stability, all poles of the closed loop must be in the left half plane, for which the criterion is by Routh. Thus the characteristic equation is

$$s(s^2 + 2\zeta s + 1) + K(s + a) = 0$$

and the Routh array is

$$\begin{array}{ll} 1 & 1+K \\ 2\zeta & aK \\ \dfrac{2\zeta(1+K)-aK}{2\zeta} & \\ aK & \end{array}$$

Thus the requirements are

$$\begin{aligned} 2\zeta(1+K)-aK &> 0 \\ aK &> 0 \end{aligned}$$

(c) If we assume that $\zeta > 0$ and, for this part, that $a > 0$ also, the requirements can be reduced to

$$\begin{aligned} K &> 0 \\ 2\zeta + K(2\zeta - a) &> 0 \end{aligned}$$

If $a < 2\zeta$, inspection of these conditions shows that the system will be *stable for all positive values of* K. On the other hand, if $a > 2\zeta$, then the requirement is

$$0 < K < \frac{2\zeta}{a - 2\zeta}$$

Extra credit: work out the case for $\zeta < 0$. Note to the Instructor: you might come back to this problem in chapter 5 and verify this point using the rule of asymptotes.

12. Consider the system shown in Fig. 4.28(a).

(a) What is the system type? Compute the steady-state tracking error due to a ramp input $r(t) = r_o t1(t)$.

(b) For the modified system with a feed forward path shown in Fig. 4.28(b), give the value of H_f so the system is Type 2 for reference inputs and compute the K_a in this case.

(c) Is the resulting Type 2 property of this system robust with respect to changes in H_f? i.e., will the system remain Type 2 if H_f changes slightly?

Solution:

(a) System is Type 1 since it is unity feedback and has a pole at $s = 0$in the forward path. Also,.

$$E(s) = [1 - T(s)]R(s)$$

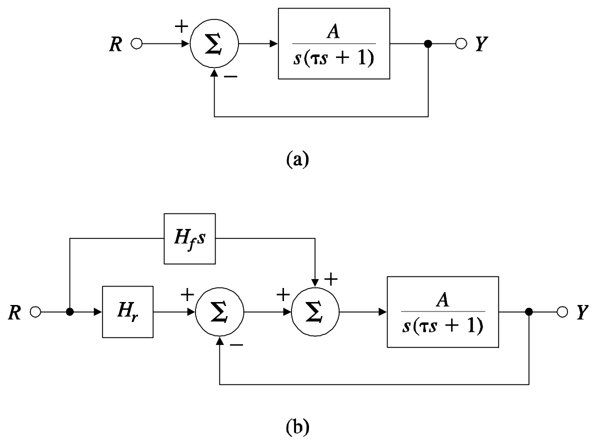


Figure 4.28: Control system for Problem 12

$$= \left[\frac{1}{1+G(s)}\right] R(s)$$

$$= \frac{s(\tau s+1)}{s(\tau s+1)+A} \frac{r_o}{s^2}$$

The steady-state tracking error using the FVT (assuming stability) is

$$e_{ss} = \lim_{s\to 0} sE(s) = \frac{r_o}{A}.$$

(b)

$$Y(s) = \frac{A}{s(\tau s+1)} U(s)$$

$$U(s) = H_f sR(s) + H_r R(s) - Y(s)$$

$$Y(s) = \frac{A(H_f s + H_r)}{s(\tau s+1)+A} R(s)$$

The tracking error is,

$$E(s) = R(s) - Y(s)$$

$$= \frac{s(\tau s+1) + A - A(H_f s + H_r)}{s(\tau s+1)+K} R(s)$$

$$= \frac{\tau s^2 + (1 - AH_f)s + A(1-H_r)}{s(\tau s+1)+A}$$

To get zero steady-state error with respect to a ramp, the numerator in the above equation must have a factor s^2. For this to happen, let

$$\begin{aligned} H_r &= 1 \\ AH_f &= 1. \end{aligned}$$

Then

$$E(s) = \frac{\tau s^2}{s(\tau s + 1) + A} R(s)$$

and, with $R(s) = \frac{r_o}{s^2}$, apply the FVT (assuming stability) to obtain

$$e_{ss} = 0.$$

Thus the system will be Type 2 with $K_a = \frac{\tau}{A}$.

(c) No, the system is not robust Type 2 because the property is lost if either H_r or H_f changes slightly.

13. A controller for a satellite attitude control with transfer function $G = 1/s^2$ has been designed with a unity feedback structure and has the transfer function $D(s) = \frac{10(s+2)}{s+5}$

    (a) Find the system type for reference tracking and the corresponding error constant for this system.

    (b) If a disturbance torque adds to the control so that the input to the process is $u + w$, what is the system type and corresponding error constant with respect to disturbance rejection?

    Solution:

    (a) There are two poles at $s = 0$ so the system is Type 2 and the error constsnts are:

$$K_p = \lim_{s \to 0} D(s)G(s) = \infty$$

$$e_{ss} = \frac{1}{1 + K_p} = 0.$$

$$K_v = \lim_{s \to 0} sD(s)G(s) = \infty$$

$$e_{ss} = \frac{1}{K_v} = 0.$$

$$K_a = \lim_{s \to 0} s^2 D(s)G(s) = 4$$

$$e_{ss} = \frac{1}{K_a} = 0.25.$$

    (b) For the disturbance input, the poles at $s = 0$ are *after* the input and therefore the system is Type 0. The error is

$$\begin{aligned}
\frac{E(s)}{W(s)} &= -\frac{G}{1 + GD} \\
&= -\frac{s + 5}{s^2(s + 5) + 10(s + 2)}
\end{aligned}$$

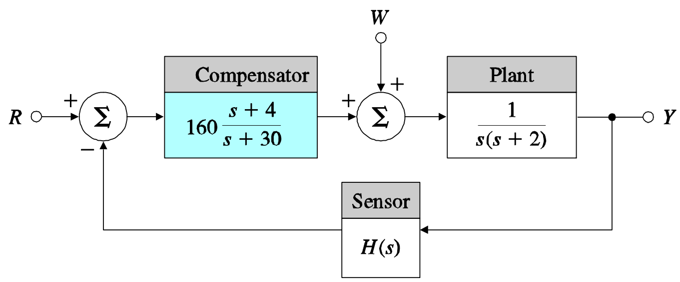


Figure 4.29: Control system for Problem 14

The steady-state error to a step is thus $e_{ss} = 0.25 = \dfrac{1}{1+K_p}$. Therefore,

$$K_p = 3$$

14. A compensated motor position control system is shown in Fig. 4.29. Assume that the sensor dynamics are $H(s) = 1$.

    (a) Can the system track a step reference input r with zero steady-state error? If yes, give the value of the velocity constant.

    (b) Can the system reject a step disturbance w with zero steady-state error? If yes, give the value of the velocity constant.

    (c) Compute the sensitivity of the closed-loop transfer function to changes in the plant pole at -2.

    (d) In some instances there are dynamics in the sensor. Repeat parts (a) to (c) for $H(s) = 20/(s+20)$ and compare the corresponding velocity constants.

    Solution:

    (a) The system is Type 1 with $H(s) = 1$.

    $$E(s) = R(s) - Y(s) = \frac{s(s+2)(s+30)}{s(s+2)(s+3) + 160(s+4)}$$

    $$e_{ss} = \lim_{s\to 0} sE(s) = 0.$$

    So the system can track a step input in the steady-state. The velocity constant is $K_v = \dfrac{4\times 160}{2\times 30} = 10.67$

    (b) The system is Type 0 with respect to the disturbance and has the steady-state error.

    $$y_{ss} = -\lim_{s\to 0} sY(s) = -\frac{s+30}{s(s+2)(s+30) + (s+4)}$$

$$= -\frac{30}{4} = -7.5.$$

So the system cannot reject a constant disturbance.

(c)

$$T(s) = \frac{160(s+4)}{s(s+A)(s+30)+160(s+4)}$$

where A was inserted for the pole at the nominal value of 2.

$$\mathcal{S}_A^T = \frac{A}{T}\frac{\partial T}{\partial A}$$

But,

$$\frac{\partial T}{\partial A} = -\frac{160(s+4)(s)(s+30))}{[s(s+30)(s+A)+160(s+4)]^2} = \frac{160(s+4)(s)(s+30))}{[*]^2}$$

therefore,

$$\begin{aligned} \mathcal{S}_A^T &= -\frac{A[*]160(s+4)(s)(s+30)}{160(s+4)[*]^2} \\ &= \frac{2s(s+30)}{s(s+30)(s+2)+160(s+4)} \end{aligned}$$

At $s = 0$ the sensitivity is zero.

(d) Because the system type is computed at $s = 0$ and at that value $H = 1$, then the system remains Type 1.with respect to the reference input. However, the K_v is changed because $H = 1 - \frac{s}{s+20}$, providing negative velocity feedback. The new expression for the error is

$$E(s) = \frac{s(s+2)(s+30(s+20)-s(160)(s+4)}{s(s+2)(s+30(s+20)+20(160)(s+4)}R(s)$$

from which $K_v = 22.86$. The system remains Type 0 with respect to the disturbance input with the same position error constant $K_p = 21.33$.

15. The general unity feedback system shown in Fig. 4.30 has disturbance inputs w_1, w_2 and w_3 and is asymptotically stable. Also,

$$G_1(s) = \frac{K_1\prod_{i=1}^{m_1}(s+z_{1i})}{s^{l_1}\prod_{i=1}^{m_1}(s+p_{1i})}, \quad G_2(s) = \frac{K_2\prod_{i=1}^{m_1}(s+z_{2i})}{s^{l_2}\prod_{i=1}^{m_1}(s+p_{2i})}.$$

(a) Show that the system is of Type 0, Type l_1, and Type $(l_1 + l_2)$ with respect to disturbance inputs w_1, w_2, and w_3 respectively.

Solution:

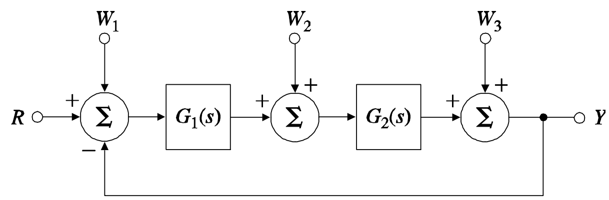


Figure 4.30: Single input-single output unity feedback system with disturbance inputs

(a)

$$Y(s) = \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)}W_1(s) = \frac{K_1K_2[\prod_i(s+z_i)]W_1(s)}{s^{l_1+l_2}\prod_i(s+p_i)+K_1K_2\prod_i(s+z_i)} \quad \text{(i)}$$

(p_i, z_i are the poles and zeros of G_1 , G_2 not at the origin).

$$-e_{ss} = y_{ss} = \lim_{s\to 0}[sY(s)] = \lim_{s\to 0}[sW_1(s)] \qquad \text{Type 0}$$

$$Y(s) = \frac{G_2(s)}{1+G_1(s)G_2(s)}W_2(s) = \frac{K_2[\prod_i(s+z_{2i})]s^{l_1}\prod_i(s+p_{1i})}{\Delta(s)}W_2(s) \qquad \text{(ii)}$$

$\Delta(s)$ is the characteristic polynomial, same as in (i) (denominator in (i)).

$$y_{ss} = [\lim_{s\to 0} s.W_2(s).s^{l_1}]\frac{\prod_i p_{1i}}{\prod_i z_{1i}} \qquad \text{Type } \ell_1$$

$$Y(s) = \frac{W_3(s)}{1+G_1(s)G_2(s)} = \frac{s^{l_1+l_2}\prod_i(s+p_i)}{\Delta(s)}W_3(s) \qquad \text{(iii)}$$

$$y_{ss} = [\lim_{s\to 0} s.W_3(s).s^{l_1+l_2}]\frac{\prod_i p_i}{\prod_i z_i} \qquad \text{Type } \ell_1+\ell_2$$

$$Y_1 = \frac{1}{s^2+s+1}R_1 + \frac{s}{s^2+s+1}W_1 + \frac{s(s+1)}{s^2+s+1}W_2$$

For constant disturbances, $R_1 = 0,\ W_1(s) = \dfrac{W_{10}}{s}$, $W_2(s) = \dfrac{W_{20}}{s}$

$$Y_1 = \frac{W_{10}+(s+1)W_{20}}{s^2+s+1}$$

Let u_2 be the signal coupling systems 1 and 2:

$$\begin{aligned} U_2 &= \frac{(s+1)(R_1-W_2)+s(s+1)W_1}{s^2+s+1} \\ Y_2 &= \frac{R_2}{s^2+3s+2} + \frac{(s+1)U_2}{s^2+3s+2} \\ &= \frac{(s+1)^2(-W_2)+s(s+1)^2W_2}{(s^2+3s+2)(s^2+s+1)} \end{aligned}$$

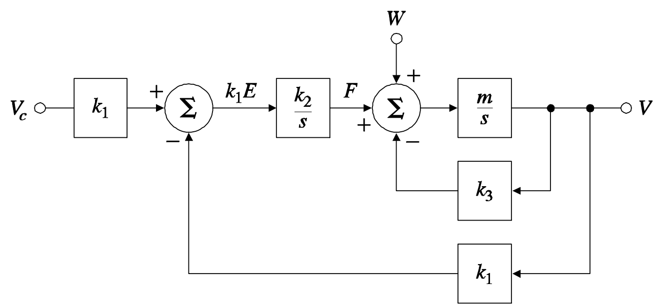


Figure 4.31: System using integral control

The system Type w.r.t. disturbances:

y_1 w.r.t. W_1	Type 1
y_1 w.r.t. W_2	Type 1
y_2 w.r.t. W_1	Type 1
y_2 w.r.t. W_2	Type 0

can be determined by applying FVT to Y_1 and Y_2 or by inspection following the rule of part (a).

16. One possible representation of an automobile speed-control system with integral control is shown in Fig. 4.31.

    (a) With a zero reference velocity input ($v_c = 0$), find the transfer function relating the output speed v to the wind disturbance w.

    (b) What is the steady-state response of v if w is a unit ramp function?

    (c) What type is this system in relation to reference inputs? What is the value of the corresponding error constant?

    (d) What is the type and corresponding error constant of this system in relation to tracking the disturbance w?

    Solution:

(a)

$$\frac{V(s)}{W(s)} = \frac{ms}{s^2 + mk_3 s + mk_1k_2}$$

(b)

$$v_{ss} = \lim_{s \to 0} s \frac{V(s)}{W(s)} \frac{W_0}{s^2} = \frac{W_0}{k_1 k_2}$$

where $W(s) = \dfrac{W_0}{s^2}$.

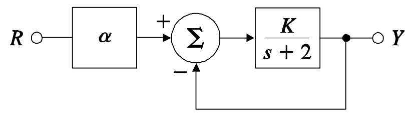


Figure 4.32: Control system for Problem 17

(c)

$$E = V_c - V = [1 - \frac{\frac{k_1 k_2 m}{s(s+mk_3)}}{1 + \frac{k_1 k_2 m}{s(s+mk_3)}}]V_c = \frac{1}{1 + \underbrace{\frac{mk_1k_2}{s(s+mk_3)}}_{G(s)}}V_c$$

$$e_{ss} = \lim_{s\to 0} \frac{sV_c}{1+G(s)}$$

$$K_p = \lim_{s\to 0} G(s) = \infty \Longrightarrow e_\infty(\text{step input}) = 0$$

$$K_v = \lim_{s\to 0} sG(s) = \frac{k_1k_2}{k_3} \Longrightarrow e_\infty(\text{ramp input}) = \frac{1}{K_v} = \frac{k_3}{k_1k_2}$$

System is Type 1.

(d) For disturbances: If the disturbance is a ramp, the result of part (a) shows that the steady state error, which is V, will be $e_{ss} = \frac{1}{k_1k_2}$. Therefore, the system is Type 1 and the velocity constant is $K_v = k_1k_2$.

17. For the feedback system shown in Fig. 4.32, find the value of α that will make the system Type 1 for $K = 5$. Give the corresponding velocity constant. Show that the system is not robust by using this value of α and computing the tracking error $e = r - y$ to a step reference for $K = 4$ and $K = 6$.

Solution:

$$Y = \frac{\alpha KR}{s+2+K} \qquad E = R - Y = \frac{s+2+K(1-\alpha)}{s+2+K}R|_{K=5} = \frac{s+7-5\alpha}{s+7}R$$

For $\alpha = \frac{7}{5}$ we have:

$$E = \frac{s}{s+7}R$$

$$e_{ss}(\text{step input}) = \lim_{s\to 0} s\frac{s}{s+7}\frac{r_0}{s} = 0$$

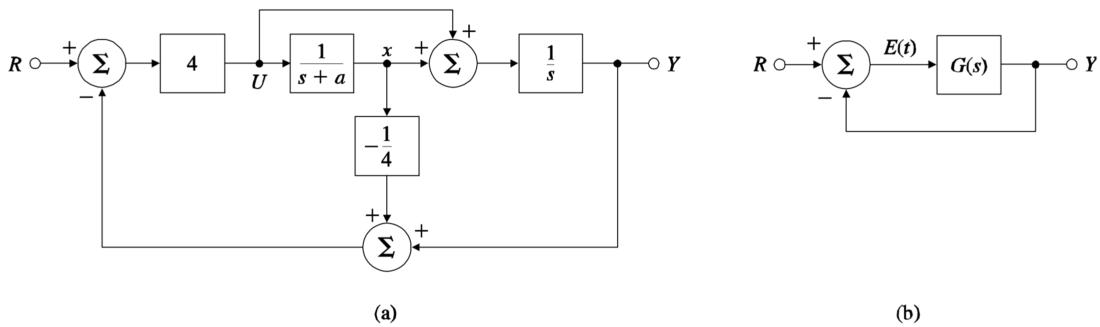


Figure 4.33: Control system for Problem 18

$$e_{ss}(\text{ramp input}) = \lim_{s\to 0} s \frac{s}{s+7}\frac{v_0}{s^2} = \frac{v_0}{7}$$

So the system is Type 1.The system is not robust:

$$e_{ss}(\text{step})|_{K=4} = \lim_{s\to 0} s \frac{s+2+4(\frac{-2}{5})}{s+6}\frac{r_0}{s} = \frac{-r_0}{15}$$

$$e_{ss}(\text{step})|_{K=6} = \frac{-r_0}{20}$$

So the system is Type 0 if $K \neq 5$.

18. Suppose you are given the system depicted in Fig. 4.33(a), where the plant parameter a is subject to variations.

   (a) Find $G(s)$ so that the system shown in Fig. 4.33(b) has the same transfer function from r to y as the system in Fig. 4.33(a).

   (b) Assume that $a = 1$ is the nominal value of the plant parameter. What is the system type and the error constant in this case?

   (c) Now assume that $a = 1 + \delta a$, where δa is some perturbation to the plant parameter. What is the system type and the error constant for the perturbed system?

   Solution:

   (a)

$$\begin{aligned} Y(s) &= \frac{1}{s}(1+\frac{1}{s+a})U(s) \\ &= \frac{1}{s}(1+\frac{1}{s+a})4(R(s)-Y(s)+\frac{1}{4(s+a)}U(s)) \qquad (1) \end{aligned}$$

$$
\begin{aligned}
U(s) &= 4R(s) - 4Y(s) + \frac{1}{s+a}U(s) \\
(1 - \frac{1}{s+a})U(s) &= 4R(s) - 4Y(s) \\
U(s) &= \frac{4(s+a)}{s+a-1}[R(s) - Y(s)] \qquad (2)
\end{aligned}
$$

Combining Eqs. (1) and (2) gives

$$
\begin{aligned}
Y(s) &= \frac{1}{s}(\frac{s+a+1}{s+a})(\frac{4(s+a)}{s+a-1})(R(s) - Y(s)) \\
&= \frac{4(s+a-1)}{s(s+a-1)}[R(s) - Y(s)]
\end{aligned}
$$

which means

$$G(s) = \frac{4(s+a+1)}{s(s+a-1)}$$

(b) $a = 1$ therefore $G(s) = \dfrac{4(s+2)}{s^2}$

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)} = \frac{s^2}{s^2+4s+8}$$

roots are in LHP so we can use the FVT,

$$e_{\text{ss,step}} = \lim_{\to 0} s(\frac{1}{s})\frac{s^2}{s^2+4s+8} = 0$$

therefore $K_p = \infty$

$$e_{\text{ss,ramp}} = \lim_{s\to 0} s(\frac{1}{s^2})\frac{s^2}{s^2+4s+8} = 0$$

and $K_v = \infty$. The error to acceleration is

$$e_{\text{ss,parabola}} = \lim_{\to 0} s(\frac{1}{s^3})\frac{s^2}{s^2+4s+8} = \frac{1}{8}$$

therefore $K_a = \frac{1}{8}$ and the system is Type 2

(c)

$$\frac{E(s)}{R(s)} = \frac{s(s+\delta a)}{s^2+(4+\delta a)s+4(2+\delta a)}$$

for small δa, roots remain in LHP.

$$e_{ss,step} = \lim_{s\to 0} s(\frac{1}{s})\frac{s(s+\delta a)}{s^2+(4+\delta a)s+4(2+\delta a)} = 0$$

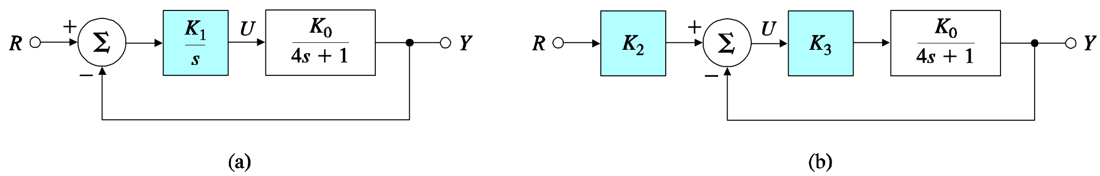


Figure 4.34: Two feedback systems for Problem 19

therefore $K_p = \infty$.

$$e_{ss,ramp} = \lim_{s \to 0} s(\frac{1}{s^2})\frac{s(s+\delta a)}{s^2 + (4+\delta a)s + 4(2+\delta a)} = \frac{\delta a}{4(2+\delta a)}.$$

Therefore, $K_v = \dfrac{4(2+\delta a)}{\delta a}$. For parabolic input, $e(t) \to \infty$, therefore, $K_a = 0$. The system is now Type 1. Plant error (parameter variation) caused the change in system type.

19. Two feedback systems are shown in Fig. 4.34.

    (a) Determine values for K_1, K_2, and K_3 so that both systems:

        i. exhibit zero steady-state error to step inputs (that is, both are Type 1), and

        ii. whose static velocity error constant $K_v = 1$ when $K_0 = 1$.

    (b) Suppose K_0 undergoes a small perturbation: $K_0 \to K_0 + \delta K_0$. What effect does this have on the system type in each case? Which system has a type which is robust? Which system do you think would be preferred?

    Solution:

    (a) System (a):

$$\begin{aligned} E &= R - Y = \frac{R(s)}{1+G(s)} \\ E(s) &= \frac{s(4s+1)}{4s^2+s+K_0K_1}R(s) \end{aligned}$$

Applying FTV:

$$e_{\text{ss,ramp}} = \frac{1}{K_1} = \frac{1}{K_v} \implies K_1 = K_v = 1$$

System (b):

$$
\begin{aligned}
E &= R - Y = \frac{1 + G - K_2 G}{1 + G} R \\
E(s) &= \frac{4s + 1 + K_3 K_0 (1 - K_2)}{4s + 1 + K_3 K_0} R(s)
\end{aligned}
$$

Applying FVT:

$$e_{\text{ss,step}} = \frac{1 + K_3 K_0 (1 - K_2)}{1 + K_3 K_0} = 0$$

for $K_0 = 1 \implies 1 + K_3(1 - K_2) = 0$

$$e_{\text{ss,ramp}} = \frac{4}{1 + K_3} = \frac{1}{K_v}$$

for $K_v = 1 \implies K_3 = 3$

$$\implies K_2 = \frac{4}{3}\ ,\ K_3 = 3$$

(b) Let $K_0 = K_0 + \delta K_0$ In System (a):

$$e_{\text{ss,step}} = \lim_{s \to 0} s \frac{s(4s + 1)}{(4s^2 + s + K_0 + \delta K_0)} \frac{1}{s} = 0$$

regardless of K_0 value. In System (b):

$$e_{\text{ss,step}} = \frac{1 + K_3 (K_0 + \delta K_0)(1 - K_2)}{1 + K_3 (K_0 + \delta K_0)} |_{K_0 = 1} = \frac{-\delta K_0}{1 + 3(1 + \delta K_0} \neq 0$$

Thus the system type of System (b) is not robust (it is a "calibrated" system type.) Control engineers prefer system (a) over (b) because it is more robust to parameter changes. (This can be expected for a closed-loop with feedback to the input while (b) has an open-loop stage to entering the feedback loop.)

20. You are given the system shown in Fig. 4.35, where the feedback gain β is subject to variations. You are to design a controller for this system so that the output $y(t)$ accurately tracks the reference input $r(t)$.

(a) Let $\beta = 1$. You are given the following three options for the controller $D_i(s)$:

$$D_1(s) = k_p, \quad D_2(s) = \frac{k_p s + k_I}{s}, \quad D_3(s) = \frac{k_p s^2 + k_I s + k_2}{s^2}.$$

Choose the controller (including particular values for the controller constants) that will result in a Type 1 system with a steady-state error to a unit reference ramp of less than $\frac{1}{10}$.

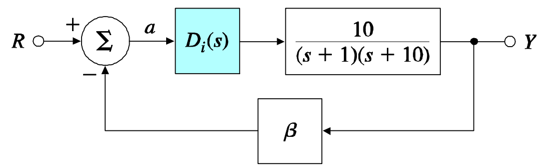


Figure 4.35: Control system for Problem 20

(b) Next, suppose that there is some attenuation in the feedback path that is modeled by $\beta = 0.9$. Find the steady-state error due to a ramp input for your choice of $D_i(s)$ in part (a).

(c) If $\beta = 0.9$, what is the system type for part (b)? What are the values of the appropriate error constant?

Solution:

(a) Need an integrator in the loop - choose $D_2(s)$

$$T(s) = \frac{Y(s)}{R(s)} = \frac{\dfrac{10(k_p s + k_I)}{s(s+1)(s+10)}}{1 + \beta \dfrac{10(k_p s + k_I)}{s(s+1)(s+10)}}$$

$$E(s) = (1 - T(s))R(s) = [\frac{s(s+1)(s+10) + \ 10(k_p s + k_I)\beta - 10(k_p s + k_I)}{s(s+1)(s+10) + 10(k_p s + k_I)\beta}]\frac{1}{s^2}$$

For $\beta = 1$,

$$\begin{aligned} e_{ss} &= \lim_{s \to 0} s[\frac{s(s+1)(s+10)}{s(s+1)(s+10) + 10(k_p s + k_I)}]\frac{1}{s^2} \\ &= \frac{10}{10k_I} = \frac{1}{k_I} \end{aligned}$$

Therefore $k_I \geq 10$ will meet the steady-state specifications The closed-loop poles are the roots of $s^3 + 11s^2 + 10s + 10(k_p s + k_I) = 0$. The Routh's array is,

$$\begin{array}{lcc} s^3: & 1 & 10(1+k_p) \\ s^2: & 11 & 10k_I \\ s^1: & \frac{110(1+k_p) - 10k_I}{11} & \\ s^0: & 10k_I & \end{array}$$

which requires $k_I > 0$ and $11(1 + k_p) - k_I > 0$ for stability.

(b) From above, with $\beta = 0.9$

$$\begin{aligned} E(s) &= [\frac{s(s+1)(s+10) + \ 9(k_p s + k_I) - 10(k_p s + k_I)}{s(s+1)(s+10) + 9(k_p s + k_I)}]R(s) \\ &= \frac{s(s+1)(s+10) - k_p - k_I}{s(s+1)(s+10) + 9(k_p s + k_I)}R(s) \end{aligned}$$

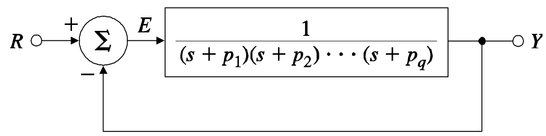


Figure 4.36: Control system for Problem 21

If k_p and k_I are chosen so that the system is stable, applying the FVT for $R(s) = \dfrac{1}{s^2}$ results in $e_{ss} \to \infty$. The system is no longer Type 1.

(c) Try $R(s) = \dfrac{1}{s}$

$$
\begin{aligned}
\lim_{s\to 0} sE(s) &= \lim_{s\to 0} \frac{s(s+1)(s+10) - k_p s - k_I}{s(s+1)(s+10) + 9(k_p s + k_I)} \\
&= -\frac{k_I}{9k_I} = -\frac{1}{9}
\end{aligned}
$$

and the system is Type 0. K_p is defined such that $|e_{ss}| = \dfrac{1}{1+K_p}$. Thus, $K_p = 8$. Without the magnitude an equivalent result is that $K_p = -10$.

21. Consider the system shown in Fig. 4.36.

(a) Find the transfer function from the reference input to the tracking error.

(b) For this system to respond to inputs of the form $r(t) = t^n 1(t)$ (where $n < q$) with zero steady-state error, what constraint is placed on the open-loop poles $p_1, p_2, \cdots, p_q$?

Solution:

(a)

$$
\frac{E(s)}{R(s)} = \frac{1}{1+G(s)} = \frac{\prod_{i=1}^{q}(s+p_i)}{\prod_{i=1}^{q}(s+p_i)+1}
$$

(b)

$$
r(t) = t^n \implies R(s) = \frac{n!}{s^{n+1}}
$$

$$
e_{ss} = \lim_{s\to 0} s\frac{n!}{s^{n+1}} \frac{\prod_{i=1}^{q}(s+p_i)}{\prod_{i=1}^{q}(s+p_i)+1}
$$

If e_{ss} is to be zero the system must have at least $n+1$ poles at the origin:

$$e_{ss} = \lim_{s \to 0} s \frac{n!}{s^{n+1}} \frac{s^{n+1} \prod_{i=1}^{q}(s+p_i)}{s^{n+1} \prod_{i=1}^{q}(s+p_i) + 1} = 0$$

22. A linear ODE model of the DC motor with negligible armature inductance ($L_a = 0$) and with a disturbance torque w was given earlier in the chapter; it is restated here, in slightly different form, as

$$\frac{JR_a}{K_t}\ddot{\theta}_m + K_e\dot{\theta}_m = \upsilon_a + \frac{R_a}{K_t}w,$$

where θ_m is measured in radians. Dividing through by the coefficient of $\ddot{\theta}_m$, we obtain

$$\ddot{\theta}_m + a_1\dot{\theta}_m = b_0\upsilon_a + c_0 w,$$

where

$$a_1 = \frac{K_t K_e}{JR_a}, \quad b_0 = \frac{K_t}{JR_a}, \quad c_0 = \frac{1}{J}.$$

With rotating potentiometers, it is possible to measure the positioning error between θ and the reference angle θ_r or $e = \theta_{ref} - \theta_m$. With a tachometer we can measure the motor speed $\dot{\theta}_m$. Consider using feedback of the error e and the motor speed $\dot{\theta}_m$ in the form

$$\upsilon_a = K(e - T_D\dot{\theta}_m),$$

where K and T_D are controller gains to be determined.

(a) Draw a block diagram of the resulting feedback system showing both θ_m and $\dot{\theta}_m$ as variables in the diagram representing the motor.

(b) Suppose the numbers work out so that $a_1 = 65$, $b_0 = 200$, and $c_0 = 10$. If there is no load torque ($w = 0$), what speed (in rpm) results from $\upsilon_a = 100$ V?

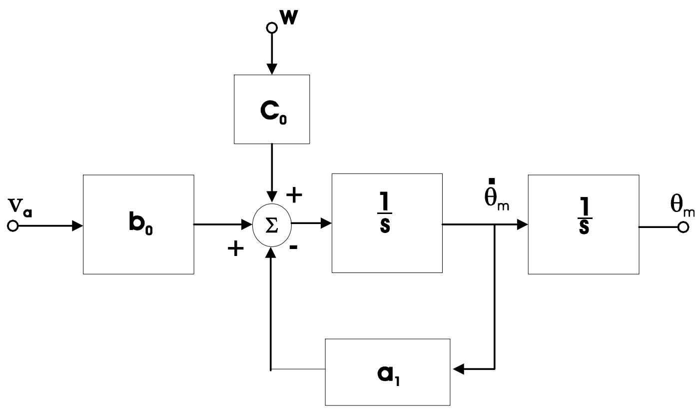
(c) Using the parameter values given in part (b), let the control be $D = k_p + k_D s$ and find k_p and k_D so that, using the results of Chapter 3, a step change in θ_{ref} with zero load torque results in a transient that has an approximately 17% overshoot and that settles to within 5% of steady-state in less than 0.05 sec.

(d) Derive an expression for the steady-state error to a reference angle input, and compute its value for your design in part (c) assuming $\theta_{ref} = 1$ rad.

(e) Derive an expression for the steady-state error to a constant disturbance torque when $\theta_{ref} = 0$, and compute its value for your design in part (c) assuming $w = 1.0$.

Solution:

(a) Block diagram:



Block diagram for Problem 4.22

(b) If V_a=constant the system is in steady state:

$$\dot{\theta} = \frac{b_0}{a_1} V_a = \frac{200 \times 100}{65} \frac{60}{2\pi} \quad \frac{\text{rad.sec}^{-1}}{\text{rpm}} = 2938 \ \text{ rpm}$$

(c)

$$\frac{\theta}{\theta_r} = \frac{Kb_0}{s^2 + s(a_1 + T_D K b_0) + Kb_0} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$M_p = 17\%, \Longrightarrow: \zeta = 0.5 \quad t_s = 0.05$ sec. to 5% :

$$\Longrightarrow e^{-\zeta\omega_n t_s} = 0.05 \Longrightarrow \zeta\omega_n = 60 \Longrightarrow \omega_n = 120$$

Comparing coefficients:

$$K = 72 \quad , \quad T_D = 3.8 \times 10^{-3}$$

(d) Steady-state error:

$$E(s) = \theta_r - \theta = \frac{s(s + a_1 + T_D K b_0)}{s^2 + s(a_1 + T_D K b_0) + Kb_0} \theta_r$$

For $\theta_r = \dfrac{1}{s}$:

$$e_{ss} = \lim_{s \to 0} sE(s) = 0 \quad \text{(Type 1)}$$

(e) Response to torque:

$$\frac{\theta}{Q_L} = \frac{c_0}{s^2 + s(a_1 + T_D K b_0) + Kb_0}$$

$$\theta_{ss} = \lim_{s \to 0} s.\theta(s) = \lim_{s \to 0} s \frac{c_0}{s^2 + \dots} \frac{1}{s} = \frac{c_0}{Kb_0} = \frac{1}{1440} \ \text{ rad}$$

23. We wish to design an automatic speed control for an automobile. Assume that (1) the car has a mass m of 1000 kg, (2) the accelerator is the control U and supplies a force on the automobile of 10 N per degree of accelerator motion, and (3) air drag provides a friction force proportional to velocity of 10 N · sec/m.

    (a) Obtain the transfer function from control input U to the velocity of the automobile.

    (b) Assume the velocity changes are given by

    $$V(s) = \frac{1}{s+0.02}U(s) + \frac{0.05}{s+0.02}W(s),$$

    where V is given in meters per second, U is in degrees, and W is the percent grade of the road. Design a proportional control law $U = -k_p V$ that will maintain a velocity error of less than 1 m/sec in the presence of a constant 2% grade.

    (c) Discuss what advantage (if any) integral control would have for this problem.

    (d) Assuming that pure integral control (that is, no proportional term) is advantageous, select the feedback gain so that the roots have critical damping ($\zeta = 1$).

    Solution:

    a.

    $$\begin{aligned} m\ddot{x} &= \sum F = K_a u - D\dot{x} \\ \mathcal{L}\{m\dot{v} &= K_a u - Dv\} \\ \frac{V}{U} &= \frac{K_a}{ms+D} = \frac{0.01}{s+0.01} \end{aligned}$$

    b. Error:

    $$\begin{aligned} E(s) &= V_d - V = V_d - \frac{\frac{k_p}{s+0.02}}{1+\frac{k_p}{s+0.02}}V_d + \frac{0.05\frac{1}{s+0.02}}{1+\frac{k_p}{s+0.02}}G(s) \\ &= \frac{(s+0.02)V_d - 0.05G}{s+0.02+k_p} \end{aligned}$$

    If we want error < 1 m/sec in presence of grade, we in fact need $|e_{ss}(\text{step})| < 1$. Assume no input : ($V_d = 0$)

    $$e_{ss}(\text{step}) = \lim_{s\to 0} s\left(\frac{-0.05}{s+0.02+k_p}\right)\frac{2}{s} = \frac{-0.1}{0.02+k_p}$$

    $$\left|\frac{-0.1}{0.02+k_p}\right| < 1$$

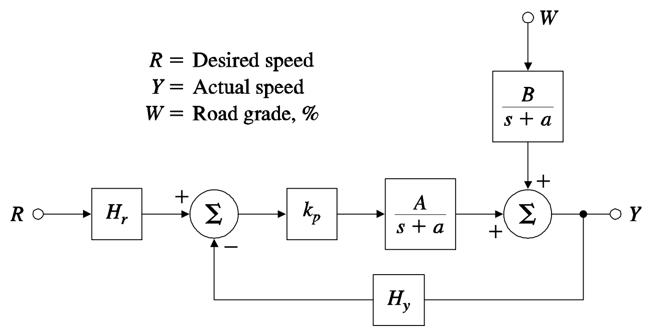


Figure 4.37: Automobile speed-control system

While solving the inequality apply (or check) restriction that poles are in LHP.

$$\Longrightarrow k_p > 0.08$$

c. The obvious advantage of integral control would be zero s.s. error for step input (Type 1 system would result).

d. Pure integral control: $k_p \rightarrow \dfrac{k_I}{s}$

$$E(s) = \frac{s(s+0.02)V_d - 0.05sG(s)}{s^2 + 0.02s + k_I}$$

$$\zeta = 1 \Longrightarrow \omega_n = 0.01 \Longrightarrow k_I = 0.0001$$

24. Consider the automobile speed control system depicted in Fig. 4.37.

(a) Find the transfer functions from $W(s)$ and from $R(s)$ to $Y(s)$.

(b) Assume that the desired speed is a constant reference r, so that $R(s) = r_o/s$. Assume that the road is level, so $w(t) = 0$. Compute values of the gains K, H_r, and H_f to guarantee that

$$\lim_{t\rightarrow\infty} y(t) = r_o.$$

Include both the open-loop (assuming $H_y = 0$) and feedback cases ($H_y \neq 0$) in your discussion.

(c) Repeat part (b) assuming that a constant grade disturbance $W(s) = w_o/s$ is present *in addition to* the reference input. In particular, find the variation in speed due to the grade change for both the feed forward and feedback cases. Use your results to explain (1) why feedback control is necessary and (2) how the gain k_p should be chosen to reduce steady-state error.

(d) Assume that $w(t) = 0$ and that the gain A undergoes the perturbation $A + \delta A$. Determine the error in speed due to the gain change for both the feed forward and feedback cases. How should the gains be chosen in this case to reduce the effects of δA?

Solution:

(a)

$$Y(s) = \frac{B}{s + a + k_p A H_y} W(s) + \frac{k_p A H_r}{s + a + k_p A H_y} R(s)$$

(b) Feedforward:$(H_y = 0)$

$$\lim_{s \to 0} sY(s) = \lim_{s \to 0} \frac{k_p A H_r}{s + a + 0} r = r$$

therefore,

$$k_p = \frac{a}{AH_r}.$$

Feedback:

$$\lim_{t \to \infty} y(t) = r$$

results in

$$\frac{k_p A H_r}{a + k_p A H_y} r = r$$

Choose k_p for performance and H_y for sensor characteristics, and set

$$H_r = \frac{a + A k_p H_y}{k_p A}$$

(c) Feedforward:

$$\begin{aligned}\lim_{t \to \infty} y(t) &= \frac{Bw}{a} + \frac{ar}{a} \\ &= r + \frac{Bw}{a}\end{aligned}$$

Therefore,

$$\delta y_{ff}(\infty) = \frac{Bw}{a},$$

all quantities are fixed- no way to reduce effect of disturbance.

Feedback:

$$\begin{aligned}\lim_{t \to \infty} y(t) &= \frac{B}{a + k_p A H_y} w + \frac{k_p A H_r}{a + k_p A H_y} r \\ &= \frac{B}{a + k_p A H_y} w + r\end{aligned}$$

(if H_r is chosen as in part **(b)**). Therefore,

$$\delta y_{fb}(\infty) = \frac{B}{a + k_p A H_y} w$$

Effect of disturbance can be made small by choosing k_p large.

(d) Feedforward: using $k_p = \frac{a}{AH_r}$ as derived in part (b),

$$y_{ff}(\infty) = (1 + \frac{\delta A}{A})r$$

therefore,

$$\delta y_{ff}(\infty) = \frac{\delta A}{A}r$$

or

$$\frac{\delta y_{ff}(\infty)}{r} = \frac{\delta A}{A}$$

which means that 5% error in A results in 5% error in tracking.
Feedback:

$$y_{fb}(\infty) = \frac{(A + \delta A)k_p H_r}{a + (A + \delta A)k_p H_y}r$$

using value for H_r chosen in part **(b)** gives

$$y_{fb}(\infty) = [\frac{(A + \delta A)}{a + (A + \delta A)k_p H_y}\frac{a + k_p A H_y}{A}]r$$

$$= r + \frac{a\delta A}{aA + (A + \delta A)k_p A H_y}r$$

$$\cong r + \frac{a}{a + k_p A H_y}\frac{\delta A}{A}$$

$$\frac{\delta y_{fb}(\infty)}{r} = \frac{a}{a + k_p A H_y}\frac{\delta A}{A}$$

Tracking error due to parameter variation can be reduced by choosing k_p large.

25. Consider the multivariable system shown in Fig. 4.38. Assume that the system is stable. Find the transfer functions from each disturbance input to each output and determine the steady-state values of y_1 and y_2 for constant disturbances. We define a multivariable system to be Type k with respect to polynomial inputs at w_i if the steady-state value of *every* output is zero for any combination of inputs of degree less than k and at least one input is a non-zero constant for an input of degree k. What is the system type with respect to disturbance rejection at w_1? At w_2?

Solution:

(a)

$$Y_1 = \frac{1}{s^2 + s + 1}R_1 + \frac{s}{s^2 + s + 1}W_1 + \frac{s(s + 1)}{s^2 + s + 1}W_2$$

For constant disturbances, $R_1 = 0,\ W_1(s) = \frac{W_{10}}{s}$, $W_2(s) = \frac{W_{20}}{s}$

$$Y_1 = \frac{W_{10} + (s + 1)W_{20}}{s^2 + s + 1}$$

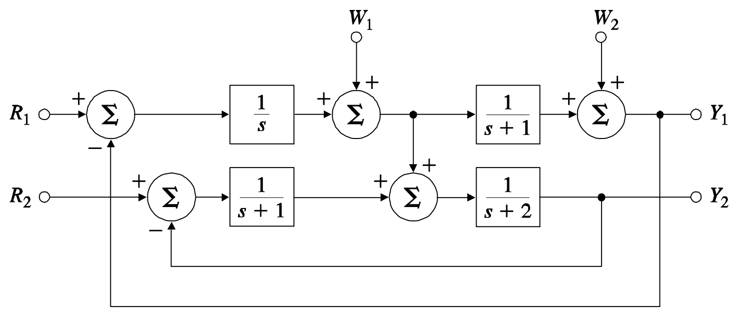


Figure 4.38: Multivariable system

Let u_2 be the signal coupling systems 1 and 2:

$$
\begin{aligned}
U_2 &= \frac{(s+1)(R_1 - W_2) + s(s+1)W_1}{s^2+s+1} \\
Y_2 &= \frac{R_2}{s^2+3s+2} + \frac{(s+1)U_2}{s^2+3s+2} \\
&= \frac{(s+1)^2(-W_2) + s(s+1)^2 W_2}{(s^2+3s+2)(s^2+s+1)}
\end{aligned}
$$

The system type w.r.t. disturbances:

y_1 w.r.t. W_1	Type 1
y_1 w.r.t. W_2	Type 1
y_2 w.r.t. W_1	Type 1
y_2 w.r.t. W_2	Type 0

can be determined by applying FVT to Y_1 and Y_2 or by inspection.

Problems and Solutions for Section 4.3: The Three Term controller: PID control

26. The transfer functions of speed control for a magnetic tape-drive system are shown in Fig. 4.39. The speed sensor is fast enough that its dynamics can be neglected and the diagram shows the equivalent unity feedback system.

    (a) Assuming the reference is zero, what is the steady-state error due to a step disturbance torque of 1 N · m? What must the amplifier gain K be in order to make the steady-state error $e_{ss} \leq 0.001$ rad/sec.?

    (b) Plot the roots of the closed-loop system in the complex plane, and accurately sketch the time response of the output for a step reference input using the gain K computed in part (a).

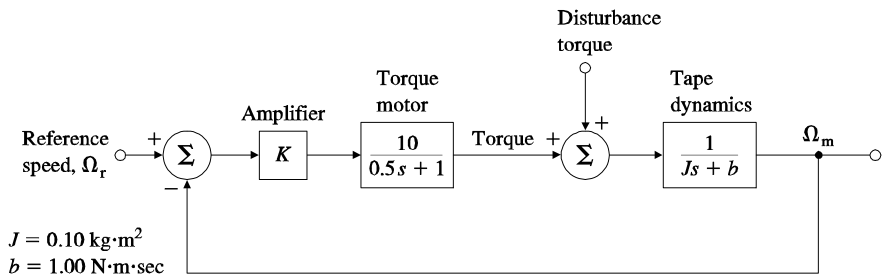


Figure 4.39: Speed-control system for a magnetic tape drive

(c) Plot the region in the complex plane of acceptable closed-loop poles corresponding to the specifications of a 1% settling time of $t_s \leq 0.1$ sec. and an overshoot $M_p \leq 5\%$.

(d) Give values for k_p and k_D for a PD controller which will meet the specifications.

(e) How would the disturbance-induced steady-state error change with the new control scheme in part (d)? How could the steady-state error to a disturbance torque be eliminated entirely?

Solution:

(a) TF for disturbance:

$$\frac{Y}{W} = \frac{\frac{1}{Js+b}}{1 + \frac{1}{Js+b} \cdot \frac{10k_p}{0.5s+1}} \qquad b = 1 \ , \ \ J = 0.1$$

$$e_{ss}(\text{step in W}) = \lim_{s \to 0} s \frac{1}{s} \frac{Y}{W} = \frac{1}{1 + 10k_p}$$

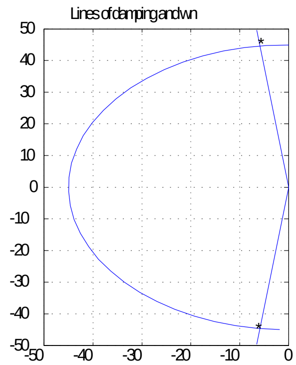
$e_{ss} \leq 0.01$, $k_p \geq 9.9$ pick $k_p = 10$.

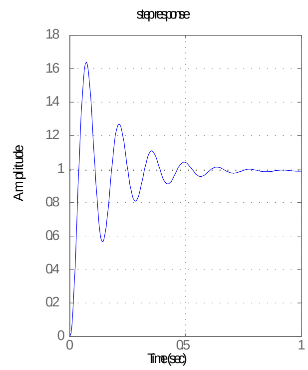
(b)

$$\frac{Y(s)}{\Omega_r(s)} = \frac{\frac{10k_p}{0.5s+1} \cdot \frac{1}{Js+b}}{1 + \frac{1}{Js+b} \cdot \frac{10k_p}{0.5s+1}} = \frac{2000}{s^2 + 12s + 2020}$$

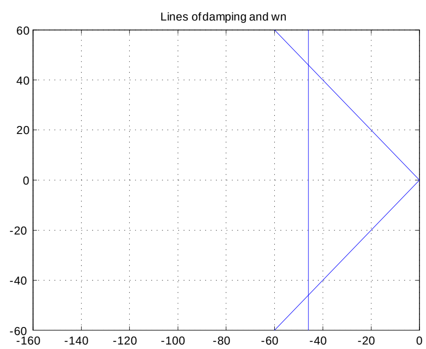
$$\omega_n = \sqrt{2020} \simeq 45 \ , \ \zeta = \frac{12}{2\sqrt{2020}} \cong 0.13$$

The roots are undesirable (damping too low, high overshoot).





(c) For $t_s \le 0.1 \Longrightarrow \sigma \ge 46$ For $M_p \le 0.05 \Longrightarrow \zeta \ge 0.7$



s-plane for part(c)

(a) d. We know that larger ω_n and ζ are needed. This can be achieved by increasing k_p and adding derivative feedback as in Fig. 4.12 (

$$\frac{Y(s)}{\Omega_r(s)} = \frac{\dfrac{10k_p}{0.5s+1}.\dfrac{1}{Js+b}}{1+\dfrac{10k_p(k_D s+1)}{(0.5s+1)(Js+b)}} = \frac{200k_p}{s^2+(12+200k_p.k_D)s+20(1+10k_p)}$$

By choosing k_p and k_D any ζ and ω_n may be achieved.

e. The TF to disturbance with new control:

$$\frac{Y}{W} = \frac{\dfrac{1}{Js+b}}{1+\dfrac{1}{Js+b}.\dfrac{10k_p(k_D s+1)}{(0.5s+1)}} = \frac{20(0.5s+1)}{s^2+(12+200k_p k_D)s+20(1+10k_p)}$$

$$e_{ss}(\text{step in W}) = \frac{1}{1+10k_p}$$

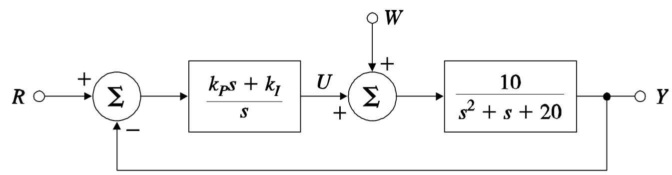


Figure 4.40: Control system for Problem 27

As before derivative feedback affects transient response only. To eliminate steady-state error we can add an integrator to the loop. This can be represented by replacing k_p with $k_p + \dfrac{k_I}{s}$ in the forward loop and still keeping PD control in the feedback loop to obtain

$$\frac{Y}{W} = \frac{20(0.5s+1)s}{s^3 + (12 + 200k_p k_D)s^2 + (20 + 200k_p + 200k_I k_D)s + 200k_I}$$

$$e_{ss}(\text{step in W}) = 0.$$

27. Consider the system shown in Fig. 4.40 with PI control.

(a) Determine the transfer function from R to Y.

(b) Determine the transfer function from W to Y.

(c) What is the system type and error constant with respect to reference tracking?

(d) What is the system type and error constant with respect to disturbance rejection?

Solution:

(a)

$$\frac{Y(s)}{R(s)} = \frac{10(k_I + k_p s)}{s[s(s+1) + 20] + 10(k_I + k_p s)}.$$

(b)

$$\frac{Y(s)}{W(s)} = \frac{10s}{s[s(s+1) + 20] + 10(k_I + k_p s)}.$$

(c) The characteristic equation is $s^3 + s^2 + (10k_p + 20)s + 10k_I = 0$. The Routh's array is

$$\begin{array}{lcc} s^3: & 1 & 10k_p + 20 \\ s^2: & 1 & 10k_I \\ s^1: & 10k_p + 20 - 10k_I & \\ s^0: & 10k_I & \end{array}$$

For stability we must have $k_I > 0$ and $k_p > k_I - 2$.

(d) System is Type 1 with respect to both r and w. The velocity constant with respect to reference tracking is $K_v = k_I/2$ and with respect to disturbance rejection is k_I.

28. Consider the second-order plant with transfer function

$$G(s) = \frac{1}{(s+1)(5s+1)}.$$

and in a unity feedback structure.

(a) Determine the system type and error constant with respect to tracking polynomial reference inputs of the system for P $[D = k_p]$, PD $[D = k_p + k_D s]$, and PID $[D = k_p + \frac{k_I}{s} + k_D s]$ controllers. Let $k_p = 19,\ k_I = 0.5$, and $k_D = \frac{4}{19}$.

(b) Determine the system type and error constant of the system with respect to disturbance inputs for each of the three regulators in part (a) with respect to rejecting polynomial disturbances $w(t)$ at the *input* to the plant.

(c) Is this system better at tracking references or rejecting disturbances? Explain your response briefly.

(d) Verify your results for parts (a) and (b) using MATLAB by plotting unit step and ramp responses for both tracking and disturbance rejection.

Solution:

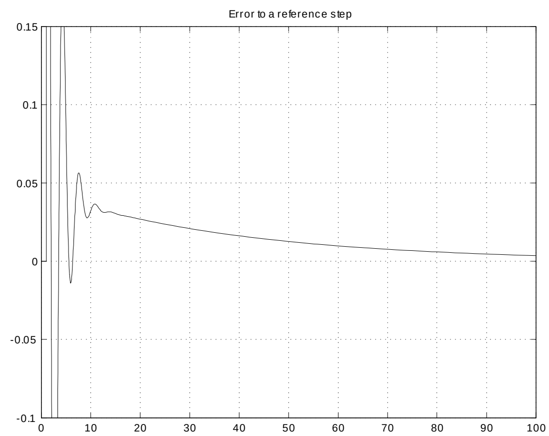
a. This plant has no pole at the origin and DC gain of 1 so, unless the controller has such a pole, the system will be Type 0.

Thus, we have: P and PD are Type 0, $K_p = k_p = 19$;PID is Type 1, $K_v = k_I = 0.5$

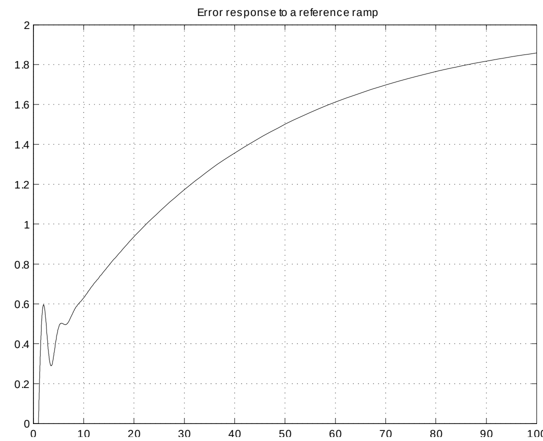
b. Again, P and PD are Type 0, $K_p = k_p = 19$;PID is Type 1, $K_v = k_I = 0.5$.

c. Because the Types and error constants are the same, this system does the same with references as with disturbances.

d. We expect the steady state error to steps to be 0 and to unit ramps to be $1/k_p = 1/0.5 = 2.0$. Note that steady-state is after a long time!

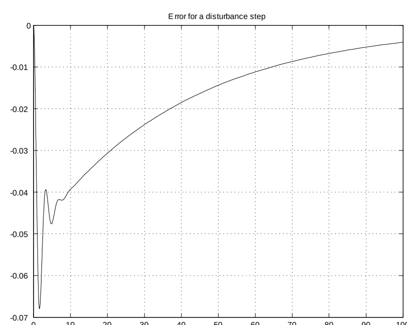


Error to a reference step.

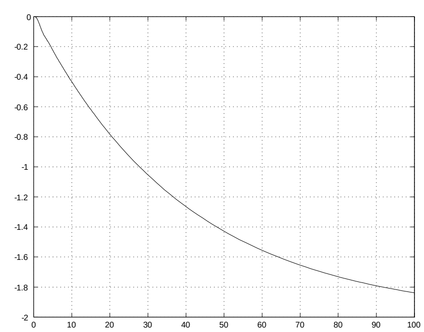


Error to a reference ramp

Notice that these transients are *very* slow. They are the consequence of a pole at $s = -0.0252$. A good rule of thumb is that a transient is over in 5 time constants. In this case the time constant is $1/0.0252 = 39.68$. Therefore we'd expect the transient to go on for about 200 seconds! The responses to disturbances are similar:.



Error to a disturbance step



Error to a disturbance ramp

. .In this case, the disturbance ramp does not excite the fast roots very much at all.

29. The DC-motor speed control shown in Fig. 4.41 is described by the differential equation

$$\dot{y} + 60y = 600v_a - 1500w,$$

where y is the motor speed, v_a is the armature voltage, and w is the load torque. Assume the armature voltage is computed using the PI control law

$$v_a = -\left(k_p e + k_I \int_0^t e dt\right).$$

where $e = r - y$.

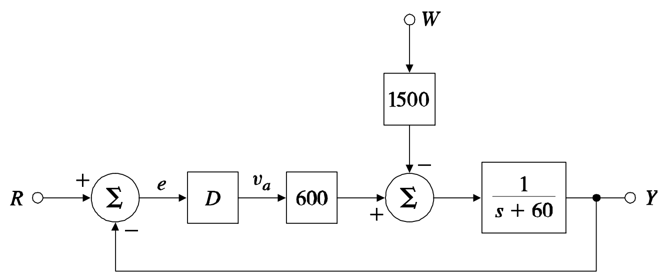


Figure 4.41: D.C. Motor speed control block diagram for Problems 29 and 30

(a) Compute the transfer function from W to Y as a function of k_p and k_I.

(b) Compute values for k_p and k_I so that the characteristic equation of the closed-loop system will have roots at $-60 \pm 60j$.

Solution:

(a) Transfer function: Set $R = 0$, then $E = -Y$

$$\begin{aligned}(s+60)Y(s) &= -600[k_pY(s) + \frac{k_I}{s}Y(s)] - 1500W(s) \\ \frac{Y(s)}{W(s)} &= \frac{-1500s}{s^2 + 60(1+10k_p)s + 600k_I}\end{aligned}$$

(b) For roots at $-60 \pm j60$: comparing to the standard form:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \implies s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$\omega_n = 60\sqrt{2}\,, \quad \zeta = 0.707$$

$$600k_I = (60\sqrt{2})^2 \implies k_I = 12$$

$$60(1+10k_p) = 2 \times 0.707 \times 60\sqrt{2} \implies k_p = 0.1$$

30. For the system in Problem 29, compute the following steady-state errors:

(a) to a unit-step reference input;

(b) to a unit-ramp reference input;

(c) to a unit-step disturbance input;

(d) for a unit-ramp disturbance input.

(e) Verify your answers to (a) and (d) using MATLAB. Note that a ramp response can be generated as a step response of a system modified by an added integrator at the reference input.

Solution:

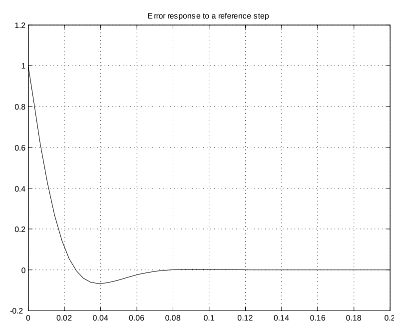
a. From Problem 21, $k_p = 0.1$ and $k_I = 12$. The DC gain of the plant is 10 so the $K_v = 10k_I$. The system is Type 1 so the error to a step is 0.

b. To a unit ramp, the error is $\dfrac{1}{K_v} = \dfrac{1}{10k_I} = \dfrac{1}{120}$.

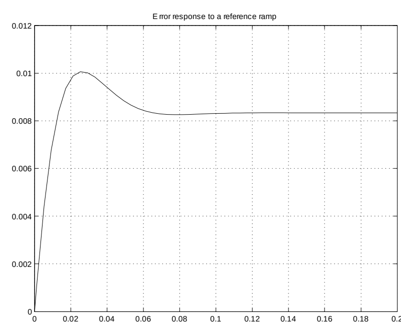
c. For a disturbance input, the system is also Type 1. The error to a step will be 0.

d. For a unit ramp disturbance input the error equals the output and is given by

$$\begin{aligned} E &= -\frac{1500}{s + 60 + 600D} W \\ &= -\frac{1500s}{s^2 + 60s + 600(k_P s + k_I)} W \\ e_{ss} &= -\frac{5}{24} \qquad\qquad for\ W = 1/s^2 \end{aligned}$$

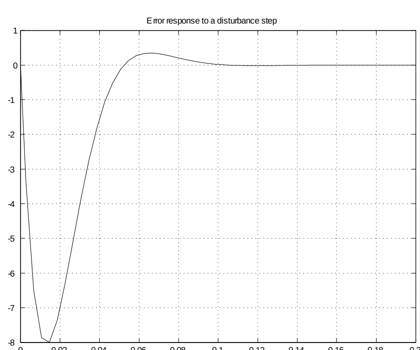


Error Response to a Reference Step

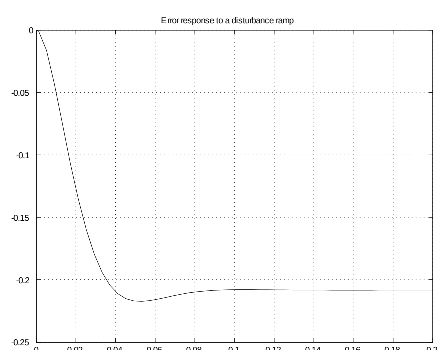


Error Response to a Reference Ramp

e. As these figures show, the error to a step goes to zero and that to a ramp goes to $1/k_I = 1/120$.



Error to a Disturbance Step



Error to a Disturbance Ramp

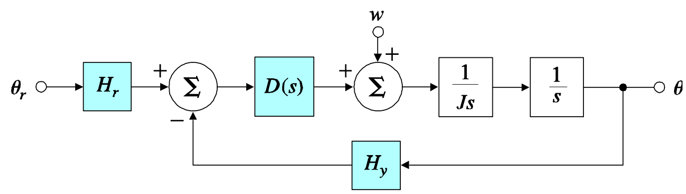


Figure 4.42: Satellite attitude control

And in this case, the error to a disturbance step goes to zero and the error to a disturbance ramp goes to $e_{ss} = 1/k_I = -0.208$.

31. Consider the satellite-attitude control problem shown in Fig. 4.42 where the normalized parameters are

$$\begin{aligned} J &= 10 \quad \text{spacecraft inertia, N-m-sec}^2/rad \\ \theta_r &= \text{reference satellite attitude, rad.} \\ \theta &= \text{actual satellite attitude, rad.} \\ H_y &= 1 \quad \text{sensor scale, factor volts/rad.} \\ H_r &= 1 \quad \text{reference sensor scale factor, volts/rad.} \\ w &= \text{disturbance torque. N-m} \end{aligned}$$

(a) Use proportional control, **P**, with $D(s) = k_p$, and give the range of values for k_p for which the system will be stable.

(b) Use **PD** control and let $D(s) = (k_p + k_D s)$ and determine the system type and error constant with respect to reference inputs.

(c) Use **PD** control, let $D(s) = (k_p + k_D s)$ and determine the system type and error constant with respect to disturbance inputs.

(d) Use **PI** control, let $D(s) = (k_p + k_I/s)$, and determine the system type and error constant with respect to reference inputs.

(e) Use **PI** control, let $D(s) = (k_p + k_I/s)$, and determine the system type and error constant with respect to disturbance inputs.

(f) Use **PID** control, let $D(s) = (k_p + k_I/s + k_D s)$ and determine the system type and error constant with respect to reference inputs.

(g) Use **PID** control, let $D(s) = (k_p + k_I/s + k_D s)$ and determine the system type and error constant with respect to disturbance inputs.

Solution:

(a) $D(s) = k_p$; The characteristic equation is

$$1 + H_y D(s) \frac{1}{Js^2} = 0$$

$$Js^2 + H_y k_p = 0$$

or $s = \pm j\sqrt{\dfrac{H_y k_p}{J}}$ so that no additional damping is provided. The system cannot be made stable with proportional control alone.

(b) Steady-state error to reference steps.

$$\begin{aligned}\frac{\Theta(s)}{\Theta_r(s)} &= H_r \frac{D(s)\frac{1}{Js^2}}{1 + D(s)H_y\frac{1}{Js^2}} \\ &= H_r \frac{(k_p + k_D s)}{Js^2 + (k_p + k_D s)H_y}\end{aligned}$$

The parameters can be selected to make the (closed-loop) system stable. If $\Theta_r(s) = \dfrac{1}{s}$ then using the FVT (assuming the system is stable)

$$\theta_{ss} = \frac{H_r}{H_y}$$

and there is zero steady-state error if $H_r = H_y$ (i.e., unity feedback).

(c) Steady-state error to disturbance steps

$$\frac{\Theta(s)}{W(s)} = \frac{1}{Js^2 + (k_p + k_D s)H_y}$$

If $W(s) = \dfrac{1}{s}$ then using the FVT (assuming system is stable), the error is $\theta_{ss} = -\dfrac{1}{k_p H_y}$.

(d) The characteristic equation is

$$1 + H_y D(s)\frac{1}{Js^2} = 0$$

With PI control,

$$Js^3 + H_y k_p s + H_y k_I = 0$$

From the Hurwitz's test, with the s^2 term missing the system will always have (at least) one pole not in the LHP. Hence, this is not a good control strategy.

(e) See d above.

(f) The characteristic equation with PID control is

$$1 + H_y(k_p + \frac{k_I}{s} + k_D s)\frac{1}{Js^2} = 0$$

or

$$Js^3 + H_y k_D s^2 + H_y k_p s + H_y k_I = 0$$

There is now control over all the three poles and the system can be made stable.

$$\begin{aligned}
\frac{\Theta(s)}{\Theta_r(s)} &= H_r \frac{D(s)\frac{1}{Js^2}}{1 + D(s)H_y\frac{1}{Js^2}} \\
&= \frac{H_r(k_p + \frac{k_I}{s} + k_D s)}{Js^2 + (k_p + \frac{k_I}{s} + k_D s)H_y} \\
&= \frac{H_r(k_D s^2 + k_p s + k_I)}{Js^3 + (k_D s^2 + k_p s + k_I)H_y}
\end{aligned}$$

If $\Theta_r(s) = \frac{1}{s}$ then using the FVT (assuming system is stable)

$$\theta_{ss} = \frac{H_r}{H_y}$$

and there is zero steady-state error if $H_r = H_y$ (i.e., unity feedback). In that case, the system is Type 3 and the (Jerk!) error constant is $K_J = \frac{k_I}{J}$.

(g) The error to a disturbance is found from

$$\frac{\Theta(s)}{W(s)} = \frac{s}{Js^3 + H_y(k_D s^2 + k_p s + k_I)}$$

If $W(s) = \frac{1}{s}$ then using the FVT (assuming the system is stable), $\theta_{ss} = 0$, the system is Type 1 and the error constant is $K_v = H_y k_p$.

32. The unit-step response of a paper machine is shown in Fig. 4.43(a) where the input into the system is stock flow onto the wire and the output is basis weight (thickness). The time delay and slope of the transient response may be determined from the figure.

(a) Find the proportional, PI, and PID-controller parameters using the Zeigler–Nichols transient-response method.

(b) Using proportional feedback control, control designers have obtained a closed-loop system with the unit impulse response shown in Fig. 4.43(b). When the gain $K_u = 8.556$, the system is on the verge of instability. Determine the proportional-, PI-, and PID-controller parameters according to the Zeigler–Nichols ultimate sensitivity method.

Solution:

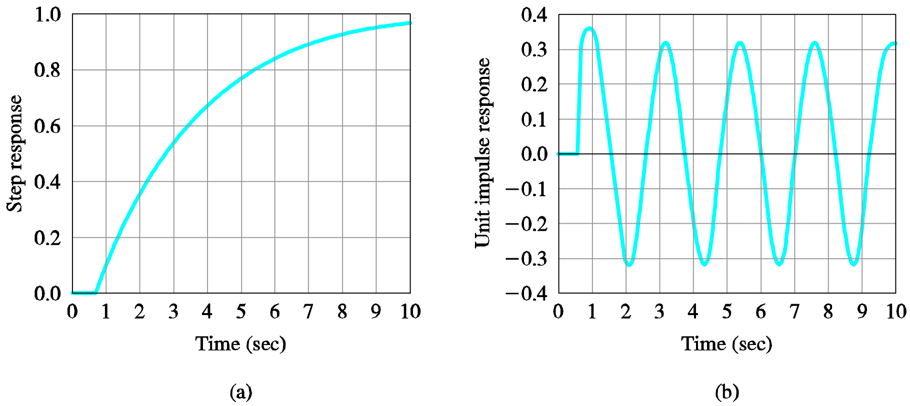


Figure 4.43: Paper-machine response data for problem 32

(a) From step response: $L = \tau_d \simeq 0.65$ sec

$$R = \frac{1}{\tau} \simeq \frac{0.2}{1.25 - 0.65} = 0.33 \quad \sec^{-1}$$

From Table 4.1:

$$\begin{array}{llll} \text{Controller Gain} & P & K = \frac{1}{RL} = 4.62 & \\ & PI & K = \frac{0.9}{RL} = 4.15 & T_I = \dfrac{L}{0.3} = 2.17 \\ & PID & K = \frac{1.2}{RL} = 5.54 & T_I = 2L = 1.3 T_D = 0.5L = 0.33 \end{array}$$

(b) From the impulse response: $P_u \simeq 2.33$ sec. and from Table 4.2:

$$\begin{array}{llll} \text{Controller Gain} & P & K = 0.5K_u = 4.28 & \\ & PI & K = 0.45K_u = 3.85 & T_I = \dfrac{1}{1.2}P_u = 1.86 \\ & PID & K = 0.6K_u = 5.13 & T_I = \dfrac{1}{2}P_u = 1.12 T_D = \dfrac{1}{8}P_u = 0.28 \end{array}$$

For the unity feedback system with proportional control $D = k_p$ and process transfer function $G(s) = \dfrac{A}{s(\tau s + 1)}$,

33. A paper machine has the transfer function

$$G(s) = \frac{e^{-2s}}{3s + 1},$$

where the input is stock flow onto the wire and the output is basis weight or thickness.

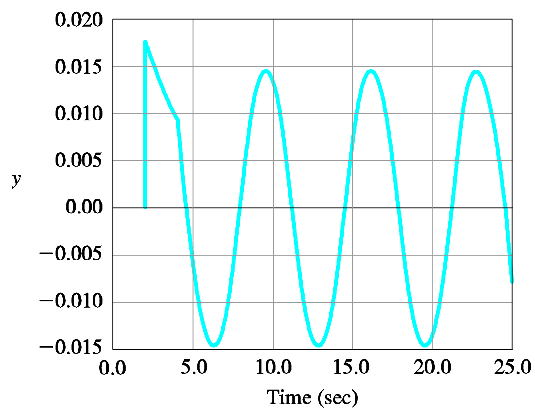


Figure 4.44: Unit impulse response for paper-machine in Problem 33

(a) Find the PID-controller parameters using the Zeigler–Nichols tuning rules.

(b) The system becomes marginally stable for a proportional gain of $K_u = 3.044$ as shown by the unit impulse response in Fig. 4.44. Find the optimal PID-controller parameters according to the Zeigler–Nichols tuning rules.

Solution:

(a) From the transfer function: $L = \tau_d \simeq 2$ sec

$$R = \frac{1}{3} \simeq 0.33 \quad \text{sec}^{-1}$$

From Table 4.1:

Controller Gain	P	$K = \frac{1}{RL}1.5$	
	PI	$K = \frac{0.9}{RL} = 1.35$	$T_I = \frac{L}{0.3} = 6.66$
	PID	$K = \frac{1.2}{RL} = 1.8$	$T_I = 2L = 4$ $T_D = 0.5L = 1.0$

(b) From the impulse response: $P_u \simeq 7$ sec From Table 4.2:

Controller Gain	P	$K = 0.5K_u = 1.52$	
	PI	$K = 0.45K_u = 1.37$	$T_I = \frac{1}{1.2}P_u = 5.83$
	PID	$K = 0.6K_u = 1.82$	$T_I = \frac{1}{2}P_u = 3.5 T_D = \frac{1}{8}P_u = 0.875$

Problems and Solutions for Section 4.4: Introduction to Digital Control

34. Compute the discrete equivalents for the following possible controllers using the trapezoid rule of Eq. (4.98). Let $T_s = 0.05$ in each case.

(a) $D_1(s) = (s+2)/2$,

(b) $D_2(s) = 2\dfrac{s+2}{s+4}$,

(c) $D_3(s) = 5\dfrac{(s+2)}{s+10}$,

(d) $D_4(s) = 5\dfrac{(s+2)(s+0.1)}{(s+10)(s+0.01)}$

Solution:

(a) Using the formula $s \leftarrow \dfrac{2}{T_s}\dfrac{z-1}{z+1}$ we find $D_1(z) = \dfrac{21z-19}{z+1}$

(b) $D_2(z) = \dfrac{1.909z - 1.727}{z - 0.8182}$

(c) $D_3(z) = \dfrac{4.2z - 3.8}{z - 0.6}$

(d) $D_4(z) = \dfrac{4.209z^2 - 7.997z + 3.79}{z^2 - 1.6z + 0.5997}$

35. Give the difference equations corresponding to the discrete controllers found in Problem 34 respectively

(a) part 1.

(b) part 2.

(c) part 3.

(d) part 4.

Solution:

(a) Reduce the $z-$ transforms to be in terms of z^{-1} if you want the equations in terms of past values. We have divided by the coefficient of the highest power if z in the denominator to get the coefficient of $u(k)$ to be 1.0 in each case. For part (a), $\dfrac{U}{E} = \dfrac{21z-19}{z+1} = \dfrac{21-19z^{-1}}{1+z^{-1}}$ and thus $[1+z^{-1}]U(z) = [21-19z^{-1}]E(z)$ from which, as $z^{-1}U(z) \Longrightarrow u(k-1)$ we get
$u(k) = -u(k-1) + 21e(k) - 19e(k-1)$. We have suppressed the T_s in these equations. It should properly be $u(kT_s)$, $u([k-1]T_s)$, etc.

(b) $u(k) = 0.8182u(k-1) + 1.909e(k) - 1.727e(k-1)$.

(c) $u(k) = 0.6u(k-1) + 4.2e(k) - 3.8e(k-1)$

(d) $u(k) = 1.6u(k-1) - 0.5997u(k-2) + 4.209e(k) - 7.997e(k-1) + 3.79e(k-2)$

Chapter 5

The Root-Locus Design Method

Problems and solutions for Section 5.1

1. Set up the following characteristic equations in the form suited to Evans's root-locus method. Give $L(s)$, $a(s)$, and $b(s)$ and the parameter, K, in terms of the original parameters in each case. Be sure to select K so that $a(s)$ and $b(s)$ are monic in each case and the degree of $b(s)$ is not greater than that of $a(s)$.

   (a) $s + (1/\tau) = 0$ versus parameter τ

   (b) $s^2 + cs + c + 1 = 0$ versus parameter c

   (c) $(s + c)^3 + A(Ts + 1) = 0$

      i. versus parameter A,

      ii. versus parameter T,

      iii. versus the parameter c, if possible. Say why you can or can not. Can a plot of the roots be drawn versus c for given constant values of A and T by any means at all

   (d) $1 + [k_p + \dfrac{k_I}{s} + \dfrac{k_D s}{\tau s + 1}]G(s) = 0$. Assume that $G(s) = A\dfrac{c(s)}{d(s)}$ where $c(s)$ and $d(s)$ are monic polynomials with the degree of $d(s)$ greater than that of $c(s)$.

      i. versus k_p

      ii. versus k_I

      iii. versus k_D

      iv. versus τ

Solution:

(a) $K = 1/\tau$; $a = s$; $b = 1$

(b) $K = c$; $a = s^2 + 1$; $b = s + 1$

(c) Part (c)

i. $K = AT$; $a = (s + c)^3$; $b = s + 1/T$

ii. $K = AT$; $a = (s + c)^3 + A$; $b = s$

iii. The parameter c enters the equation in a nonlinear way and a standard root locus does not apply. However, using a polynomial solver, the roots can be plotted versus c.

(d) Part (d)

i. $K = k_p A\tau$; $a = s(s + 1/\tau)d(s) + k_I(s + 1/\tau)c(s) + \dfrac{k_D}{\tau}s^2Ac(s)$; $b = s(s + 1/\tau)c(s)$

ii. $K = Ak_I$; $a = s(s + 1/\tau)d(s) + Ak_p s(s + 1/\tau) + \dfrac{k_D}{\tau}s^2Ac(s)$; $b = s(s + 1/\tau)c(s)$

iii. $K = \dfrac{Ak_D}{\tau}$; $a = s(s + 1/\tau)d(s) + Ak_p s(s + 1/\tau)c(s) + Ak_I(s + 1/\tau)c(s)$; $b = s^2c(s)$

iv. $K = 1/\tau$; $a = s^2d(s) + k_pAs^2c(s) + k_IAsc(s)$; $b = sd(s) + k_psAc(s) + k_IAc(s) + k_Ds^2Ac(s)$

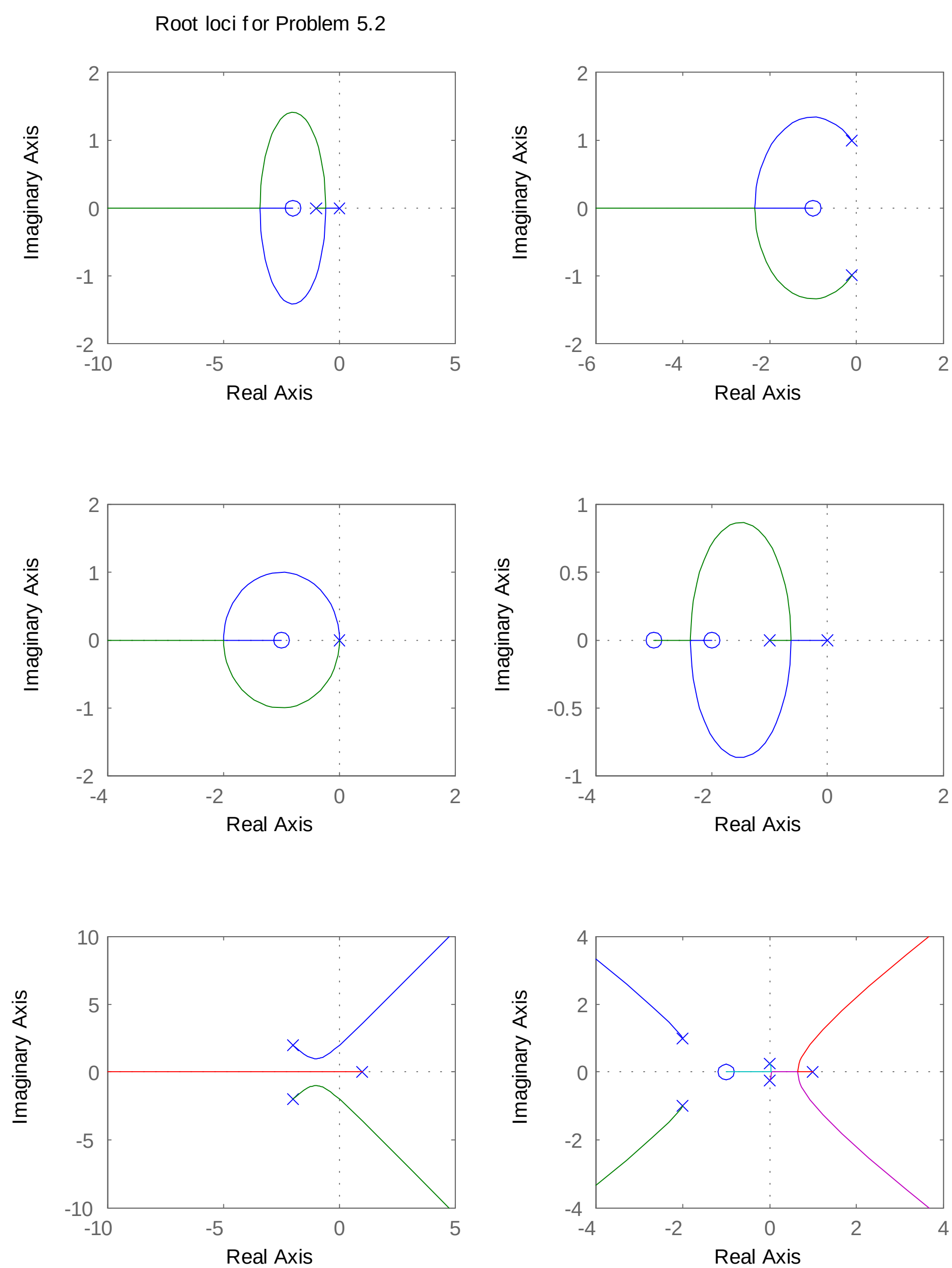
Problems and solutions for Section 5.2

2. Roughly sketch the root loci for the pole-zero maps as shown in Fig. 5.51. Show your estimates of the center and angles of the asymptotes, a rough evaluation of arrival and departure angles for complex poles and zeros, and the loci for positive values of the parameter K. Each pole-zero map is from a characteristic equation of the form

$$1 + K\frac{b(s)}{a(s)} = 0,$$

where the roots of the numerator $b(s)$ are shown as small circles o and the roots of the denominator $a(s)$ are shown as $\times' s$ on the s-plane. Note that in Fig. 5.51(c), there are two poles at the origin and there are two poles on the imaginary axis in (f), slightly off the real axis.

Solution:


Root loci f or Problem 5.2
Imaginary Axis
Real Axis
2
1
0
-1
-2
-10
-5
0
5
Imaginary Axis
Real Axis
-6
-4
-2
0
2
Imaginary Axis
Real Axis
-4
-2
0
2
Imaginary Axis
Real Axis
0.5
-0.5
Imaginary Axis
Real Axis
10
5
-5
-10
Imaginary Axis
Real Axis
4
-4

We had to make up some numbers to do it on Matlab, so the results partly depend on what was dreamed up, but the idea here is just get the basic rules right.

(a)$a(s) = \ s^2 + s$; $b(s) = s + 1$

Breakin(s)   -3.43; Breakaway(s)   -0.586

(b) $a(s) = s^2 + 0.2s + 1$; $b(s) = s + 1$

Angle of departure: 135.7

Breakin(s)   -4.97

(c) $a(s) = s^2$; $b(s) = (s + 1)$

Breakin(s)   -2

(d) $a(s) = s^2 + 5s + 6$; $b(s) = s^2 + s$

Breakin(s)   -2.37

Breakaway(s)   -0.634

(e) $a(s) = s^3 + 3s^2 + 4s - 8$

Center of asymptotes -1

Angles of asymptotes ± 60, 180

Angle of departure: -56.3

(f) $a(s) = s^3 + 3s^2 + s - 5$; $b(s) = s + 1$

Center of asymptotes -.667

Angles of asymptotes ± 60, -180

Angle of departure: -90

Breakin(s)   -2.06

Breakaway(s)   0.503

But, to get this one right, all you have to do is get the real axis segments and the 4 asymptotes going out at the right angles. You don't know, really, where it breaks away from the real axis nor where the center of asymptotes are.

3. For the characteristic equation

$$1 + \frac{K}{s(s+1)(s+5)} = 0 :$$

(a) Draw the real-axis segments of the corresponding root locus.

(b) Sketch the asymptotes of the locus for $K \to \infty$.

(c) For what value of K are the roots on the imaginary axis?

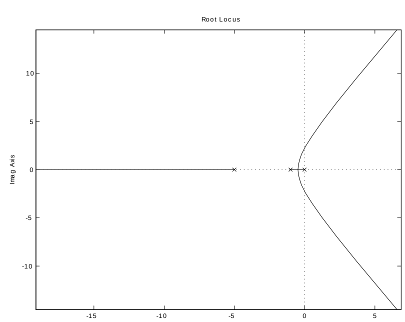
(d) Verify your sketch with a MATLAB plot.

Solution:

(a) The real axis segments are $0 > \sigma > -1; -5 > \sigma$

(b) $\alpha = -6/3 = -2$; $\phi_i = \pm 60$, 180

(c) $K_o = 30$



Solution for Problem 5.3

4. *Real poles and zeros.* Sketch the root locus with respect to K for the equation $1 + KL(s) = 0$ and the following choices for $L(s)$. Be sure to give the asymptotes, arrival and departure angles at any complex zero or pole, and the frequency of any imaginary-axis crossing. After completing each hand sketch verify your results using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

(a) $L(s) = \dfrac{(s+2)}{s(s+1)(s+5)(s+10)}$

(b) $L(s) = \dfrac{1}{s(s+1)(s+5)(s+10)}$

(c) $L(s) = \dfrac{(s+2)(s+6)}{s(s+1)(s+5)(s+10)}$

(d) $L(s) = \dfrac{(s+2)(s+4)}{s(s+1)(s+5)(s+10)}$

Solution:

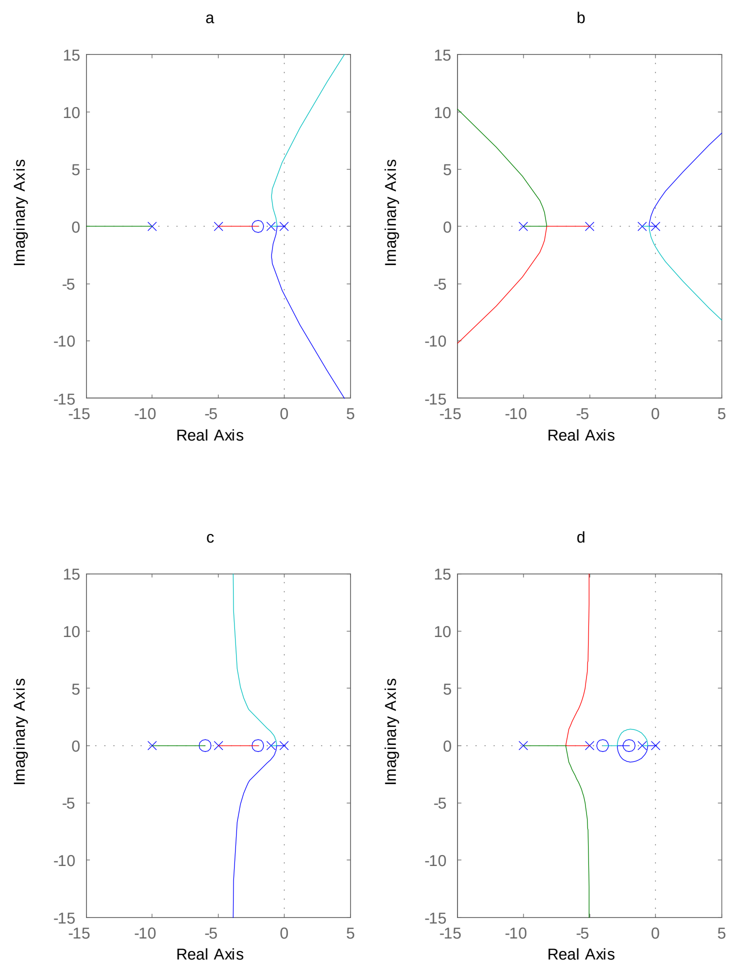
All the root locus plots are displayed at the end of the solution set for this problem.

(a) $\alpha = -4.67$; $\phi_i = \pm 60$; ± 180; $\omega_o = 5.98$

(b) $\alpha = -4$; $\phi_i = \pm 45$; ± 135; $\omega_o = 1.77$

(c) $\alpha = -4$; $\phi_i = \pm 90$; $\omega_o -> none$

(d) $\alpha = -5$; $\phi_i = \pm 90$; $\omega_o -> none$



Root loci for Problem 5.4

5. *Complex poles and zeros* Sketch the root locus with respect to K for the equation $1 + KL(s) = 0$ and the following choices for $L(s)$. Be sure to give the asymptotes, arrival and departure angles at any complex zero or pole, and the frequency of any imaginary-axis crossing. After completing each hand sketch verify your results using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

(a) $L(s) = \dfrac{1}{s^2 + 3s + 10}$

(b) $L(s) = \dfrac{1}{s(s^2 + 3s + 10)}$

(c) $L(s) = \dfrac{(s^2 + 2s + 8)}{s(s^2 + 2s + 10)}$

(d) $L(s) = \dfrac{(s^2 + 2s + 12)}{s(s^2 + 2s + 10)}$

(e) $L(s) = \dfrac{(s^2 + 1)}{s(s^2 + 4)}$

(f) $L(s) = \dfrac{(s^2 + 4)}{s(s^2 + 1)}$

Solution:

All the root locus plots are displayed at the end of the solution set for this problem.

(a) $\alpha = -3$; $\phi_i = \pm 90$; $\theta_d = \pm 90$ $\omega_o -> none$

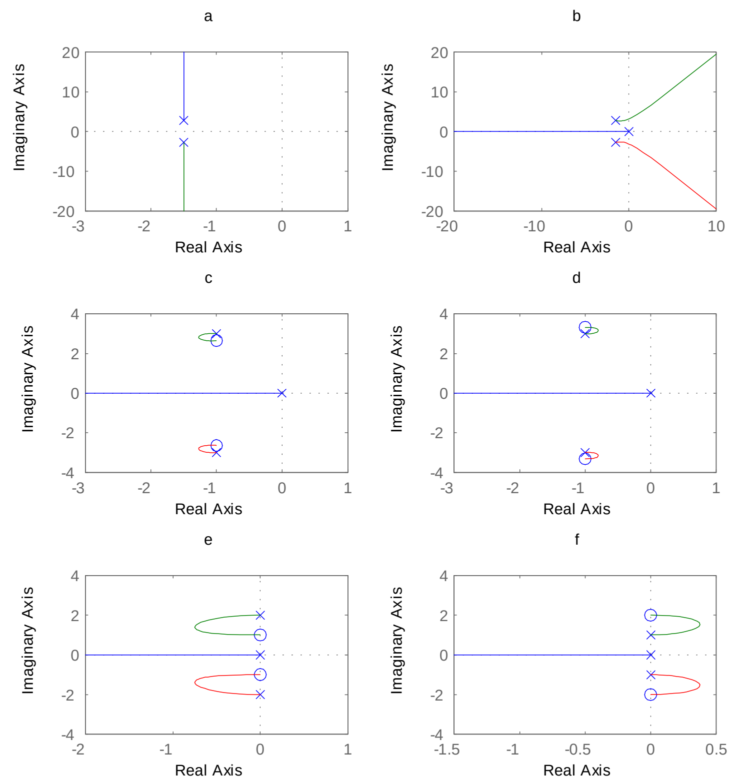
(b) $\alpha = -3$; $\phi_i = \pm 60, \pm 180$; $\theta_d = \pm 28.3$ $\omega_o = 3.16$

(c) $\alpha = -2$; $\phi_i = \pm 180$; $\theta_d = \pm 161.6$; $\theta_a = \pm 200.7$; $\omega_o -> none$

(d) $\alpha = -2$; $\phi_i = \pm 180$; $\theta_d = \pm 18.4$; $\theta_a = \pm 16.8$; $\omega_o -> none$

(e) $\alpha = 0$; $\phi_i = \pm 180$; $\theta_d = \pm 180$; $\theta_a = \pm 180$; $\omega_o -> none$

(f) $\alpha = 0$; $\phi_i = \pm 180$; $\theta_d = 0$; $\theta_a = 0$; $\omega_o -> none$



Root loci for Problem 5.5

6. *Multiple poles at the origin* Sketch the root locus with respect to K for the equation $1+KL(s)=0$ and the following choices for $L(s)$. Be sure to give the asymptotes, arrival and departure angles at any complex zero or pole, and the frequency of any imaginary-axis crossing. After completing each hand sketch verify your results using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

   (a) $L(s) = \dfrac{1}{s^2(s+8)}$

   (b) $L(s) = \dfrac{1}{s^3(s+8)}$

   (c) $L(s) = \dfrac{1}{s^4(s+8)}$

   (d) $L(s) = \dfrac{(s+3)}{s^2(s+8)}$

   (e) $L(s) = \dfrac{(s+3)}{s^3(s+4)}$

   (f) $L(s) = \dfrac{(s+1)^2}{s^3(s+4)}$

   (g) $L(s) = \dfrac{(s+1)^2}{s^3(s+10)^2}$

   Solution:

   All the root locus plots are displayed at the end of the solution set for this problem.

   (a) $\alpha = -2.67$; $\phi_i = \pm 60$; ± 180; $w_0 -> none$

   (b) $\alpha = -2$; $\phi_i = \pm 45$; ± 135; $w_0 -> none$

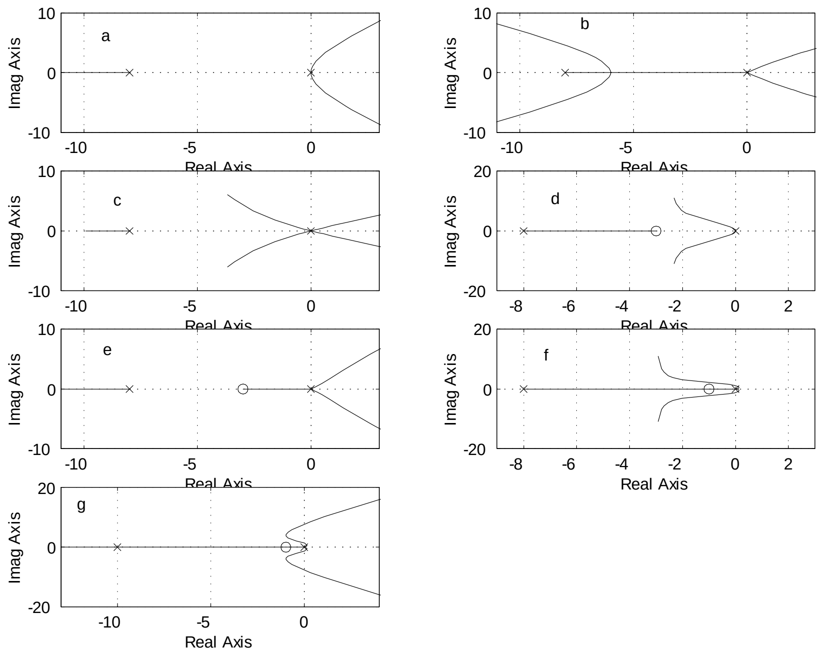
   (c) $\alpha = -1.6$; $\phi_i = \pm 36$; ± 108; $w_0 -> none$

   (d) $\alpha = -2.5$; $\phi_i = \pm 90$; $w_0 -> none$

   (e) $\alpha = -0.33$; $\phi_i = \pm 60$; ± 180; $w_0 -> none$

   (f) $\alpha = -3$; $\phi_i = \pm 90$; $w_0 = \pm 1.414$

   (g) $\alpha = -6$; $\phi_i = \pm 60; 180$; $w_0 = \pm 1.31; \pm 7.63$



Solution for Problem 5.6

7. *Mixed real and complex poles* Sketch the root locus with respect to K for the equation $1 + KL(s) = 0$ and the following choices for $L(s)$. Be sure to give the asymptotes, arrival and departure angles at any complex zero or pole, and the frequency of any imaginary-axis crossing. After completing each hand sketch verify your results using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

(a) $L(s) = \dfrac{(s+2)}{s(s+10)(s^2+2s+2)}$

(b) $L(s) = \dfrac{(s+2)}{s^2(s+10)(s^2+6s+25)}$

(c) $L(s) = \dfrac{(s+2)^2}{s^2(s+10)(s^2+6s+25)}$

(d) $L(s) = \dfrac{(s+2)(s^2+4s+68)}{s^2(s+10)(s^2+4s+85)}$

(e) $L(s) = \dfrac{[(s+1)^2+1]}{s^2(s+2)(s+3)}$

Solution:

All the plots are attached at the end of the solution set.

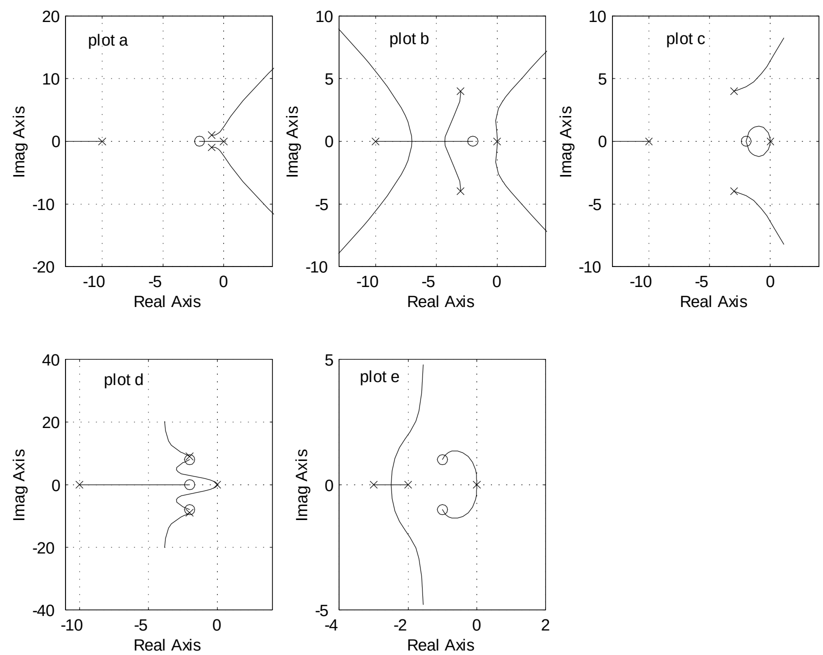
(a) $\alpha = -3.33$; $\phi_i = \pm 60$; ± 180; $w_0 = \pm 2.32$; $\quad \theta_d = \pm 6.34$

(b) $\alpha = -3.5$; $\phi_i = \pm 45$; ± 135; $w_0 -> none$; $\quad \theta_d = \pm 103.5$

(c) $\alpha = -4$; $\phi_i = \pm 60$; ± 180; $w_0 = \pm 6.41$; $\quad \theta_d = \pm 14.6$

(d) $\alpha = -4$; $\phi_i = \pm 90$; $w_0 -> none$; $\quad \theta_d = \pm 106$; $\quad \theta_a = \pm 253.4$

(e) $\alpha = -1.5$; $\phi_i = \pm 90$; $w_0 -> none$; $\quad \theta_a = \pm 71.6$



Solution for Problem 5.7

8. *Right half plane poles and zeros* Sketch the root locus with respect to K for the equation $1+KL(s)=0$ and the following choices for $L(s)$. Be sure to give the asymptotes, arrival and departure angles at any complex zero or pole, and the frequency of any imaginary-axis crossing. After completing each hand sketch verify your results using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

   (a) $L(s) = \dfrac{s+2}{s+10}\dfrac{1}{s^2-1}$; The model for a case of magnetic levitation with lead compensation.

   (b) $L(s) = \dfrac{s+2}{s(s+10)}\dfrac{1}{(s^2-1)}$; The magnetic levitation system with integral control and lead compensation.

   (c) $L(s) = \dfrac{s-1}{s^2}$

   (d) $L(s) = \dfrac{s^2+2s+1}{s(s+20)^2(s^2-2s+2)}$. What is the largest value that can be obtained for the damping ratio of the stable complex roots on this locus?

   (e) $L(s) = \dfrac{(s+2)}{s(s-1)(s+6)^2}$,

   (f) $L(s) = \dfrac{1}{(s-1)[(s+2)^2+3]}$

   Solution:

   (a) $\alpha=-4$; $\phi_i=\pm 90$; $w_0->none$

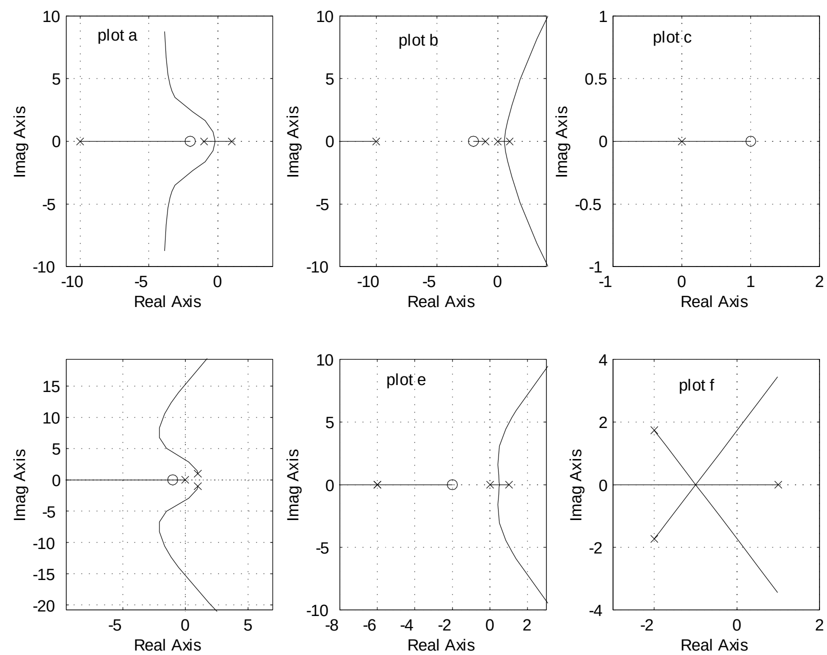
   (b) $\alpha=-4$; $\phi_i=\pm 60; 180$; $w_0->none$

   (c) $\alpha=-1$; $\phi_i=\pm 180$; $w_0->none$

   (d) $\alpha=-12$; $\phi_i=\pm 60; 180$; $w_0=\pm 3.24; \pm 15.37$; $\theta_d=\pm 92.4$

   (e) $\alpha=-3$; $\phi_i=\pm 60; 180$; $w_0->none$

   (f) $\alpha=-1$; $\phi_i=\pm 60; 180$; $w_0=\pm 1.732$; $\theta_d=\pm 40.9$



Solution for Problem 5.8

9. Put the characteristic equation of the system shown in Fig. 5.52 in root locus form with respect to the parameter α and identify the corresponding $L(s)$, $a(s)$, and $b(s)$. Sketch the root locus with respect to the parameter α, estimate the closed-loop pole locations and sketch the corresponding step responses when $\alpha = 0$, 0.5, and 2. Use MATLAB to check the accuracy of your approximate step responses.

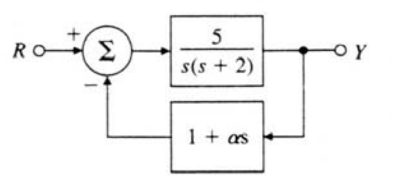
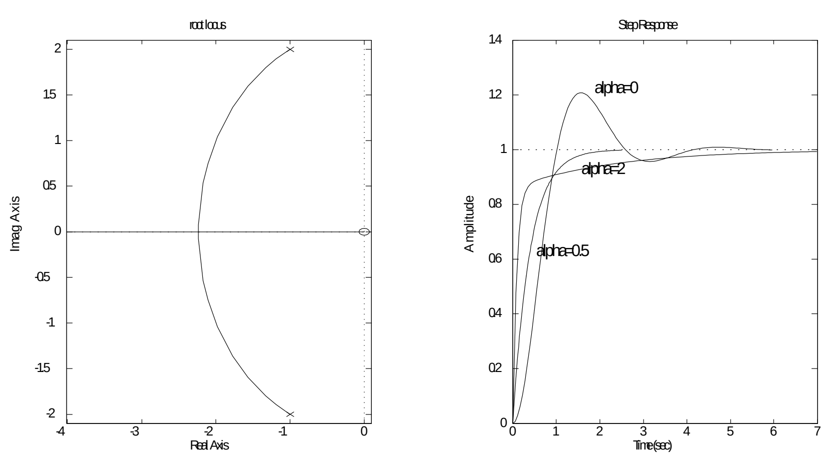


Figure 5.52: Control system for problem 9

Solution:

The characteristic equation is $s^2+2s+5+5\alpha s = 0$ and $L(s) = \dfrac{s}{s^2+2s+5}$. the root locus and step responses are plotted below.



10. Use the MATLAB function rltool to study the behavior of the root locus of $1 + KL(s)$ for

$$L(s) = \frac{(s+a)}{s(s+1)(s^2+8s+52)}$$

as the parameter a is varied from 0 to 10, paying particular attention to the region between 2.5 and 3.5. Verify that a multiple root occurs at a complex value of s for some value of a in this range.

Solution:

For small values of α, the locus branch from $0, -1$ makes a circular path around the zero and the branches from the complex roots curve off toward the asymptotes. For large values of α the branches from the complex roots break into the real axis and those from 0, -1 curve off toward the asymptotes. At about $\alpha = 3.11$ these loci touch corresponding to complex multiple roots.

11. Use the Routh criterion to find the range of the gain K for which the systems in Fig. 5.53 are stable and use the root locus to confirm your calculations.

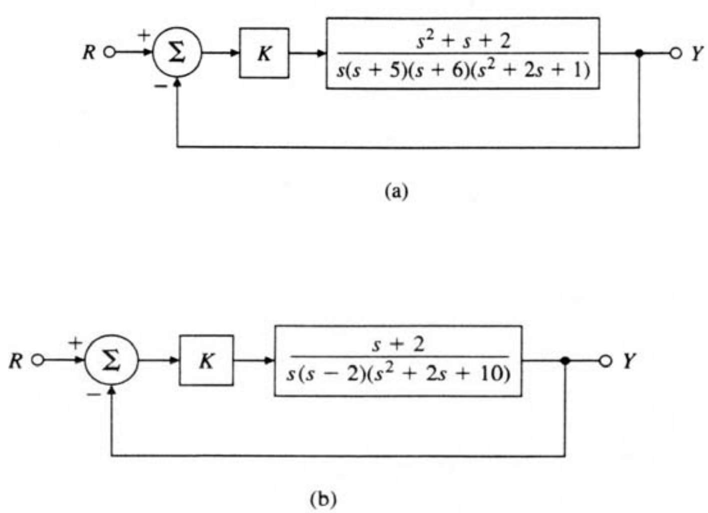
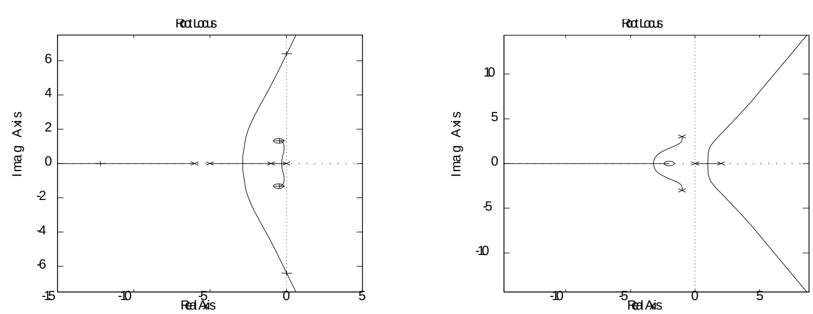


Figure 5.53: Feedback systems for problem 11

Solution:

(a) The system is stable for $0 \le K \le 478.226$ The root locus of the system and the location of the roots at the crossover points are shown in the plots

(b) There is a pole in the right hand plane thus the system is unstable for all values of K as shown in the last plot.



Solution for Problem 11

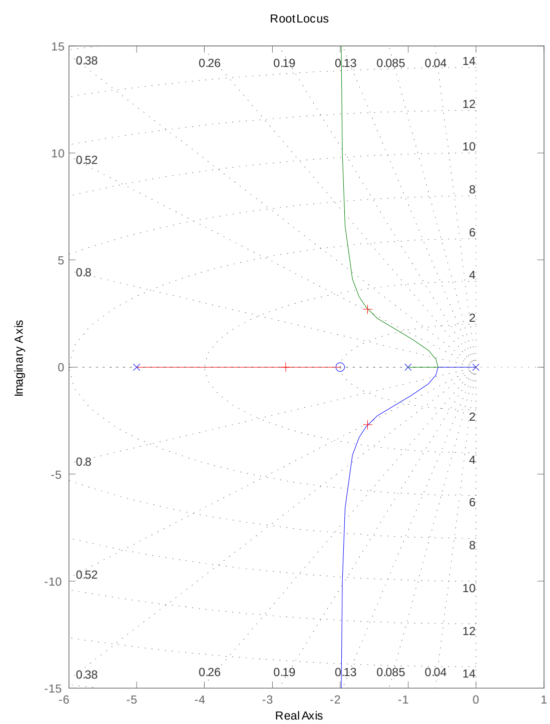
12. Sketch the root locus for the characteristic equation of the system for which

$$L(s) = \frac{(s+2)}{s(s+1)(s+5)},$$

and determine the value of the root-locus gain for which the complex conjugate poles have a damping ratio of 0.5.

Solution:

Plot the system on Matlab using rlocus(sys), and use [K]= rlocfind(sys) to pick the gain where the damping ratio = 0.5. Find that K = 14 (approximately).



Root locus with 0.5 damping marked

13. For the system in Fig. 5.54:

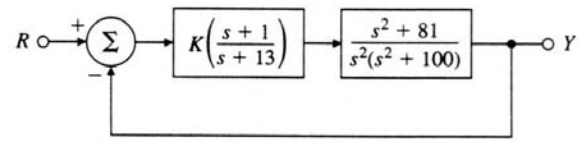


Figure 5.54: Feedback system for problem 13

(a) Find the locus of closed-loop roots with respect to K.

(b) Is there a value of K that will cause all roots to have a damping ratio greater than 0.5?

(c) Find the values of K that yield closed-loop poles with the damping ratio $\zeta = 0.707$.

(d) Use MATLAB to plot the response of the resulting design to a reference step.

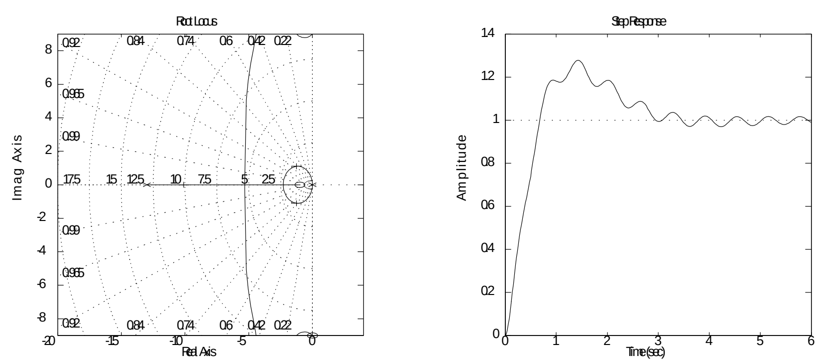
Solution:

(a) The locus is plotted below

(b) There is a K which will make the 'dominant' poles have damping 0.5 but none that will make the poles from the resonance have that much damping.

(c) Using rlocfind, the gain is about 35.

(d) The step response shows the basic form of a well damped response with the vibration of the response element added.



Root locus and step response for Problem 5.13

14. For the feedback system shown in Fig. 5.55, find the value of the gain K that results in dominant closed-loop poles with a damping ratio $\zeta = 0.5$.

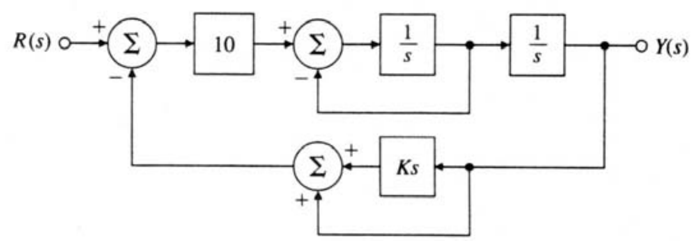
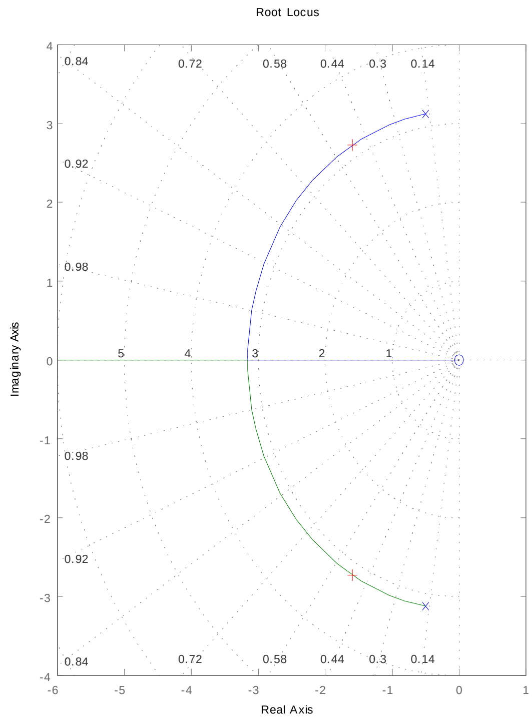


Figure 5.55: Feedback system for Problem 14

Solution:

Use block diagram reduction to find the characteristic equation of the closed loop system, then divide that up into terms with and without K to find the root locus form, where $L(s) = \dfrac{10s}{s^2 + s + 10}$. Plugging into Matlab and using rlocfind produces the required gain to be $K = 0.22$.The locus is



Root locus with 0.5 damping marked

Problems and solutions for Section 5.3

15. A simplified model of the longitudinal motion of a certain helicopter near hover has the transfer function

$$G(s) = \frac{9.8(s^2 - 0.5s + 6.3)}{(s + 0.66)(s^2 - 0.24s + 0.15)}.$$

and the characteristic equation $1 + D(s)G(s) = 0$. Let $D(s) = k_p$ at first.

(a) Compute the departure and arrival angles at the complex poles and zeros.

(b) Sketch the root locus for this system for parameter $K = 9.8k_p$.Use axes -4 $\leq x \leq 4$. $-3 \leq y \leq 3$;

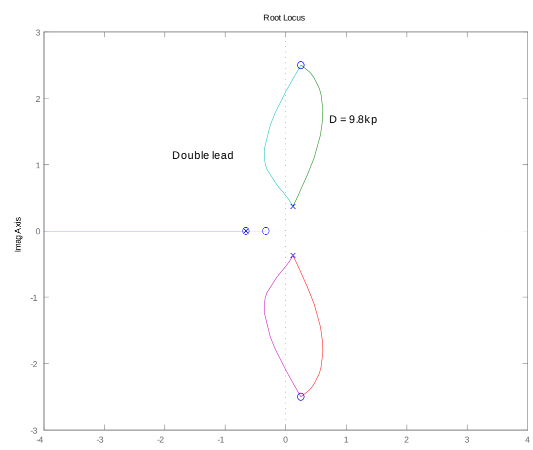
(c) Verify your answer using MATLAB. Use the command axis([-4 4 -3 3]) to get the right scales.

(d) Suggest a practical (at least as many poles as zeros) alternative compensation $D(s)$ which will at least result in a stable system.

Solution:

(a) $\alpha = .92$; $\phi = 180$; $\varphi = 63.83$; $\psi = -26.11$

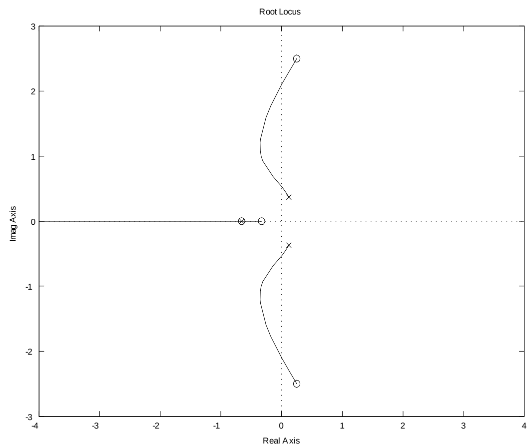
(b)



Problem 5.15(b)

(c) For this problem a double lead is needed to bring the roots into the left half-plane. The plot shows the rootlocus for control for. Let $D = \dfrac{(s + .66)(s + .33)}{(s + 5)^2}$.

(d)



Problem 5.15(d)

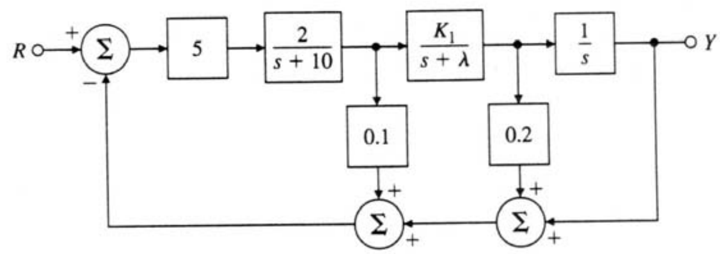


Figure 5.56: Control system for problem 5.16

16. For the system given in Fig. 5.56,

(a) plot the root locus of the characteristic equation as the parameter K_1 is varied from 0 to ∞ with $\lambda = 2$. Give the corresponding $L(s)$, $a(s)$, and $b(s)$.

(b) Repeat part (a) with $\lambda = 5$. Is there anything special about this value?

(c) Repeat part (a) for fixed $K_1 = 2$ with the parameter $K = \lambda$ varying from 0 to ∞.

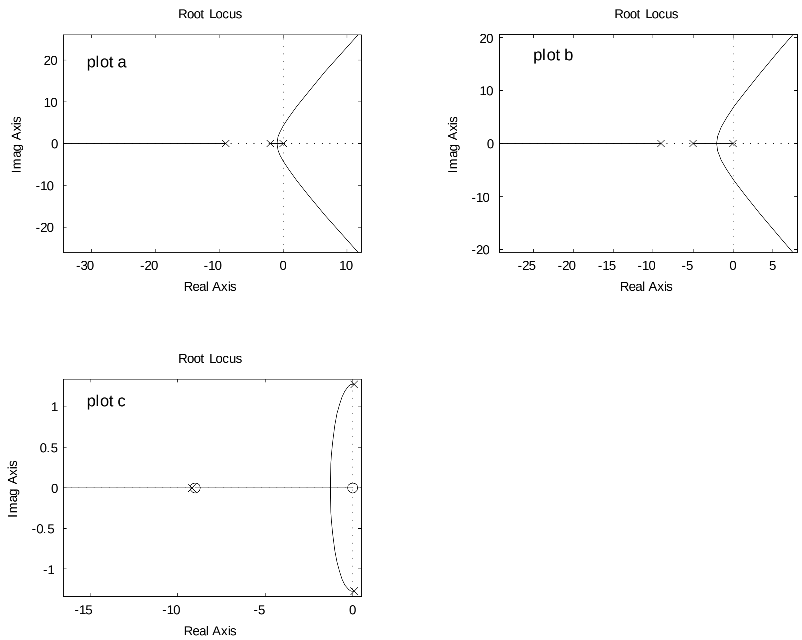
Solution:

The root locus for each part is attached at the end.

(a) $L(s) = \frac{0.75}{S(0.1S^2+1.1S+1.8)} = \frac{a(s)}{b(s)}$

(b) L(s)=$\frac{0.75}{S(0.1S^2+1.4S+4.5)} = \frac{a(s)}{b(s)}$

(c) L(s)=$\frac{S(0.1S+0.9)}{0.1S\hat{}3+0.9S+1.5} = \frac{a(s)}{b(s)}$



Solution for problem 5.16

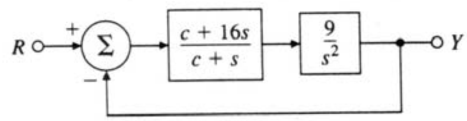
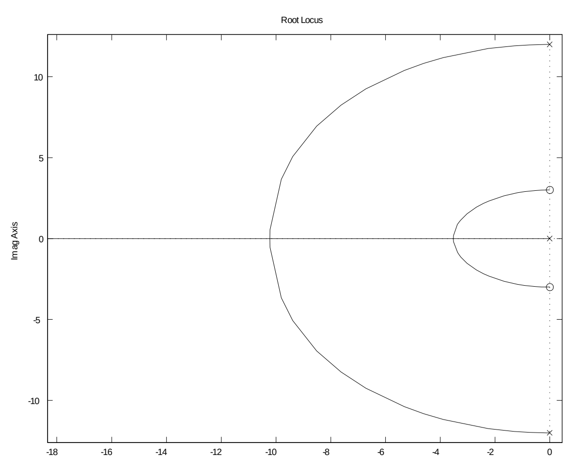


Figure 5.57: Control system for problem 17

17. For the system shown in Fig. 5.57, determine the characteristic equation and sketch the root locus of it with respect to positive values of the parameter c. Give $L(s)$, $a(s)$, and $b(s)$ and be sure to show with arrows the direction in which c increases on the locus.

    (a) Solution:

$$L(s) = \frac{s^2 + 9}{s^3 + 144s} = \frac{a(s)}{b(s)}$$



Solution for problem 5.17

18. Suppose you are given a system with the transfer function

$$L(s) = \frac{(s+z)}{(s+p)^2},$$

where z and p are real and $z > p$. Show that the root-locus for $1+KL(s) = 0$ with respect to K is a circle centered at z with radius given by

$$r = (z-p)$$

Hint. Assume $s + z = re^{j\phi}$ and show that $L(s)$ is real and negative for real ϕ under this assumption.

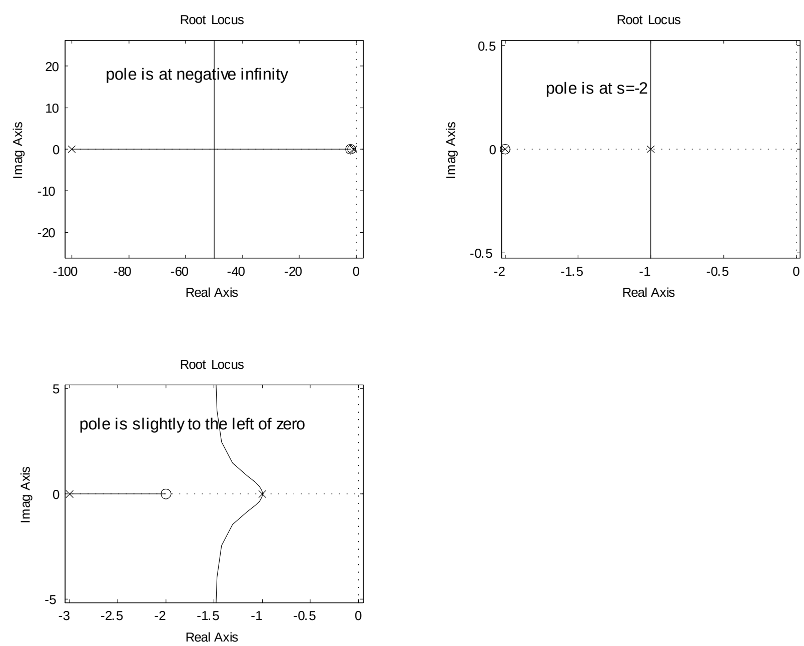
Solution:

$s + z = (z-p)e^{j\phi}$

$$G = \frac{(z-p)e^{j\phi}}{((z-p)e^{j\phi} + p - z)^2} = \frac{(z-p)e^{j\phi}}{(z-p)^2(e^{j\phi}-1)^2} = \frac{1}{(z-p)(-4)(\frac{e^{j\phi/2}-e^{-j\phi/2}}{2j})^2}$$

$= \dfrac{1}{-4(z-p)}\dfrac{1}{(\sin(\phi/2))^2}$ Because $z > p$, this function is real and negative for real ϕ and therefore these points are on the locus.

19. The loop transmission of a system has two poles at $s = -1$ and a zero at $s = -2$. There is a third real-axis pole p located somewhere to the *left* of the zero. Several different root loci are possible, depending on the exact location of the third pole. The extreme cases occur when the pole is located at infinity or when it is located at $s = -2$. Give values for p and sketch the three distinct types of loci.



Solution for problem 5.19

20. For the feedback configuration of Fig. 5.58, use asymptotes, center of asymptotes, angles of departure and arrival, and the Routh array to sketch root loci for the characteristic equations of the following feedback control systems versus the parameter K. Use MATLAB to verify your results.

(a) $G(s) = \dfrac{1}{s(s+1+3j)(s+1-3j)}$, $\quad H(s) = \dfrac{s+2}{s+8}$

(b) $G(s) = \dfrac{1}{s^2}$, $\quad H(s) = \dfrac{s+1}{s+3}$

(c) $G(s) = \dfrac{(s+5)}{(s+1)}$, $\quad H(s) = \dfrac{s+7}{s+3}$

(d) $G(s) = \dfrac{(s+3+4j)(s+3-4j)}{s(s+1+2j)(s+1-2j)}$, $\quad H(s) = 1+3s$

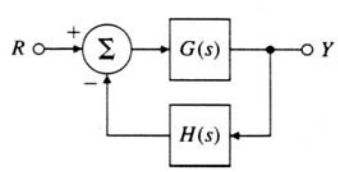
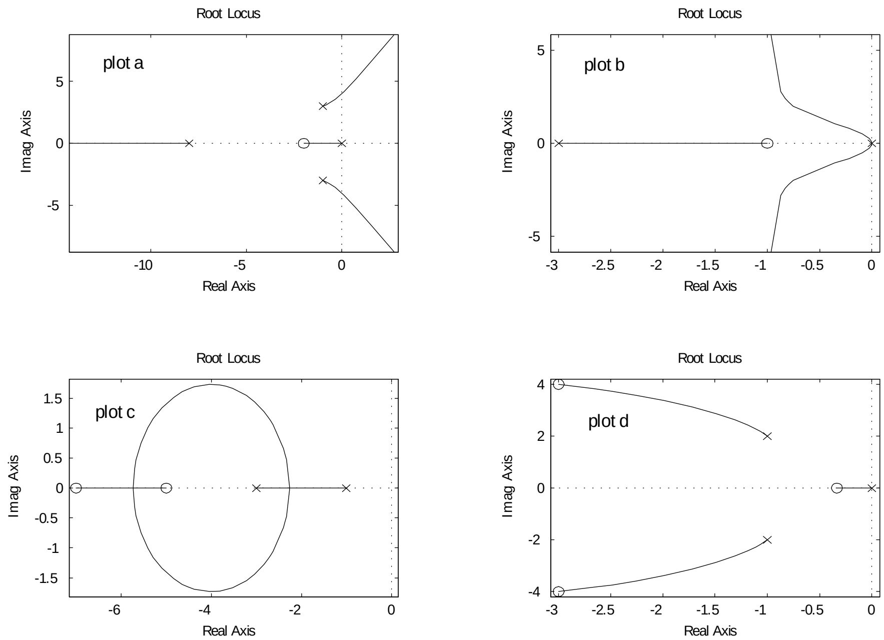


Figure 5.58: Feedback system for problem 20

Solution:



Solution for problem 5.20

21. Consider the system in Fig. 5.59.

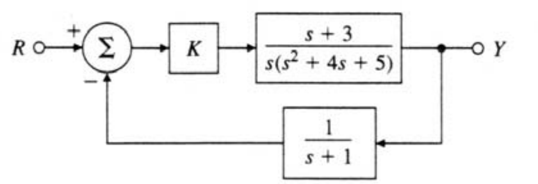


Figure 5.59: Feedback system for problem 5.21

(a) Using Routh's stability criterion, determine all values of K for which the system is stable.

(b) Use Matlab to find the root locus versus K. Find the values for K at imaginary-axis crossings.

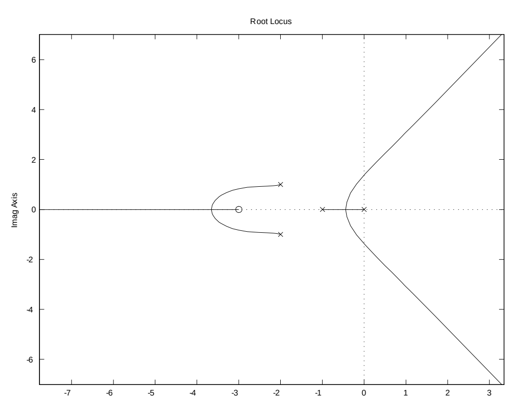
Solution:

(a) a. $0 \leq K \leq 40$

(b) $\theta_d = \pm 161.6°$ $\theta_a = 0°$

At imaginary axis crossing s=$\pm j1.8186$ $k = 6.2758$

Root locus is attached for reference.



Root locus for problem 21

Problems and solutions for Section 5.4

22. Let

$$G(s) = \frac{1}{(s+2)(s+3)} \quad \text{and} \quad D(s) = K\frac{s+a}{s+b}.$$

Using root-locus techniques, find values for the parameters a, b, and K of the compensation $D(s)$ that will produce closed-loop poles at $s = -1 \pm j$ for the system shown in Fig. 5.60.

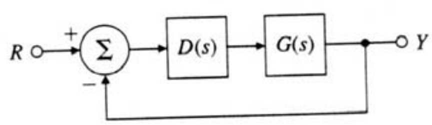


Figure 5.60: Unity feedback system for Problems 5.22 to 5.28 and 5.33

Solution:

Since the desired poles are slower than he plant, we will use PI control. The solution is to cancel the pole at -3 with the zero and set the gain to $K = 2$. Thus, $p = 0$, $z = -3$, $K = 2$.

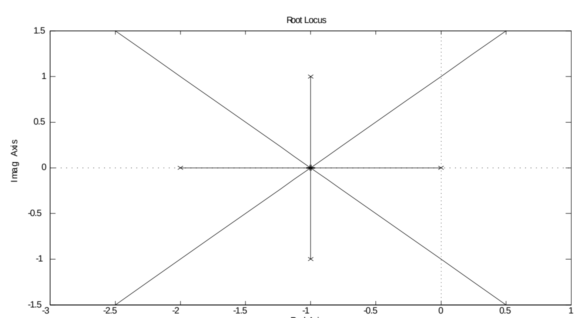
23. Suppose that in Fig. 5.60,

$$G(s) = \frac{1}{s(s^2 + 2s + 2)} \quad \text{and} \quad D(s) = \frac{K}{s+2}.$$

Sketch the root-locus with respect to K of the characteristic equation for the closed-loop system, paying particular attention to points that generate multiple roots if $KL(s) = D(s)G(s)$.

Solution:

The locus is plotted below. The roots all come together at $s = -1$ at $K = 1$.

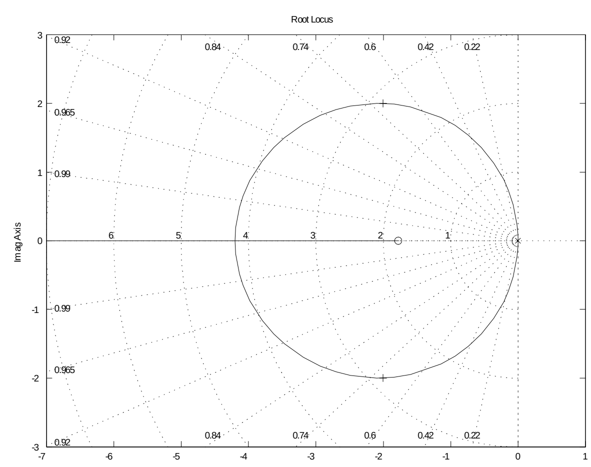


Root locus for Problem 23

24. Suppose the unity feedback system of Fig. 5.60 has an open-loop plant given by $G(s) = 1/s^2$. Design a lead compensation $D(s) = K\dfrac{s+z}{s+p}$ to be added in series with the plant so that the dominant poles of the closed-loop system are located at $s = -2 \pm 2j$.

Solution:

Setting the pole of the lead to be at $p = -20$, the zero is at $z = -1.78$ with a gain of $K = 72$. The locus is plotted below.



Root locus for Problem 24

25. Assume that the unity feedback system of Fig. 5.60 has the open-loop plant

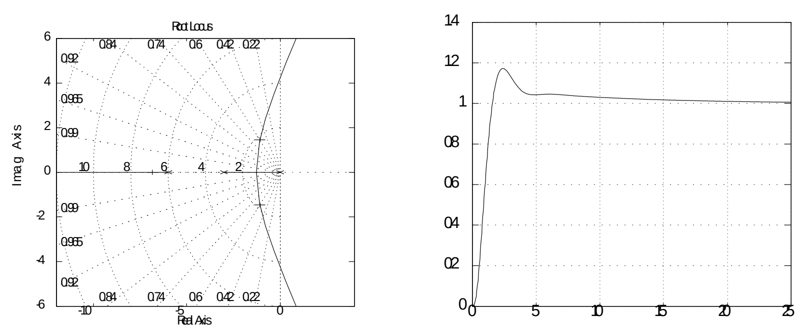
$$G(s) = \frac{1}{s(s+3)(s+6)}.$$

Design a lag compensation to meet the following specifications:

- The step response settling time is to be less than 5 sec.
- The step response overshoot is to be less than 17%.
- The steady-state error to a unit ramp input must not exceed 10%.

Solution:

The overshoot specification requires that damping be 0.5 and the settling time requires that $\omega_n > 1.8$. From the root locus plotted below, these can be met at $K = 28$ where the $\omega_n = 2$. With this gain, the $K_v = 28/18 = 1.56$. To get a $K_v = 10$, we need a lag gain of about 6.5. Selecting the lag zero to be at 0.1 requires the pole to be at $0.1/6.5 = 0.015$. To meet the overshoot specifications, it is necessary to select a smaller K and set $p = 0.01$. Other choices are of course possible. The step response of this design is plotted below.



Root locus and step response for Problem 5.25

26. A numerically controlled machine tool positioning servomechanism has a normalized and scaled transfer function given by

$$G(s) = \frac{1}{s(s+1)}.$$

Performance specifications of the system in the unity feedback configuration of Fig. 5.60 are satisfied if the closed-loop poles are located at $s = -1 \pm j\sqrt{3}$.

(a) Show that this specification cannot be achieved by choosing proportional control alone, $D(s) = k_p$.

(b) Design a lead compensator $D(s) = K\dfrac{s+z}{s+p}$ that will meet the specification.

Solution:

(a) With proportional control, the poles have real part at $s = -.5$.

(b) To design a lead, we select the pole to be at $p = -10$ and find the zero and gain to be $z = -3$, $k = 12$.

27. A servomechanism position control has the plant transfer function

$$G(s) = \frac{10}{s(s+1)(s+10)}.$$

You are to design a series compensation transfer function $D(s)$ in the unity feedback configuration to meet the following closed-loop specifications:

- The response to a reference step input is to have no more than 16% overshoot.
- The response to a reference step input is to have a rise time of no more than 0.4 sec.
- The steady-state error to a unit ramp at the reference input must be less than 0.02

(a) Design a lead compensation that will cause the system to meet the dynamic response specifications.

(b) If $D(s)$ is proportional control, $D(s) = k_p$, what is the velocity constant K_v?

(c) Design a lag compensation to be used in series with the lead you have designed to cause the system to meet the steady-state error specification.

(d) Give the MATLAB plot of the root locus of your final design.

(e) Give the MATLAB response of your final design to a reference step .

Solution:

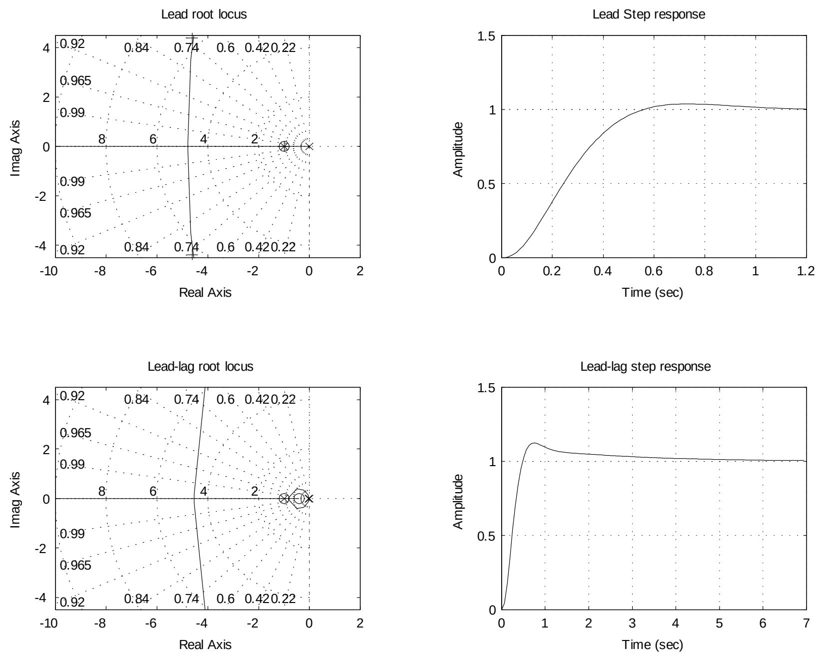
(a) Setting the lead pole at $p = -60$ and the zero at $z = -1$, the dynamic specifications are met with a gain of 245 resulting in a $K_v = 4$.

(b) Proportional control will not meet the dynamic spec. The K_v of the lead is given above.

(c) To meet the steady-state requirement, we need a new $K_v = 50$, which is an increase of 12.5. If we set the lag zero at $z = -.4$, the pole needs to be at $p = -0.032$.

(d) The root locus is plotted below.

(e) The step response is plotted below.



Solution to Problem 27

28. Assume the closed-loop system of Fig. 5.60 has a feed forward transfer function $G(s)$ given by

$$G(s) = \frac{1}{s(s+2)}.$$

Design a lag compensation so that the dominant poles of the closed-loop system are located at $s = -1 \pm j$ and the steady-state error to a unit ramp input is less than 0.2.

Solution:

The poles can be put in the desired location with proportional control alone, with a gain of $k_p = 2$ resulting in a $K_v = 1$. To get a $K_v = 5$, we add a compensation with zero at 0.1 and a pole at 0.02. $D(s) = 2\dfrac{s+0.1}{s+0.02}$.

29. An elementary magnetic suspension scheme is depicted in Fig. 5.61. For small motions near the reference position, the voltage e on the photo detector is related to the ball displacement x (in meters) by $e = 100x$. The upward force (in newtons) on the ball caused by the current i (in amperes) may be approximated by $f = 0.5i + 20x$. The mass of the ball is 20 g, and the gravitational force is 9.8 N/kg. The power amplifier is a voltage-to-current device with an output (in amperes) of $i = u + V_0$.

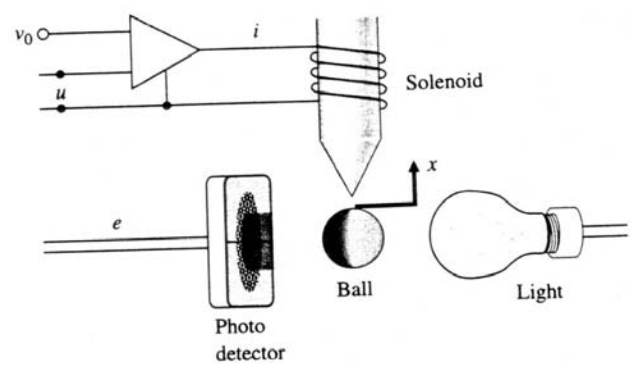


Figure 5.61: Elementary magnetic suspension

(a) Write the equations of motion for this setup.

(b) Give the value of the bias V_0 that results in the ball being in equilibrium at $x = 0$.

(c) What is the transfer function from u to e?

(d) Suppose the control input u is given by $u = -Ke$. Sketch the root locus of the closed-loop system as a function of K.

(e) Assume that a lead compensation is available in the form $\frac{U}{E} = D(s) = K\frac{s+z}{s+p}$. Give values of K, z, and p that yields improved performance over the one proposed in part (d).

Solution:

(a) $m\ddot{x} = 20x + 0.5i - mg$. Substituting numbers, $0.02\ddot{x} = 20x + 0.5(u + V_o) - 0.196$.

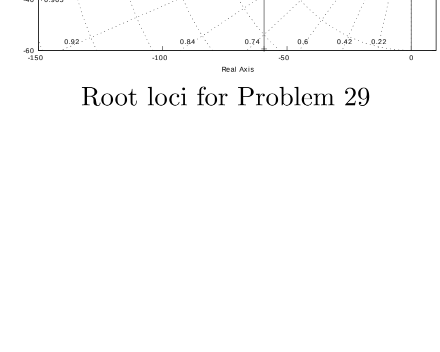
(b) To have the bias cancel gravity, the last two terms must add to zero. Thus $V_o = 0.392$.

(c) Taking transforms of the equation and substituting $e = 100x$,

$$\frac{E}{U} = \frac{2500}{s^2 - 1000}$$

(d) The locus starts at the two poles symmetric to the imaginary axis, meet at the origin and cover the imaginary axis. The locus is plotted below.

(e) The lead can be used to cancel the left-hand-plane zero and the pole at m-150 which will bring the locus into the left-hand plane where K can be selected to give a damping of, for example 0.7. See the plot below.



Root loci for Problem 29

\

30. A certain plant with the non minimum phase transfer function

$$G(s) = \frac{4 - 2s}{s^2 + s + 9},$$

is in a unity positive feedback system with the controller transfer function $D(s)$.

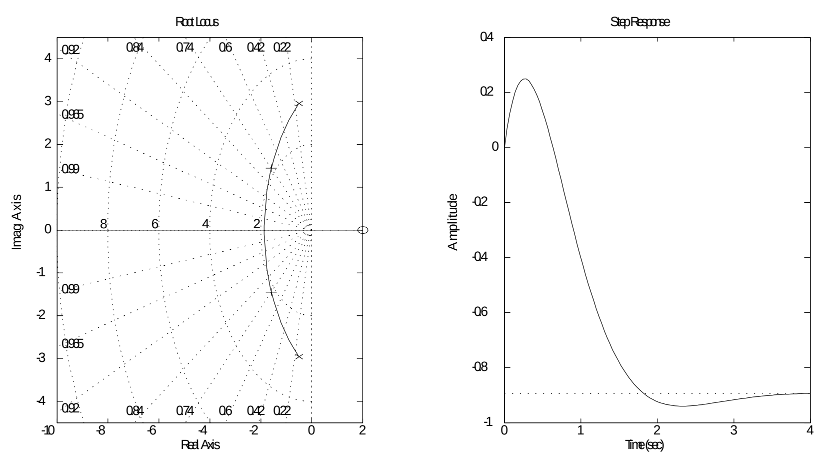
(a) Use MATLAB to determine a (negative) value for $D(s) = K$ so that the closed-loop system with negative feedback has a damping ratio $\zeta = 0.707$.

(b) Use MATLAB to plot the system's response to a reference step.

Solution:

(a) With all the negatives, the problem statement might be confusing. With the $G(s)$ as given, MATLAB needs to plot the negative locus, which is the regular positive locus for $-G$. The locus is plotted below. The value of gain for closed loop roots at damping of 0.7 is $k = -1.04$

(b) The final value of the step response plotted below is -0.887. To get a positive output we would use a positive gain in positive feedback.



Solutions for Problem 30

31. Consider the rocket-positioning system shown in Fig. 5.62.

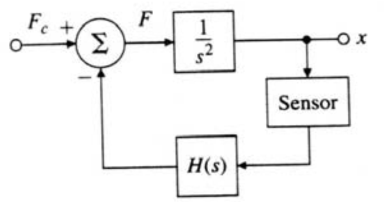


Figure 5.62: Block diagram for rocket-positioning control system

(a) Show that if the sensor that measures x has a unity transfer function, the lead compensator

$$H(s) = K\frac{s+2}{s+4}$$

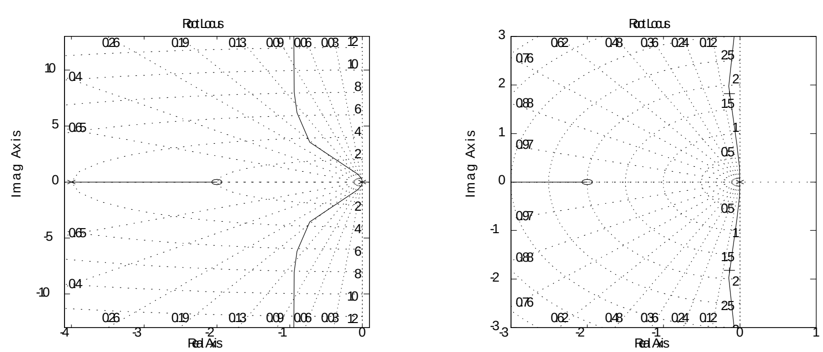
stabilizes the system.

(b) Assume that the sensor transfer function is modeled by a single pole with a 0.1 sec time constant and unity DC gain. Using the root-locus procedure, find a value for the gain K that will provide the maximum damping ratio.

Solution:

(a) The root locus is plotted below and lies entirely in the left-half plane. However the maximum damping is 0.2.

(b) At maximum damping, the gain is $K = 6.25$ but the damping of the complex poles is only 0.073. A practical design would require much more lead.



Loci for problem 31

32. For the system in Fig. 5.63:

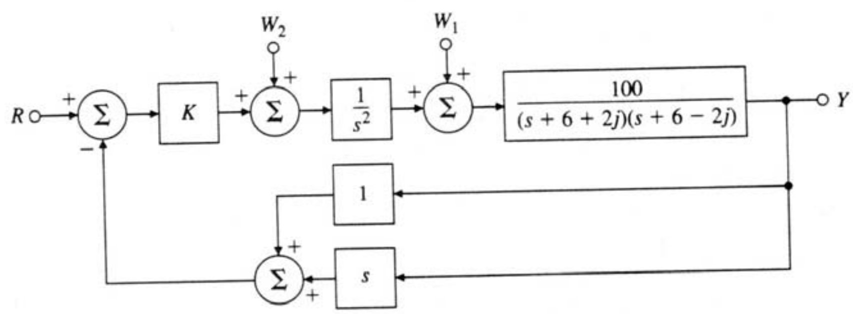


Figure 5.63: Control system for Problem 32

(a) Sketch the locus of closed-loop roots with respect to K.

(b) Find the maximum value of K for which the system is stable. Assume $K = 2$ for the remaining parts of this problem.

(c) What is the steady-state error ($e = r - y$) for a step change in r?

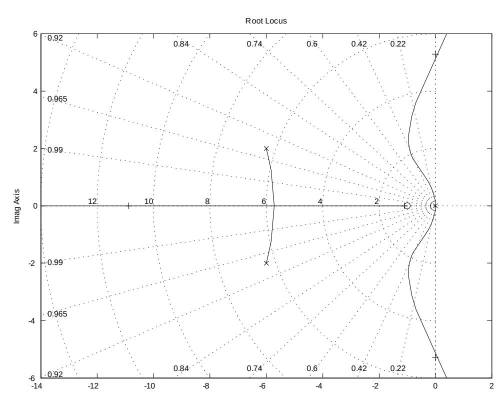
(d) What is the steady-state error in y for a constant disturbance w_1?

(e) What is the steady-state error in y for a constant disturbance w_2?

(f) If you wished to have more damping, what changes would you make to the system?

Solution:

(a) For the locus, $L(s) = \dfrac{100(s+1)}{s^2(s^2+12s+40)}$. The locus is plotted below.



Locus for Problem 32

(b) The maximum value of K for stability is $K = 3.35$.

(c) The equivalent plant with unity feedback is $G' = \dfrac{200}{s^2(s^2+12+40)+200s}$. Thus the system is **type** 1 with $K_v = 1$. If the velocity feedback were zero, the system would be type 2 with $K_a = \dfrac{200}{40} = 5$.

(d) The transfer function $\dfrac{Y}{W_1} = \dfrac{100s^2}{s^2(s^2+12s+40)+200(s+1)}$. The system is thus **type 2** with $K_a = 100$.

(e) The transfer function $\dfrac{Y}{W_2} = \dfrac{100}{s^2(s^2+12s+40)+200(s+1)}$. The system here is **type 0** with $K_p = 1$.

(f) To get more damping in the closed-loop response, the controller needs to have a lead compensation.

33. Consider the plant transfer function

$$G(s) = \frac{bs+k}{s^2[mMs^2 + (M+m)bs + (M+m)k]}$$

to be put in the unity feedback loop of Fig. 5.60. This is the transfer function relating the input force $u(t)$ and the position $y(t)$ of mass M in the non-collocated sensor and actuator problem. In this problem we will use root-locus techniques to design a controller $D(s)$ so that the closed-loop step response has a rise time of less than 0.1 sec and an overshoot of less than 10%. You may use MATLAB for any of the following questions.

(a) Approximate $G(s)$ by assuming that $m \cong 0$, and let $M = 1$, $k = 1$, $b = 0.1$, and $D(s) = K$. Can K be chosen to satisfy the performance specifications? Why or why not?

(b) Repeat part (a) assuming $D(s) = K(s+z)$, and show that K and z can be chosen to meet the specifications.

(c) Repeat part (b) but with a practical controller given by the transfer function

$$D(s) = K\,\frac{p(s+z)}{s+p},$$

and pick p so that the values for K and z computed in part (b) remain more or less valid.

(d) Now suppose that the small mass m is not negligible, but is given by $m = M/10$. Check to see if the controller you designed in part (c) still meets the given specifications. If not, adjust the controller parameters so that the specifications are met.

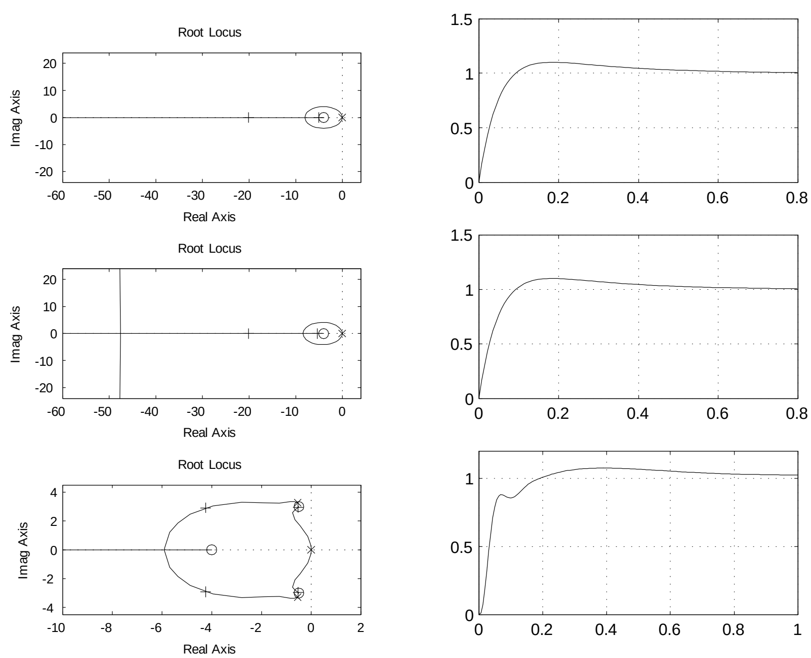
Solution:

(a) The locus in this case is the imaginary axis and cannot meet the specs for any K.

(b) The specs require that $\zeta > 0.6$, $\omega_n > 18$. Select $z = 15$ for a start. The locus will be a circle with radius 15. Because of the zero, the overshoot will be increased and Figure 3.32 indicates that we'd better make the damping greater than 0.7. As a matter of fact, experimentation shows that we can lower the overshoot of less than 10% only by setting the zero at a low value and putting the poles on the real axis. The plot shows the result if $D = 25(s + 4)$.

(c) In this case, we take $D(s) = 20\dfrac{s+4}{.01s+1}$.

(d) With the resonance present, the only chance we have is to introduce a notch as well as a lead. The compensation resulting in the plots shown is $D(s) = 11\dfrac{s+4}{(.01s+1)}\dfrac{s^2/9.25 + s/9.25 + 1}{s^2/3600 + s/30 + 1}$. The design gain was obtained by a cycle of repeated loci, root location finding, and step responses. Refer to the file `ch5p35.m` for the design aid.



Root loci and step responses for Problem 33

34. Consider the type 1 system drawn in Fig. 5.64. We would like to design the compensation $D(s)$ to meet the following requirements: (1) The steady-state value of y due to a constant unit disturbance w should be less than $\frac{4}{5}$, and (2) the damping ratio $\zeta = 0.7$. Using root-locus techniques:

(a) Show that proportional control alone is not adequate.

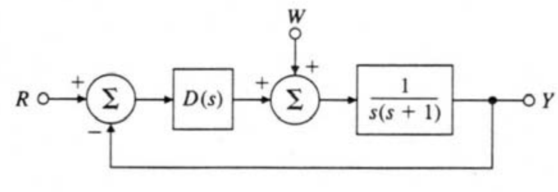


Figure 5.64: Control system for problem 34

(b) Show that proportional-derivative control will work.

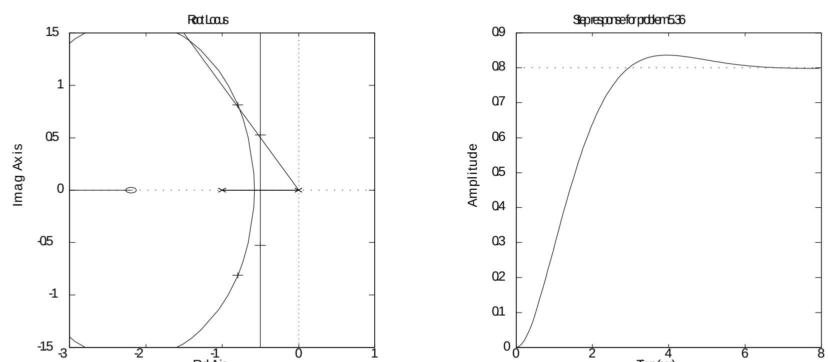
(c) Find values of the gains k_p and k_D for $D(s) = k_p + k_D\, s$ that meet the design specifications.

Solution:

(a) To meet the error requirements, the input to $D(s)$ is -0.8 and the output must be 1.0 to cacel the disturbance. Thus the controller dc gain must be at least 1.25. With proportional control and a closed loop damping of 0.70, the gain is 0.5 which is too low.

(b) With PD control, the characteristic equation is $s^2 + (1 + k_D)s + k_p$. Setting $k_p = 1.25$ and damping 0.7, we get $k_D = 0.57$. The root loci and disturbance step response are plotted below.

(c) The gains are $k_p = 1.25$, $k_D = 0.57$.



Solution for problem 34

Problems and solutions for Section 5.5

35. Consider the positioning servomechanism system shown in Fig. 5.65, where

$$
\begin{aligned}
e_i &= K_{\text{pot}}\theta_i, \quad e_o = K_{\text{pot}}\theta_o, \quad K_{\text{pot}} = 10\text{V/rad}, \\
T &= \text{motor torque} = K_t i_a, \\
K_t &= \text{torque constant} = 0.1\ \text{N}\cdot\text{m/A}, = K_e \\
R_a &= \text{armature resistance} = 10\Omega, \\
\text{Gear ratio} &= 1:1, \\
J_L + J_m &= \text{total inertia} = 10^{-3}\ \text{kg}\cdot\text{m}^2, \\
C &= 200\mu\text{F}, \\
v_a &= K_A(e_i - e_f).
\end{aligned}
$$

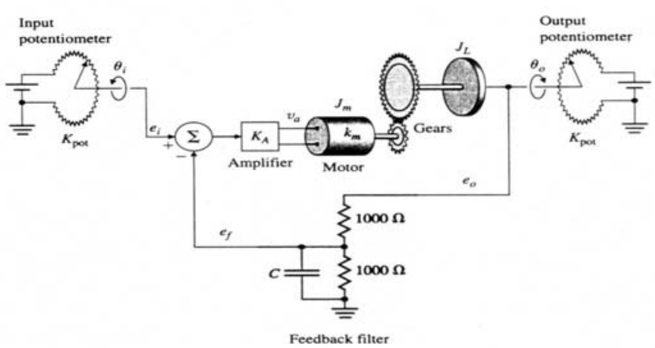


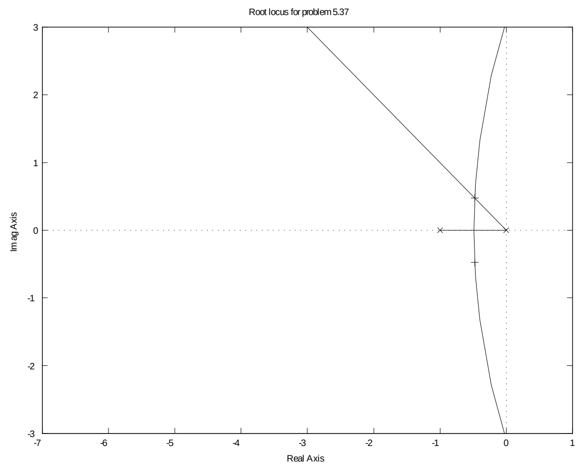
Figure 5.65: Positioning servomechanism

(a) What is the range of the amplifier gain K_A for which the system is stable? Estimate the upper limit graphically using a root-locus plot.

(b) Choose a gain K_A that gives roots at $\zeta = 0.7$. Where are all three closed-loop root locations for this value of K_A?

Solution:

(a) $0 < K < 110$



Root locus for problem 35

$K = 10.$; poles are at $s = -10.05,\ -0.475 \pm j0.475$.

36. We wish to design a velocity control for a tape-drive servomechanism. The transfer function from current $I(s)$ to tape velocity $\Omega(s)$ (in millimeters per millisecond per ampere) is

$$\frac{\Omega(s)}{I(s)} = \frac{15(s^2 + 0.9s + 0.8)}{(s+1)(s^2 + 1.1s + 1)}.$$

We wish to design a type 1 feedback system so that the response to a reference step satisfies

$$t_r \leq 4\text{msec}, \quad t_s \leq 15\text{msec}, \quad M_p \leq 0.05$$

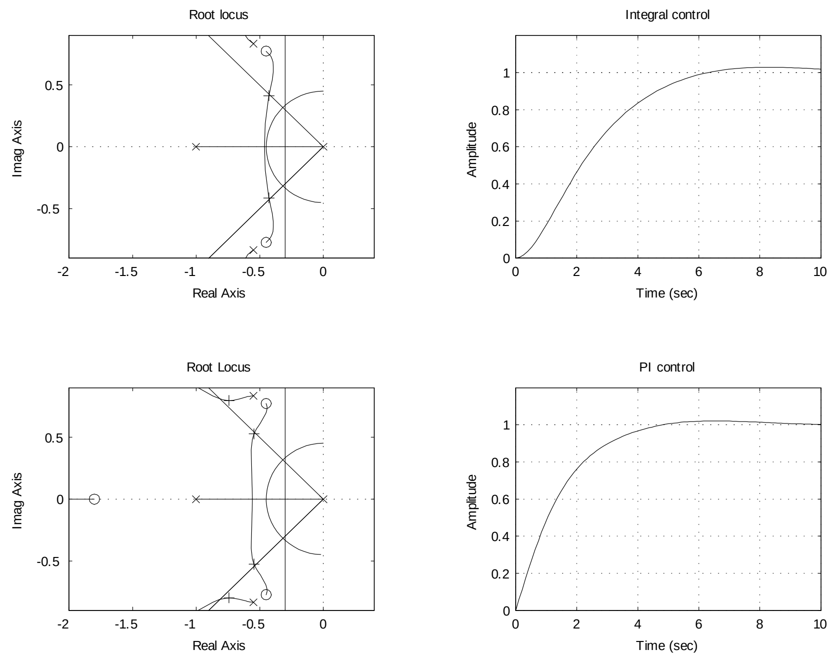
(a) Use the integral compensator k_I/s to achieve type 1 behavior, and sketch the root-locus with respect to k_I. Show on the same plot the region of acceptable pole locations corresponding to the specifications.

(b) Assume a proportional-integral compensator of the form $k_p(s+\alpha)/s$, and select the best possible values of k_p and α you can find. Sketch the root-locus plot of your design, giving values for k_p and α, and the velocity constant K_v your design achieves. On your plot, indicate the closed-loop poles with a dot $\bullet$, and include the boundary of the region of acceptable root locations.

Solution:

(a) The root locus is plotted with the step response below in the first row.

(b) The zero was put at $s = -1.7$ and the locus and step response are plotted in the second row below



Solution for problem 36

37. The normalized, scaled equations of a cart as drawn in Fig. 5.66 of mass m_c holding an inverted uniform pendulum of mass m_p and length ℓ with no friction are

$$\begin{aligned} \ddot{\theta} - \theta &= -v \\ \ddot{y} + \beta\theta &= v \end{aligned} \tag{5.1}$$

where $\beta = \dfrac{3m_p}{4(m_c + m_p)}$ is a mass ratio bounded by $0 < \beta < 0.75$. Time is measured in terms of $\tau = \omega_o t$ where $\omega_o^2 = \dfrac{3g(m_c + m_p)}{\ell(4m_c + m_p)}$. The cart motion, y, is measured in units of pendulum length as $y = \dfrac{3x}{4\ell}$ and the input is force normalized by the system weight, $v = \dfrac{u}{g(m_c + m_p)}$. These equations can be used to compute the transfer functions

$$\frac{\Theta}{V} = -\frac{1}{s^2 - 1} \tag{5.2}$$

$$\frac{Y}{V} = \frac{s^2 - 1 + \beta}{s^2(s^2 - 1)} \tag{5.3}$$

In this problem you are to design a control for this system by first closing a loop around the pendulum, Eq.(5.2) and then, with this loop closed,

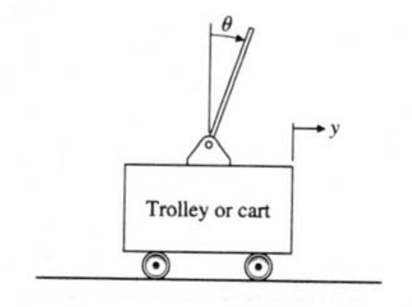


Figure 5.66: Figure of cart-pendulum for Problem 37

closing a second loop around the cart plus pendulum Eq.(5.3). For this problem, let the mass ratio be $m_c = 5m_p$.

(a) Draw a block diagram for the system with V input and both Y and Θ as outputs.

(b) Design a lead compensation $D_p(s) = K_p \frac{s+z}{s+p}$ for the Θ loop to cancel the pole at $s = -1$ and place the two remaining poles at $-4 \pm j4$. The new control is $U(s)$ where the force is $V(s) = U(s) + D(s)\Theta(s)$. Draw the root locus of the angle loop.

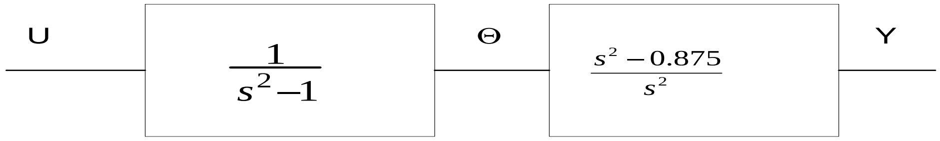
(c) Compute the transfer function of the new plant from U to Y with $D(s)$ in place.

(d) Design a controller $D_c(s)$ for the cart position with the pendulum loop closed. Draw the root locus with respect to the gain of $D_c(s)$

(e) Use MATLAB to plot the control, cart position, and pendulum position for a unit step change in cart position.

Solution:

(a)

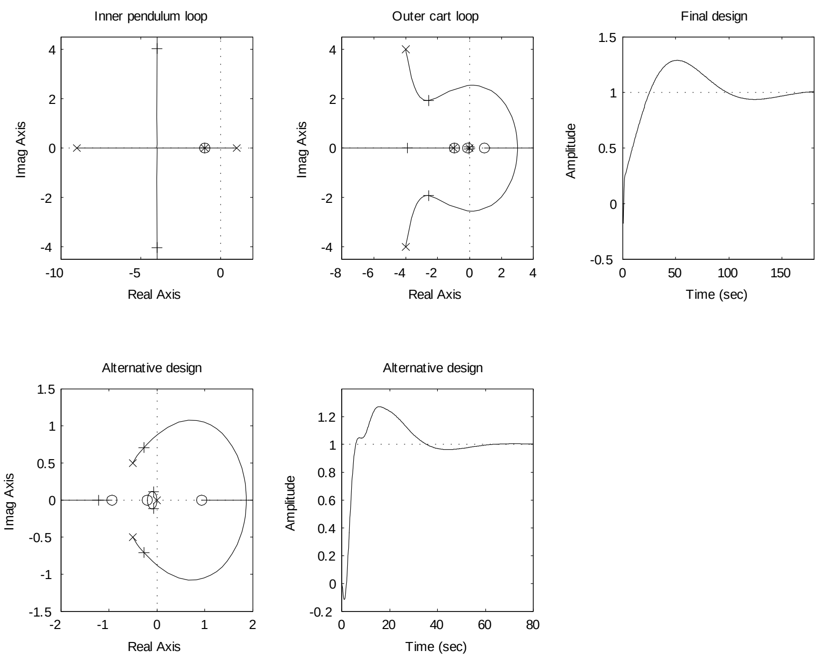


(b) $D_p(s) = 41\dfrac{s+1}{s+9}$ The root locus is shown below.

(c) $G_1 = \dfrac{-41}{s^2+8s+32}\dfrac{s^2-0.875}{s^2}$

(d) $D_c = k_c\dfrac{s^2+0.2s+0.01}{s^2+2s+1}$. The root locus is shown below.

(e) The step responses are shown below. The pendulum position control is rather fast for this problem. A more reasonable alternative choice would be to place the pendulum roots at $s = -0.5 \pm j0.5$.



Root loci and step responses for Problem 37

38. Consider the 270-ft U.S. Coast Guard cutter *Tampa* (902) shown in Fig. 5.67. Parameter identification based on sea-trials data (Trankle, 1987) was used to estimate the hydrodynamic coefficients in the equations of motion. The result is that the response of the heading angle of the ship ψ to rudder angle δ and wind changes w can be described by the second-order transfer functions

$$G_\delta(s) = \frac{\psi(s)}{\delta(s)} = \frac{-0.0184(s+0.0068)}{s(s+0.2647)(s+0.0063)},$$
$$G_w(s) = \frac{\psi(s)}{w(s)} = \frac{0.0000064}{s(s+0.2647)(s+0.0063)},$$

where

$$
\begin{aligned}
\psi &= \text{ heading angle, rad} \\
\psi_r &= \text{reference heading angle, rad.} \\
r &= \dot{\psi} \text{ yaw rate, rad/sec,} \\
\delta &= \text{rudder angle, rad,} \\
w &= \text{wind speed, m/sec.}
\end{aligned}
$$

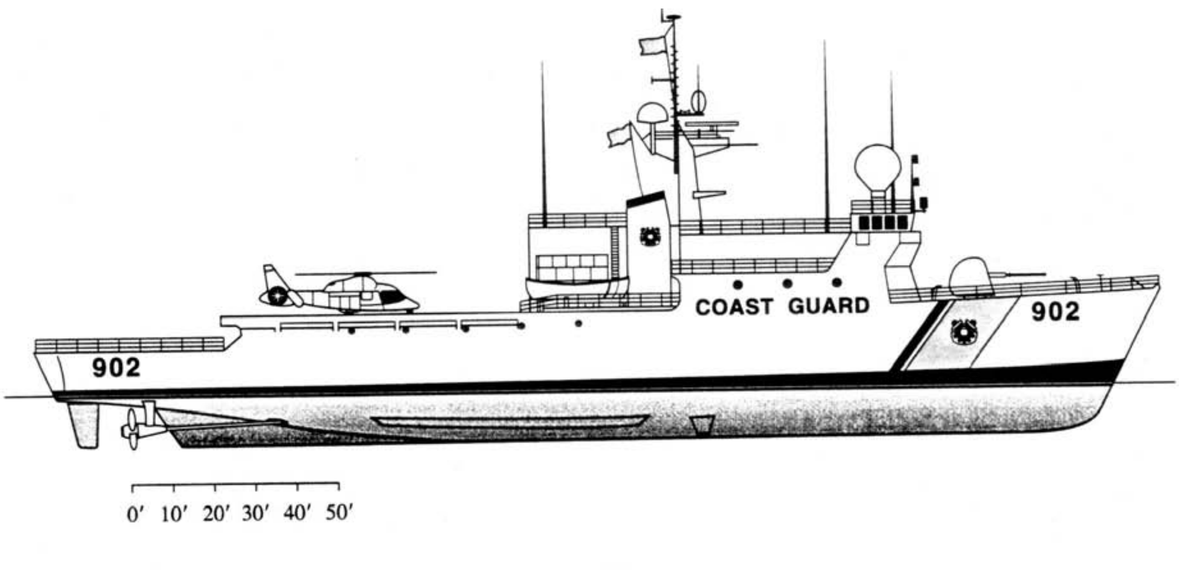


Figure 5.67: USCG cutter *Tampa* (902)

(a) Determine the open-loop settling time of r for a step change in δ.

(b) In order to regulate the heading angle ψ , design a compensator that uses ψ and the measurement provided by a yaw-rate gyroscope (that is, by $\dot{\psi} = r$). The settling time of ψ to a step change in ψ_r is specified to be less than 50 *sec*, and, for a 5° change in heading the maximum allowable rudder angle deflection is specified to be less than 10°.

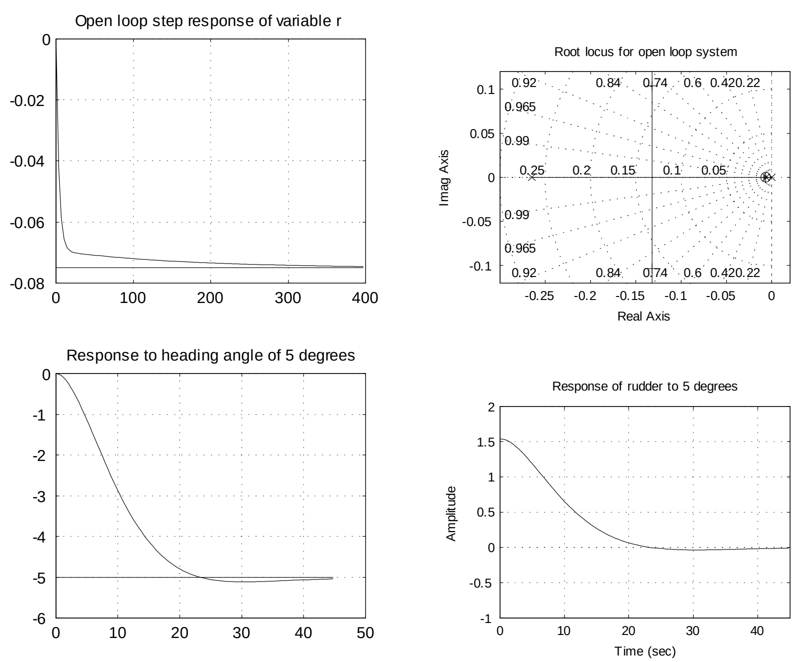
(c) Check the response of the closed-loop system you designed in part (b) to a wind gust disturbance of 10 m/sec (model the disturbance as a step input). If the *steady-state* value of the heading due to this wind gust is more than 0.5°, modify your design so that it meets this specification as well.

Solution:

(a) From the transfer function final value theorem, the final value is 0.075. Using the step function in MATLAB, the settling time to 1% of the final value is $t_s = 316.11$ sec.

(b) The maximum deflection of the rudder is almost surely at the initial instant, when it is $\delta(0) = K\Psi_r(0)$. Thus to keep δ below $10°$ for a step of $5°$, we need $K < 2.and$ for a settling time less than $50\,\text{sec}$. we need $\sigma > 4.6/50 = 0.092$. Drawing the root locus versus K and using the function rlocfind we find that $K = 1.56$ gives roots with real parts less than 0.13. The step response shows that this proportional control is adequate for the problem.

(c) The steady-state error to a disturbance of $10m/\sec$ is less than 0.35.



39. Golden Nugget Airlines has opened a free bar in the tail of their airplanes in an attempt to lure customers. In order to automatically adjust for the sudden weight shift due to passengers rushing to the bar when it first opens, the airline is mechanizing a pitch-attitude auto pilot. Figure 5.68 shows the block diagram of the proposed arrangement. We will model the passenger moment as a step disturbance $M_p(s) = M_0/s$, with a maximum expected value for M_0 of 0.6.

(a) What value of K is required to keep the steady-state error in θ to less than 0.02 rad($\cong 1°$)? (Assume the system is stable.)

(b) Draw a root locus with respect to K.

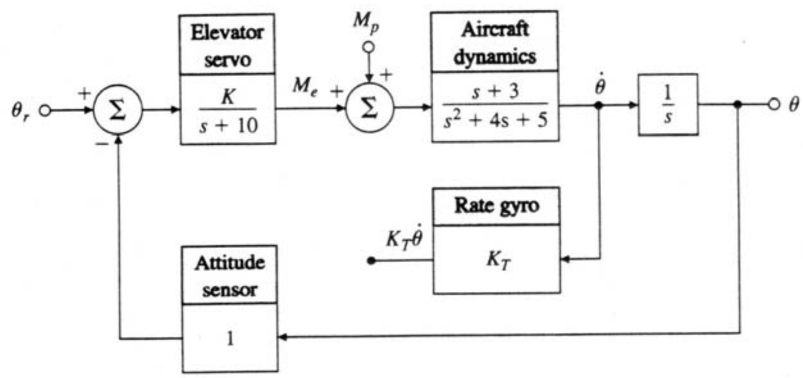


Figure 5.68: Golden Nugget Airlines Autopilot

(c) Based on your root locus, what is the value of K when the system becomes unstable?

(d) Suppose the value of K required for acceptable steady-state behavior is 600. Show that this value yields an unstable system with roots at

$$s = -2.9, -13.5, +1.2 \pm 6.6j.$$

(e) You are given a black box with *rate gyro* written on the side and told that when installed, it provides a perfect measure of $\dot\theta$, with output $K_T\dot\theta$. Assume $K = 600$ as in part (d) and draw a block diagram indicating how you would incorporate the rate gyro into the auto pilot. (Include transfer functions in boxes.)

(f) For the rate gyro in part (e), sketch a root locus with respect to K_T.

(g) What is the maximum damping factor of the complex roots obtainable with the configuration in part (e)?

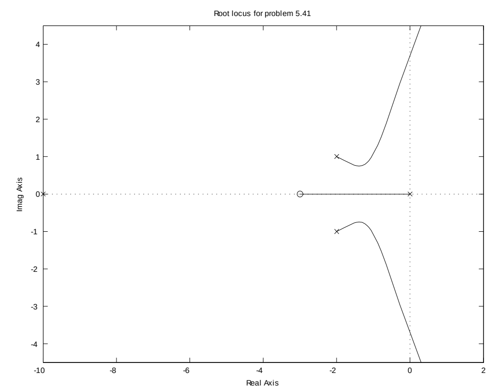
(h) What is the value of K_T for part (g)?

(i) Suppose you are not satisfied with the steady-state errors and damping ratio of the system with a rate gyro in parts (e) through (h). Discuss the advantages and disadvantages of adding an integral term and extra lead networks in the control law. Support your comments using MATLAB or with rough root-locus sketches.

Solution:

(a) $K = 300$.

(b) $K = 144$

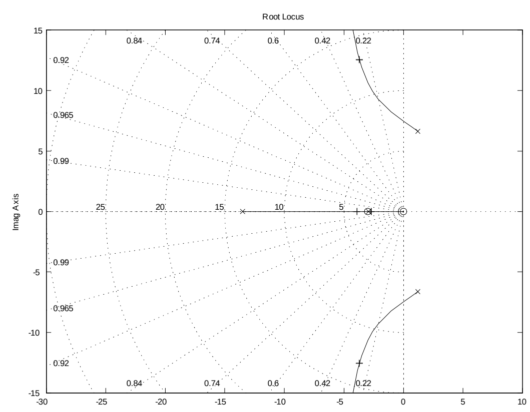


(c) The characteristic equation is $s^4 + 14s^3 + 45s^2 + 650s + 1800$. The exact roots are $-13.5, -2.94, -1.22 \pm 6.63$.

(d) The output of the rate gyro box would be added at the same spot as the attitude sensor output.

(e) $\zeta = 0.28$

(f) $K_T = 185/600 = 0.31$



Root locus for problem 39f

(g) Integral (PI) control would reduce the steady-state error to the moment to zero but would make the damping less and the settling time longer. A lead network could improve the damping of the response.

40. Consider the instrument servomechanism with the parameters given in Fig. 5.69. For each of the following cases, draw a root locus with respect to the parameter K, and indicate the location of the roots corresponding to your final design.

(a) *Lead network* : Let

$$H(s) = 1, \quad D(s) = K\frac{s+z}{s+p}, \quad \frac{p}{z} = 6.$$

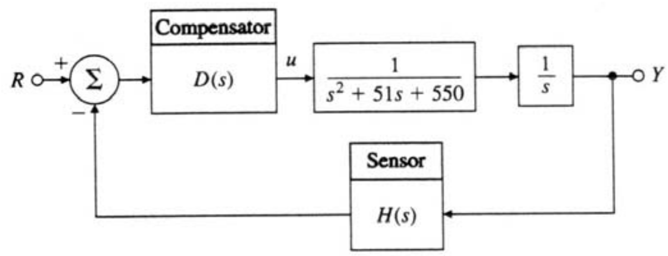


Figure 5.69: Control system for problem 40

Select z and K so that the roots nearest the origin (the dominant roots) yield

$$\zeta \geq 0.4, \quad -\sigma \leq -7, \quad K_v \geq 16\frac{2}{3}\text{sec}^{-1}.$$

(b) *Output-velocity (tachometer) feedback*: Let

$$H(s) = 1 + K_T s \quad \text{and} \quad D(s) = K.$$

Select K_T and K so that the dominant roots are in the same location as those of part (a). Compute K_v. If you can, give a physical reason explaining the reduction in K_v when output derivative feedback is used.

(c) *Lag network* : Let

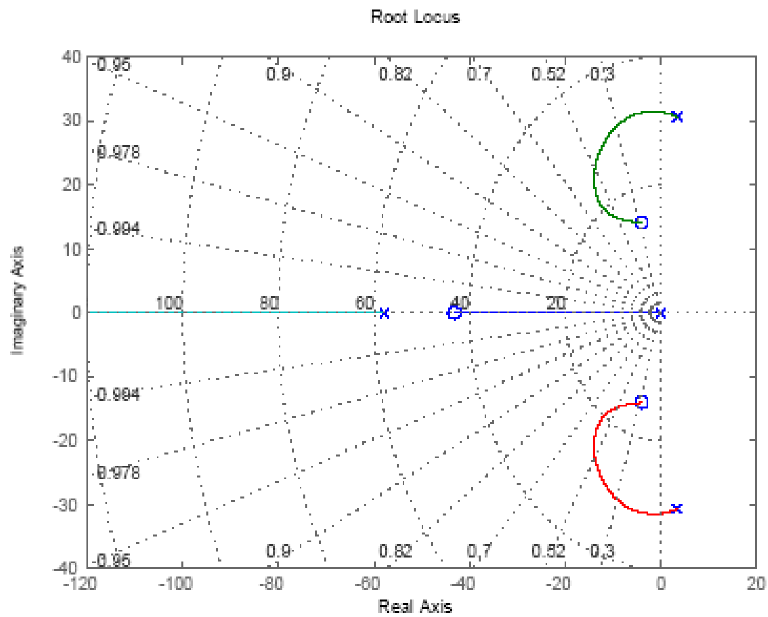
$$H(s) = 1 \quad \text{and} \quad D(s) = K\frac{s+1}{s+p}.$$

Using proportional control, it is possible to obtain a $K_v = 12$ at $\zeta = 0.4$. Select K and p so that the dominant roots correspond to the proportional-control case but with $K_v = 100$ rather than $K_v = 12$.

Solution:

(a) The K_v requirement leads to $K \geq 55000$. With this value, a root locus can be drawn with the parameter z by setting $p = 6z$.

$$1 + z\frac{\left[6s(s^2 + 51s + 550) + K\right]}{s^2(s^2 + 51s + 550) + Ks} = 0$$

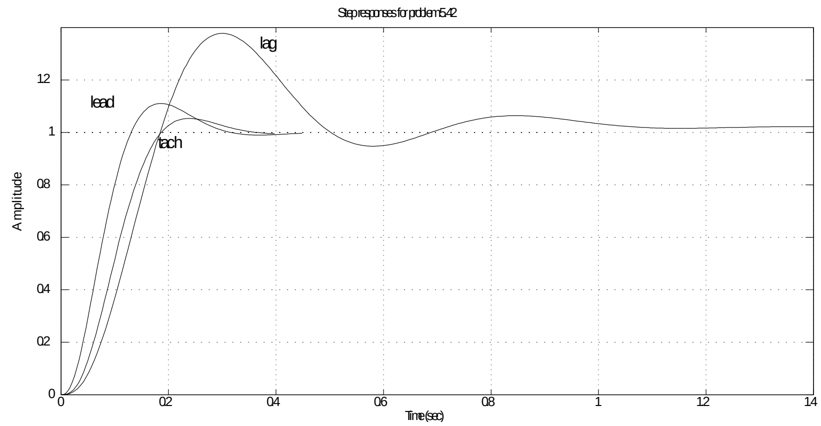


Root locus for Problem 5.40(a)

At the point of maximum damping, the values are $z = 17$ and the dominant roots are at about $-13 \pm j17$.

(b) To find the values of K and K_v, we compute a polynomial with roots at $-13 \pm j17$ and a third pole such that the coefficient of s^2 is 51,which is at $s = -25.15$ This calculation leads to $K = 11785$, $K_T = 0.0483$ and $K_v = 20.81$.

(c) The K_v needs to be increased by a factor of 100/12. Thus, we have $p = 0.12$. The step responses of these designs are given in the plots below.



Problems and solutions for Section 5.6

41. Plot the loci for the 0° locus or negative K for

   (a) The examples given in Problem 3

   (b) The examples given in Problem 4

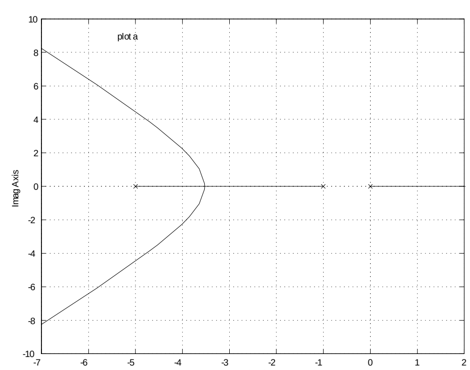
   (c) The examples given in Problem 5

   (d) The examples given in Problem 6

   (e) The examples given in Problem 7

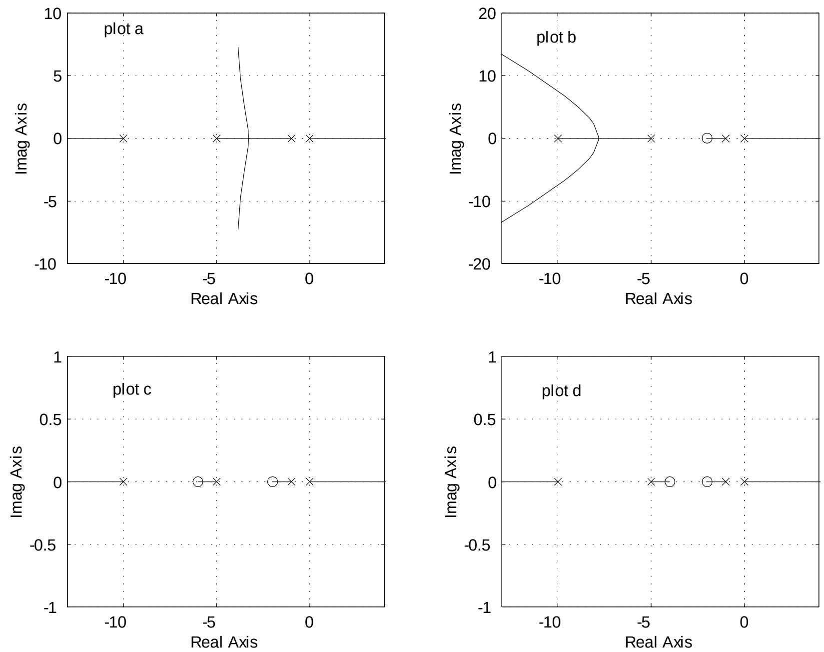
   (f) The examples given in Problem 8

   Solution:

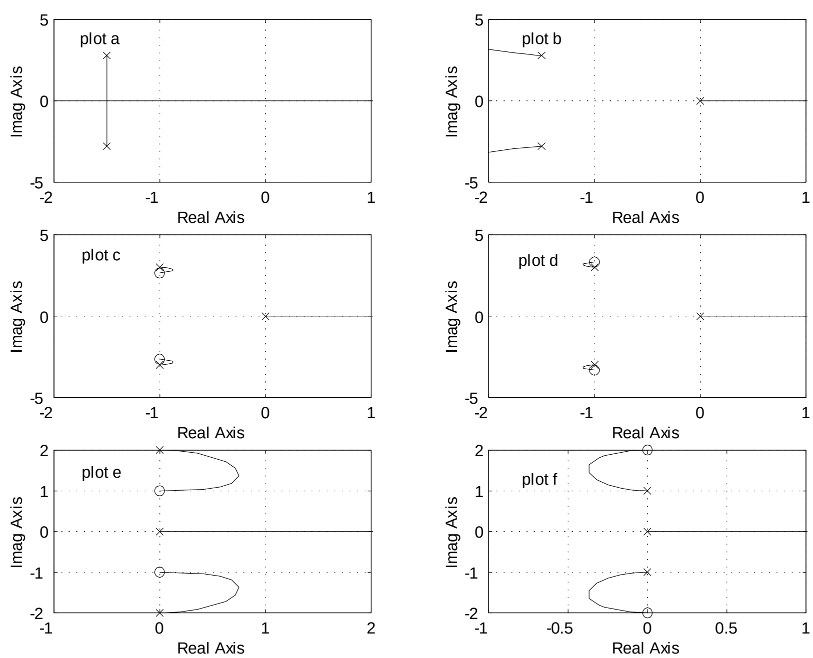


(a)

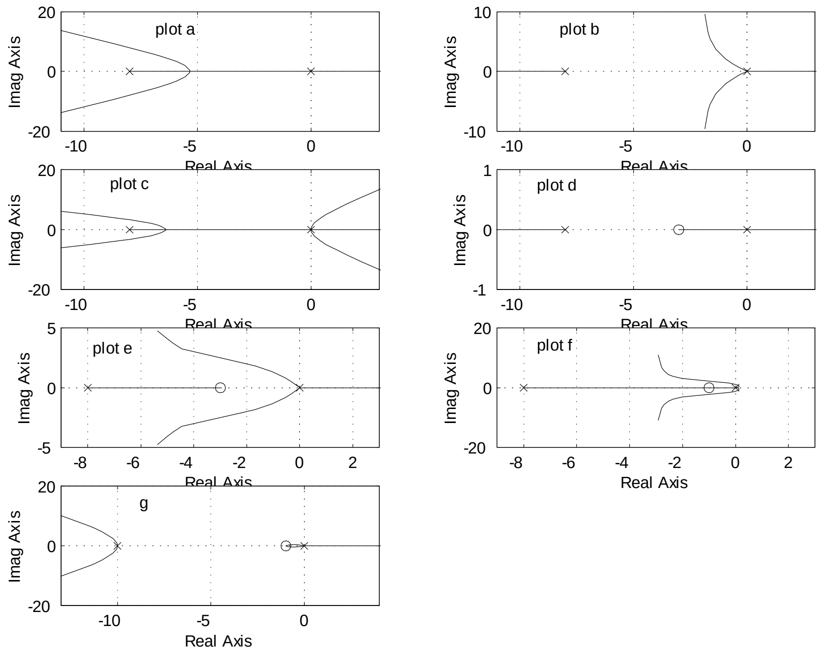
Problem 41(a)



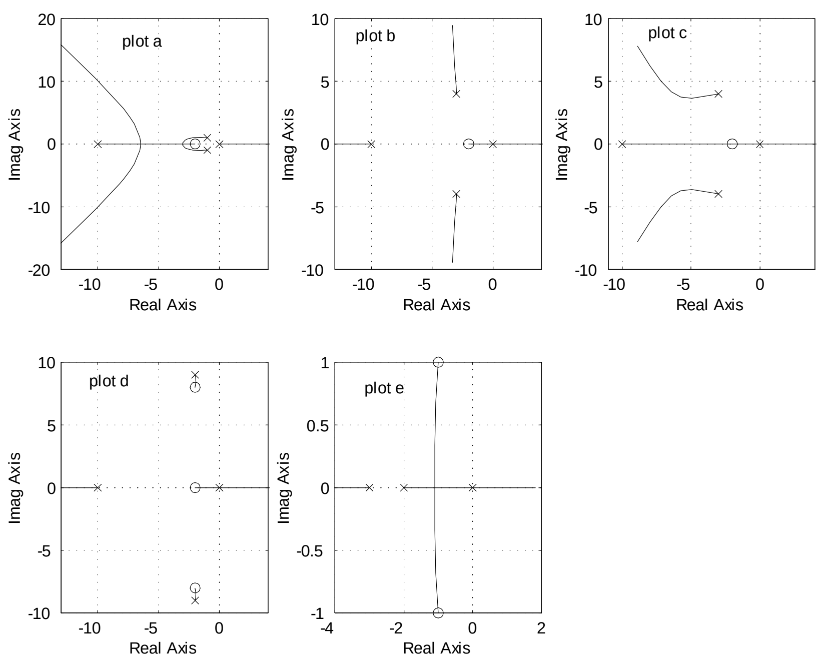
Problem 41(b)



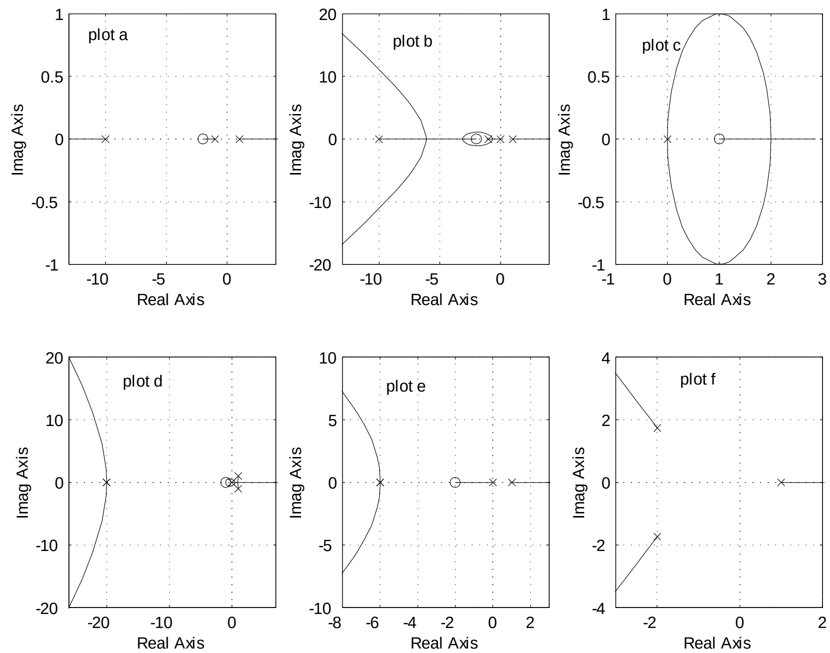
Problem 41(c)



Problem 41(d)



Problem 41(e)



Problem 41(f)

42. Suppose you are given the plant

$$L(s) = \frac{1}{s^2 + (1+\alpha)s + (1+\alpha)},$$

where α is a system parameter that is subject to variations. Use both positive and negative root-locus methods to determine what variations in α can be tolerated before instability occurs.

Solution:

$L(s) = \dfrac{s+1}{s^2+s+1}$. the system is stable for all $\alpha > -1$. The complete locus is a circle of radius 1 centered on $s = -1$.

43. Consider the system in Fig. 5.70.

(a) Use Routh's criterion to determine the regions in the (K_1, K_2) plane for which the system is stable.

(b) Use rltool to verify your answer to part (a).

Solution:

(a) Define $k_p = K_1$ and $k_I = K_1 K_2$ and the characteristic equation is

$$s^4 + 1.5s^3 + 0.5s^2 + k_p s + k_I = 0$$

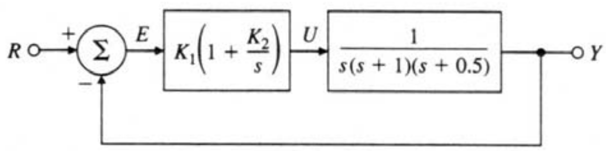


Figure 5.70: Feedback system for Problem 43

For this equation, Routh's criterion requires $k_I > 0$; $k_p < 0.75$; and $4k_p^2 - 3k_p + 9k_I < 0$. The third of these represents a parabola in the $[k_p,\ k_I]$ plane plotted below. The region of stability is the area under the parabola and above the k_p axis.

Stability region for problem 5.43

(b) When $k_I = 0$, there is obviously a pole at the origin. For points on the parabola, consider $k_p = 3/8$ and $k_I = 1/16$. The roots of the characteristic equation are -1.309, -0.191, and $\pm j0.5$.

44. The block diagram of a positioning servomechanism is shown in Fig. 5.71.

(a) Sketch the root locus with respect to K when no tachometer feedback is present ($K_T = 0$).

(b) Indicate the root locations corresponding to $K = 16$ on the locus of part (a). For these locations, estimate the transient-response parameters t_r, M_p, and t_s. Compare your estimates to measurements obtained using the step command in MATLAB.

(c) For $K = 16$, draw the root locus with respect to K_T.

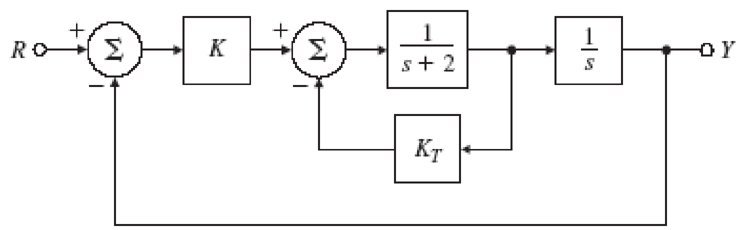


Figure 5.71: Control system for problem 44

(d) For $K = 16$ and with K_T set so that $M_p = 0.05(\zeta = 0.707)$, estimate t_r and t_s. Compare your estimates to the actual values of t_r and t_s obtained using MATLAB.

(e) For the values of K and K_T in part (d), what is the velocity constant K_v of this system?

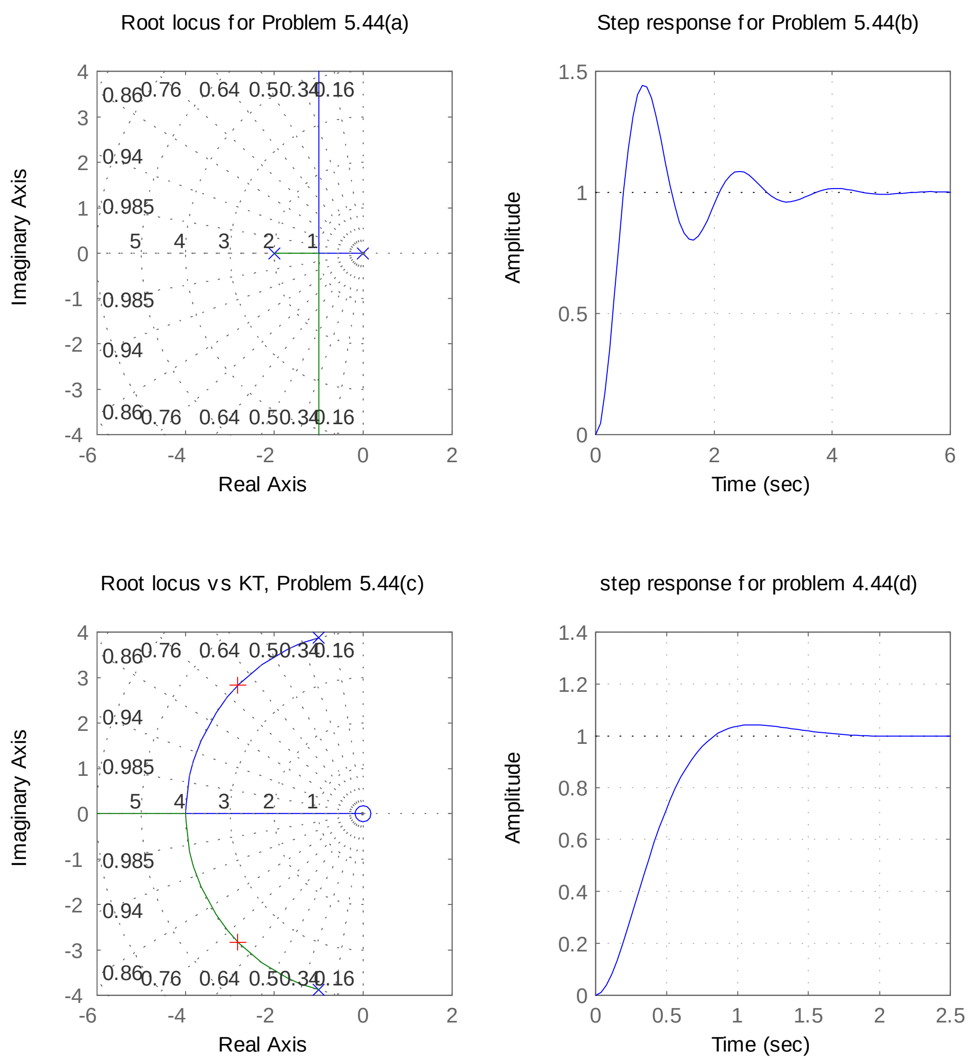
Solution:

(a) The locus is the cross centered at $s = -0.5$

(b) The roots have a damping of 0.25 and natural frequency of 4. We'd estimate the overshoot to be $M_p = 45\%$ and a rise time of less than 0.45 sec. and settling time more than 4.6 sec. The values from the plot are approximately: $t_r = 0.4$, $M_p = 45\%$, and $t_s = 5\,\text{sec}$. Not too bad.

(c) See below.

(d) Use rlocfind on the locus vs Kt to find the Kt value that yields 0.7 damping. This shows that $K_T = 3.66$. Using the formulas inside the back cover yields $M_p = 0.05$, $t_r = 0.45$, and $t_s = 1.6$.



Plots for problem 44

(e) Applying Eq. (4.33), we see that $K_v = K/(K_t + 2) = 2.83$.

45. Consider the mechanical system shown in Fig. 5.72, where g and a_0 are gains. The feedback path containing gs controls the amount of rate feedback. For a fixed value of a_0, adjusting g corresponds to varying the location of a zero in the s-plane.

(a) With $g = 0$ and $\tau = 1$, find a value for a_0 such that the poles are

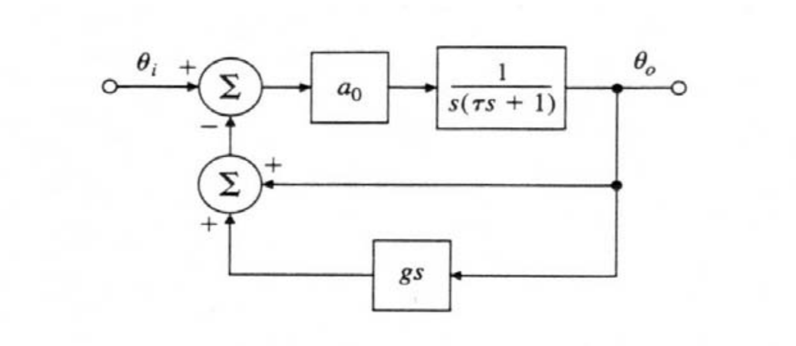


Figure 5.72: Control system for problem 5.46

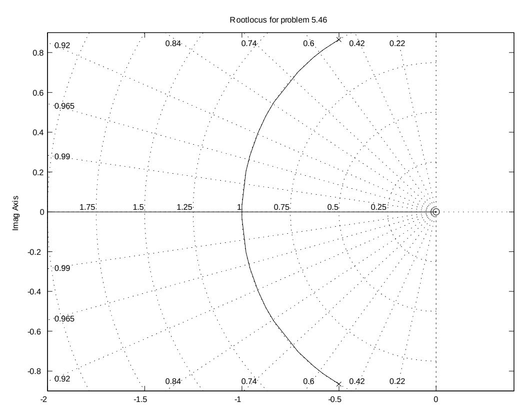
complex.

(b) Fix a_0 at this value, and construct a root locus that demonstrates the effect of varying g.

Solution:

(a) The roots are complex for $a_0 > 0.25$. We select $a_0 = 1$ and the roots are at $s = -0.5 \pm 0.866$

(b) With respect to g, the root locus equation is $s^2 + s + 1 + gs = 0$. The locus is a circle, plotted below.



46. Sketch the root locus with respect to K for the system in Fig. 5.73. What is the range of values of K for which the system is unstable?

Solution:

MATLAB cannot directly plot a root locus for a transcendental function. However, with the Pade' approximation, a locus valid for small values of s can be plotted, as shown below.

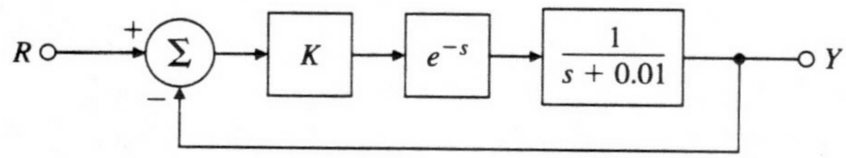
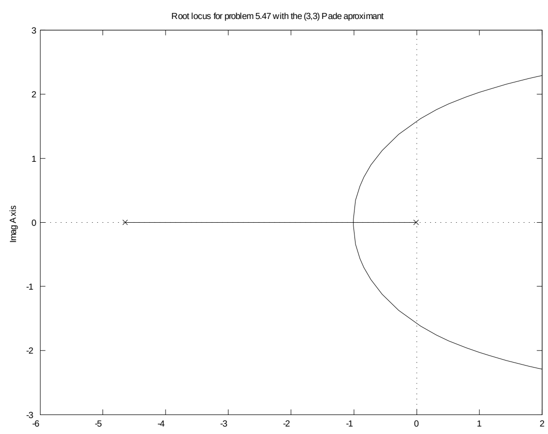


Figure 5.73: Control system for problem 5.46



Solutions for problem 5.46

47. Prove that the plant $G(s) = 1/s^3$ *cannot* be made unconditionally stable if pole cancellation is forbidden.

Solution:

The angles of departure from a triple pole are 180 and ± 60 for the negative locus and 0 and ± 120 for the positive locus. In either case, at least one pole starts out into the right-half plane. Such a system must be conditionally stable for it will be unstable if the gain is small enough.

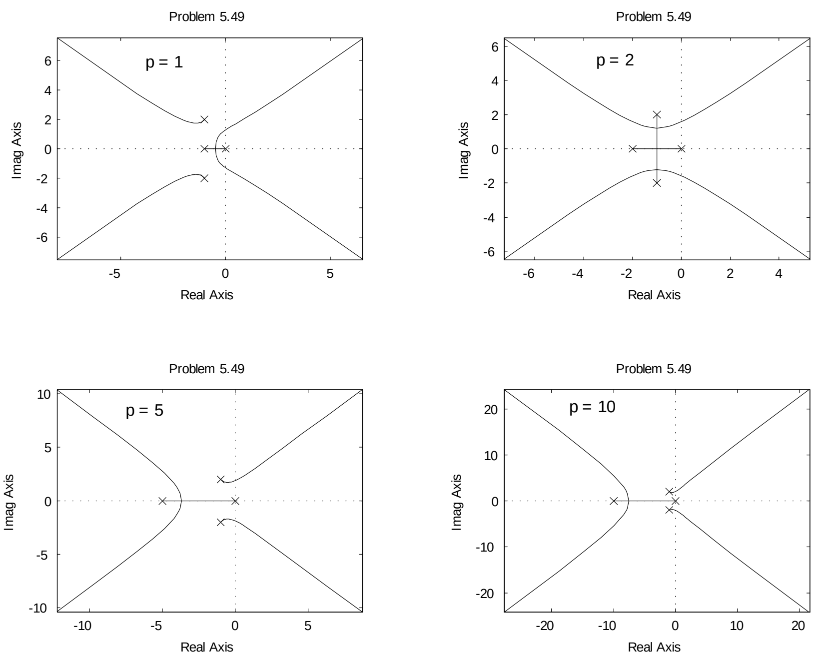
48. For the equation $1 + KG(s)$ where

$$G(s) = \frac{1}{s(s+p)[(s+1)^2+4]},$$

use MATLAB to examine the root locus as a function of K for p in the range from $p = 1$ to $p = 10$, making sure to include the point $p = 2$.

Solution:

The root loci for four values are given in the figure. The point is that the locus for $p = 2$ has multiple roots at a complex value of s.



Solutions for problem 48

Chapter 6

The Frequency-response Design Method

Problems and Solutions for Section 6.1

1. (a) Show that α_0 in Eq. (6.2) is given by

$$\alpha_0 = \left[G(s) \frac{U_0 \omega}{s - j\omega} \right]_{s=-j\omega} = -U_0 G(-j\omega) \frac{1}{2j}$$

and

$$\alpha_0^* = \left[G(s) \frac{U_0 \omega}{s + j\omega} \right]_{s=+j\omega} = U_0 G(j\omega) \frac{1}{2j}.$$

(b) By assuming the output can be written as

$$y(t) = \alpha_0 e^{-j\omega t} + \alpha_0^* e^{j\omega t},$$

derive Eqs. (6.4) - (6.6).

Solution:

(a) Eq. (6.2):

$$Y(s) = \frac{\alpha_1}{s - p_1} + \frac{\alpha_2}{s - p_2} + \cdots + \frac{\alpha_n}{s - p_n} + \frac{\alpha_o}{s + j\omega_o} + \frac{\alpha_o^*}{s - j\omega_o}$$

Multiplying this by $(s + j\omega)$:

$$Y(s)(s+j\omega) = \frac{\alpha_1}{s + a_1}(s+j\omega) + ... + \frac{\alpha_n}{s + a_n}(s+j\omega) + \alpha_o + \frac{\alpha_o^*}{s - j\omega}(s+j\omega)$$

$$
\begin{aligned}
& \Rightarrow \quad \alpha_o = Y(s)(s+j\omega) - \frac{\alpha_1}{s+a_1}(s+j\omega) - \ldots - \frac{\alpha_n}{s+a_n}(s+j\omega) - \frac{\alpha_o^*}{s-j\omega}(s+j\omega) \\
\alpha_o &= \alpha_o|_{s=-j_\omega} = \left[Y(s)(s+j\omega) - \frac{\alpha_1}{s+a_1}(s+j\omega) - \ldots - \frac{\alpha_o^*}{s-j\omega}(s+j\omega) \right]_{s=-j_\omega} \\
&= Y(s)(s+j\omega)|_{s=-j_\omega} = G(s)\frac{U_o\omega}{s^2+\omega^2}(s+j\omega)|_{s=-j_\omega} \\
&= G(s)\frac{U_o\omega}{s-j\omega}|_{s=-j_\omega} = -U_oG(-j\omega)\frac{1}{2j}
\end{aligned}
$$

Similarly, multiplying Eq. (6.2) by $(s-j\omega)$:

$$
\begin{aligned}
Y(s)(s-j\omega) &= \frac{\alpha_1}{s+a_1}(s-j\omega) + \ldots + \frac{\alpha_n}{s+a_n}(s-j\omega) + \frac{\alpha_o}{s+j\omega}(s-j\omega) + \alpha_o^* \\
\alpha_o^* &= \alpha_o^*|_{s=j\omega} = Y(s)(s-j\omega)|_{s=j_\omega} = G(s)\frac{U_o\omega}{s^2+\omega^2}(s-j\omega)|_{s=j_\omega} \\
&= G(s)\frac{U_o\omega}{s+j\omega}|_{s=j_\omega} = U_oG(j\omega)\frac{1}{2j}
\end{aligned}
$$

(b)

$$
\begin{aligned}
y(t) &= \alpha_o e^{-j\omega t} + \alpha_o^* e^{j\omega t} \\
y(t) &= -U_oG(-j\omega)\frac{1}{2j}e^{-j\omega t} + U_oG(j\omega)\frac{1}{2j}e^{j\omega t} \\
&= U_o\left[\frac{G(j\omega)e^{j\omega t} - G(-j\omega)e^{-j\omega t}}{2j}\right] \\
|G(j\omega)| &= \left\{\operatorname{Re}[G(j\omega)]^2 + \operatorname{Im}[G(j\omega)]^2\right\}^{\frac{1}{2}} = A \\
\angle G(j\omega) &= \tan^{-1}\frac{\operatorname{Im}[G(j\omega)]}{\operatorname{Re}[G(j\omega)]} = \phi \\
|G(-j\omega)| &= \left\{\operatorname{Re}[G(-j\omega)]^2 + \operatorname{Im}[G(-j\omega)]^2\right\}^{\frac{1}{2}} = |G(j\omega)| \\
&= \left\{\operatorname{Re}[G(j\omega)]^2 + \operatorname{Im}[G(j\omega)]^2\right\}^{\frac{1}{2}} = A \\
\angle G(-j\omega) &= \tan^{-1}\frac{\operatorname{Im}[G(-j\omega)]}{\operatorname{Re}[G(-j\omega)]} = \tan^{-1}\frac{-\operatorname{Im}[G(j\omega)]}{\operatorname{Re}[G(j\omega)]} = -\phi \\
&\Rightarrow \quad G(j\omega) = Ae^{j\phi},\ G(-j\omega) = Ae^{-j\phi}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
y(t) &= U_o\left[\frac{Ae^{j\phi}e^{j\omega t} - Ae^{-j\phi}e^{-j\omega t}}{2j}\right] = U_oA\left[\frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j}\right] \\
y(t) &= U_oA\sin(\omega t+\phi)
\end{aligned}
$$

where

$$A = |G(j\omega)|\,,\ \phi = \tan^{-1}\frac{\mathrm{Im}\,[G(j\omega)]}{\mathrm{Re}\,[G(j\omega)]} = \angle G(j\omega)$$

2. (a) Calculate the magnitude and phase of

$$G(s) = \frac{1}{s+10}$$

by hand for $\omega = 1$, 2, 5, 10, 20, 50, and 100 rad/sec.

(b) sketch the asymptotes for $G(s)$ according to the Bode plot rules, and compare these with your computed results from part (a).

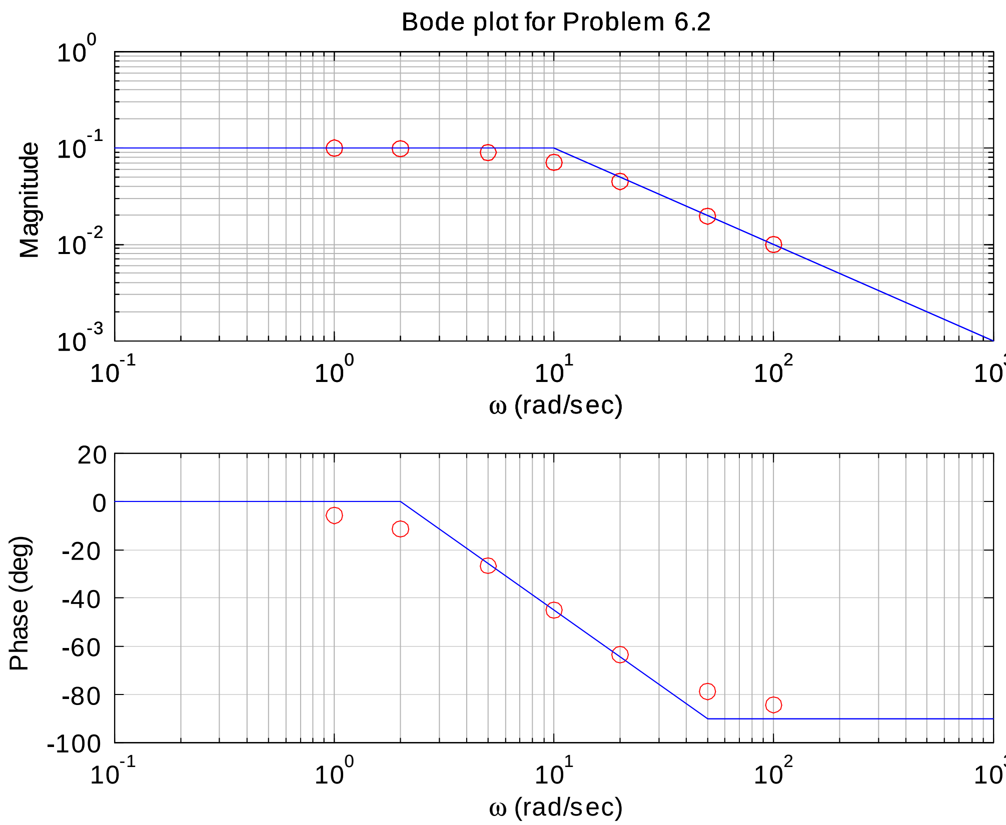
Solution:

(a)

$$\begin{aligned} G(s) &= \frac{1}{s+10},\ G(j\omega) = \frac{1}{10+j\omega} = \frac{10-j\omega}{100+\omega^2} \\ |G(j\omega)| &= \frac{1}{\sqrt{100+\omega^2}},\ \angle G(j\omega) = -\tan^{-1}\frac{\omega}{10} \end{aligned}$$

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
1	0.0995	-5.71
2	0.0981	-11.3
5	0.0894	-26.6
10	0.0707	-45.0
20	0.0447	-63.4
50	0.0196	-78.7
100	0.00995	-84.3

(b) The Bode plot is :



3. Sketch the asymptotes of the Bode plot magnitude and phase for each of the following open-loop transfer functions. After completing the hand sketches verify your result using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

(a) $L(s) = \dfrac{2000}{s(s+200)}$

(b) $L(s) = \dfrac{100}{s(0.1s+1)(0.5s+1)}$

(c) $L(s) = \dfrac{1}{s(s+1)(0.02s+1)}$

(d) $L(s) = \dfrac{1}{(s+1)^2(s^2+2s+4)}$

(e) $L(s) = \dfrac{10(s+4)}{s(s+1)(s^2+2s+5)}$

(f) $L(s) = \dfrac{1000(s+0.1)}{s(s+1)(s^2+8s+64)}$

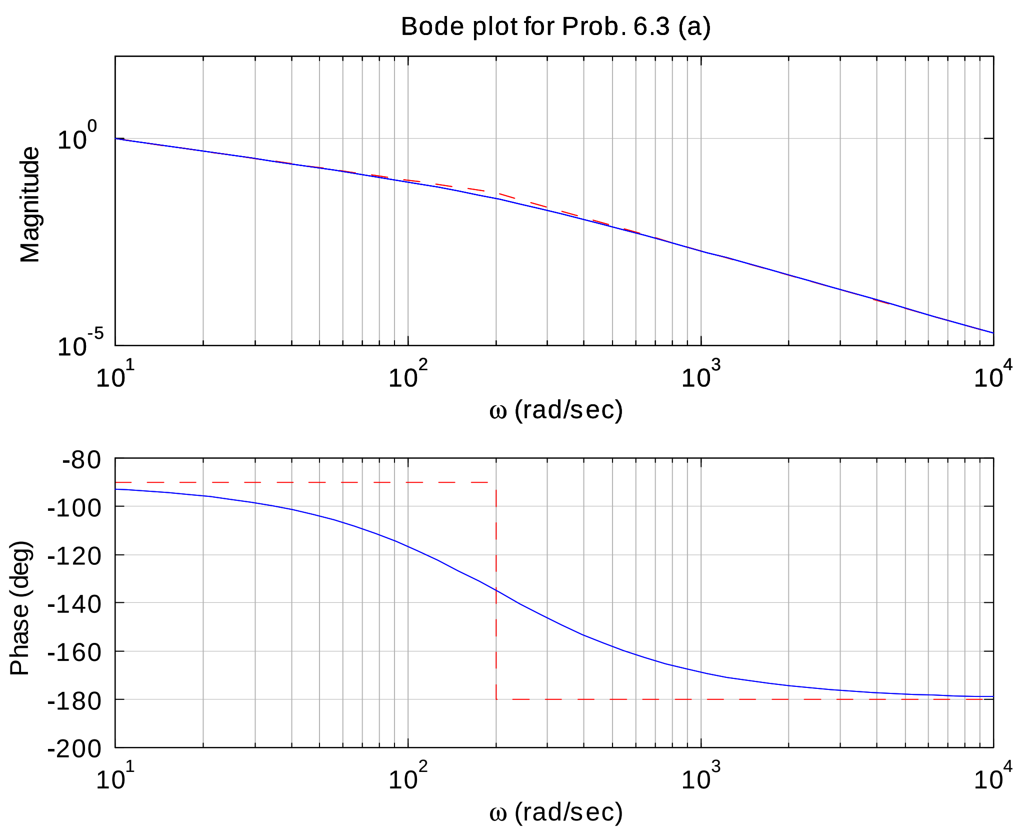
(g) $L(s) = \dfrac{(s+5)(s+3)}{s(s+1)(s^2+s+4)}$

(h) $L(s) = \dfrac{4s(s+10)}{(s+100)(4s^2+5s+4)}$

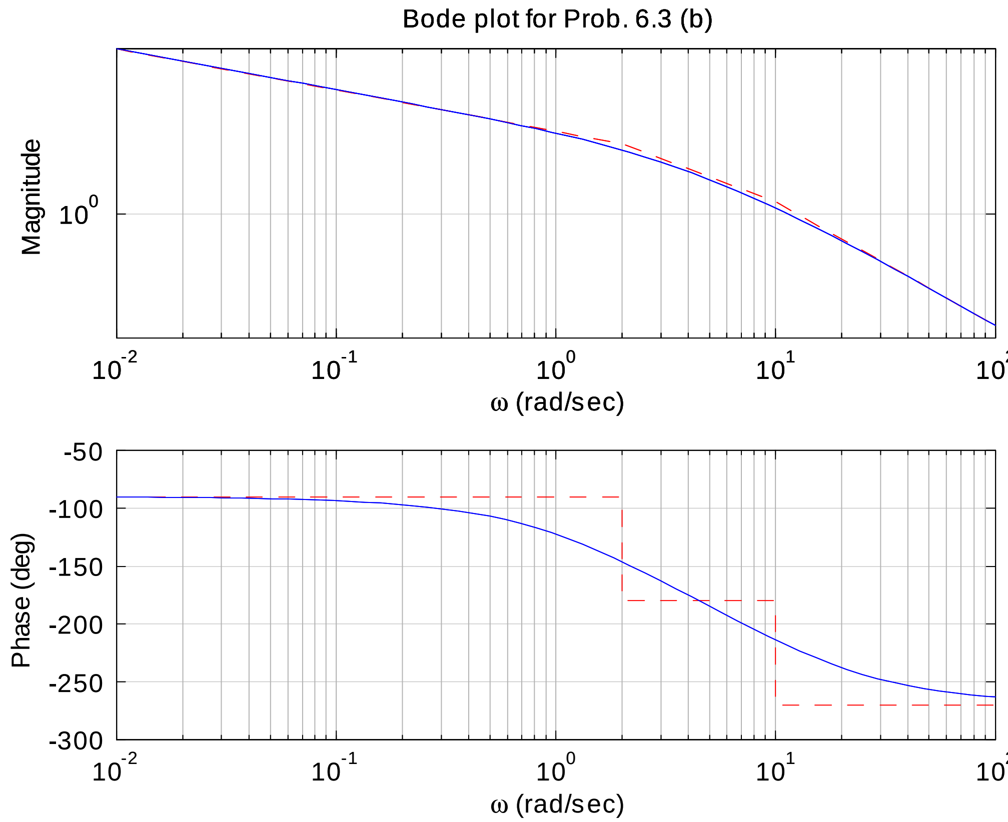
(i) $L(s) = \dfrac{s}{(s+1)(s+10)(s^2+2s+2500)}$

Solution:

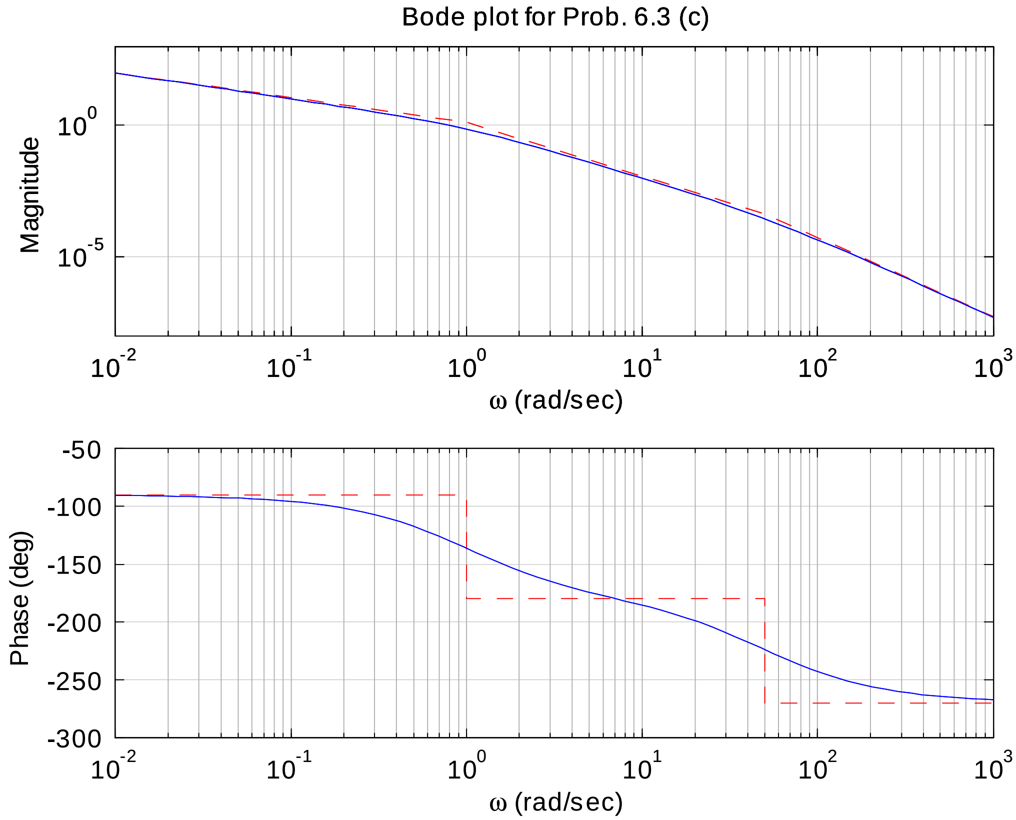
(a) $L(s) = \dfrac{10}{s\left[\frac{s}{200}+1\right]}$



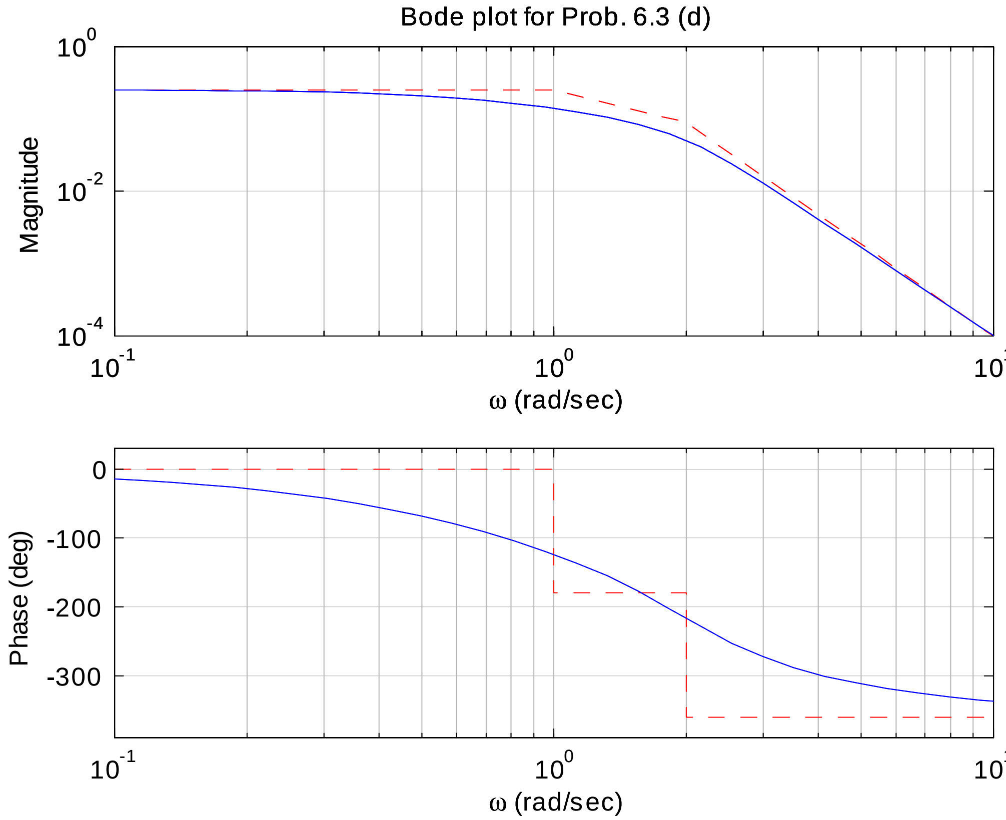
(b) $L(s) = \dfrac{100}{s\left(\frac{s}{10}+1\right)\left(\frac{s}{2}+1\right)}$



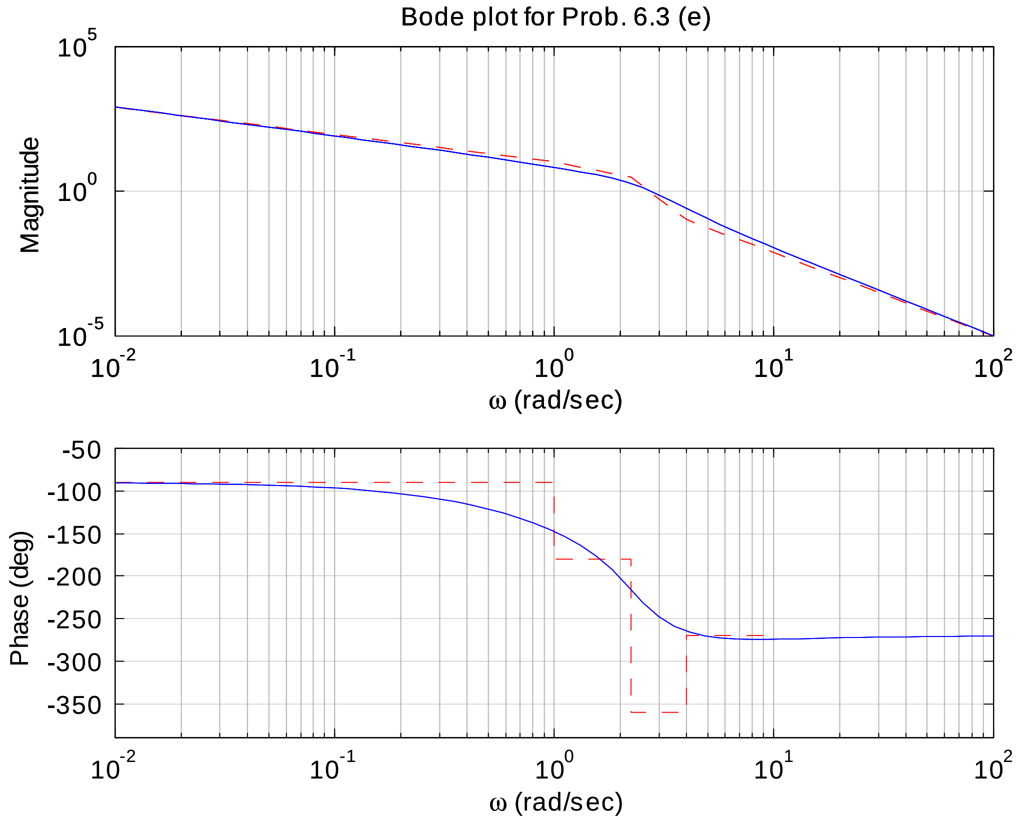
(c) $L(s) = \frac{1}{s(s+1)(0.02s+1)}$



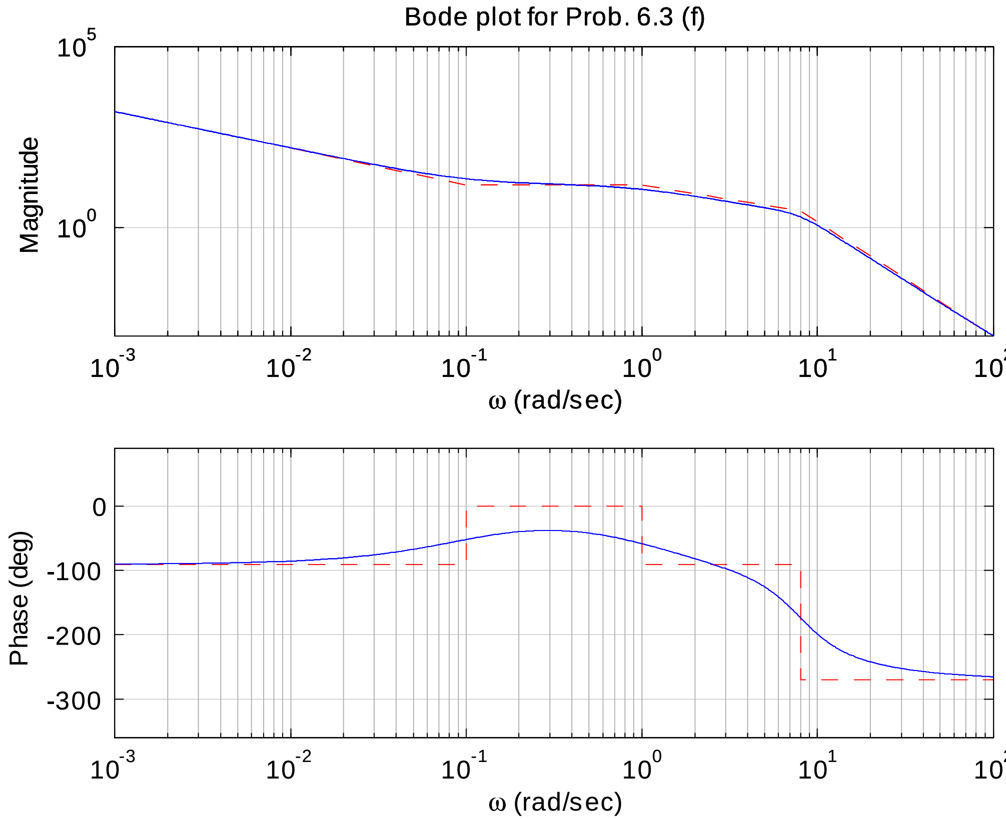
(d) $L(s) = \dfrac{\frac{1}{4}}{(s+1)^2\left[\left(\frac{s}{2}\right)^2 + \frac{s}{2} + 1\right]}$



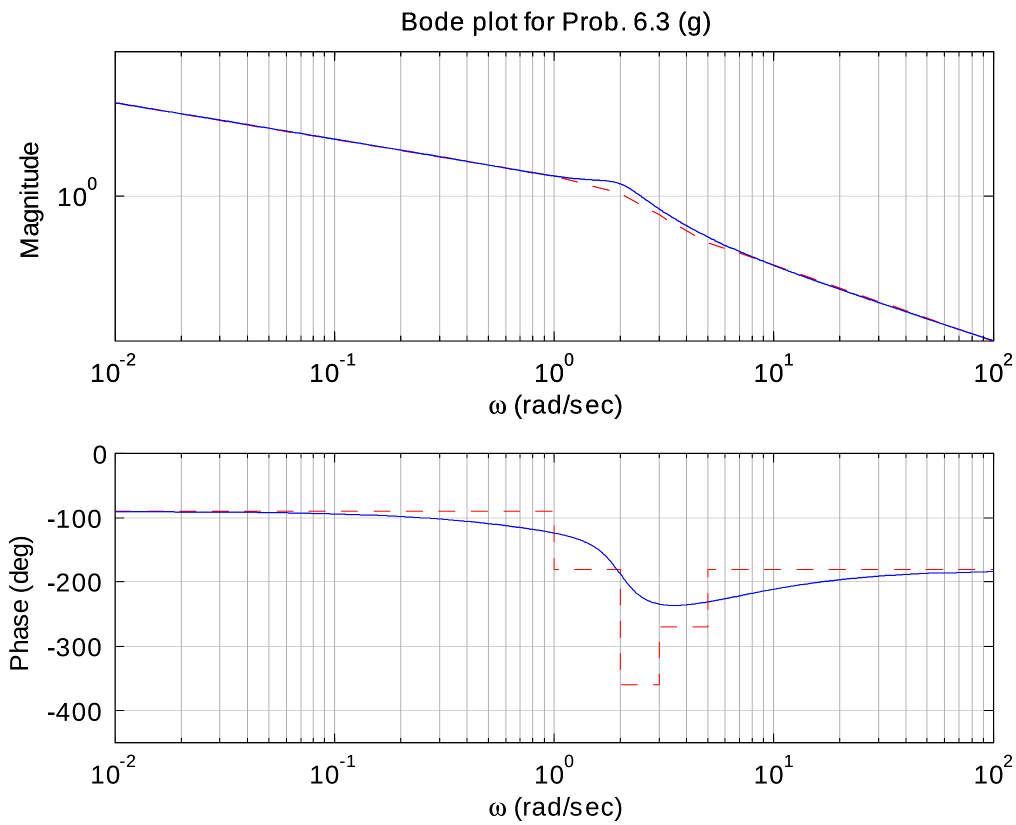
(e) $L(s) = \dfrac{8\left(\frac{s}{4}+1\right)}{s(s+1)\left[\left(\frac{s}{\sqrt{5}}\right)^2 + \frac{2}{5}s + 1\right]}$



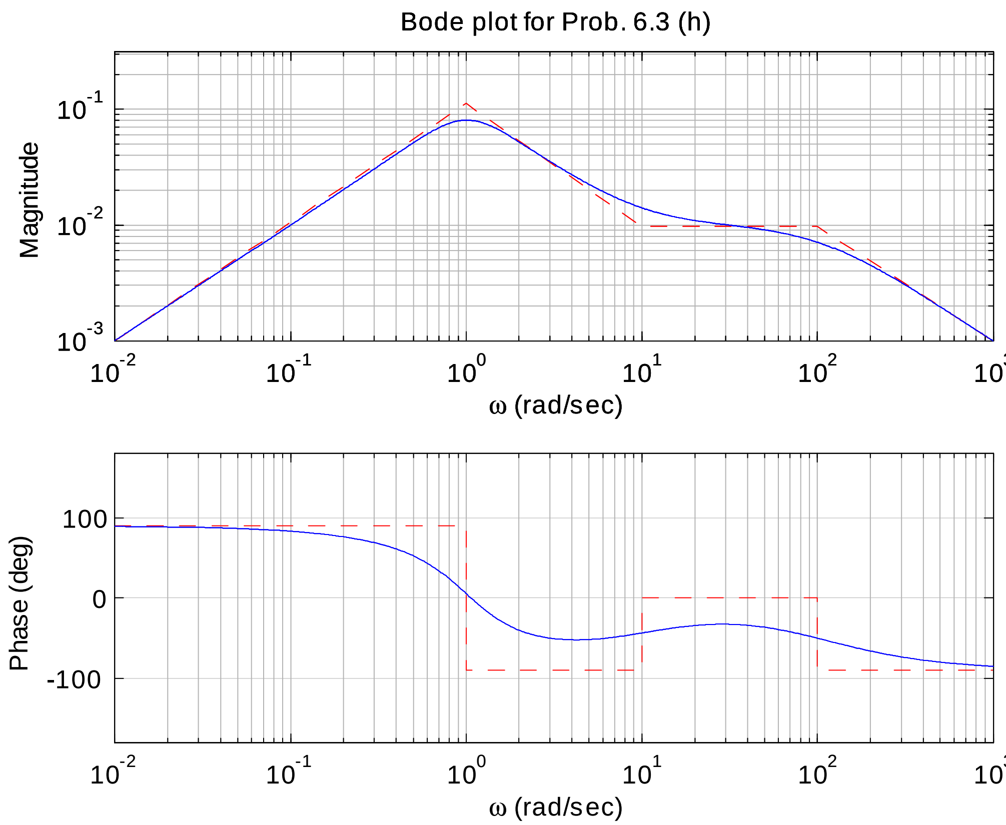
(f) $L(s) = \dfrac{\left(\frac{25}{16}\right)(10s+1)}{s(s+1)\left[\left(\frac{s}{8}\right)^2 + \frac{s}{8} + 1\right]}$



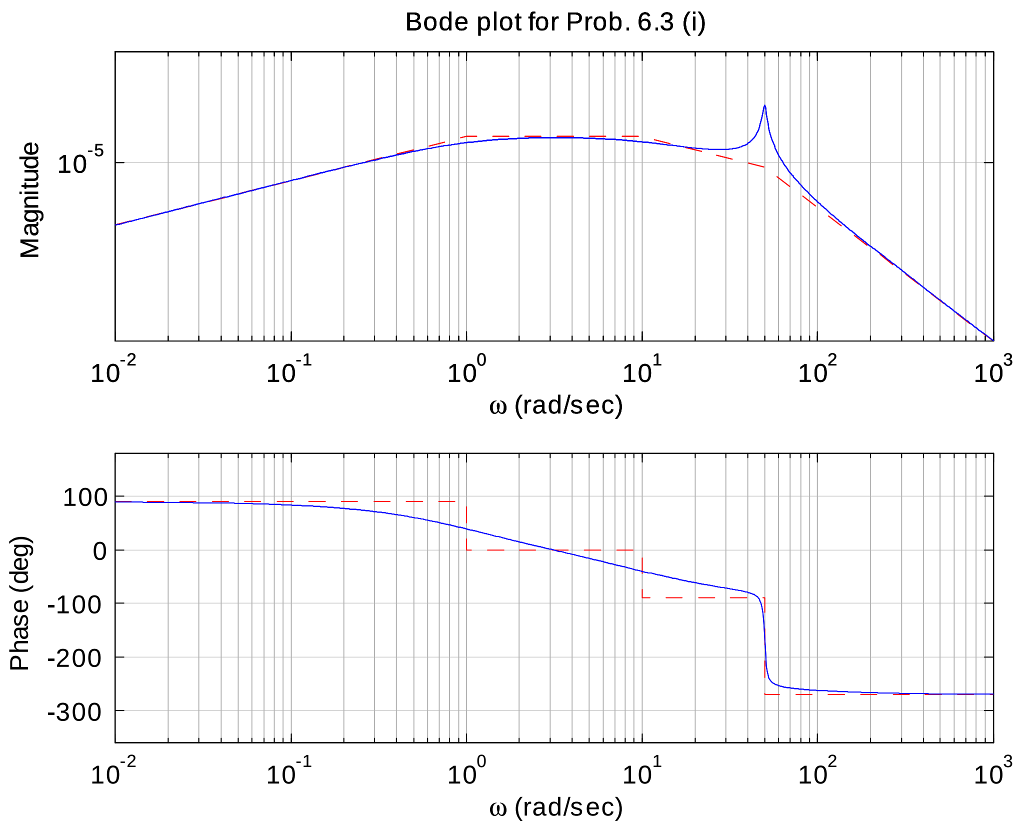
(g) $L(s) = \dfrac{\left(\frac{15}{4}\right)\left(\frac{s}{5}+1\right)\left(\frac{s}{3}+1\right)}{s(s+1)\left[\left(\frac{s}{2}\right)^2+\frac{s}{4}+1\right]}$



(h) $L(s) = \dfrac{\left(\frac{1}{10}\right) s \left(\frac{s}{10} + 1\right)}{\left(\frac{s}{100} + 1\right)\left(s^2 + \frac{5}{4}s + 1\right)}$



(i) $L(s) = \dfrac{\left(\frac{1}{25000}\right) s}{(s+1)\left(\frac{s}{10}+1\right)\left[\left(\frac{s}{50}\right)^2 + \frac{1}{1250}s + 1\right]}$



4. *Real poles and zeros.* Sketch the asymptotes of the Bode plot magnitude and phase for each of the following open-loop transfer functions. After completing the hand sketches verify your result using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

(a) $L(s) = \dfrac{1}{s(s+1)(s+5)(s+10)}$

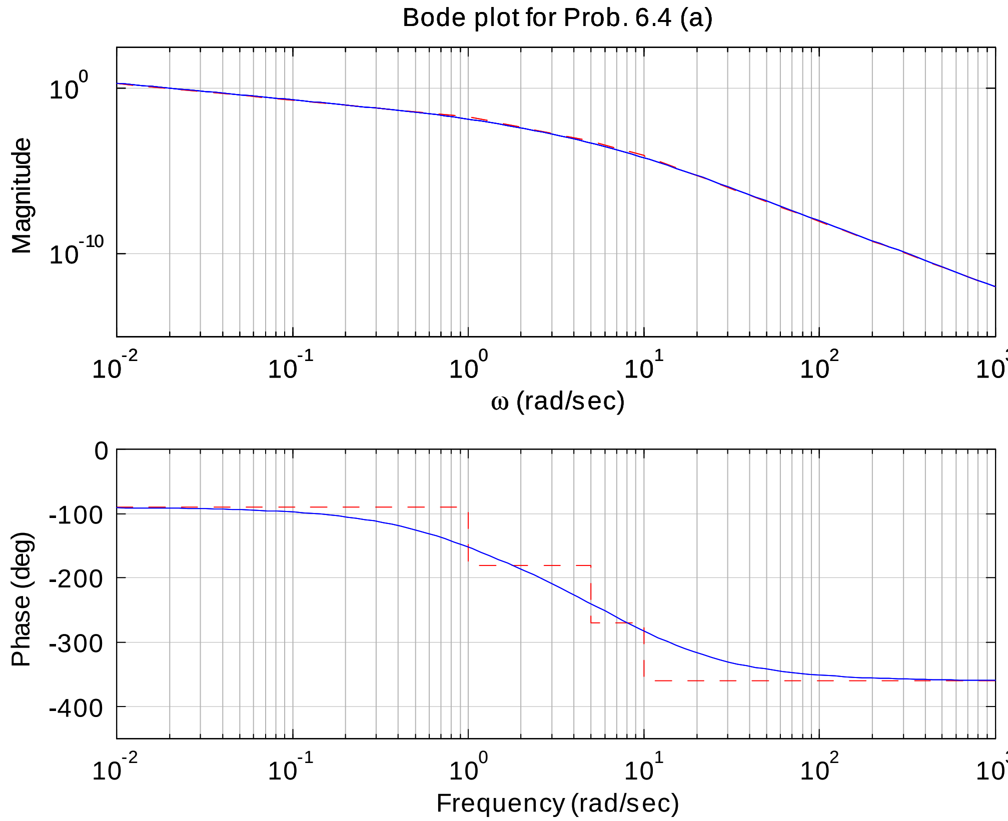
(b) $L(s) = \dfrac{(s+2)}{s(s+1)(s+5)(s+10)}$

(c) $L(s) = \dfrac{(s+2)(s+6)}{s(s+1)(s+5)(s+10)}$

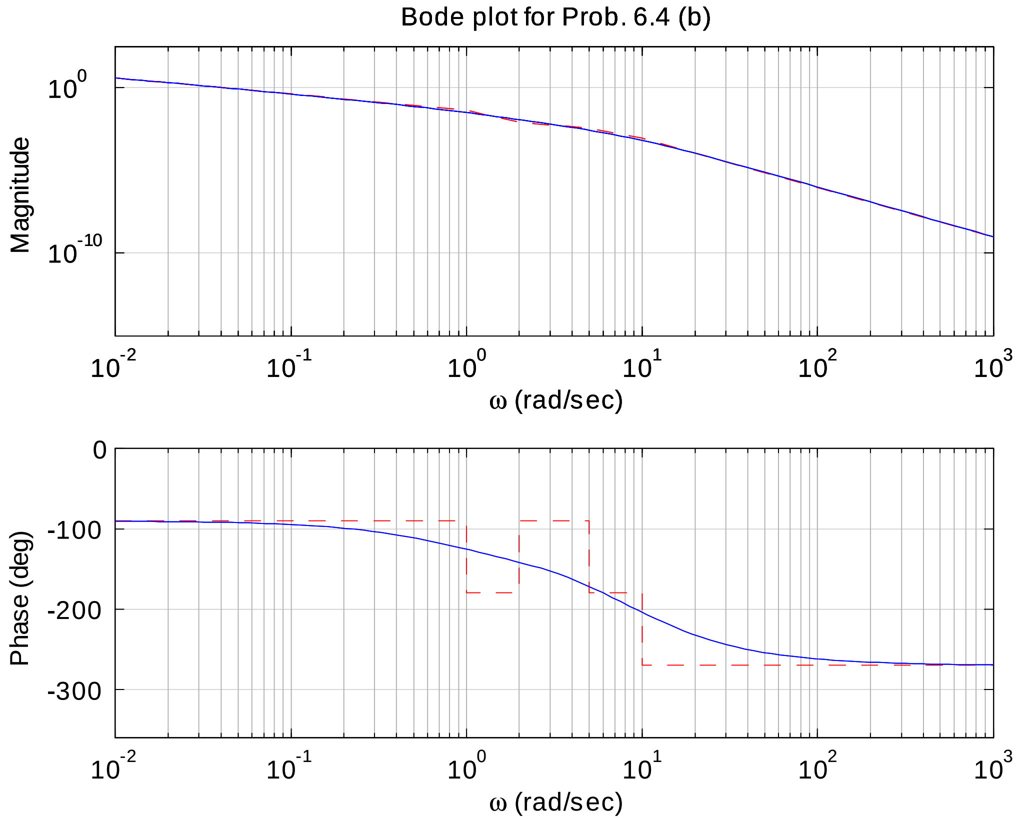
(d) $L(s) = \dfrac{(s+2)(s+4)}{s(s+1)(s+5)(s+10)}$

Solution:

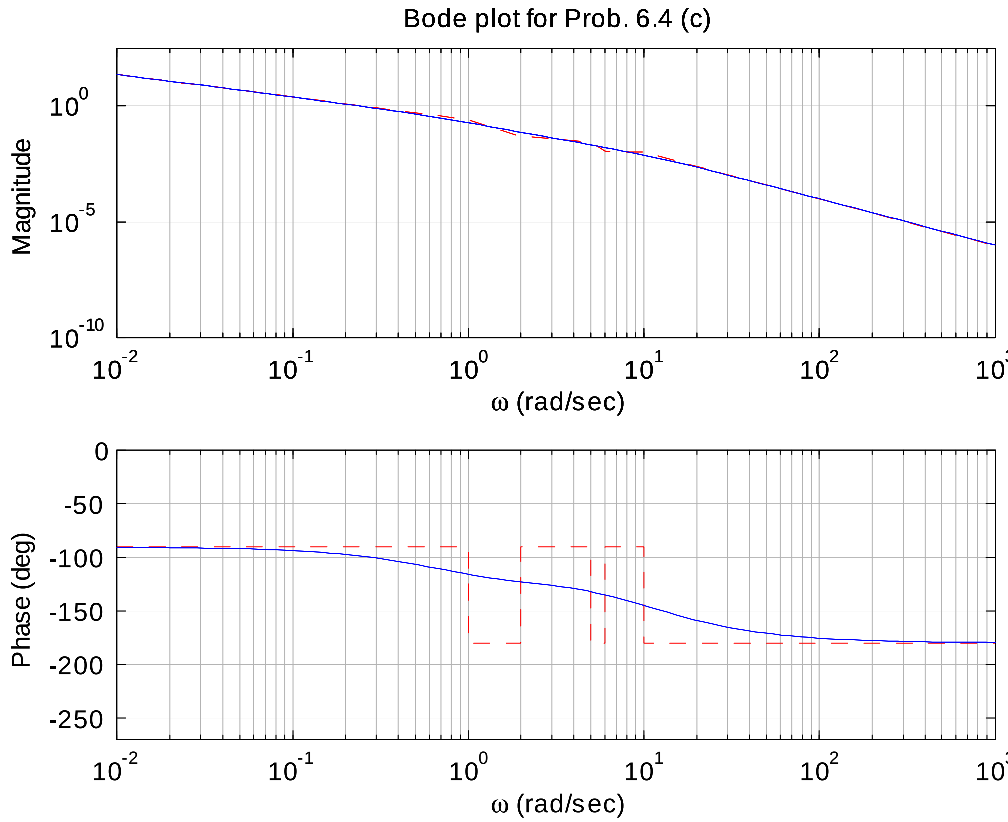
(a) $L(s) = \dfrac{\frac{1}{50}}{s(s+1)\left(\frac{s}{5}+1\right)\left(\frac{s}{10}+1\right)}$



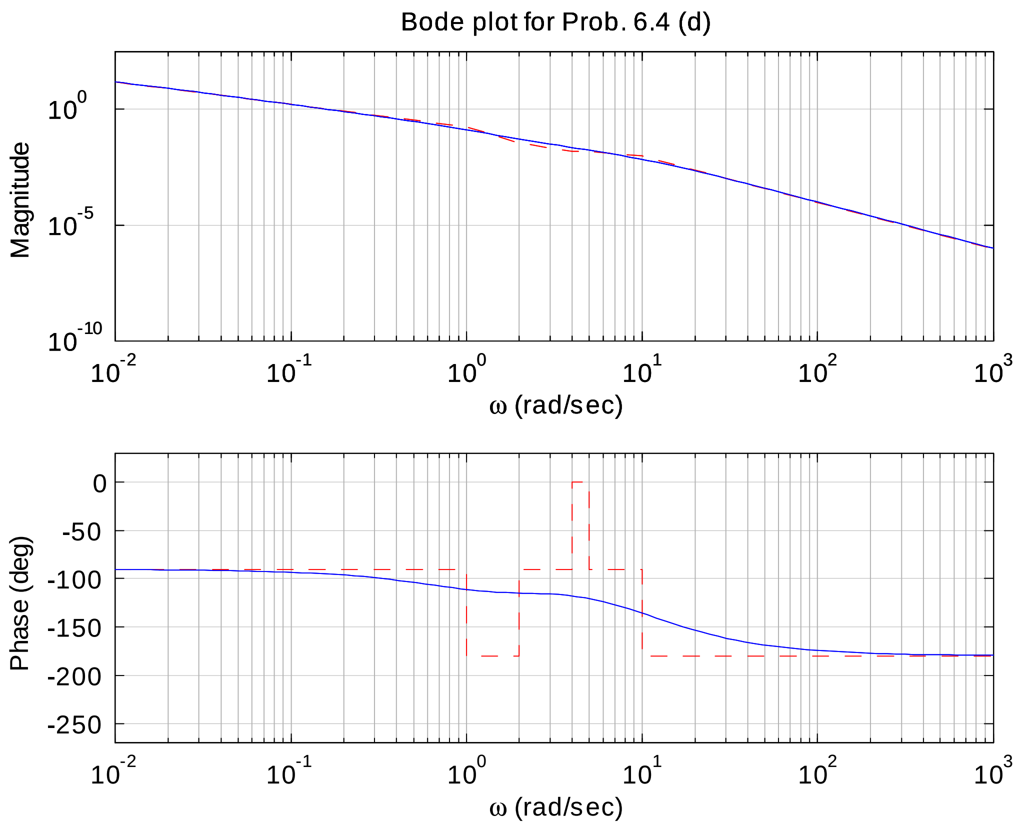
(b) $L(s) = \dfrac{\frac{1}{25}\left(\frac{s}{2}+1\right)}{s(s+1)\left(\frac{s}{5}+1\right)\left(\frac{s}{10}+1\right)}$



(c) $L(s) = \dfrac{\frac{6}{25}\left(\frac{s}{2}+1\right)\left(\frac{s}{6}+1\right)}{s(s+1)\left(\frac{s}{5}+1\right)\left(\frac{s}{10}+1\right)}$



(d) $L(s) = \dfrac{\frac{4}{25}\left(\frac{s}{2}+1\right)\left(\frac{s}{4}+1\right)}{s(s+1)\left(\frac{s}{5}+1\right)\left(\frac{s}{10}+1\right)}$



5. *Complex poles and zeros* Sketch the asymptotes of the Bode plot magnitude and phase for each of the following open-loop transfer functions and approximate the transition at the second order break point based on the value of the damping ratio. After completing the hand sketches verify your result using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

(a) $L(s) = \dfrac{1}{s^2 + 3s + 10}$

(b) $L(s) = \dfrac{1}{s(s^2 + 3s + 10)}$

(c) $L(s) = \dfrac{(s^2 + 2s + 8)}{s(s^2 + 2s + 10)}$

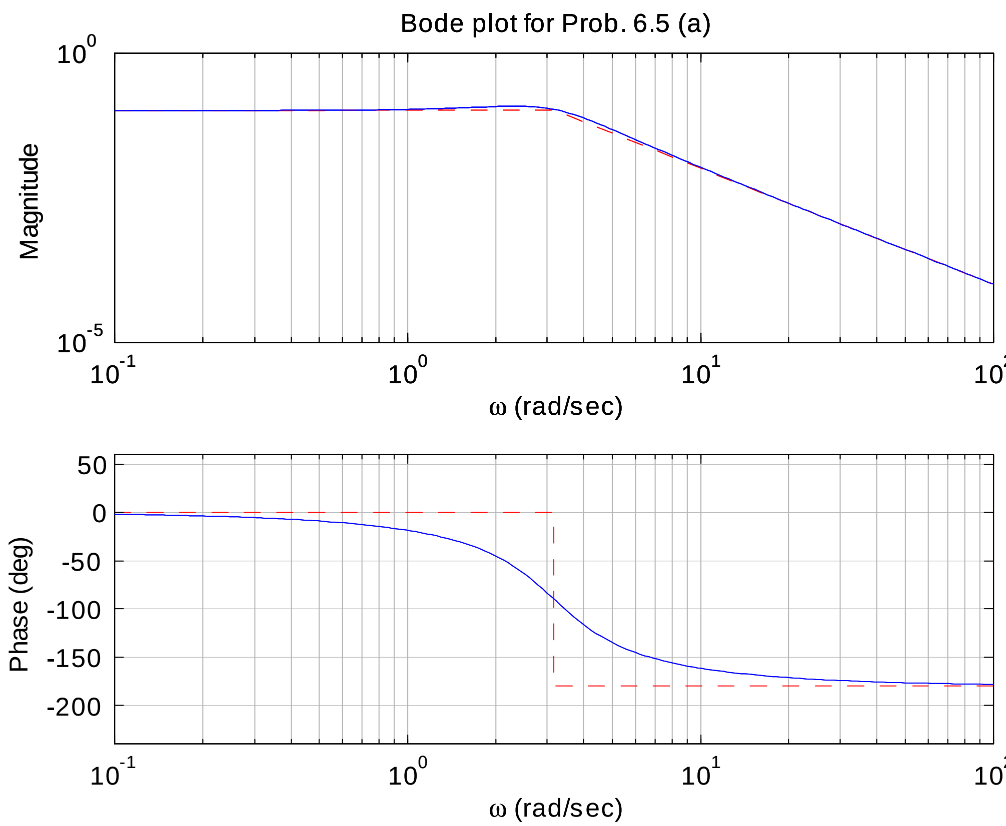
(d) $L(s) = \dfrac{(s^2 + 2s + 12)}{s(s^2 + 2s + 10)}$

(e) $L(s) = \dfrac{(s^2 + 1)}{s(s^2 + 4)}$

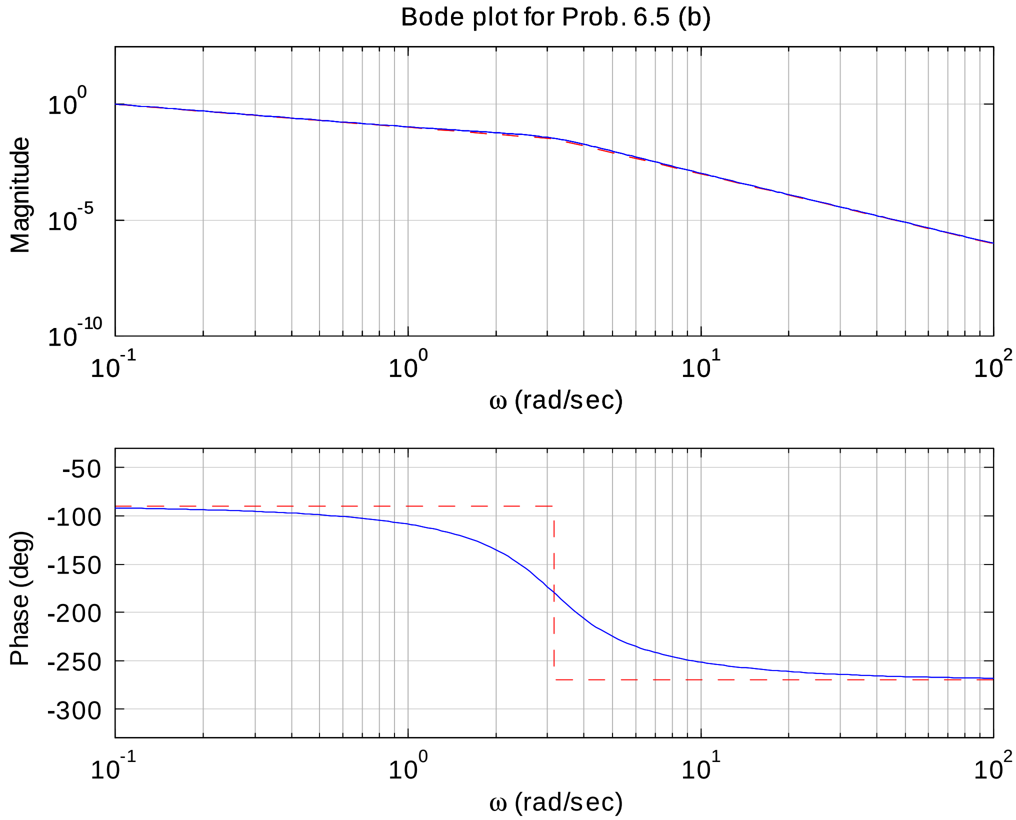
(f) $L(s) = \dfrac{(s^2+4)}{s(s^2+1)}$

Solution:

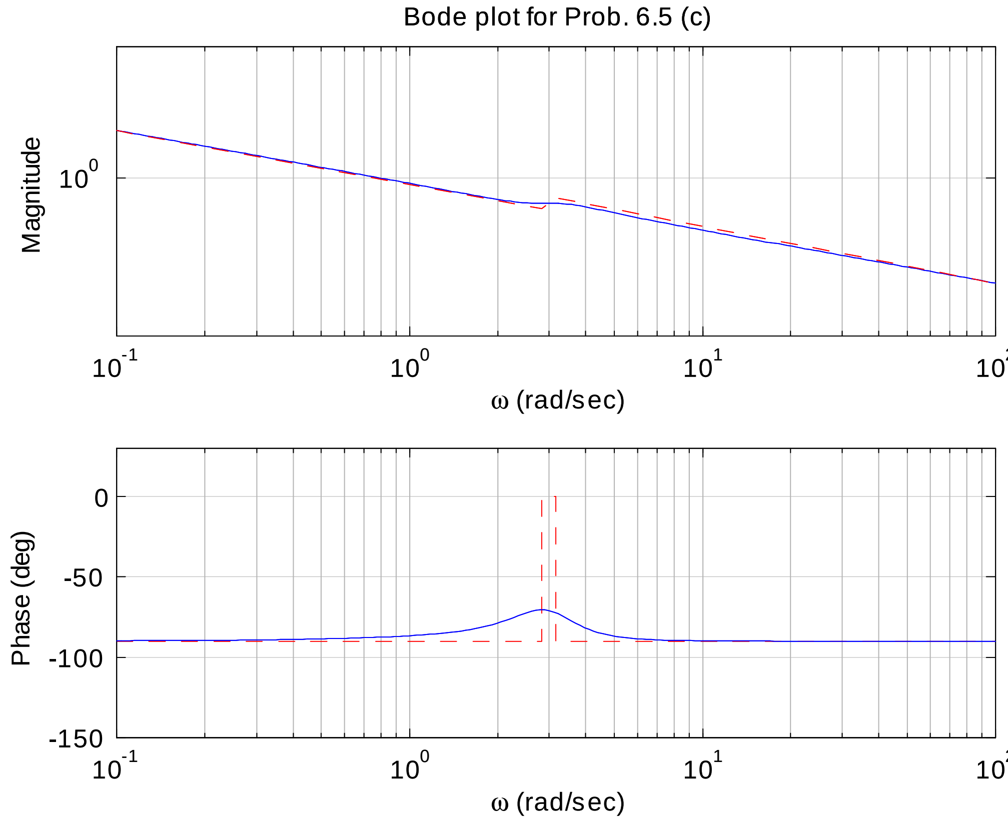
(a) $L(s) = \dfrac{\frac{1}{10}}{\left(\frac{s}{\sqrt{10}}\right)^2 + \frac{3}{10}s + 1}$



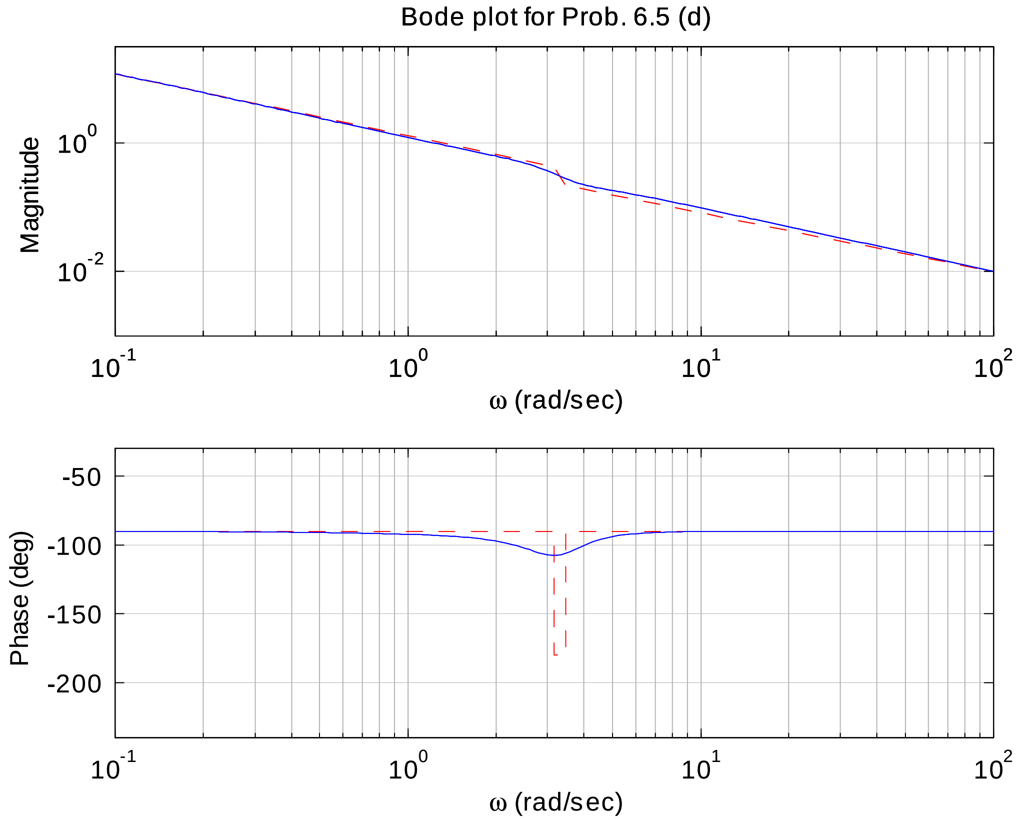
(b) $L(s) = \dfrac{\frac{1}{10}}{s\left[\left(\frac{s}{\sqrt{10}}\right)^2 + \frac{3}{10}s + 1\right]}$



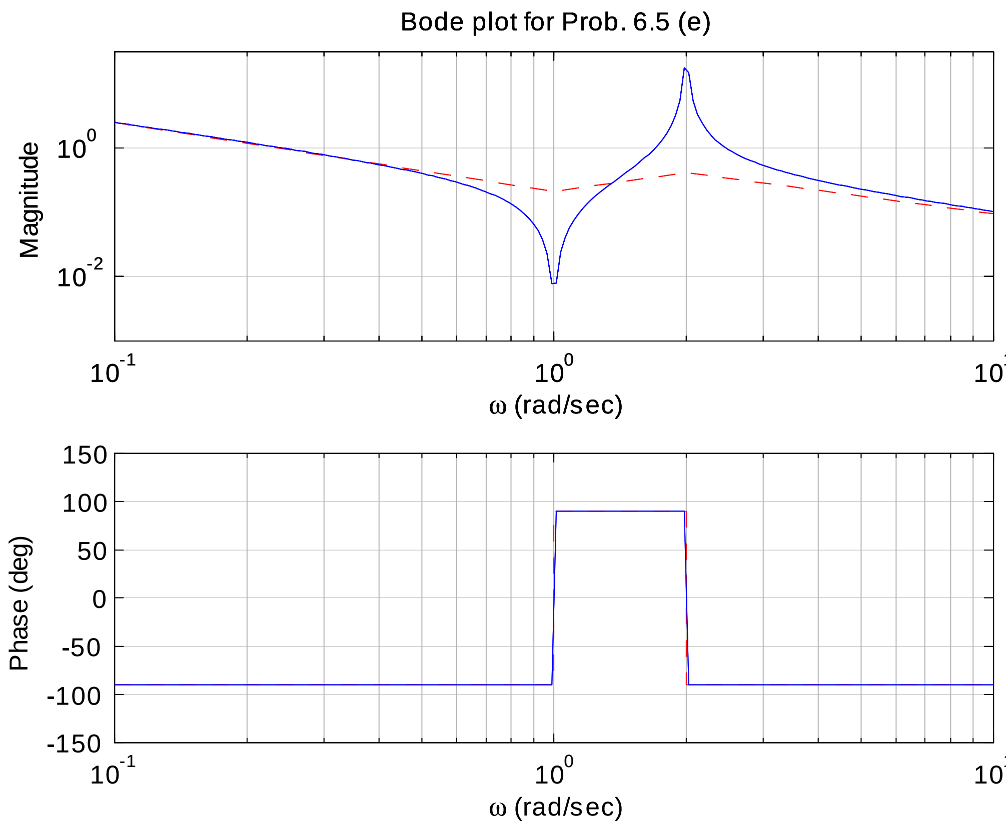
(c) $L(s) = \dfrac{\frac{4}{5}\left[\left(\frac{s}{2\sqrt{2}}\right)^2 + \frac{1}{4}s + 1\right]}{s\left[\left(\frac{s}{\sqrt{10}}\right)^2 + \frac{1}{5}s + 1\right]}$



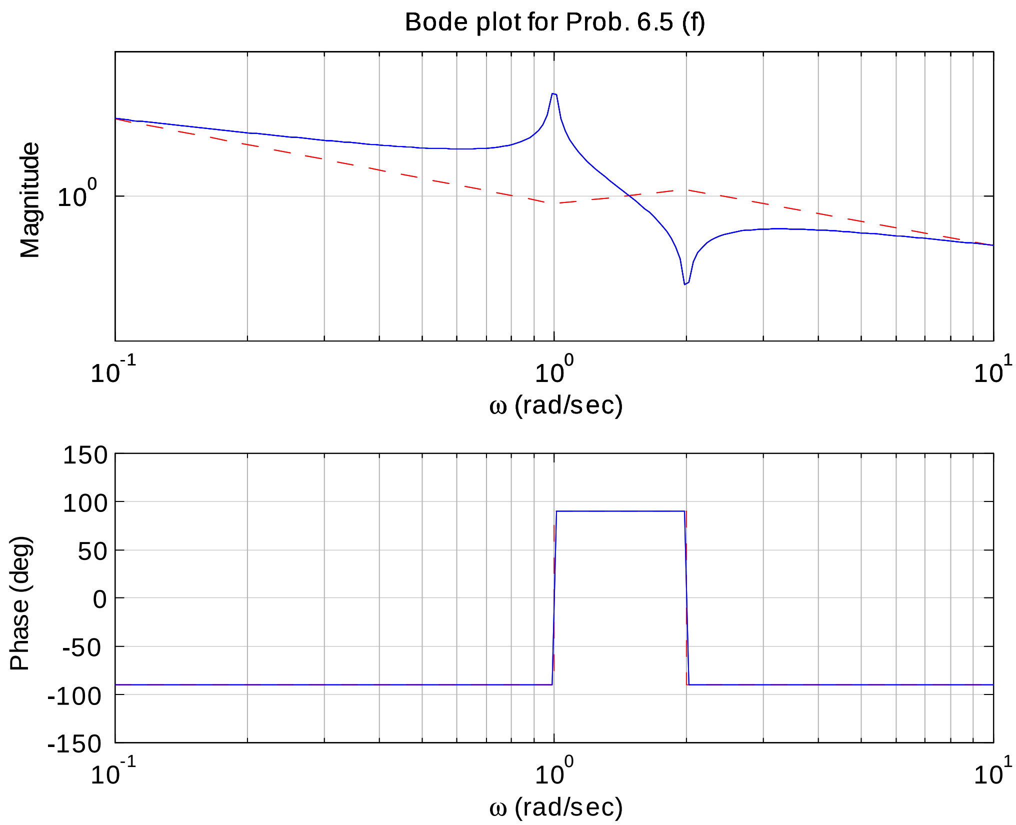
(d) $L(s) = \dfrac{\frac{6}{5}\left[\left(\frac{s}{2\sqrt{3}}\right)^2 + \frac{1}{6}s + 1\right]}{s\left[\left(\frac{s}{\sqrt{10}}\right)^2 + \frac{1}{5}s + 1\right]}$



(e) $L(s) = \dfrac{\frac{1}{4}(s^2+1)}{s\left[\left(\frac{s}{2}\right)^2+1\right]}$



(f) $L(s) = \dfrac{4\left[\left(\frac{s}{2}\right)^2 + 1\right]}{s(s^2+1)}$



6. *Multiple poles at the origin* Sketch the asymptotes of the Bode plot magnitude and phase for each of the following open-loop transfer functions. After completing the hand sketches verify your result using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

(a) $L(s) = \dfrac{1}{s^2(s+8)}$

(b) $L(s) = \dfrac{1}{s^3(s+8)}$

(c) $L(s) = \dfrac{1}{s^4(s+8)}$

(d) $L(s) = \dfrac{(s+3)}{s^2(s+8)}$

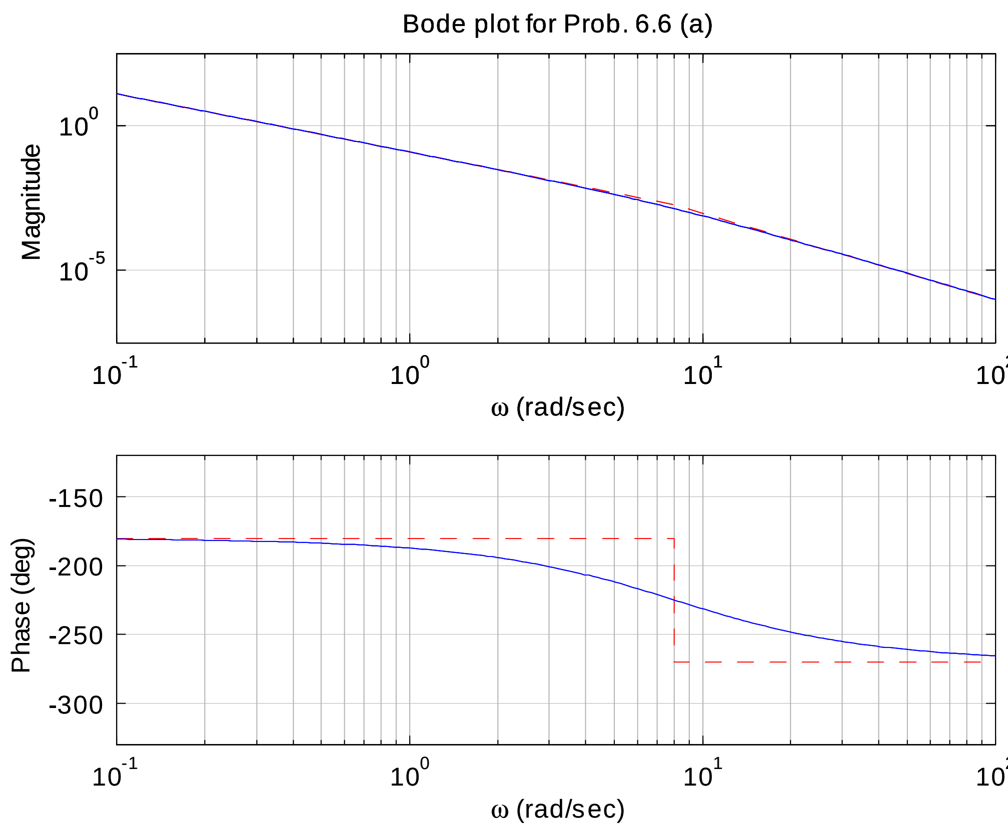
(e) $L(s) = \dfrac{(s+3)}{s^3(s+4)}$

(f) $L(s) = \dfrac{(s+1)^2}{s^3(s+4)}$

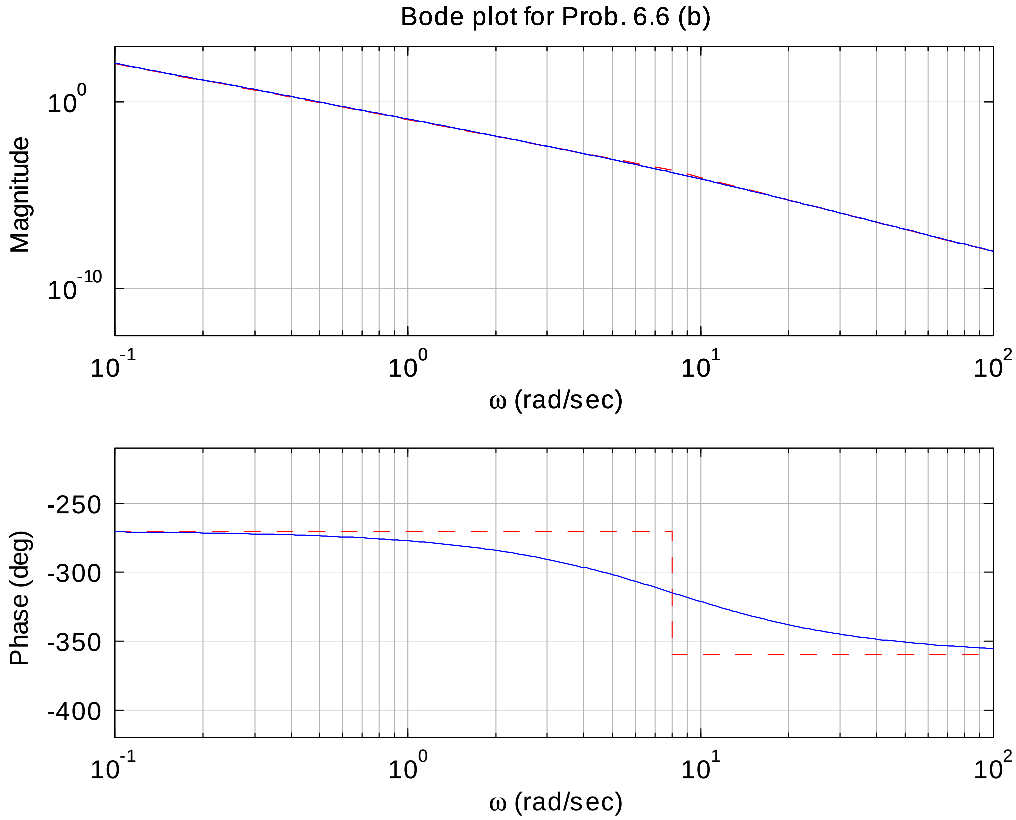
(g) $L(s) = \dfrac{(s+1)^2}{s^3(s+10)^2}$

Solution:

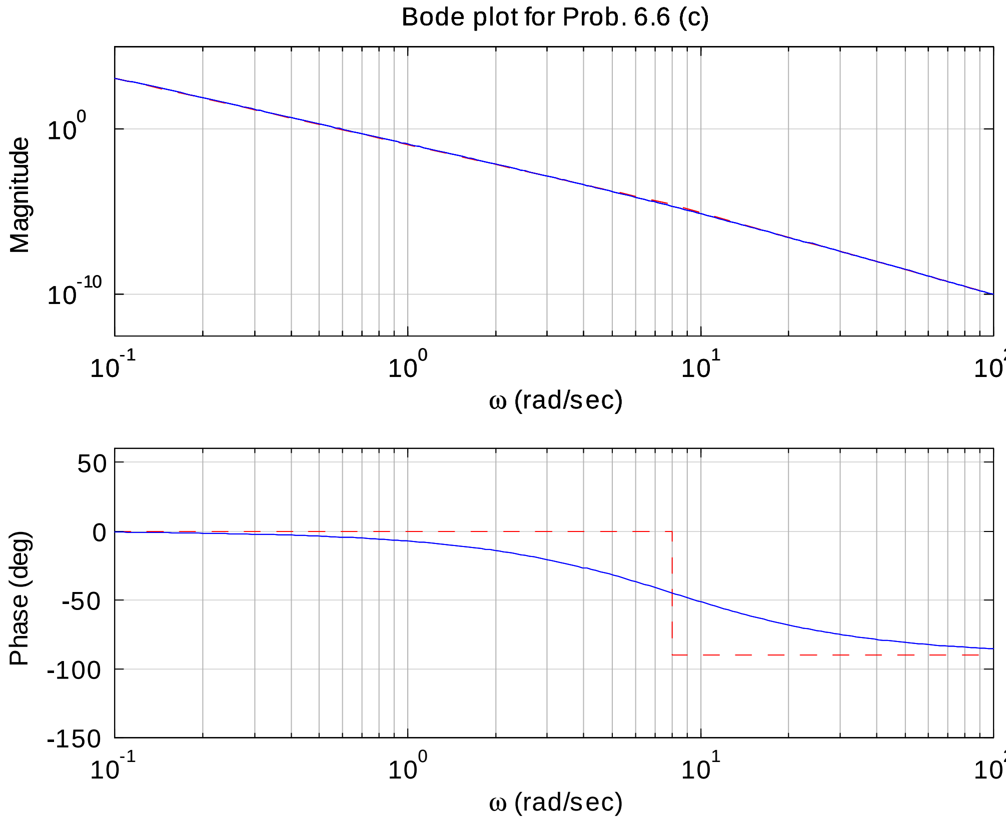
(a) $L(s) = \dfrac{\frac{1}{8}}{s^2\left(\frac{s}{8}+1\right)}$



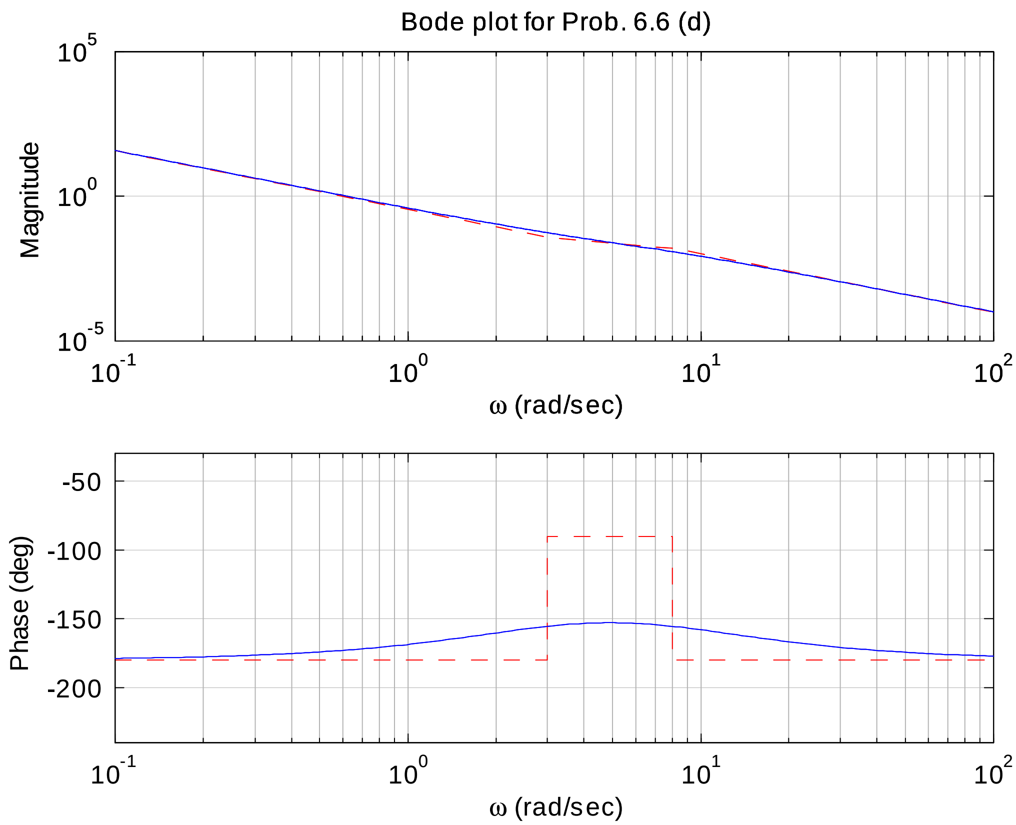
(b) $L(s) = \dfrac{\frac{1}{8}}{s^3\left(\frac{s}{8}+1\right)}$



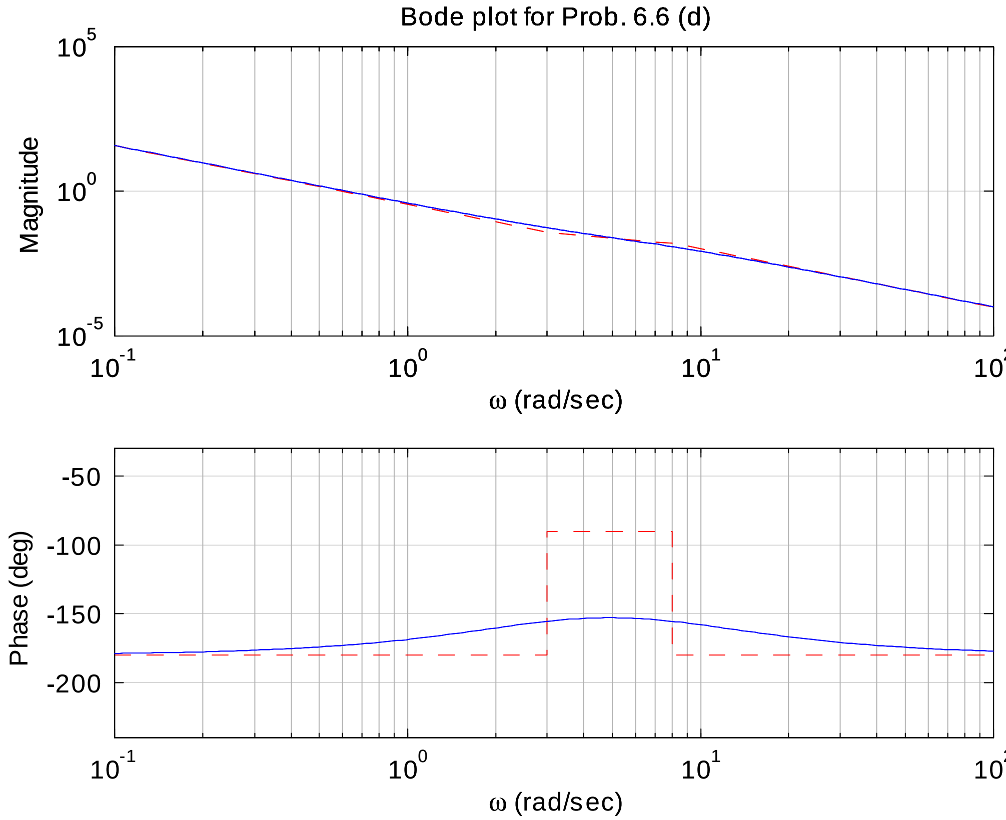
(c) $L(s) = \dfrac{\frac{1}{8}}{s^4\left(\frac{s}{8}+1\right)}$



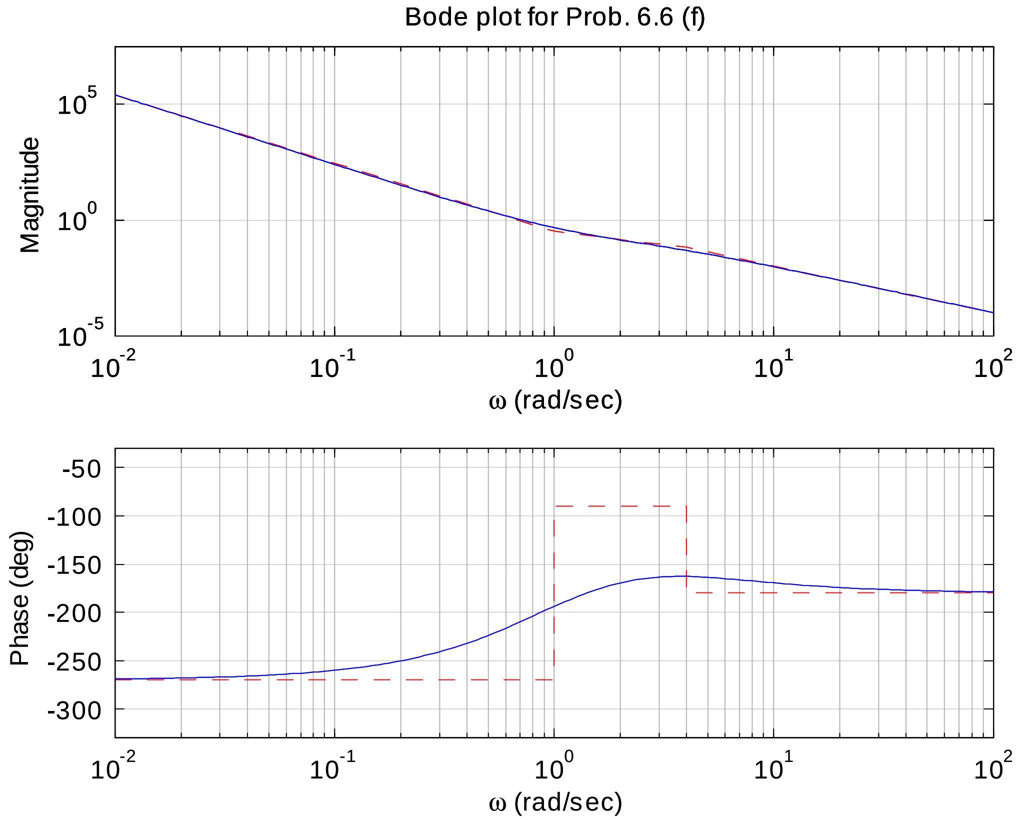
(d) $L(s) = \dfrac{\frac{3}{8}\left(\frac{s}{3}+1\right)}{s^2\left(\frac{s}{8}+1\right)}$



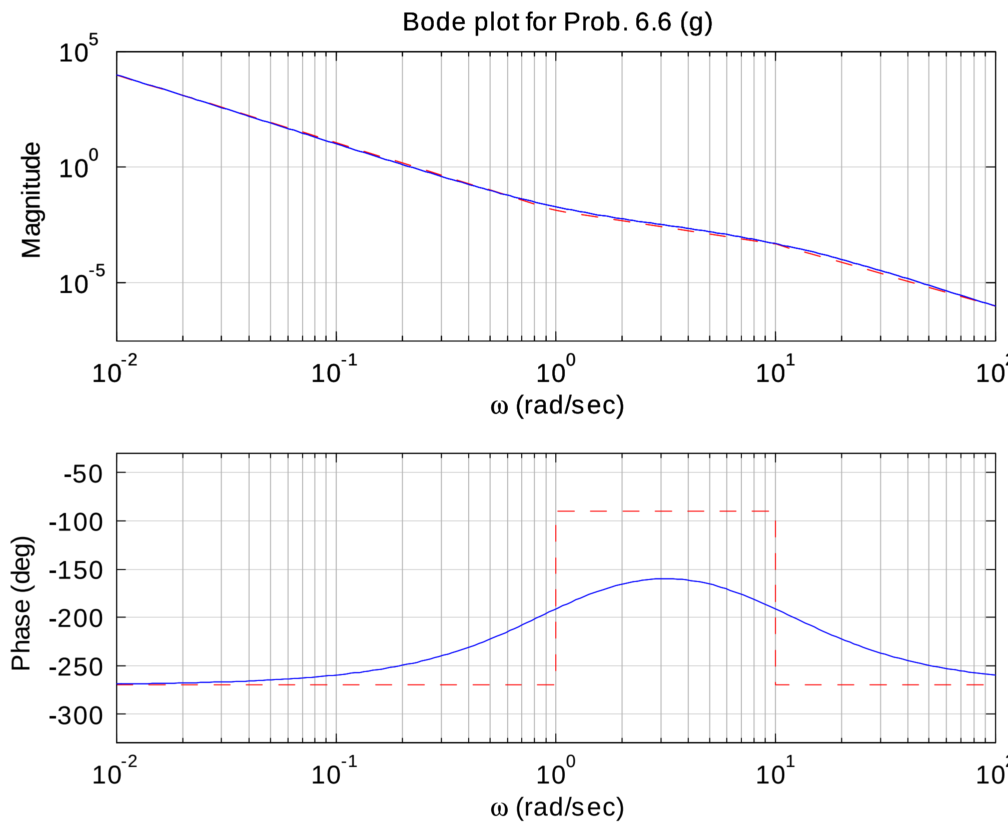
(e) $L(s) = \dfrac{\frac{3}{4}\left(\frac{s}{3}+1\right)}{s^3\left(\frac{s}{4}+1\right)}$



(f) $L(s) = \dfrac{\frac{1}{4}(s+1)^2}{s^3\left(\frac{s}{4}+1\right)}$



(g) $L(s) = \dfrac{\frac{1}{100}\,(s+1)^2}{s^3\left(\frac{s}{10}+1\right)^2}$



7. *Mixed real and complex poles* Sketch the asymptotes of the Bode plot magnitude and phase for each of the following open-loop transfer functions. After completing the hand sketches verify your result using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

   (a) $L(s) = \dfrac{(s+2)}{s(s+10)(s^2+2s+2)}$

   (b) $L(s) = \dfrac{(s+2)}{s^2(s+10)(s^2+6s+25)}$

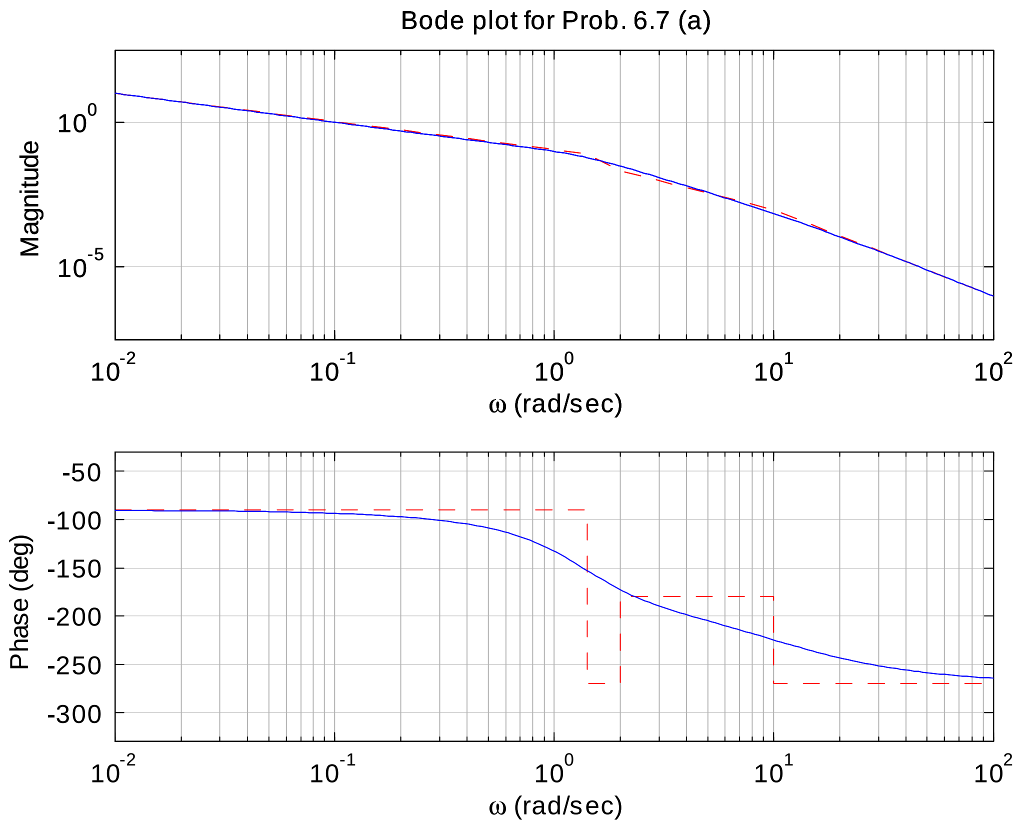
   (c) $L(s) = \dfrac{(s+2)^2}{s^2(s+10)(s^2+6s+25)}$

   (d) $L(s) = \dfrac{(s+2)(s^2+4s+68)}{s^2(s+10)(s^2+4s+85)}$

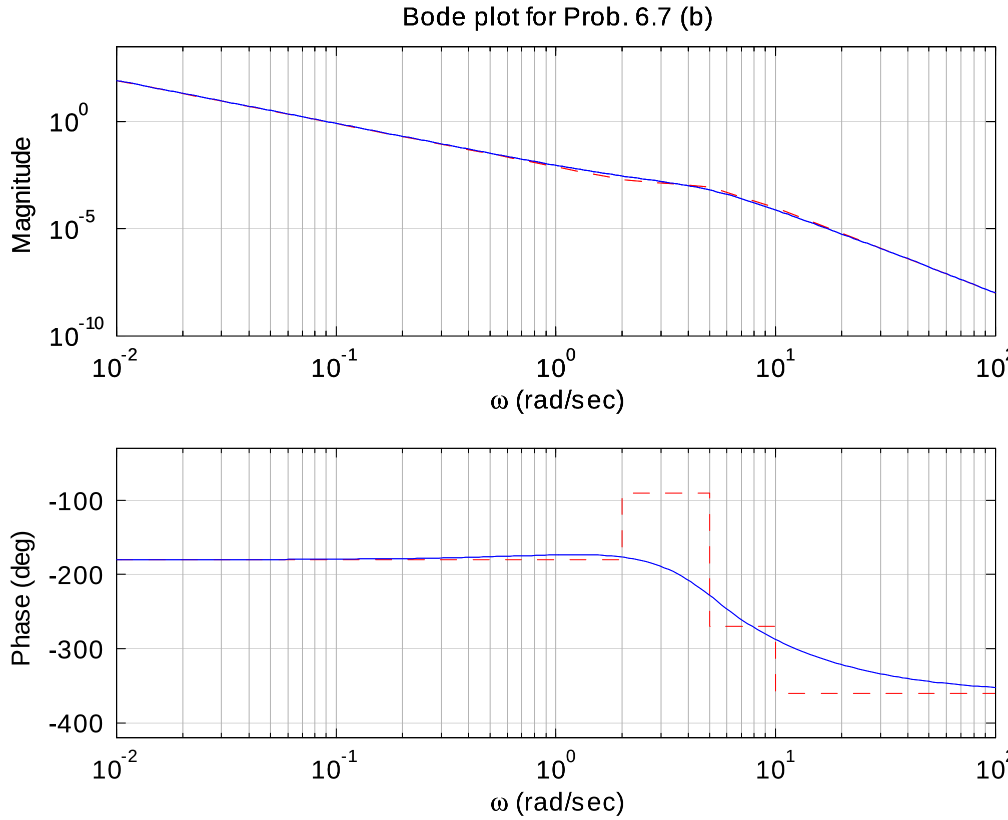
   (e) $L(s) = \dfrac{[(s+1)^2+1]}{s^2(s+2)(s+3)}$

   Solution:

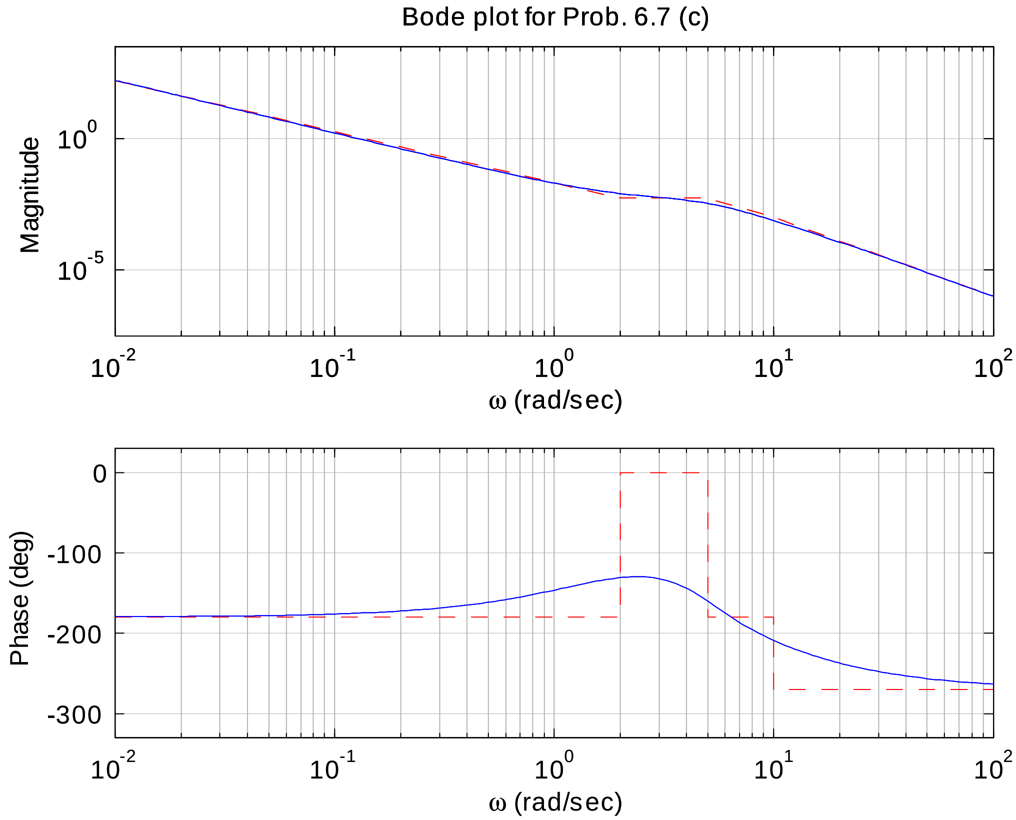
(a) $L(s) = \dfrac{\frac{1}{10}\left(\frac{s}{2}+1\right)}{s\left(\frac{s}{10}+1\right)\left[\left(\frac{s}{\sqrt{2}}\right)^2+s+1\right]}$



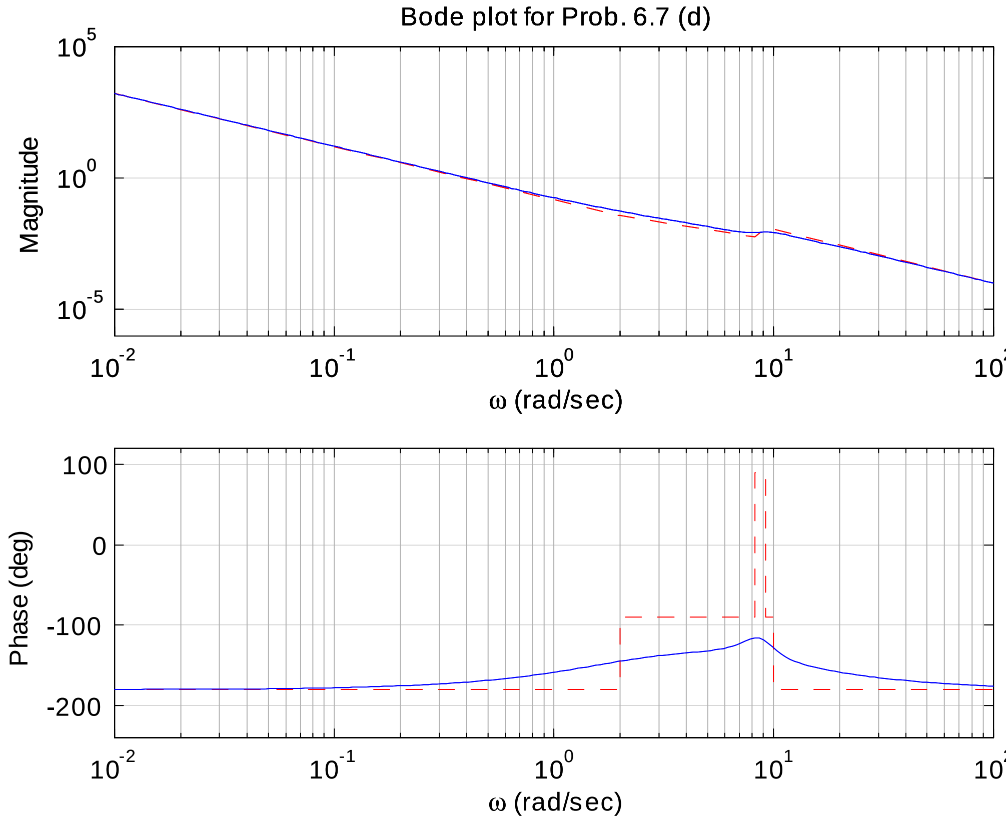
(b) $L(s) = \dfrac{\frac{1}{125}\left(\frac{s}{2}+1\right)}{s^2\left(\frac{s}{10}+1\right)\left[\left(\frac{s}{5}\right)^2+\frac{6}{25}s+1\right]}$



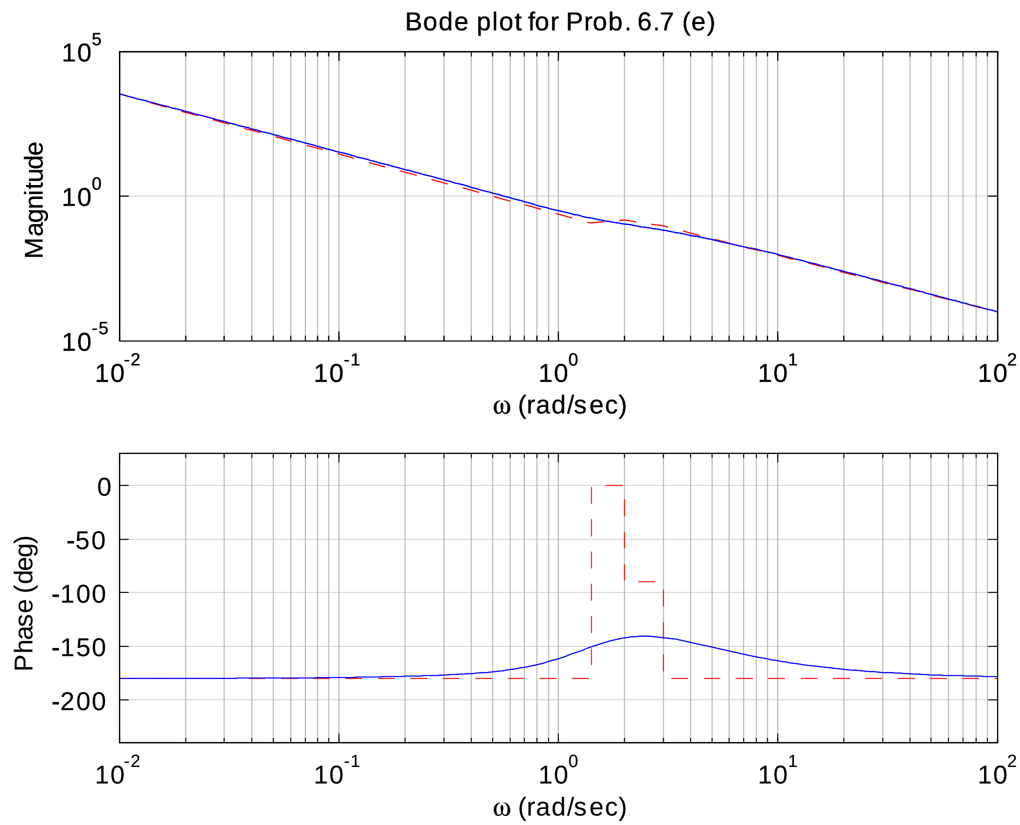
(c) $L(s) = \dfrac{\frac{2}{125}\left(\frac{s}{2}+1\right)^2}{s^2\left(\frac{s}{10}+1\right)\left[\left(\frac{s}{5}\right)^2+\frac{6}{25}s+1\right]}$



(d) $L(s) = \dfrac{\frac{4}{25}\left(\frac{s}{2}+1\right)\left[\left(\frac{s}{2\sqrt{17}}\right)^2 + \frac{1}{17}s + 1\right]}{s^2\left(\frac{s}{10}+1\right)\left[\left(\frac{s}{\sqrt{85}}\right)^2 + \frac{4}{85}s + 1\right]}$



(e) $L(s) = \dfrac{\frac{1}{3}\left[\left(\frac{s}{\sqrt{2}}\right)^2 + s + 1\right]}{s^2\left(\frac{s}{2}+1\right)\left(\frac{s}{3}+1\right)}$



8. *Right half plane poles and zeros* Sketch the asymptotes of the Bode plot magnitude and phase for each of the following open-loop transfer functions. Make sure the phase asymptotes properly take the RHP singularity into account by sketching the complex plane to see how the $\angle L(s)$ changes as s goes from 0 to $+j\infty$. After completing the hand sketches verify your result using MATLAB. Turn in your hand sketches and the MATLAB results on the same scales.

   (a) $L(s) = \dfrac{s+2}{s+10}\dfrac{1}{s^2-1}$; The model for a case of magnetic levitation with lead compensation.

   (b) $L(s) = \dfrac{s+2}{s(s+10)}\dfrac{1}{(s^2-1)}$; The magnetic levitation system with integral control and lead compensation.

   (c) $L(s) = \dfrac{s-1}{s^2}$

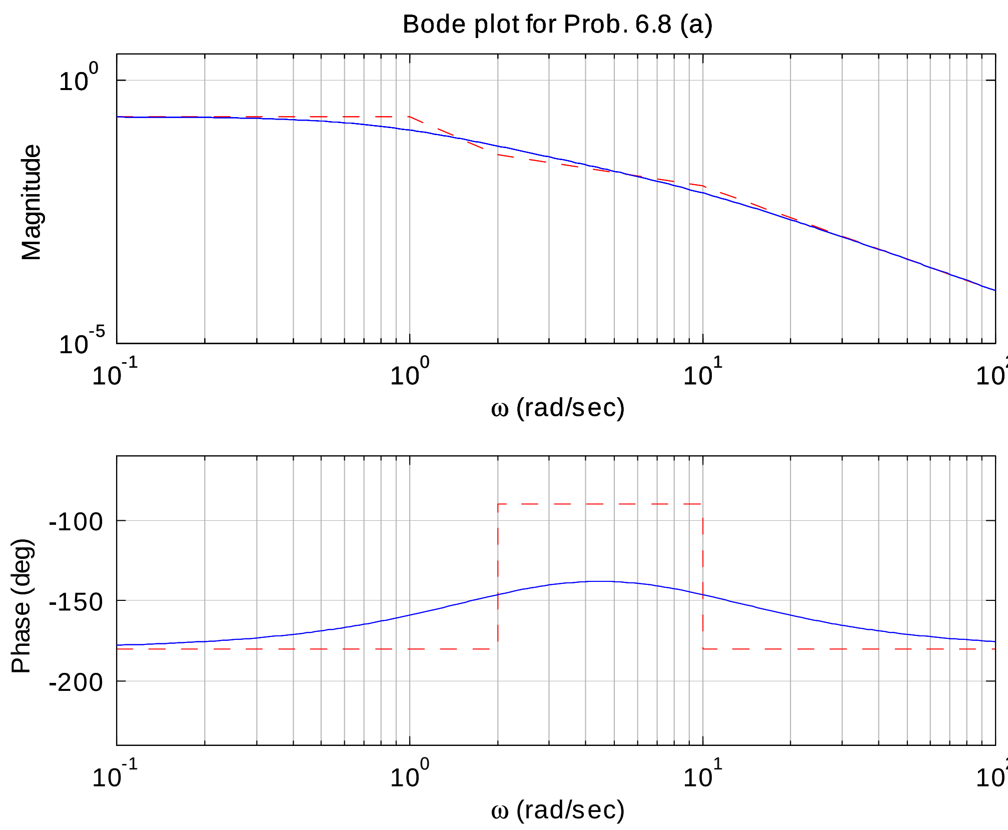
   (d) $L(s) = \dfrac{s^2+2s+1}{s(s+20)^2(s^2-2s+2)}$

(e) $L(s) = \dfrac{(s+2)}{s(s-1)(s+6)^2}$

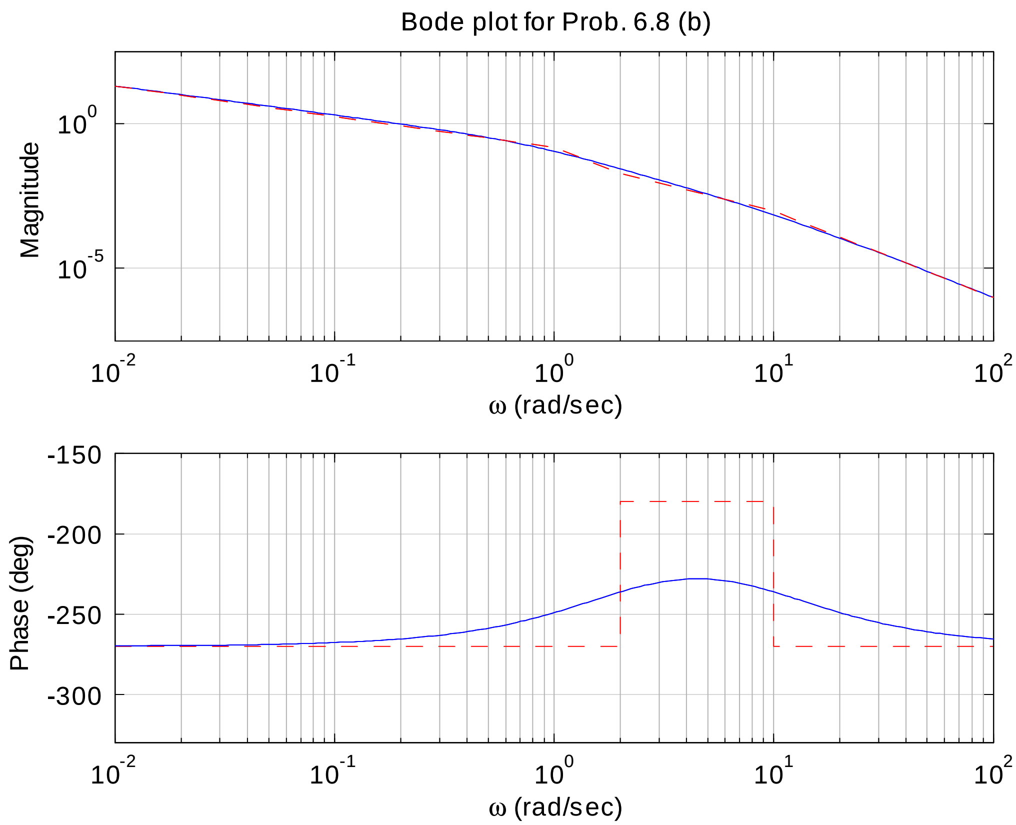
(f) $L(s) = \dfrac{1}{(s-1)[(s+2)^2+3]}$

Solution:

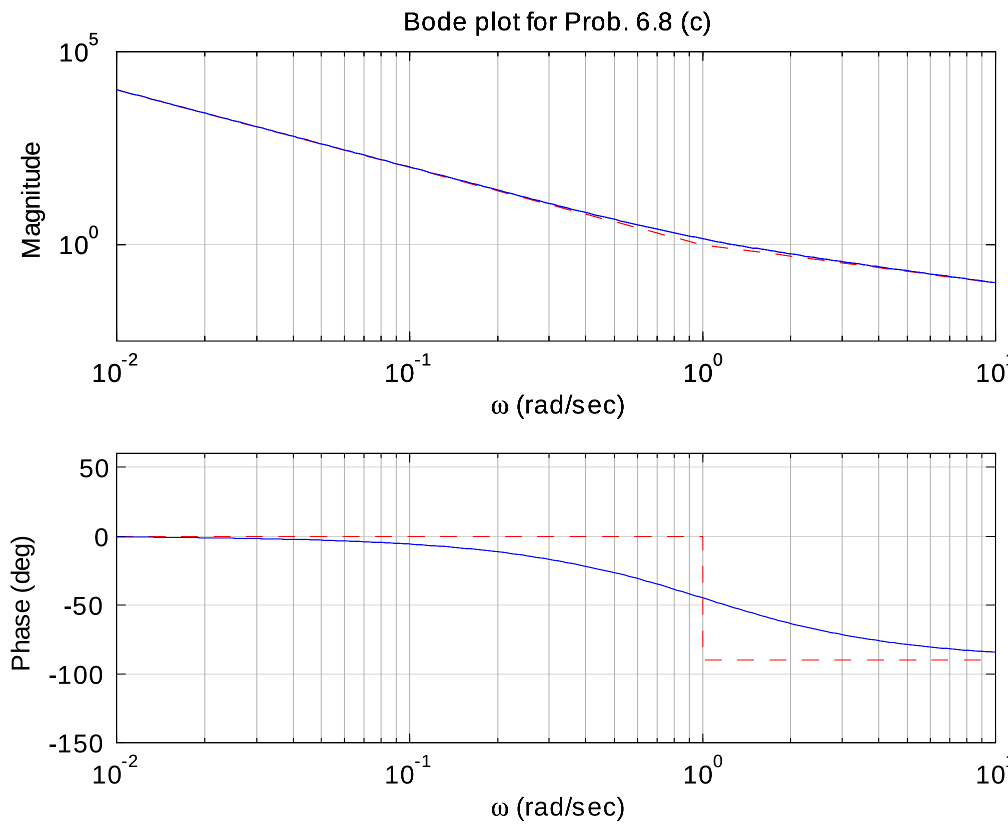
(a) $L(s) = \dfrac{\frac{1}{5}\left(\frac{s}{2}+1\right)}{s+10}\dfrac{1}{s^2-1}$



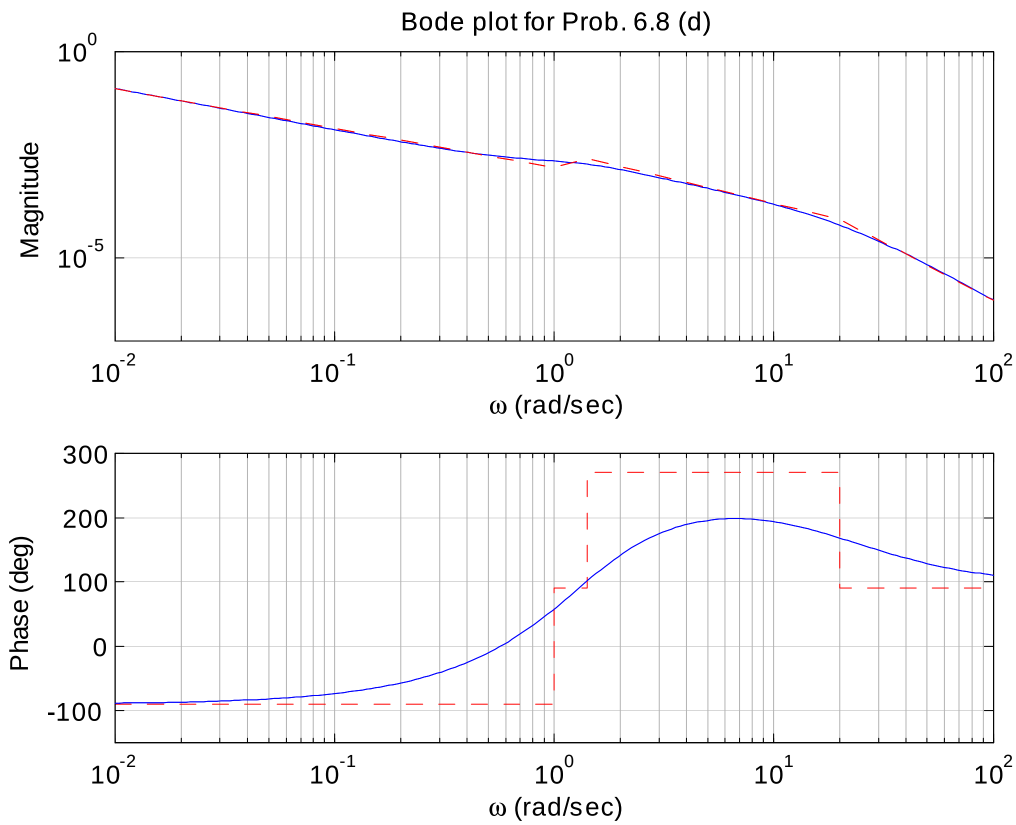
(b) $L(s) = \dfrac{\frac{1}{5}\left(\frac{s}{2}+1\right)}{s(s+10)}\dfrac{1}{s^2-1}$



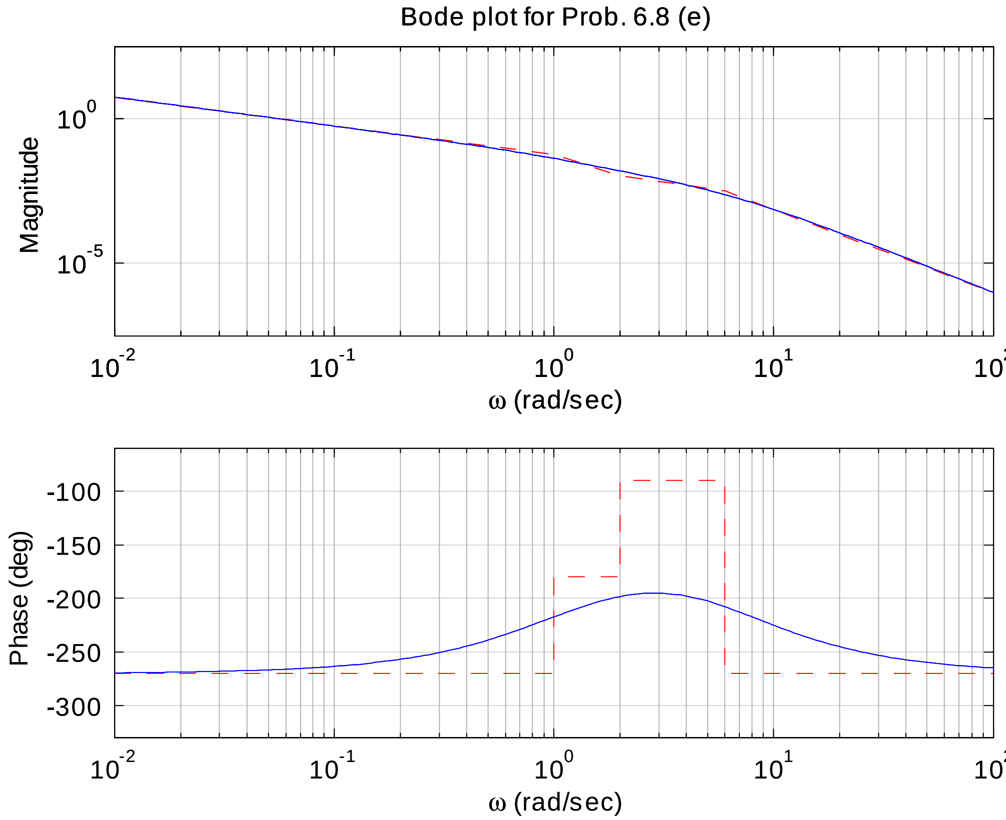
(c) $L(s) = \dfrac{s-1}{s^2}$



(d) $L(s) = \dfrac{\frac{1}{40}(s^2 + 2s + 1)}{s\left(\frac{s}{20} + 1\right)^2 \left[\left(\frac{s}{\sqrt{2}}\right)^2 - s + 1\right]}$

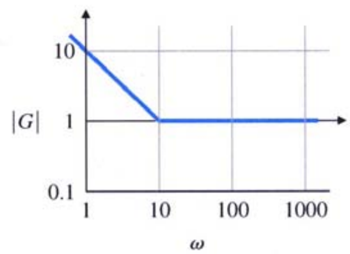


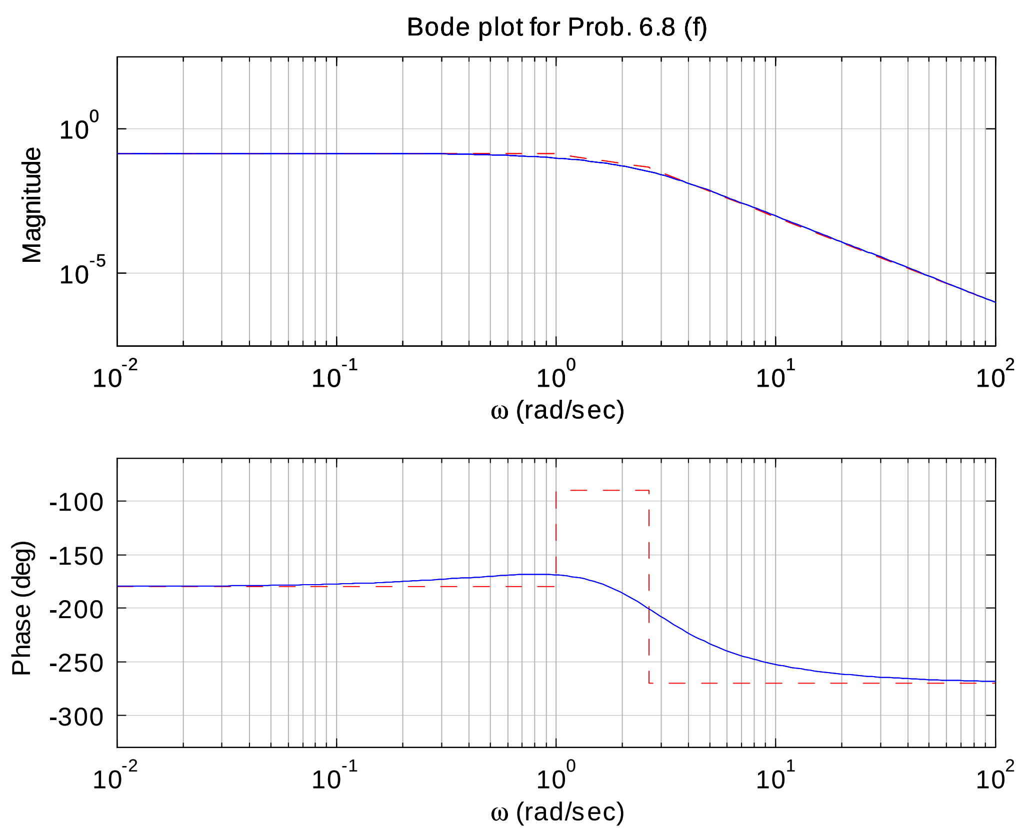
(e) $L(s) = \dfrac{\frac{1}{18}\left(\frac{s}{2}+1\right)}{s(s-1)\left(\frac{s}{6}+1\right)^2}$



(f) $L(s) = \dfrac{\frac{1}{7}}{(s-1)\left[\left(\frac{s}{\sqrt{7}}\right)^2 + \frac{4}{7}s + 1\right]}$

Figure 6.87: Magnitude portion of Bode plot for Problem 9





9. A certain system is represented by the asymptotic Bode diagram shown in Fig. 6.88. Find and sketch the response of this system to a unit step input (assuming zero initial conditions).

   Solution:

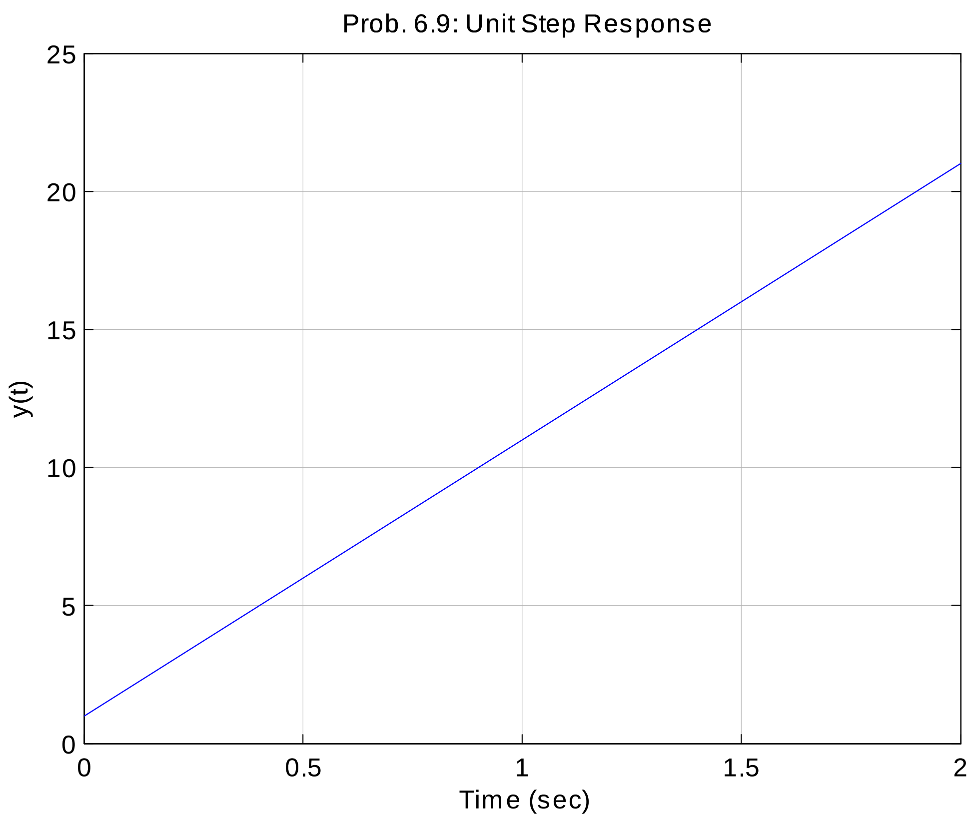
   By inspection, the given asymptotic Bode plot is from

Therefore,

$$G(s) = \frac{10(s/10+1)}{s} = \frac{s+10}{s}$$

The response to a unit step input is :

$$\begin{aligned} Y(s) &= G(s)U(s) \\ &= \frac{s+10}{s} \times \frac{1}{s} = \frac{1}{s} + \frac{10}{s^2} \\ y(t) &= \pounds^{-1}\left[Y(s)\right] \\ &= 1(t) + 10t \ \ (t \geq 0) \end{aligned}$$



10. Prove that a magnitude slope of -1 in a Bode plot corresponds to -20 db per decade or -6 db per octave.

Solution:

The definition of db is db $= 20 \log |G|$ (1)

Assume slope $= \frac{d(\log |G|)}{d(\log \omega)} = -1$ (2)

$$(2) \Longrightarrow \log|G| = -\log\omega + c \text{ (c is a constant.)} \qquad (3)$$

$$(1) \text{ and } (3) \Longrightarrow \text{db} = -20\log\omega + 20c$$

Differentiating this,

$$\frac{d\,(\text{db})}{d\,(\log\omega)} = -20$$

Thus, a magnitude slope of -1 corresponds to -20 db per decade.

Similarly,

$$\frac{d\,(\text{db})}{d\,(\log_2\omega)} = \frac{d\,(\text{db})}{d\left(\frac{\log\omega}{\log 2}\right)} \doteqdot -6$$

Thus, a magnitude slope of -1 corresponds to -6 db per octave.

11. A normalized second-order system with a damping ratio $\zeta = 0.5$ and an additional zero is given by

$$G(s) = \frac{s/a+1}{s^2+s+1}.$$

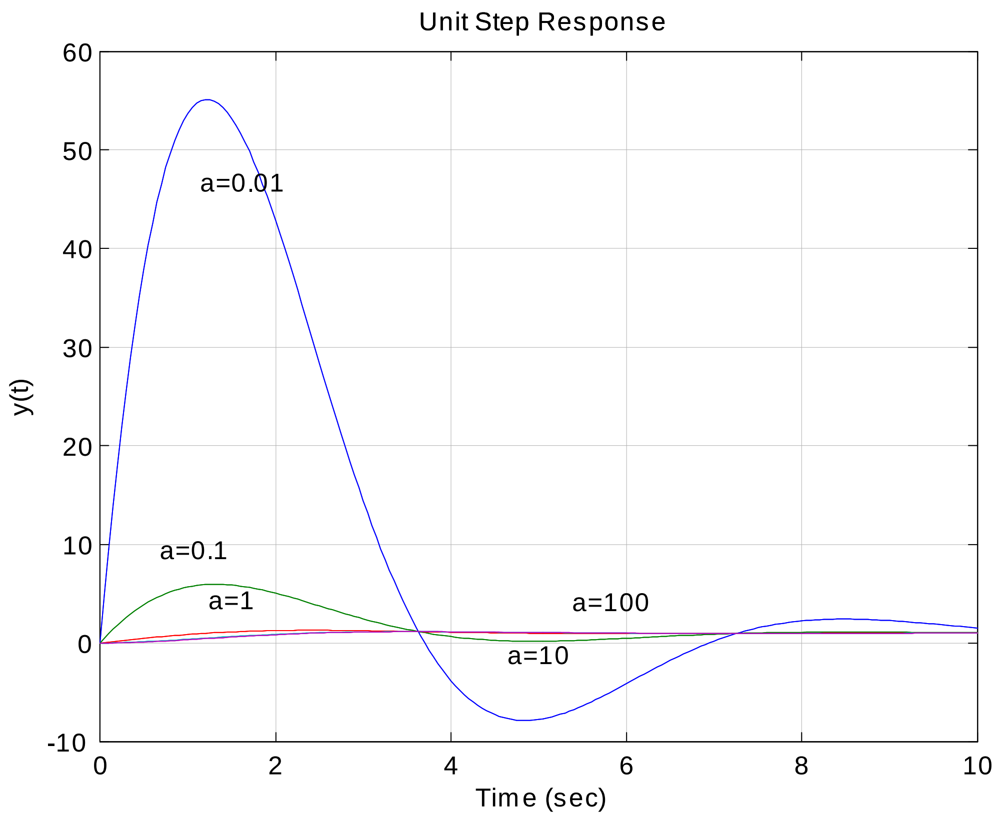
Use MATLAB to compare the M_p from the step response of the system for $a = 0.01, 0.1,$ 1, 10, and 100 with the M_r from the frequency response of each case. Is there a correlation between M_r and M_p?

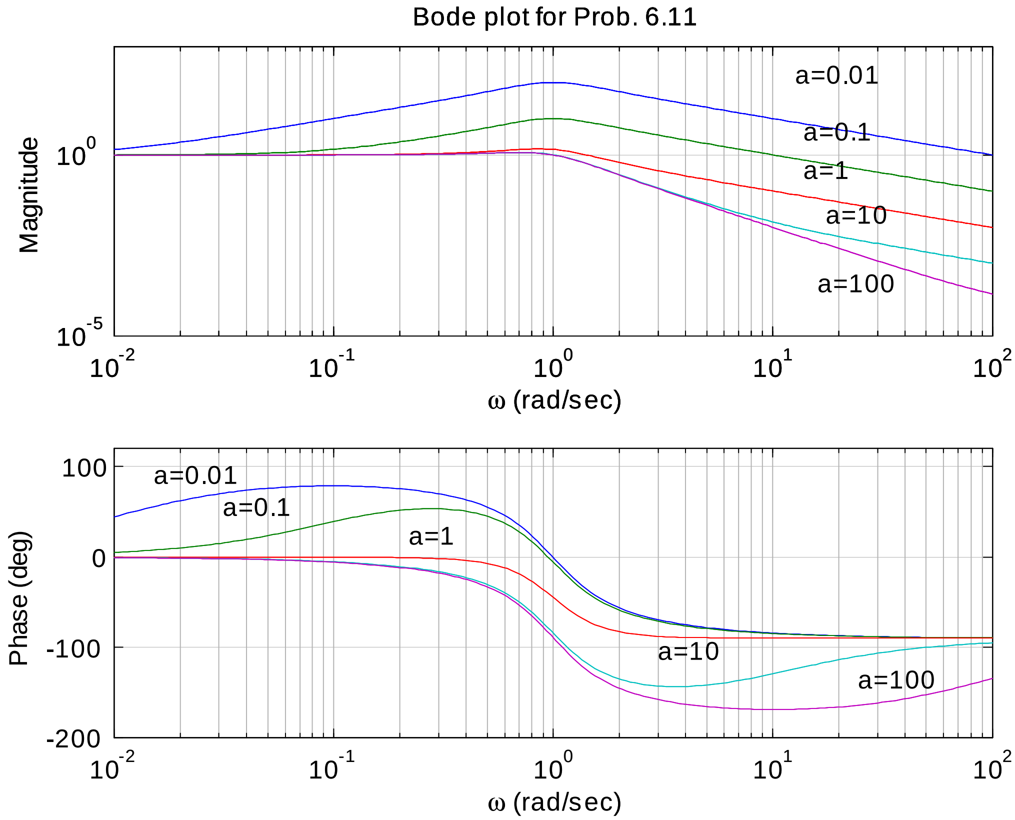
Solution:

α	Resonant peak, M_r	Overshoot, M_p
0.01	98.8	54.1
0.1	9.93	4.94
1	1.46	0.30
10	1.16	0.16
100	1.15	0.16

As α is reduced, the resonant peak in frequency response increases. This leads us to expect extra peak overshoot in transient response. This effect is significant in case of $\alpha = 0.01, 0.1, 1$, while the resonant peak in frequency response is hardly changed in case of $\alpha = 10$. Thus, we do not have considerable change in peak overshoot in transient response for $\alpha \geq 10$.

The response peak in frequency response and the peak overshoot in transient response are correlated.


Unit Step Response
60
50
40
30
20
10
0
-10
0
2
4
6
8
10
y(t)
Time (sec)
a=0.01
a=0.1
a=1
a=100
a=10



12. A normalized second-order system with $\zeta = 0.5$ and an additional pole is given by.

$$G(s) = \frac{1}{[(s/p) + 1](s^2 + s + 1)}$$

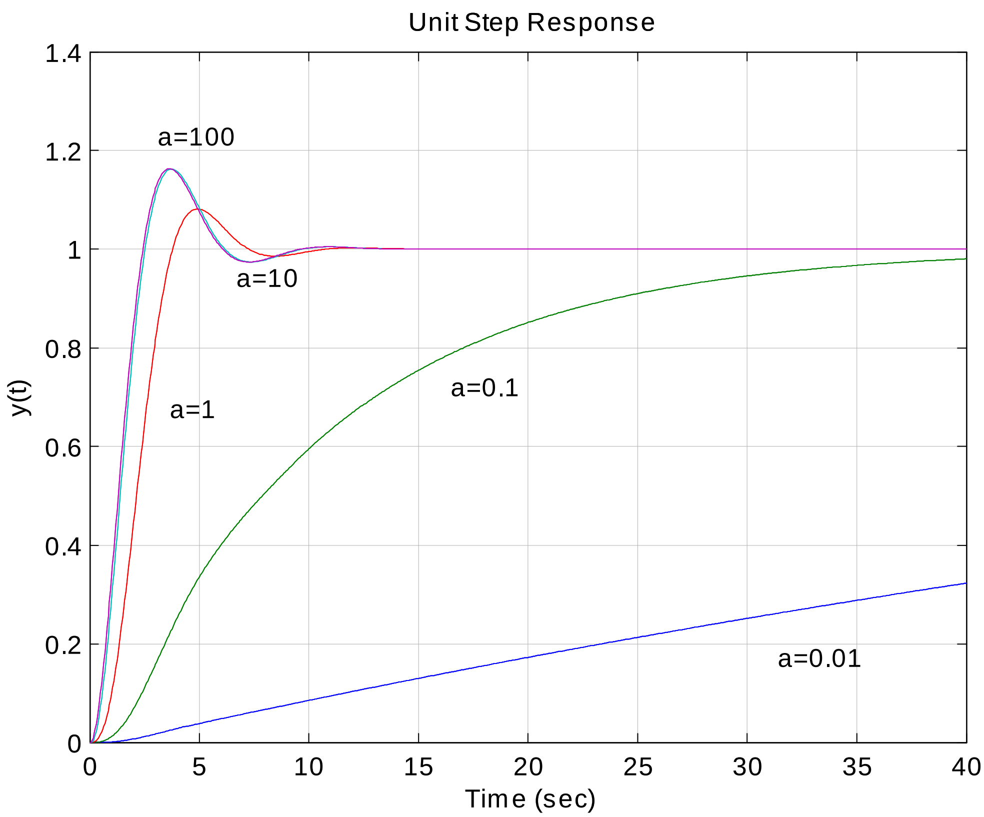
Draw Bode plots with $p = 0.01$, 0.1, 1, 10 and 100. What conclusions can you draw about the effect of an extra pole on the bandwidth compared to the bandwidth for the second-order system with no extra pole?

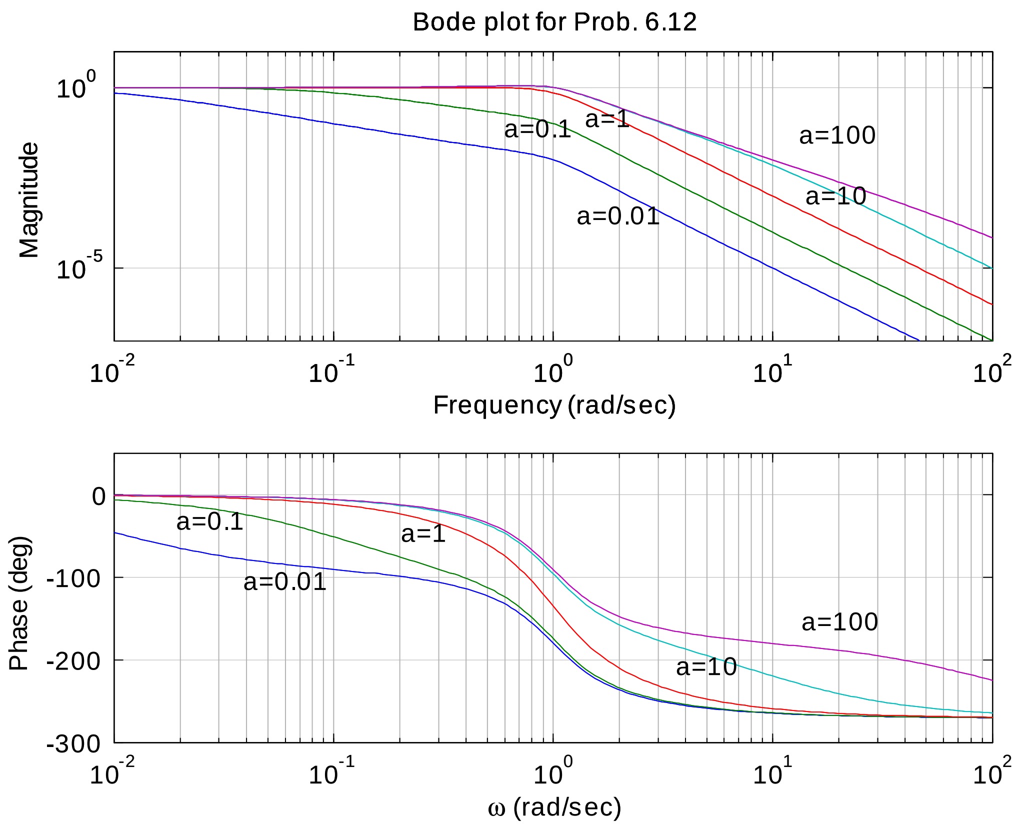
Solution:

p	Additional pole $(-p)$	Bandwidth, ω_{Bw}
0.01	-0.01	0.013
0.1	-0.1	0.11
1	-1	1.0
10	-10	1.5
100	-100	1.7

As p is reduced, the bandwidth decreases. This leads us to expect slower time response and additional rise time. This effect is significant in case of $p = 0.01,\ 0.1, 1$, while the bandwidth is hardly changed in case of $p = 10$. Thus, we do not have considerable change in rise time for $p \geq 10$.

Bandwidth is a measure of the speed of response of a system, such as rise time.





13. For the closed-loop transfer function

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

derive the following expression for the bandwidth ω_{BW} of $T(s)$ in terms of ω_n and ζ:

$$\omega_{BW} = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{2 + 4\zeta^4 - 4\zeta^2}}.$$

Assuming $\omega_n = 1$, plot ω_{BW} for $0 \le \zeta \le 1$.

Solution :

The closed-loop transfer function :

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

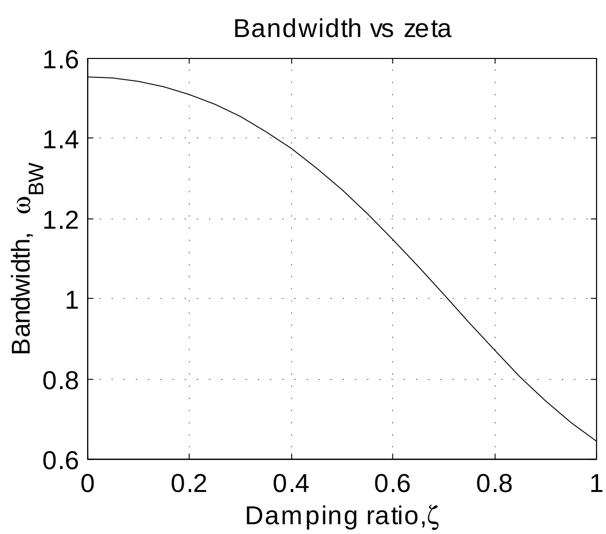
$s = j\omega$,

$$\begin{aligned}
T(j\omega) &= \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2+2\zeta\left(\frac{\omega}{\omega_n}\right)j} \\
|T(j\omega)| &= \{T(j\omega)T^*(j\omega)\}^{\frac{1}{2}} = \left[\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2\zeta^2+\left\{2\zeta\left(\frac{\omega}{\omega_n}\right)\right\}^2}\right]^{\frac{1}{2}}
\end{aligned}$$

Let $x = \dfrac{\omega_{BW}}{\omega_n}$:

$$\begin{aligned}
|T(j\omega)|_{\omega=\omega_{BW}} &= \left[\frac{1}{(1-x^2)^2+(2\zeta x)^2}\right]^{\frac{1}{2}} = 0.707 = \frac{1}{\sqrt{2}} \\
&\Longrightarrow \quad x^4+(4\zeta^2-2)x^2-1=0 \\
&\Longrightarrow \quad x = \frac{\omega_{BW}}{\omega_n} = \left[(1-2\zeta^2)+\sqrt{(1-2\zeta^2)^2+1}\right]^{\frac{1}{2}} \\
&\Longrightarrow \quad \omega_{BW} = \omega_n\sqrt{1-2\zeta^2+\sqrt{2+4\zeta^4-4\zeta^2}}
\end{aligned}$$

ζ	$x\left(=\dfrac{\omega_{BW}}{\omega_n}\right)$	ω_{BW}
0.2	1.51	$1.51\omega_n$
0.5	1.27	$1.27\omega_n$
0.8	0.87	$0.87\omega_n$



14. Consider the system whose transfer function is

$$G(s) = \frac{A_0 \omega_0 s}{Qs^2 + \omega_0 s + \omega_0^2 Q}.$$

This is a model of a tuned circuit with *quality factor* Q. (a) Compute the magnitude and phase of the transfer function analytically, and plot them for $Q = 0.5$, 1, 2, and 5 as a function of the normalized frequency ω/ω_0. (b) Define the bandwidth as the distance between the frequencies on either side of ω_0 where the magnitude drops to 3 db below its value at ω_0 and show that the bandwidth is given by

$$BW = \frac{1}{2\pi}\left(\frac{\omega_0}{Q}\right).$$

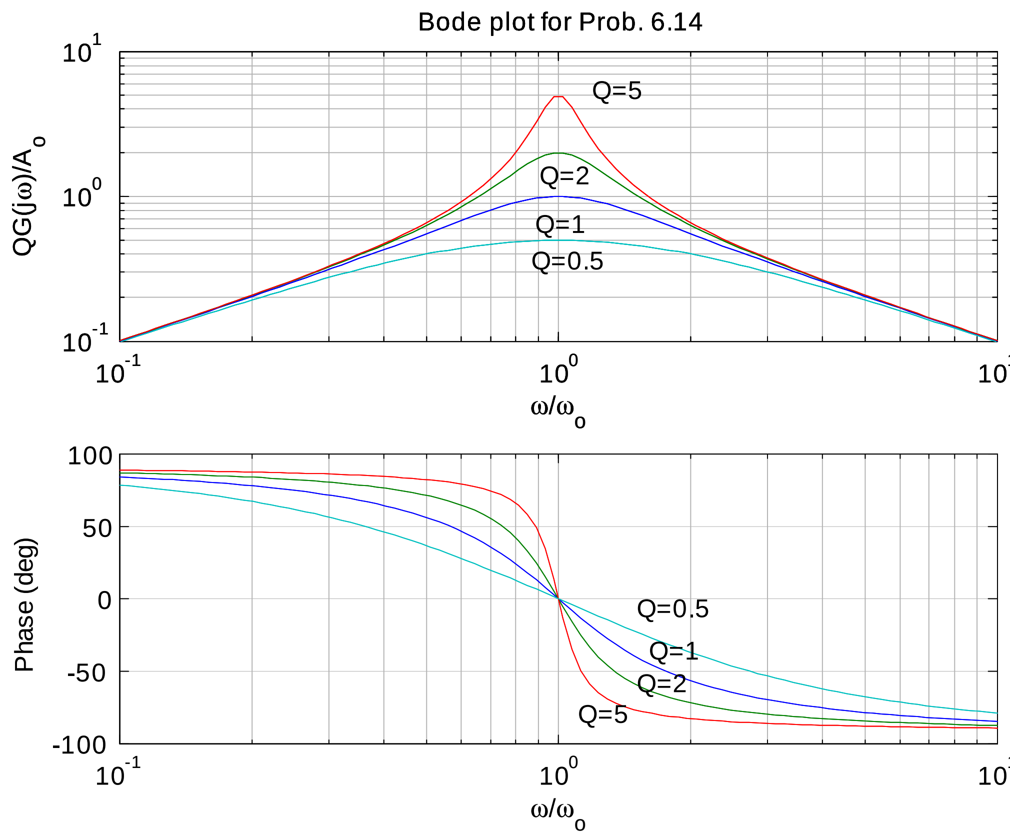
(c) What is the relation between Q and ζ?

Solution :

(a) Let $s = j\omega$,

$$\begin{aligned}
G(j\omega) &= \frac{A_o \omega_o j\omega}{-Q\omega^2 + \omega_o j\omega + \omega_o^2 Q} \\
&= \frac{A_o}{1 + \frac{Q\omega_o^2 - Q\omega^2}{j\omega_o \omega}} \\
|G(j\omega)| &= \frac{A_o}{\sqrt{1 + Q^2\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)^2}} \\
\phi &= -\tan^{-1}\left(\frac{\omega}{\omega_o} - \frac{\omega_o}{\omega}\right)
\end{aligned}$$

The normalized magnitude $\left(\frac{QG(j\omega)}{A_o}\right)$ and phase are plotted against normalized frequency $\left(\frac{\omega}{\omega_o}\right)$ for different values of Q.



(b) There is symmetry around ω_o. For every frequency $\omega_1 < \omega_o$, there exists a frequency $\omega_2 > \omega_o$ which has the same magnitude

$$|G(j\omega_1)| = |G(j\omega_2)|$$

We have that,

$$\frac{\omega_1}{\omega_o} - \frac{\omega_o}{\omega_1} = -\left(\frac{\omega_2}{\omega_o} - \frac{\omega_o}{\omega_2}\right)$$

which implies $\omega_o^2 = \omega_1\omega_2$. Let $\omega_1 < \omega_o$ and $\omega_2 > \omega_o$ be the two frequencies on either side of ω_o for which the gain drops by 3db from its value of A_o at ω_o.

$$BW = \frac{\omega_2 - \omega_1}{2\pi} = \frac{1}{2\pi}\left(\omega_2 - \frac{\omega_o^2}{\omega_2}\right) \qquad (1)$$

Now ω_2 is found from,

$$\left|\frac{G(j\omega)}{A_o}\right| = \frac{1}{\sqrt{2}}$$

or

$$1 + Q^2 \left(\frac{\omega_2}{\omega_o} - \frac{\omega_o}{\omega_2} \right)^2 = 2$$

which yields

$$Q \left(\frac{\omega_2}{\omega_o} - \frac{\omega_o}{\omega_2} \right) = 1 = \frac{Q}{\omega_o} \left(\omega_2 - \frac{\omega_o^2}{\omega_2} \right) \qquad (2)$$

Comparing (1) and (2) we find,

$$BW = \frac{1}{\sqrt{2}} \left(\frac{Q}{\omega_o} \right)$$

(c)

$$\begin{aligned} G(s) &= \frac{A_0 \omega_0 s}{Qs^2 + \omega_0 s + \omega_0^2 Q} \\ &= \frac{A_0 \omega_0 s}{Q \left(s^2 + \frac{\omega_0}{Q} s + \omega_0^2 \right)} \\ &= \frac{A_0 \omega_0 s}{Q \left(s^2 + 2\zeta \omega_0 s + \omega_0^2 \right)} \end{aligned}$$

Therefore

$$\frac{1}{Q} = 2\zeta$$

15. A DC voltmeter schematic is shown in Fig. 6.88. The pointer is damped so that its maximum overshoot to a step input is 10%.

(a) What is the undamped natural frequency of the system?

(b) What is the damped natural frequency of the system?

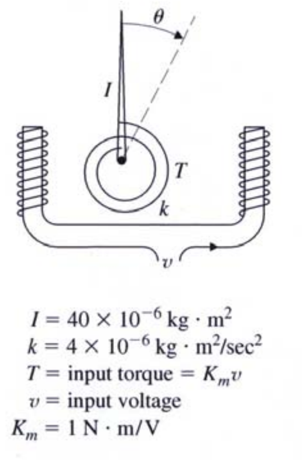
(c) Plot the frequency response using MATLAB to determine what input frequency will produce the largest magnitude output?

(d) Suppose this meter is now used to measure a 1-V AC input with a frequency of 2 rad/sec. What amplitude will the meter indicate after initial transients have died out? What is the phase lag of the output with respect to the input? Use a Bode plot analysis to answer these questions. Use the lsim command in MATLAB to verify your answer in part (d).

Solution :

The equation of motion : $I\ddot{\theta} + b\dot{\theta} + k\theta = T = K_m v$, where b is a damping coefficient.

Figure 6.88: Voltmeter schematic



Taking the Laplace transform with zero initial conditions:

$$\Theta(s) = \frac{K_m}{Is^2 + bs + k} V(s) = \frac{\frac{K_m}{I}}{s^2 + 2\zeta\omega_n s + \omega_n^2} V(s)$$

Use $I = 40 \times 10^{-6}$Kg·m^2, $k = 4 \times 10^{-6}$Kg·m^2/s^2, K_m=4×10^{-6}N·m/v

(a) Undamped natural frequency:

$$\omega_n^2 = \frac{k}{I} \Longrightarrow \omega_n = \sqrt{\frac{k}{I}} = 0.316 \text{ rad/sec}$$

(b) Since $M_p = 0.1$ and $M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$,

$$\log 0.1 = \frac{-\pi\zeta}{\sqrt{1-\zeta^2}} \Longrightarrow \zeta = 0.5911 \text{ } (\simeq 0.6 \text{ from Figure 2.44})$$

Damped natural frequency:

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 0.255 \text{ rad/sec}$$

(c)

$$
\begin{aligned}
T(j\omega) &= \frac{\Theta(j\omega)}{V(j\omega)} = \frac{K_m/I}{(j\omega)^2 + 2\zeta\omega_n j\omega + \omega_n^2} \\
|T(j\omega)| &= \frac{K_m/I}{\left[(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2\right]^{\frac{1}{2}}} \\
\frac{d\,|T(j\omega)|}{d\omega} &= \left(\frac{K_m}{I}\right) \frac{2\omega\left\{\omega_n^2 - \omega^2 - 2\zeta^2\omega_n^2\right\}}{\left[(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2\right]^{\frac{3}{2}}}
\end{aligned}
$$

When $\frac{d|T(j\omega)|}{d\omega} = 0$,

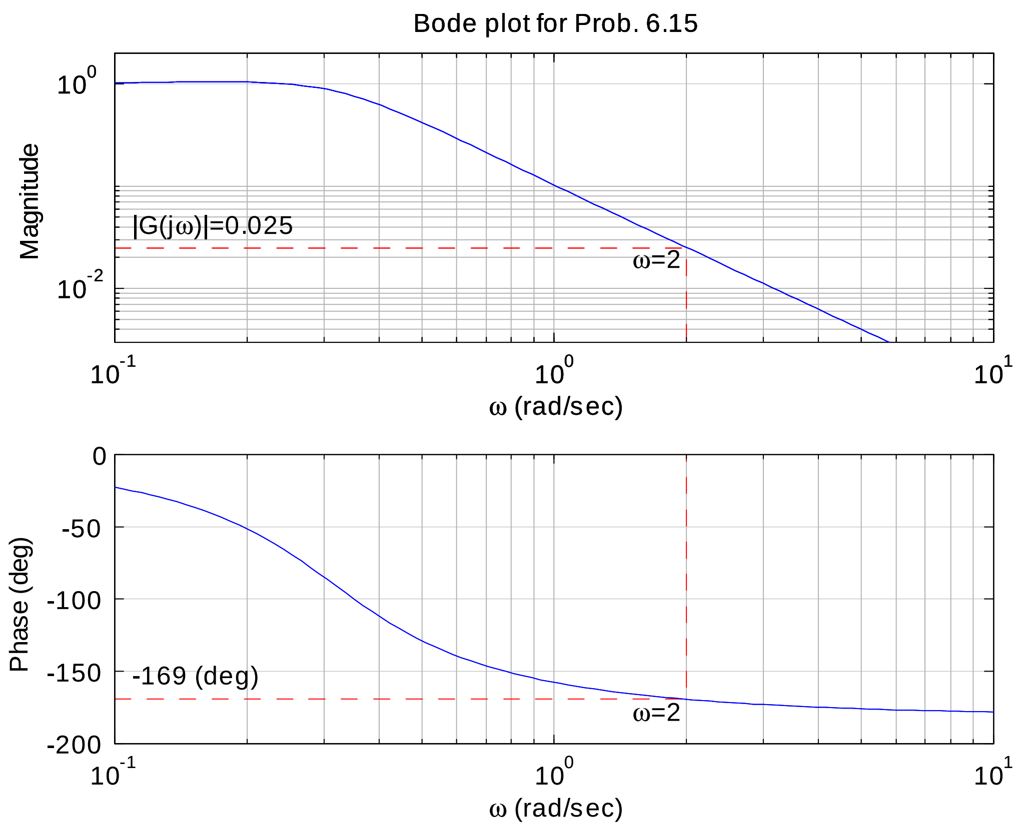
$$
\begin{aligned}
\omega^2 - (1 - 2\zeta^2)\omega_n^2 &= 0 \\
\omega &= 0.549\omega_n = 0.173
\end{aligned}
$$

Alternatively, the peak frequency can be found from the Bode plot:

$$\omega = 0.173 \text{ rad/sec}$$

(d) With $\omega = 2$ rad/sec from the Bode plot:

$$
\begin{aligned}
\text{Amplitude} &= 0.0252 \text{ rad} \\
\text{Phase} &= -169.1^\circ
\end{aligned}
$$


Bode plot for Prob. 6.15
Magnitude
10^0
10^-2
|G(jω)|=0.025
ω=2
10^-1
10^0
10^1
ω (rad/sec)
Phase (deg)
0
-50
-100
-150
-200
-169 (deg)
ω=2
10^-1
10^0
10^1
ω (rad/sec)

Problems and Solutions for Section 6.2

16. Determine the range of K for which the closed-loop systems (see Fig. 6.18) are stable for each of the cases below by making a Bode plot for $K = 1$ and imagining the magnitude plot sliding up or down until instability results. Verify your answers using a very rough sketch of a root-locus plot.

    (a) $KG(s) = \dfrac{K(s+2)}{s+20}$

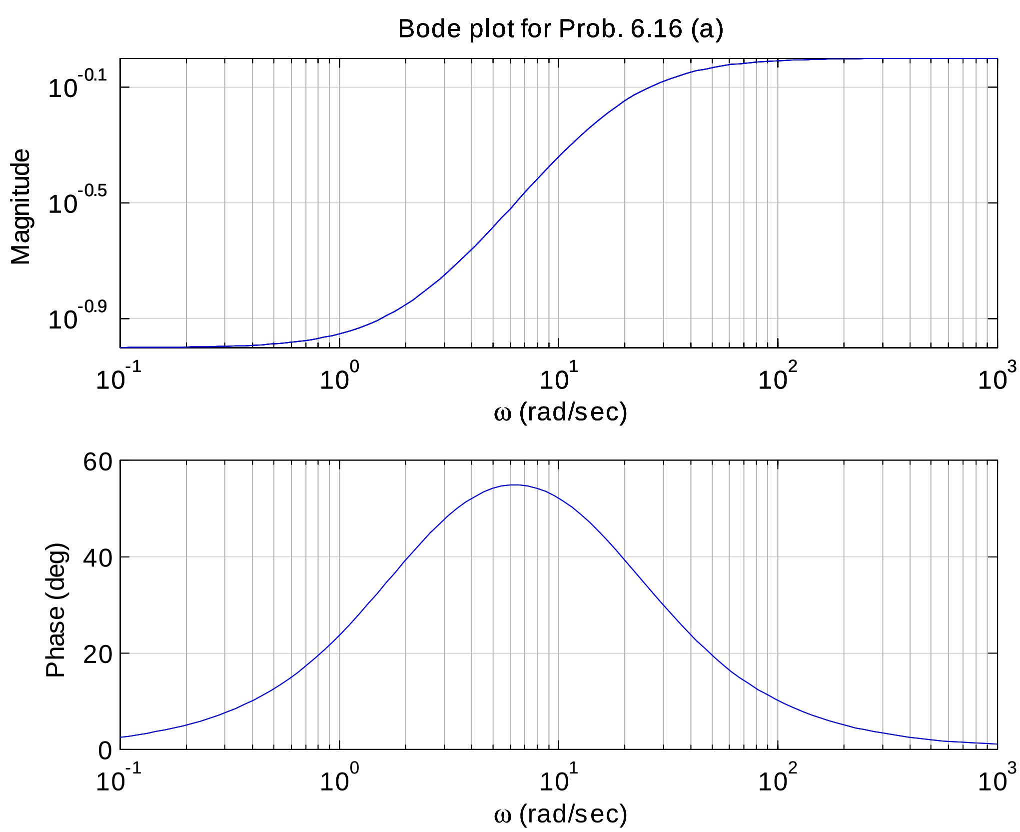
    (b) $KG(s) = \dfrac{K}{(s+10)(s+1)^2}$

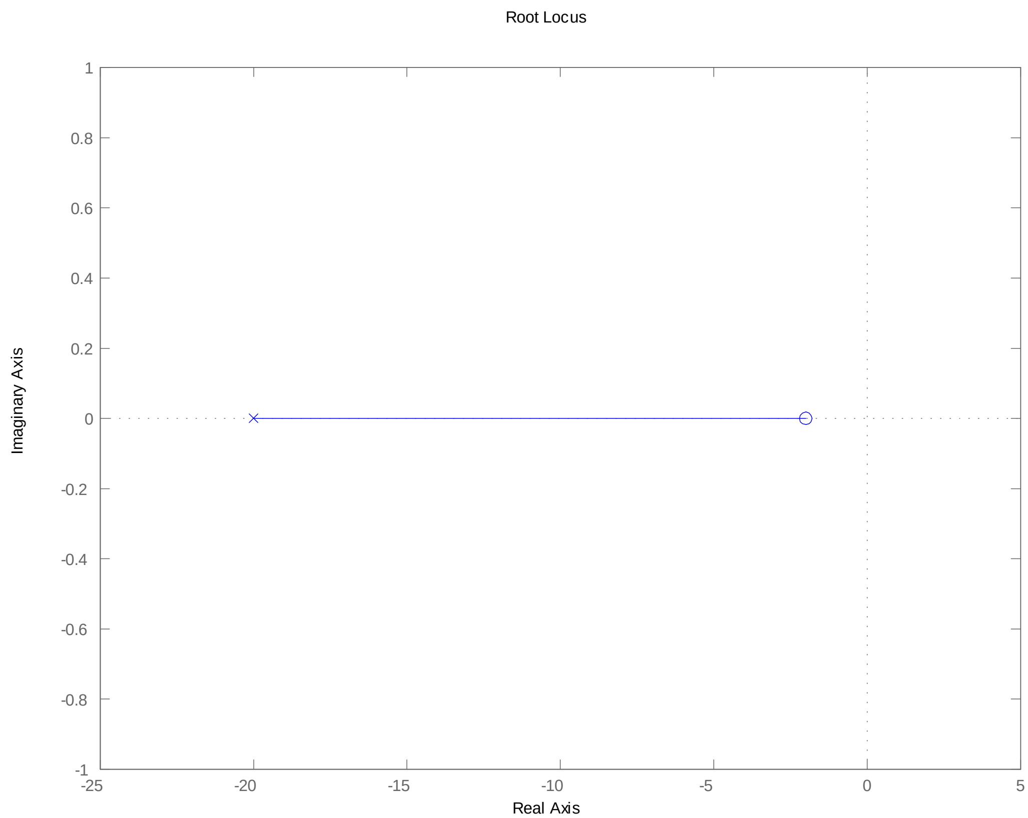
    (c) $KG(s) = \dfrac{K(s+10)(s+1)}{(s+100)(s+5)^3}$

    Solution :

    (a)

$$KG(s) = \frac{K(s+2)}{s+20} = \frac{K}{10}\frac{\left(\frac{s}{2}+1\right)}{\left(\frac{s}{20}+1\right)}$$

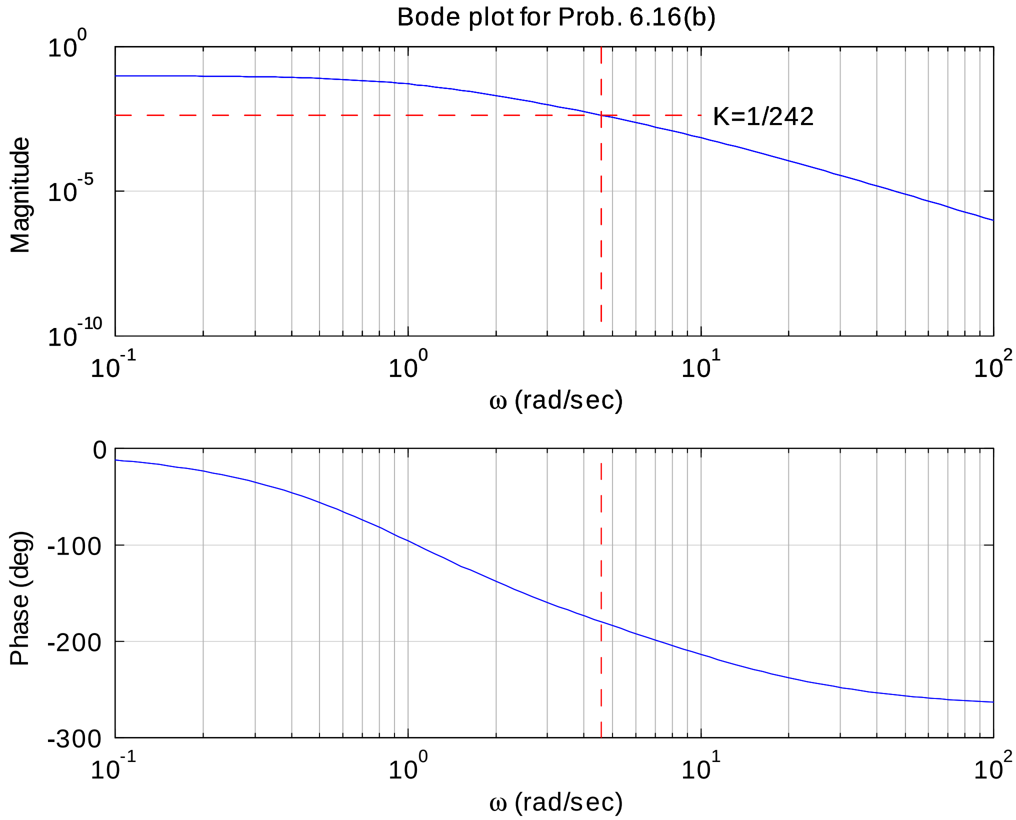




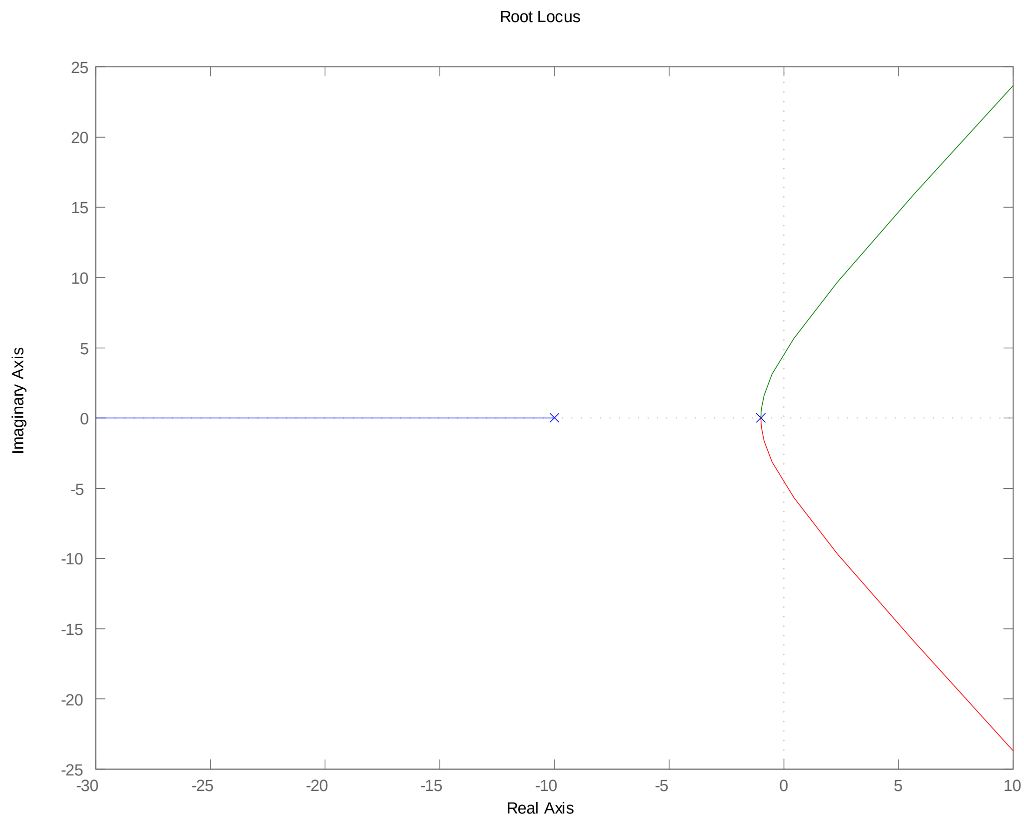
The gain can be raised or lowered on the Bode gain plot and the phase will never be less than -180^o, so the system is stable for any $K > 0$.

(b)

$$KG(s) = \frac{K}{(s+10)(s+1)^2} = \frac{K}{10} \frac{1}{\left(\frac{s}{10}+1\right)(s+1)^2}$$

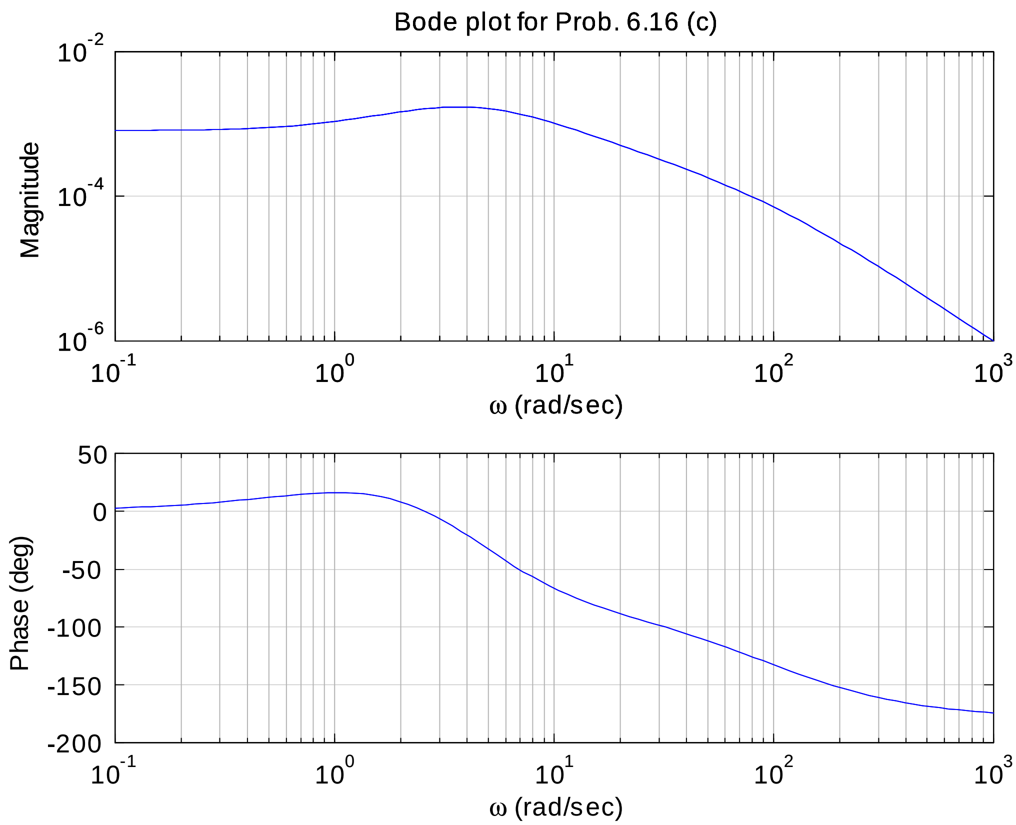


The bode plots show that the gain, K, would equal 242 when the phase crosses 180^o. So, $K < 242$ is Stable and $K > 242$ is Unstable. The phase crosses the 180^o at $\omega = 4.58$ rad/sec. The root locus below verifies the situation.



(c)

$$KG(s) = \frac{K(s+10)(s+1)}{(s+100)(s+5)^3} = \frac{K}{1250}\frac{\left(\frac{s}{10}+1\right)(s+1)}{\left(\frac{s}{100}+1\right)\left(\frac{s}{5}+1\right)^3}$$



The phase never crosses -180^o so it is stable for all $K > 0$, as confirmed by the root locus.

17. Determine the range of K for which each of the following systems is stable by making a Bode plot for $K = 1$ and imagining the magnitude plot sliding up or down until instability results. Verify your answers using a very rough sketch of a root-locus plot.

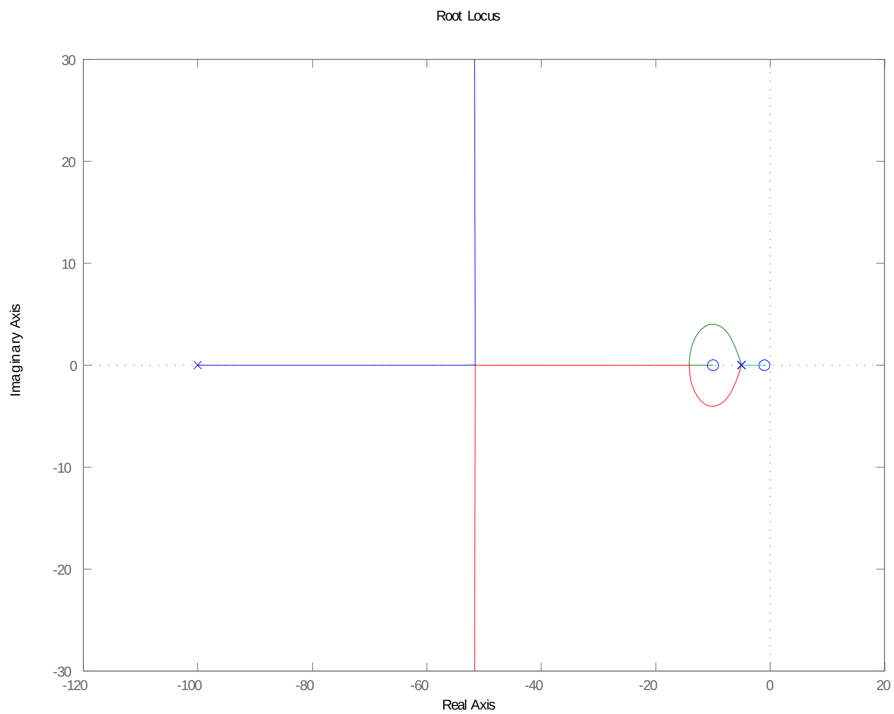
(a) $KG(s) = \dfrac{K(s+1)}{s(s+5)}$

(b) $KG(s) = \dfrac{K(s+1)}{s^2(s+10)}$

(c) $KG(s) = \dfrac{K}{(s+2)(s^2+9)}$

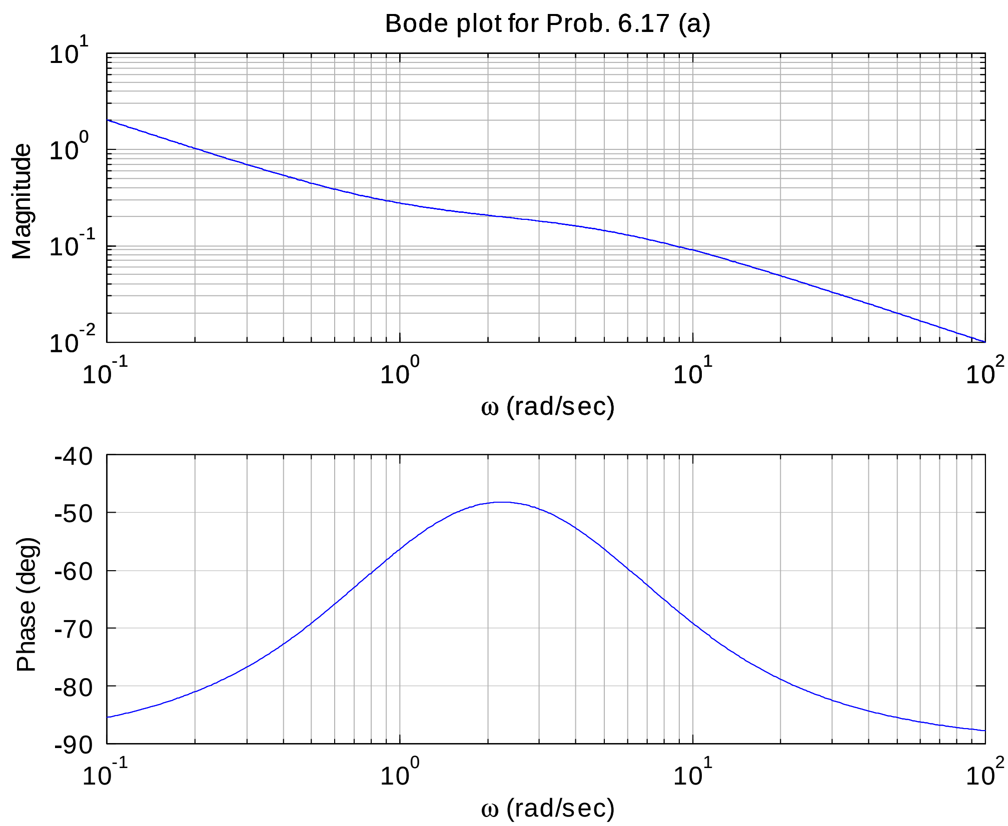
(d) $KG(s) = \dfrac{K(s+1)^2}{s^3(s+10)}$

Solution :

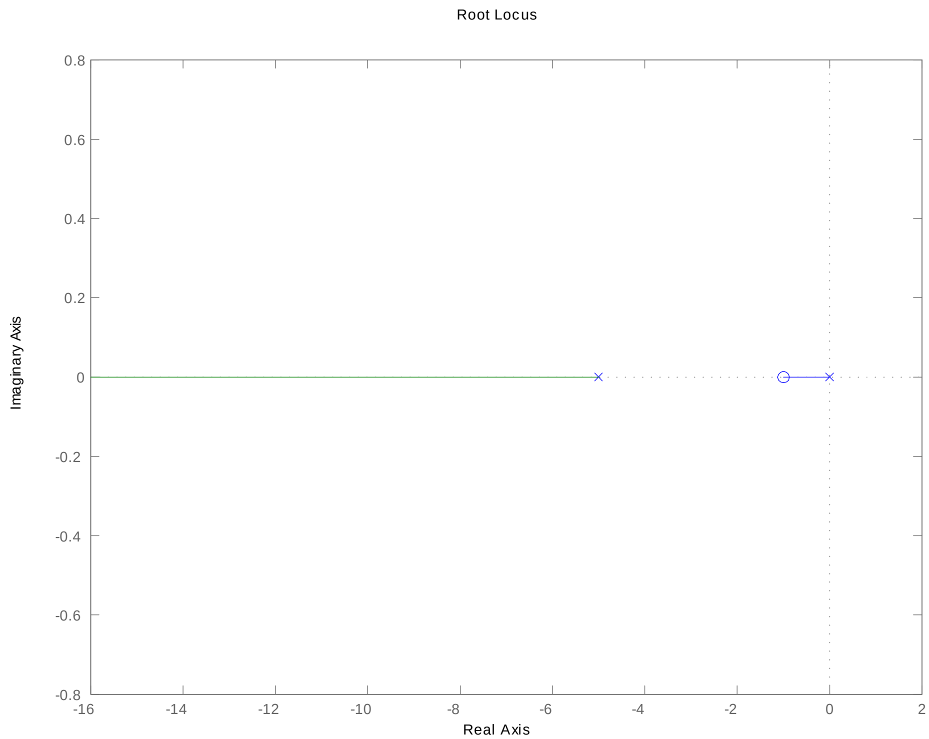

Root Locus
30
20
10
0
-10
-20
-30
-120
-100
-80
-60
-40
-20
0
20
Real Axis
Imaginary Axis

(a)

$$KG(s) = \frac{K(s+1)}{s(s+5)} = \frac{K}{5}\frac{(s+1)}{s\left(\frac{s}{5}+1\right)}$$

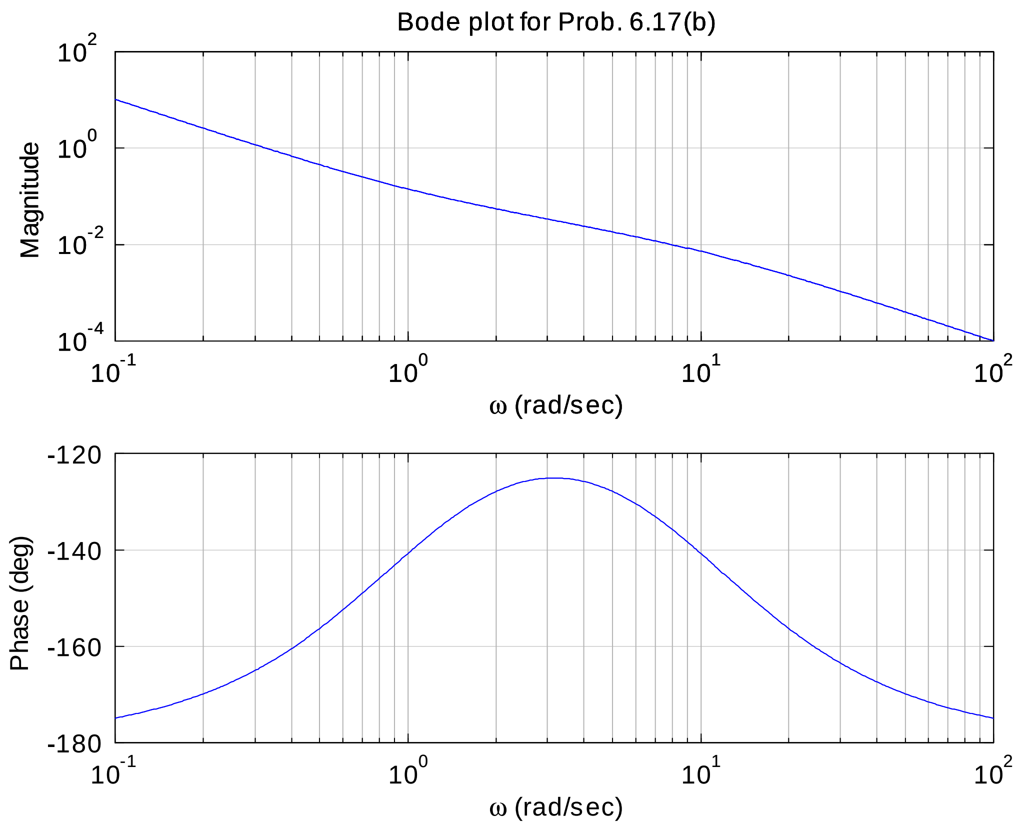


The phase never crosses -180^o so it is stable for all $K > 0$, as confirmed by the root locus.



(b)

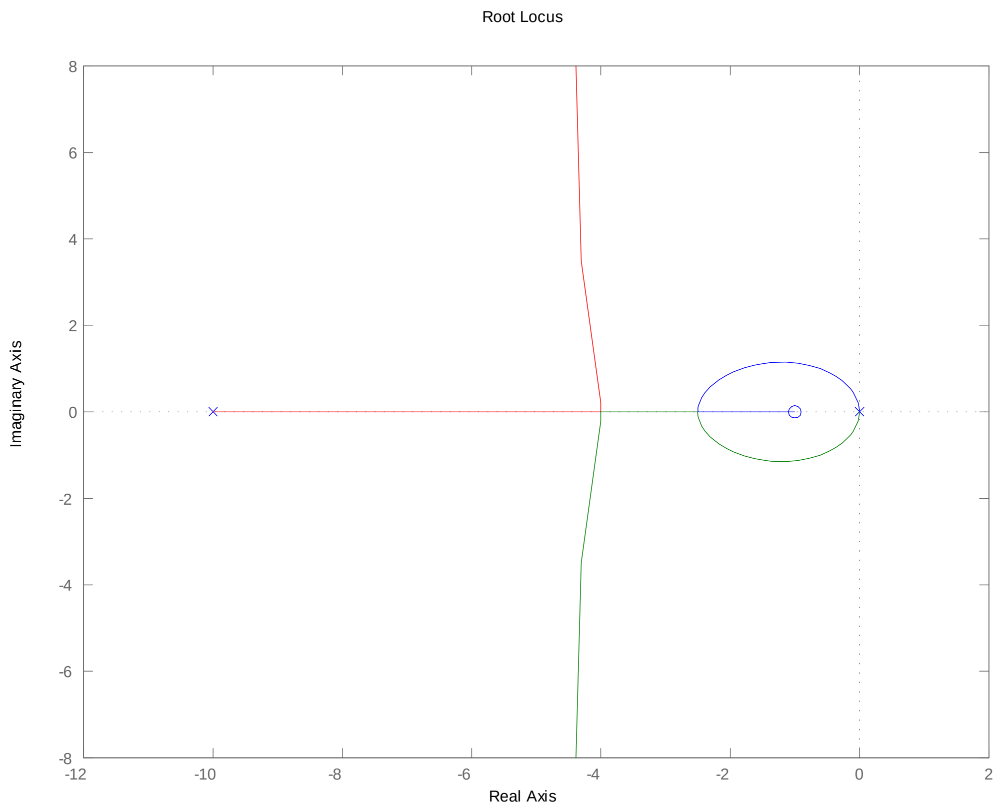
$$KG(s) = \frac{K(s+1)}{s^2(s+10)} = \frac{K}{10}\frac{s+1}{s^2\left(\frac{s}{10}+1\right)}$$

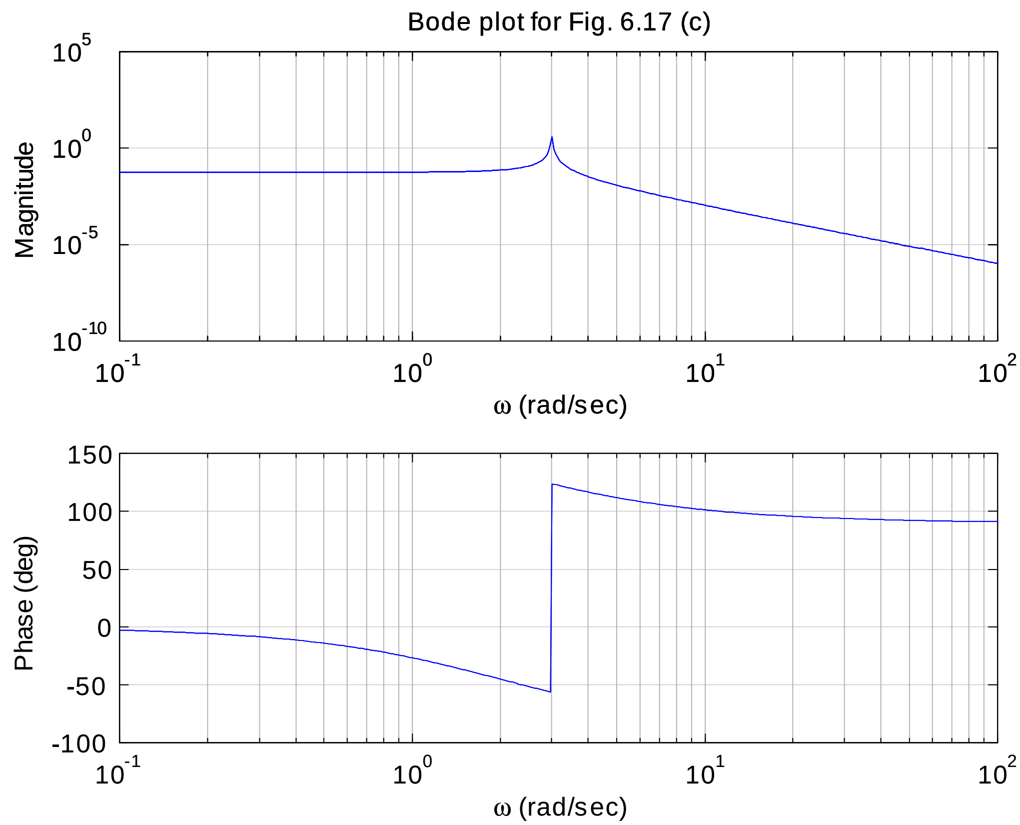


The phase never crosses -180^o so it is stable for all $K > 0$, as confirmed by the root locus. The system is stable for any $K > 0$.

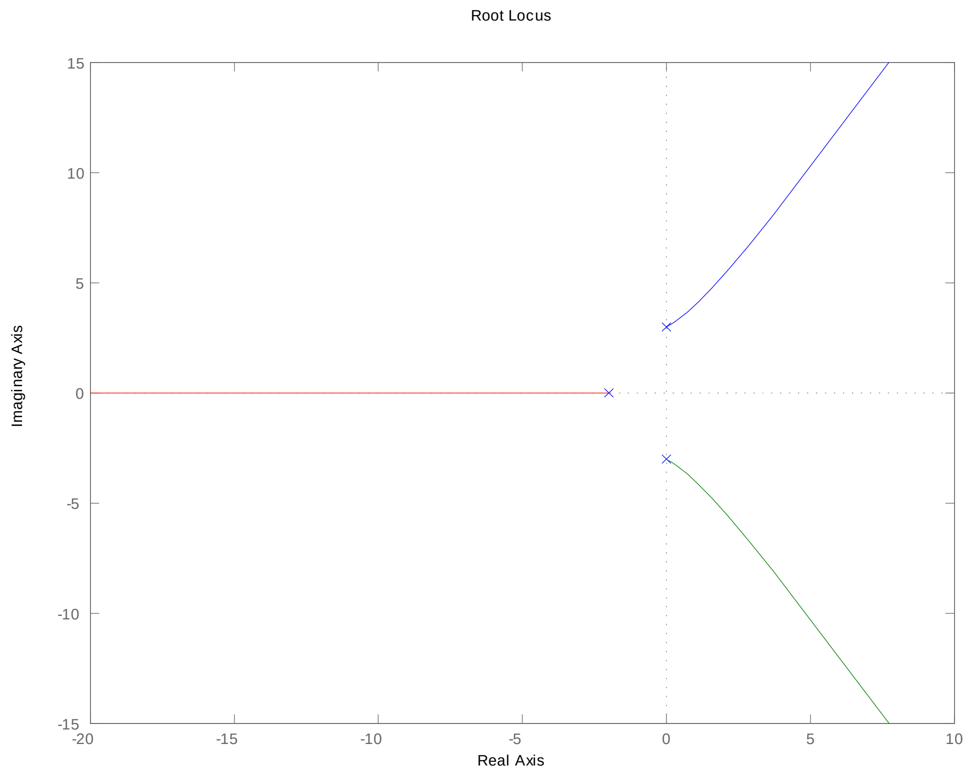
(c)

$$KG(s) = \frac{K}{(s+2)(s^2+9)} = \frac{K}{18} \frac{1}{\left(\frac{s}{2}+1\right)\left(\frac{s^2}{9}+1\right)}$$


Root Locus
Imaginary Axis
Real Axis
8
6
4
2
0
-2
-4
-6
-8
-12
-10
-8
-6
-4
-2
0
2

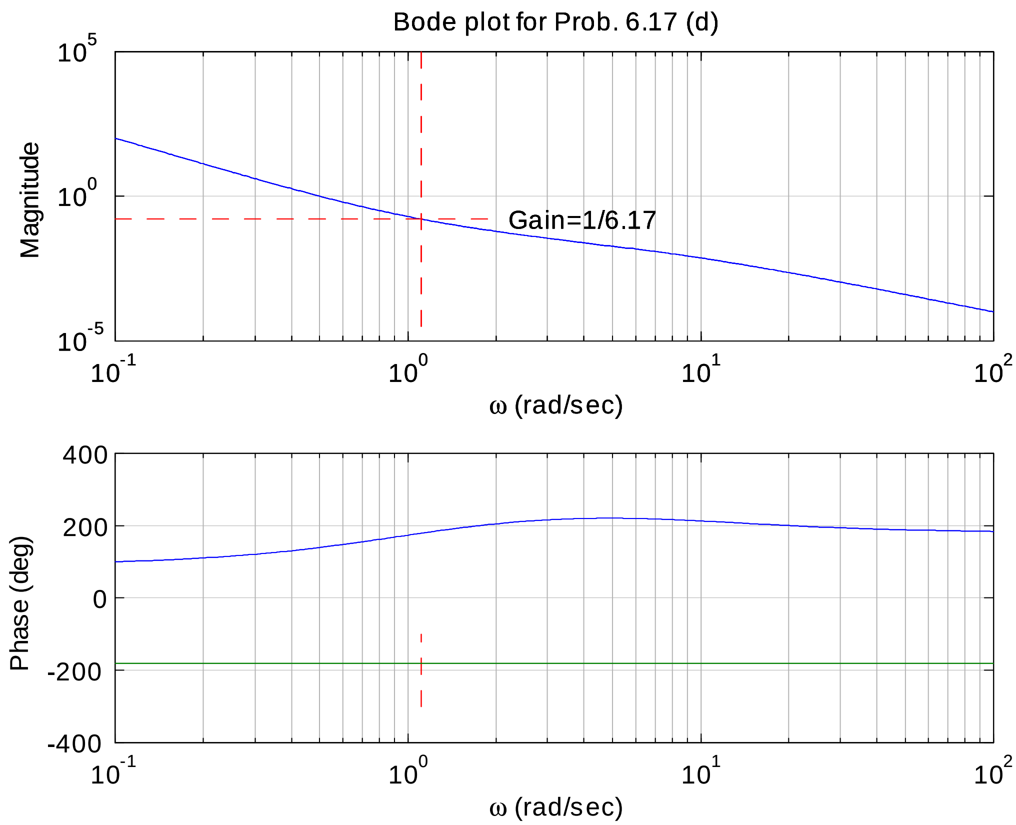


The bode is difficult to read, but the phase really dropped by 180^o at the resonance. (It appears to rise because of the quadrant action in Matlab) Furthermore, there is an infinite magnitude peak of the gain at the resonance because there is zero damping. That means that no matter how much the gain is lowered, the gain will never cross magnitude one when the phase is -180^o. So it can not be made stable for any K. This is much clearer and easier to see in the root locus below.

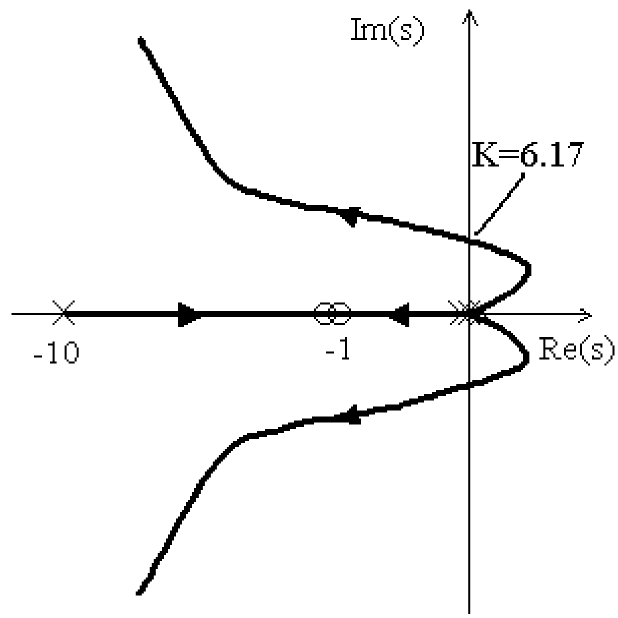


(d)

$$KG(s) = \frac{K(s+1)^2}{s^3(s+10)} = \frac{K}{10}\frac{(s+1)^2}{s^3\left(\frac{s}{10}+1\right)}$$



This is not the normal situation discussed in Section 6.2 where increasing gain leads to instability. Here we see from the root locus that K must be ≥ 6.17 in order for stability. Note that the phase is increasing with frequency here rather than the normal decrease we saw on the previous problems. It's also interesting to note that the margin command in Matlab indicates instability! (which is false.) This problem illustrates that a sketch of the root locus really helps understand what's going on... and that you can't always trust Matlab, or at least that you need good understanding to interpret what Matlab is telling you.



$K < 6.25$: Unstable

$K > 6.25$: Stable

$\omega = 1.12$ rad/sec for $K = 6.17$.

Problems and Solutions for Section 6.3

18. (a) Sketch the Nyquist plot for an open-loop system with transfer function $1/s^2$; that is, sketch

$$\frac{1}{s^2}\,|_{s=C_1},$$

where C_1 is a contour enclosing the entire RHP, as shown in Fig. 6.17. (*Hint*: Assume C_1 takes a small detour around the poles at $s = 0$, as shown in Fig. 6.27.)

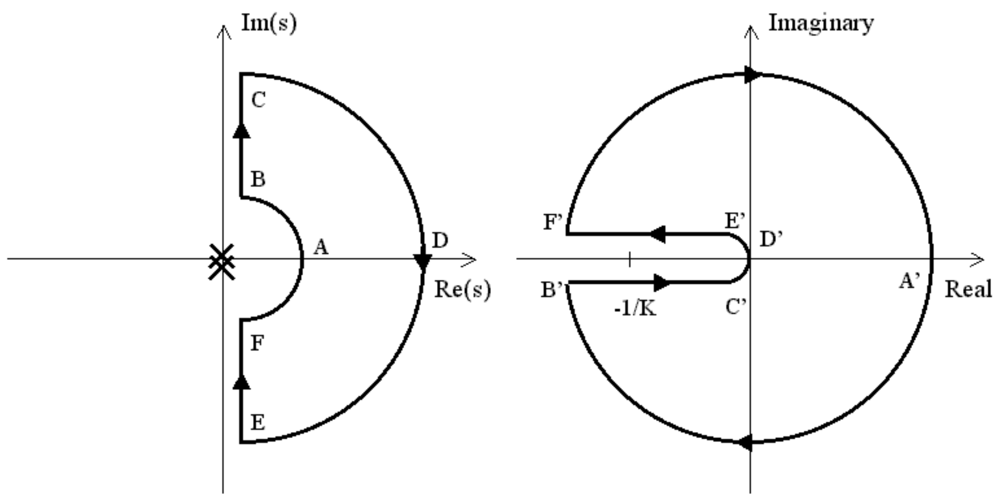
(b) Repeat part (a) for an open-loop system whose transfer function is $G(s) = 1/(s^2 + \omega_0^2)$.

Solution :

(a)

$$G(s) = \frac{1}{s^2}$$

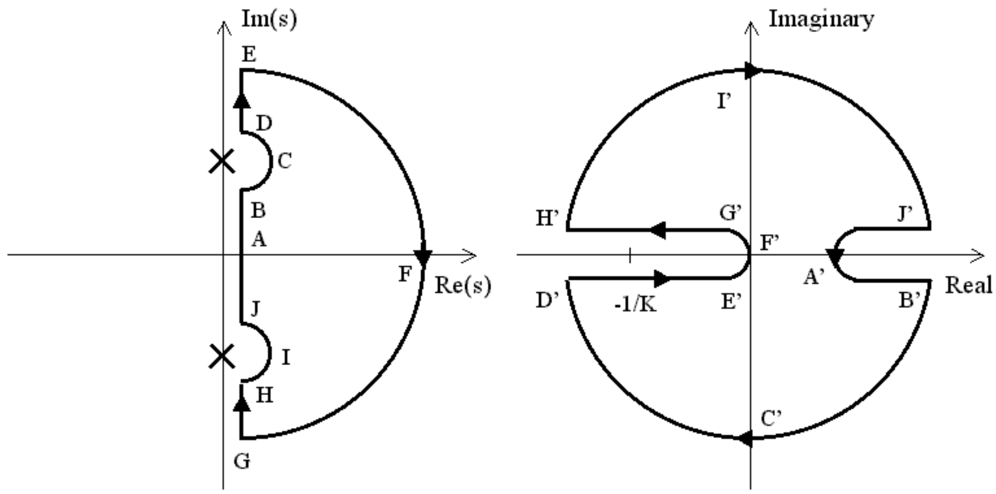
Note that the portion of the Nyquist diagram on the right side below that corresponds to the bode plot is from B' to C'. The large loop from F' to A' to B' arises from the detour around the 2 poles at the origin.



(b)

$$G(s) = \frac{1}{s^2 + \omega_0^2}$$

Note here that the portion of the Nyquist plot coming directly from a Bode plot is the portion from A' to E'. That portion includes a 180^o arc that arose because of the detour around the pole on the imaginary axis.



19. Sketch the Nyquist plot based on the Bode plots for each of the following systems, then compare your result with that obtained using the MATLAB command `nyquist`:

(a) $KG(s) = \dfrac{K(s+2)}{s+10}$

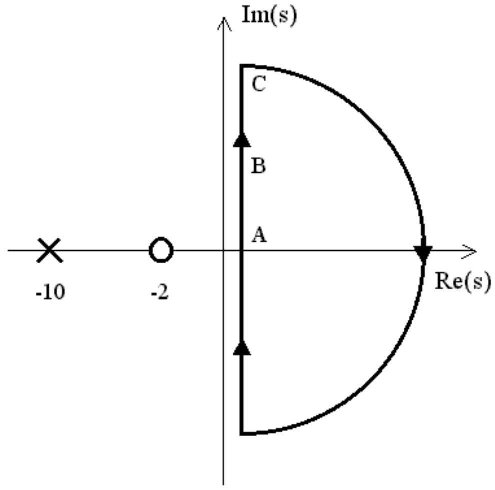
(b) $KG(s) = \dfrac{K}{(s+10)(s+2)^2}$

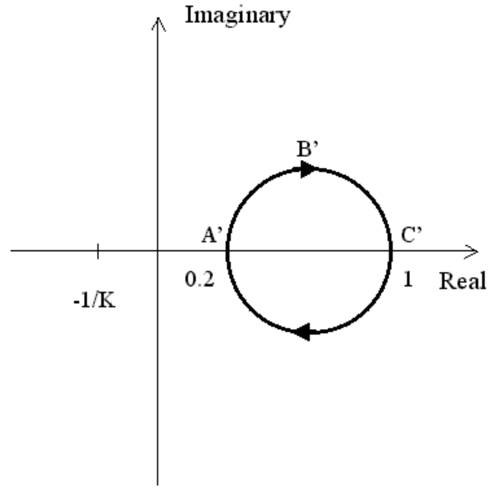
(c) $KG(s) = \dfrac{K(s+10)(s+1)}{(s+100)(s+2)^3}$

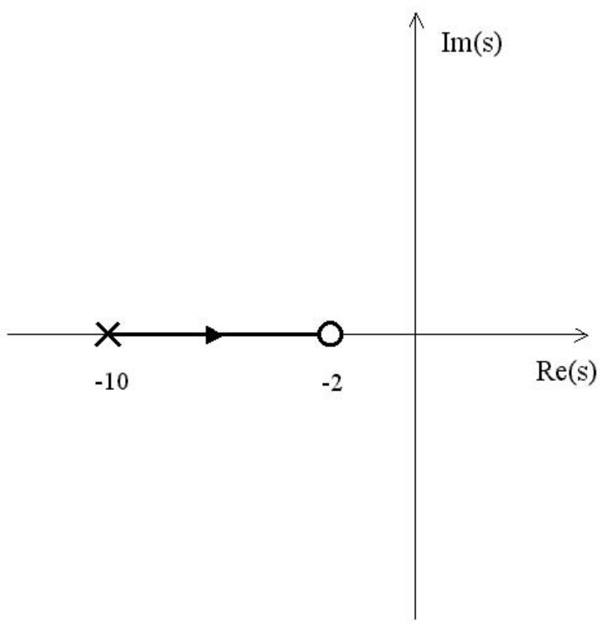
(d) Using your plots, estimate the range of K for which each system is stable, and qualitatively verify your result using a rough sketch of a root-locus plot.

Solution :

(a)



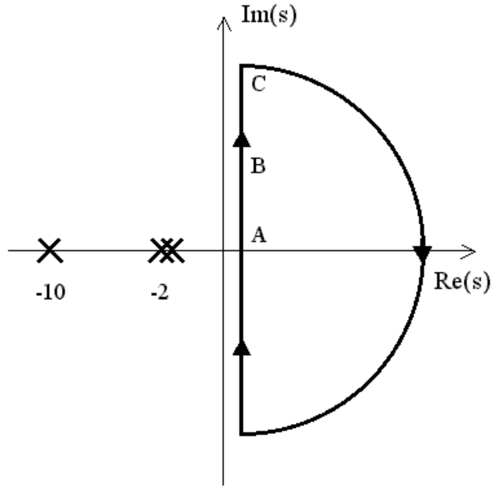


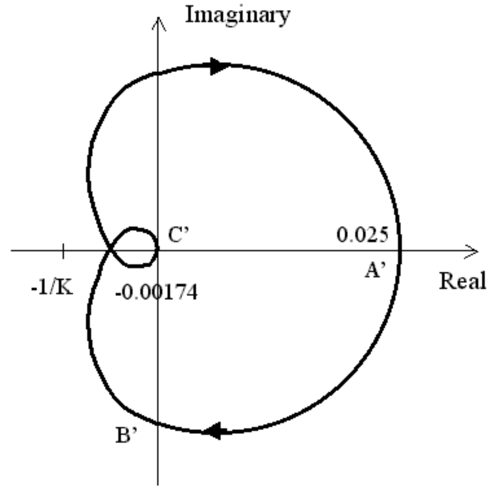


$N = 0, P = 0 \Longrightarrow Z = N + P = 0$

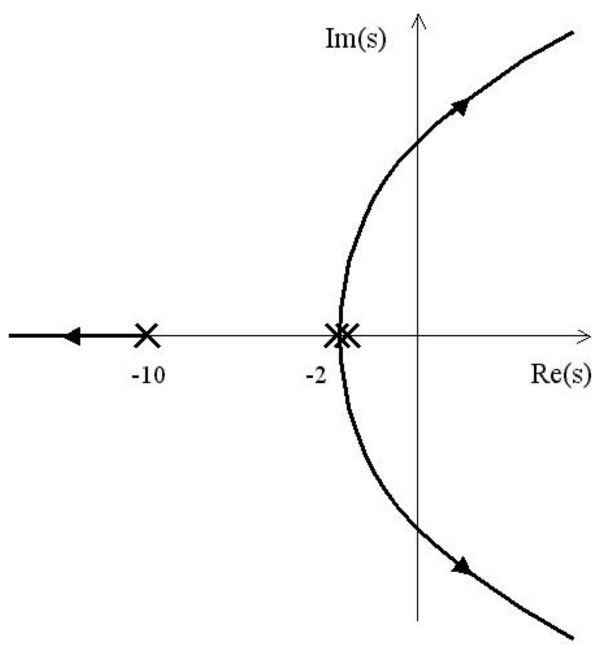
The closed-loop system is stable for any $K > 0$.

(b) The Bode plot shows an initial phase of 0^o hence the Nyquist starts on the positive real axis at A'. The Bode ends with a phase of -270^o hence the Nyquist ends the bottom loop by approaching the origin from the positive imaginary axis (or an angle of -270^o).





The magnitude of the Nyquist plot as it crosses the negative real axis is 0.00174. It will not encircle the $-1/K$ point until $K = 1/0.00174 = 576$.

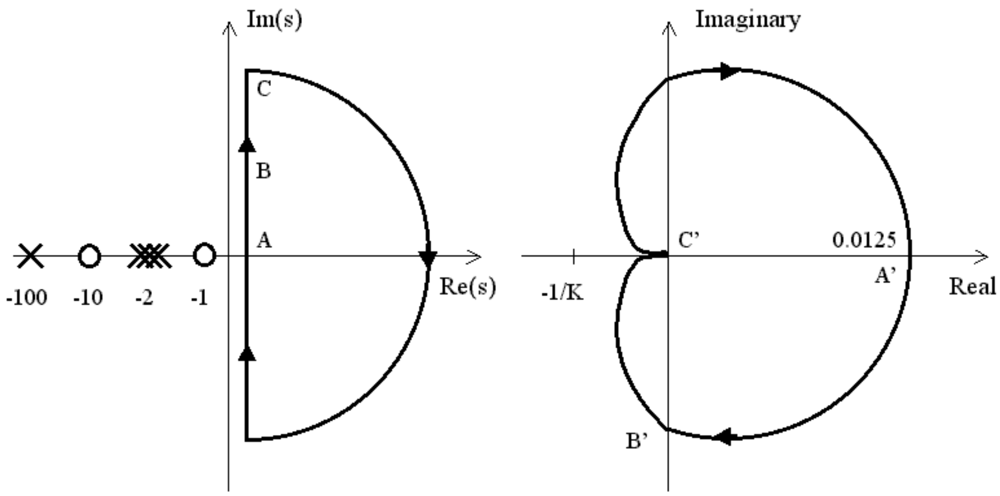


i. $0 < K < 576$
$N = 0, P = 0 \Longrightarrow Z = N + P = 0$
The closed-loop system is stable.

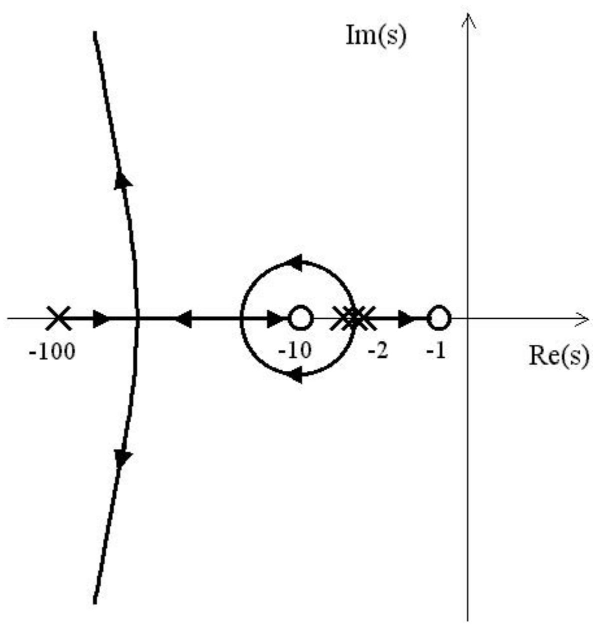
ii. $K > 576$
$N = 2, P = 0 \Longrightarrow Z = N + P = 2$

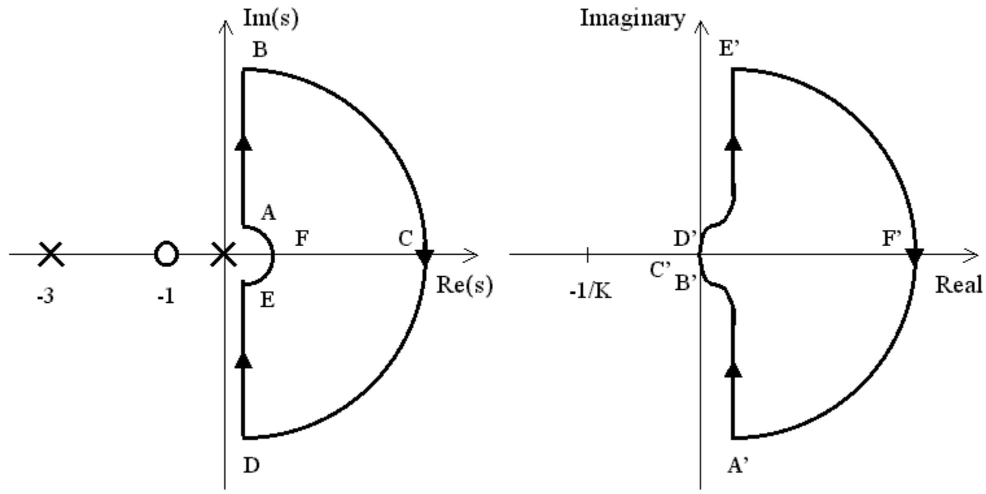
The closed-loop system has two unstable roots as verified by the root locus.

(c) The Bode plot shows an initial phase of 0^o hence the Nyquist starts on the positive real axis at A'. The Bode ends with a phase of -180^o hence the Nyquist ends the bottom loop by approaching the origin from the negative real axis (or an angle of -180^o).



It will never encircle the -1/K point, hence it is always stable. The root locus below confirms that.





$N = 0, P = 0 \Longrightarrow Z = N + P = 0$

The closed-loop system is stable for any $K > 0$.

20. Draw a Nyquist plot for

$$KG(s) = \frac{K(s+1)}{s(s+3)} \tag{1}$$

choosing the contour to be to the right of the singularity on the $j\omega$-axis. and determine the range of K for which the system is stable using the Nyquist Criterion. Then redo the Nyquist plot, this time choosing the contour to be to the left of the singularity on the imaginary axis and again check the range of K for which the system is stable using the Nyquist Criterion. Are the answers the same? Should they be?

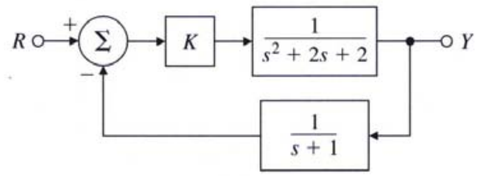
Solution :

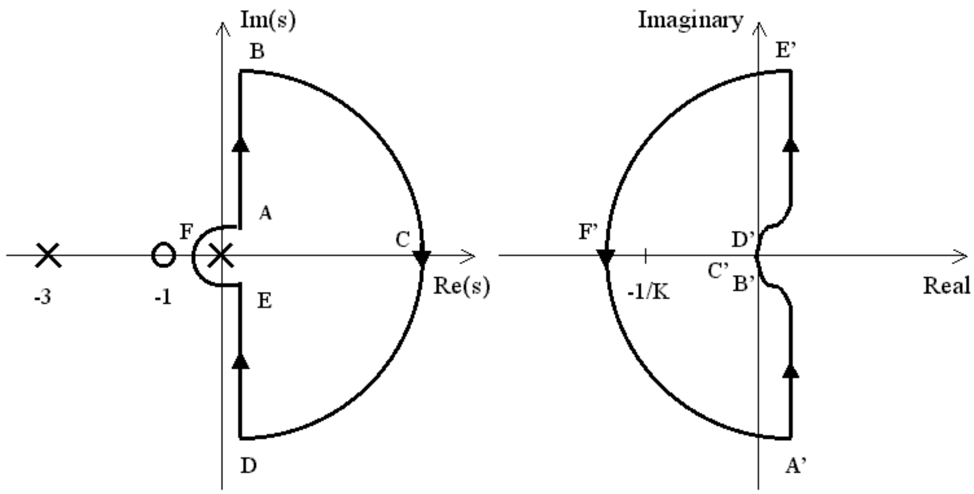
If you choose the contour to the right of the singularity on the origin, the Nyquist plot looks like this :

From the Nyquist plot, the range of K for stability is $-\frac{1}{K} < 0$ ($N = 0, P = 0 \Longrightarrow Z = N + P = 0$). So the system is stable for $K > 0$.

Similarly, in the case with the contour to the left of the singularity on the origin, the Nyquist plot is:

Figure 6.89: Control system for Problem 21





From the Nyquist plot, the range of K for stability is $-\frac{1}{K} < 0$ ($N = -1, P = 1 \Longrightarrow Z = N + P = 0$). So the system is stable for $K > 0$.

The way of choosing the contour around singularity on the $j\omega$-axis does not affect its stability criterion. The results should be the same in either way. However, it is somewhat less cumbersome to pick the contour to the right of a pole on the imaginary axis so that there are no unstable poles within the contour, hence P=0.

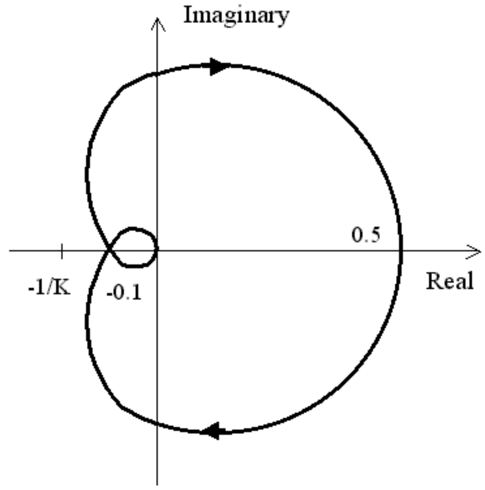
21. Draw the Nyquist plot for the system in Fig. 6.89. Using the Nyquist stability criterion, determine the range of K for which the system is stable. Consider both positive and negative values of K.

Solution :

The characteristic equation:

$$1 + K\frac{1}{(s^2+2s+2)}\frac{1}{(s+1)} = 0$$

$$G(s) = \frac{1}{(s+1)(s^2+2s+2)}$$



For positive K, note that the magnitude of the Nyquist plot as it crosses the negative real axis is 0.1, hence $K < 10$ for stability. For negative K, the entire Nyquist plot is essentially flipped about the imaginary axis, thus the magnitude where it crosses the negative real axis will be 0.5 and the stability limit is that $|K| < 2$ Therefore, the range of K for stability is $-2 < K < 10$.

22. (a) For $\omega = 0.1$ to 100 rad/sec, sketch the phase of the minimum-phase system

$$\left| G(s) = \frac{s+1}{s+10} \right|_{s=j\omega}$$

and the nonminimum-phase system

$$\left| G(s) = -\frac{s-1}{s+10} \right|_{s=j\omega},$$

noting that $\angle(j\omega - 1)$ decreases with ω rather than increasing.

(b) Does a RHP zero affect the relationship between the -1 encirclements on a polar plot and the number of unstable closed-loop roots in Eq. (6.28)?

(c) Sketch the phase of the following unstable system for $\omega = 0.1$ to 100 rad/sec:

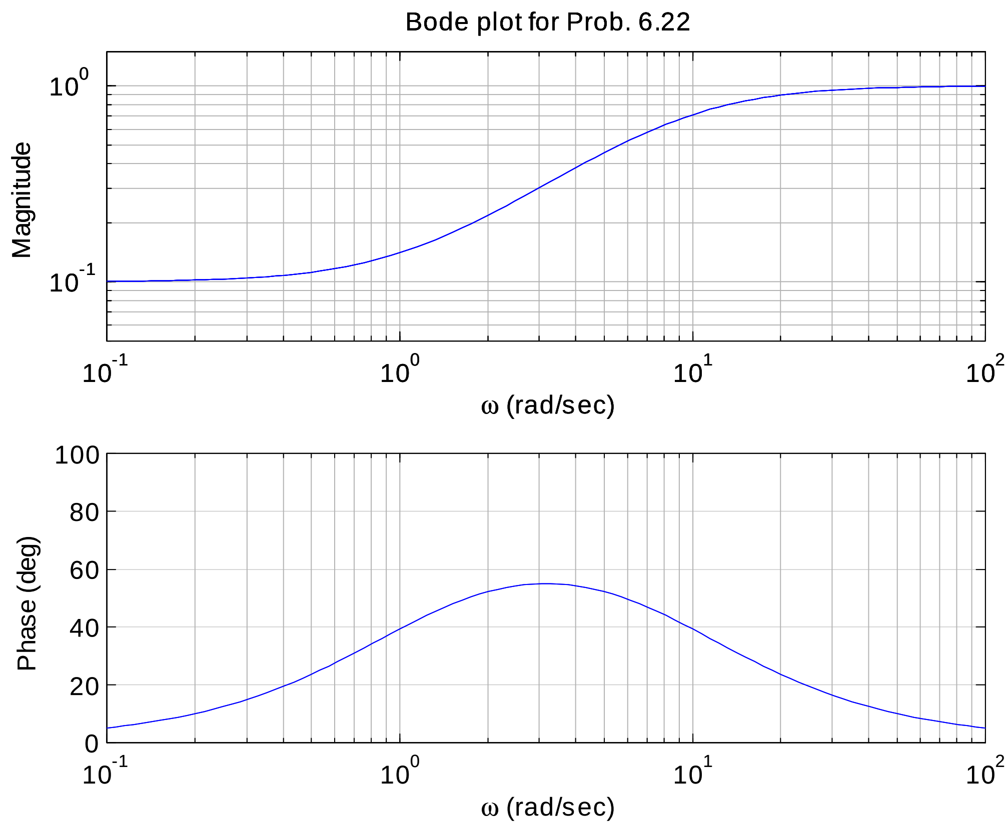
$$G(s) = \left| \frac{s+1}{s-10} \right|_{s=j\omega}.$$

(d) Check the stability of the systems in (a) and (c) using the Nyquist criterion on $KG(s)$. Determine the range of K for which the closed-loop system is stable, and check your results qualitatively using a rough root-locus sketch.

Solution :

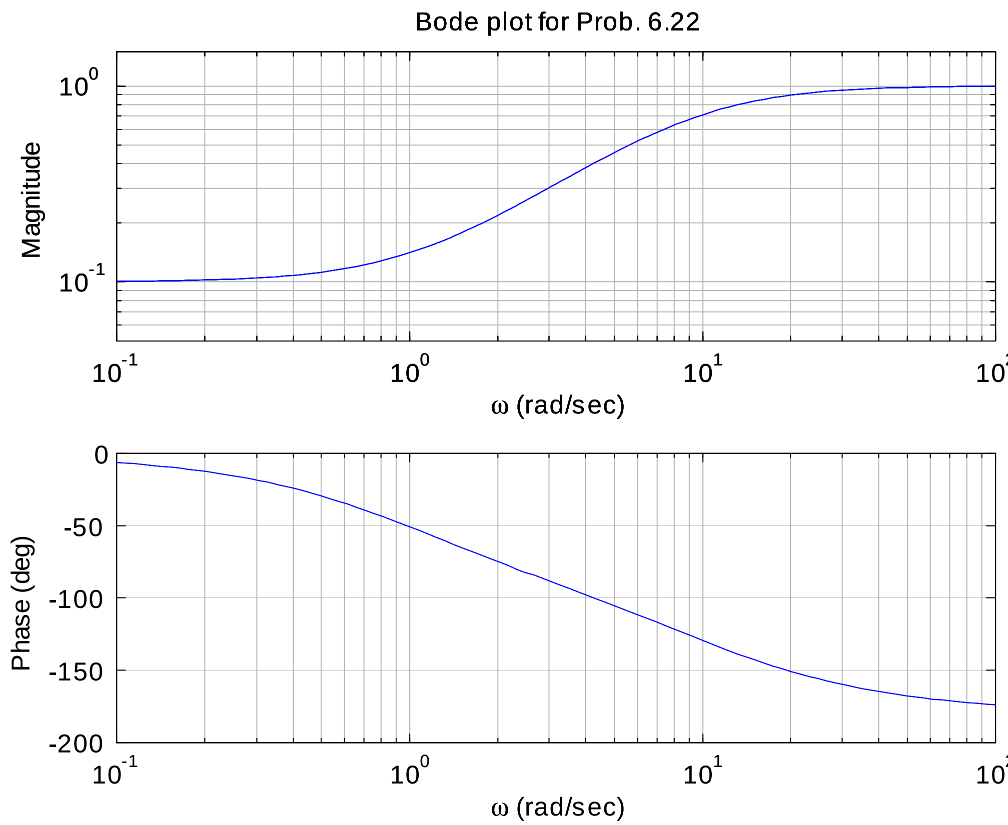
(a) Minimum phase system,

$$G_1(j\omega) = \frac{s+1}{s+10}|_{s=j\omega}$$



Non-minimum phase system,

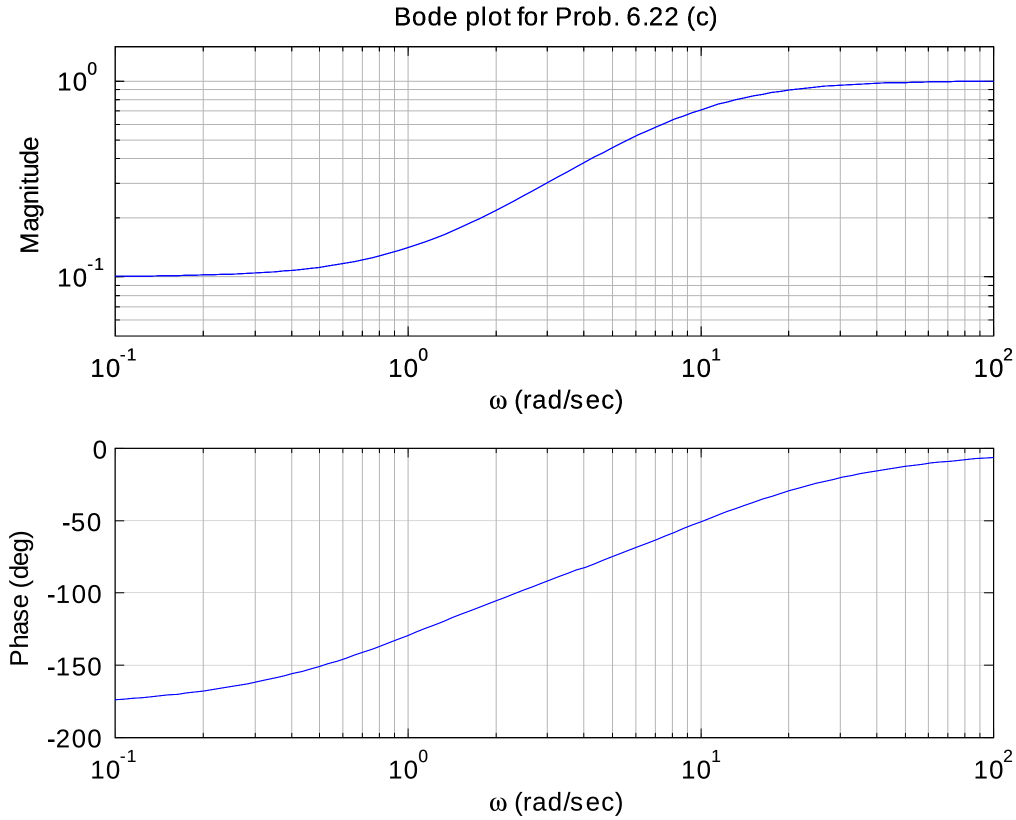
$$G_2(j\omega) = -\frac{s-1}{s+10}|_{s=j\omega}$$



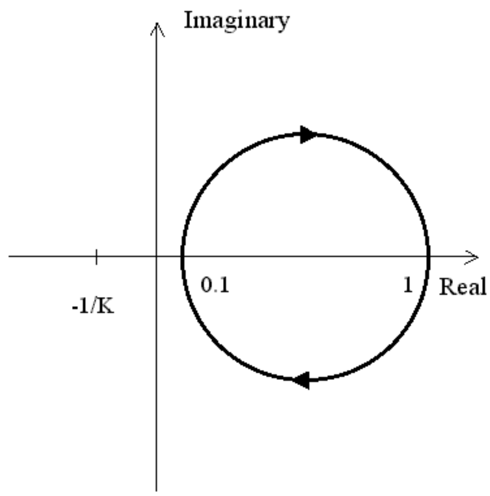
(b) No, a RHP zero doesn't affect the relationship between the -1 encirclements on the Nyquist plot and the number of unstable closed-loop roots in Eq. (6.28).

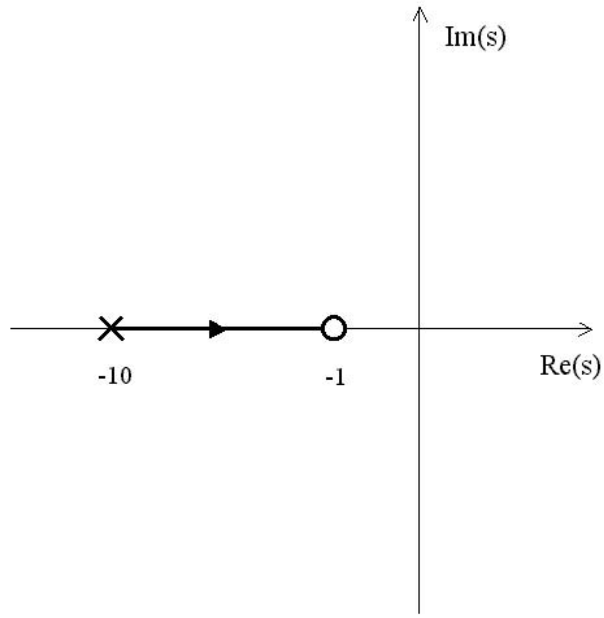
(c) Unstable system:

$$G_3(j\omega) = \frac{s+1}{s-10}|_{s=j\omega}$$



i. Minimum phase system $G_1(j\omega)$:
For any $K > 0$, $N = 0$, $P = 0 \Longrightarrow Z = 0 \Longrightarrow$ The system is stable, as verified by the root locus being entirely in the LHP.

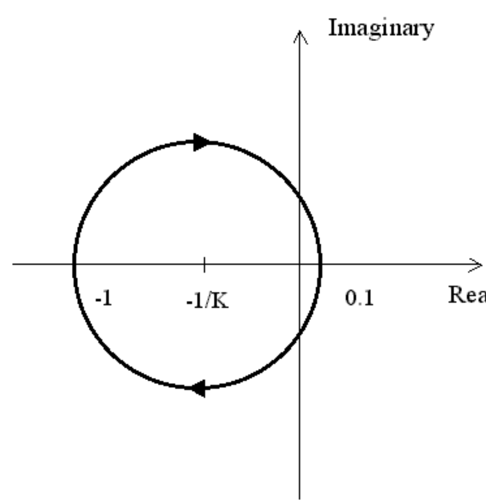




ii. Non-minimum phase system $G_2(j\omega)$: the $-1/K$ point will not be encircled if $K < 1$.

$$\begin{array}{ll} 0 < K < 1 & N = 0,\ P = 0 \Longrightarrow Z = 0 \Longrightarrow \text{Stable} \\ 1 < K & N = 1,\ P = 0 \Longrightarrow Z = 1 \Longrightarrow \text{Unstable} \end{array}$$

This is verified by the Root Locus shown below where the branch

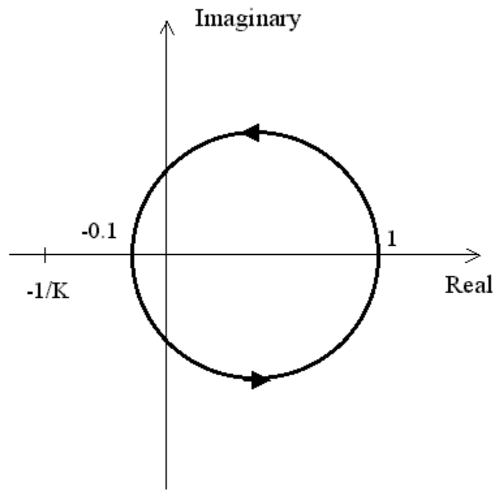


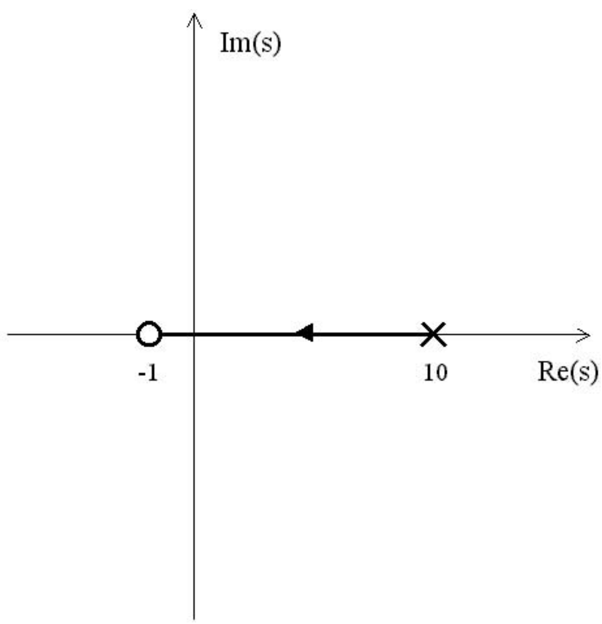
of the locus to the left of the pole is from $K < 1$.

iii. Unstable system $G_3(j\omega)$: The $-1/K$ point will be encircled if $K > 10$, however, $P = 1$, so

$$\begin{array}{rl} 0 < K < 10: & N = 0,\ P = 1 \Longrightarrow Z = 1 \Longrightarrow \text{Unstable} \\ 10 < K\ : & N = -1,\ P = 1 \Longrightarrow Z = 0 \Longrightarrow \text{Stable} \end{array}$$

This is verified by the Root Locus shown below right, where the locus crosses the imaginary axis when $K = 10$, and stays in the LHP for $K > 10$.





23. *Nyquist plots and their classical plane curves:* Determine the Nyquist plot using Matlab for the systems given below with $K = 1$ and verify that

the beginning point and end point for the $j\omega > 0$ portion have the correct magnitude and phase:

(a) the classical curve called Cayley's Sextic, discovered by Maclaurin in 1718

$$KG(s) = K\frac{1}{(s+1)^3}$$

(b) the classical curve called the Cissoid, meaning ivy-shaped

$$KG(s) = K\frac{1}{s(s+1)}$$

(c) the classical curve called the Folium of Kepler, studied by Kepler in 1609.

$$KG(s) = K\frac{1}{(s-1)(s+1)^2}$$

(d) the classical curve called the Folium (not Kepler's)

$$KG(s) = K\frac{1}{(s-1)(s+2)}$$

(e) the classical curve called the Nephroid, meaning kidney-shaped.

$$KG(s) = K\frac{2(s+1)(s^2-4s+1)}{(s-1)^3}$$

(f) the classical curve called Nephroid of Freeth, named after the English mathematician T. J. Freeth.

$$KG(s) = K\frac{(s+1)(s^2+3)}{4(s-1)^3}$$

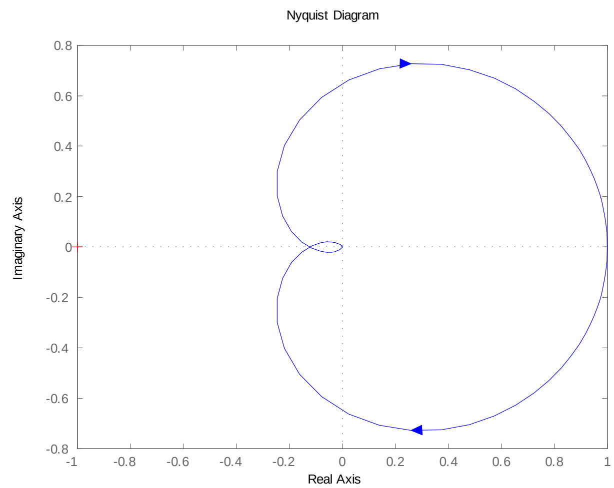
(g) a shifted Nephroid of Freeth

$$KG(s) = K\frac{(s^2+1)}{(s-1)^3}$$

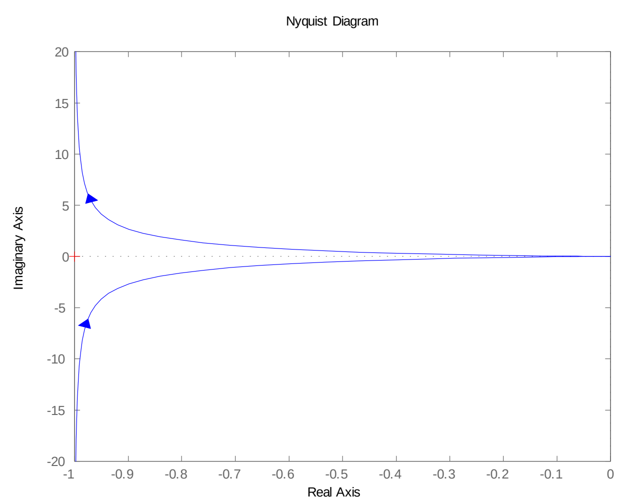
Solution :

These are all accomplished by using Matlab's Nyquist function. All interesting shapes. To check the magnitude and phase for each, plug in $s = 0$ and $s = \inf$ and then compare those values with the beginning and end points on the Nyquist diagrams.

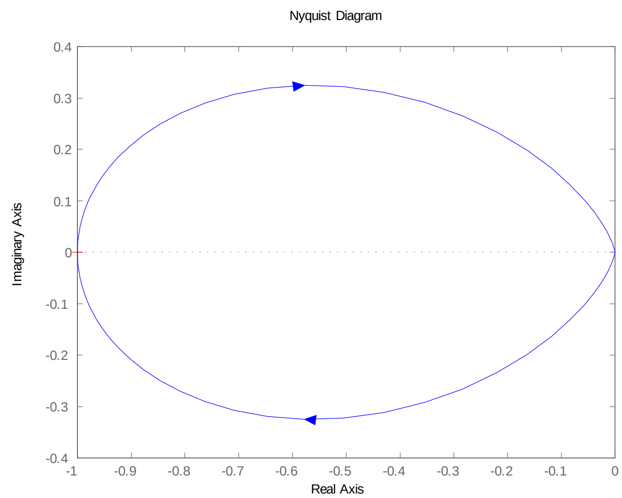
(a)


Nyquist Diagram
Imaginary Axis
Real Axis
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

(b)

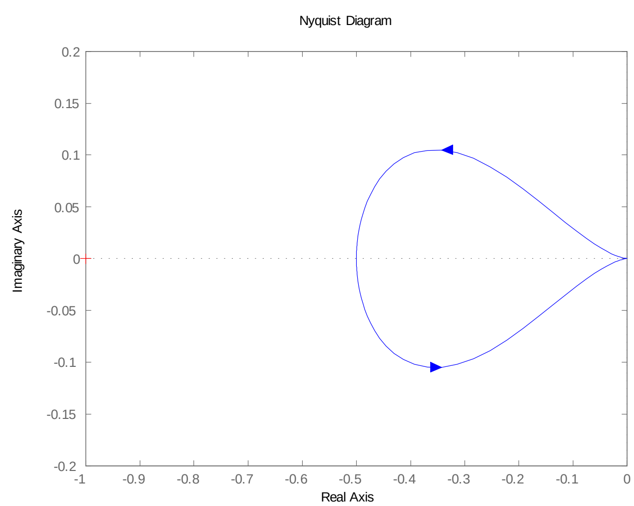

Nyquist Diagram
Imaginary Axis
Real Axis
20
15
10
5
0
-5
-10
-15
-20
-1
-0.9
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0

(c)



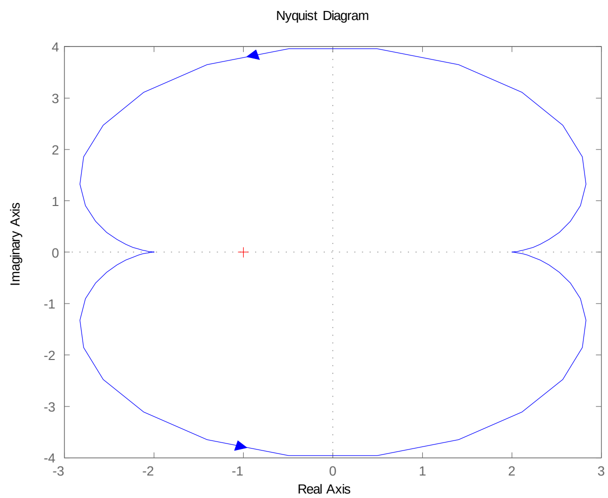
Nyquist Diagram
Imaginary Axis
Real Axis
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-1
-0.9
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0

(d)

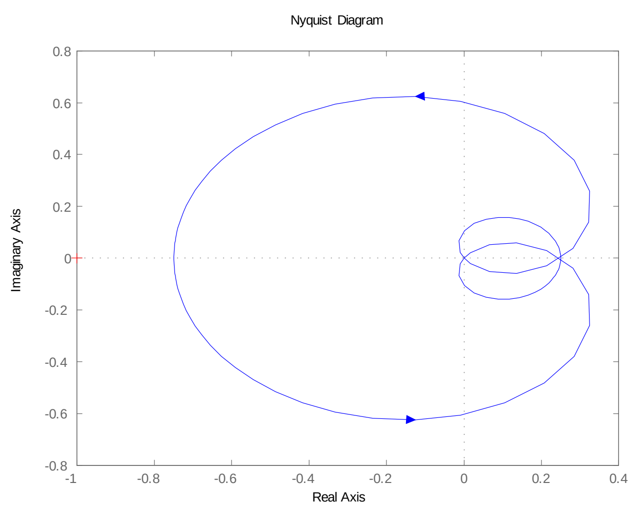


Nyquist Diagram
Imaginary Axis
Real Axis
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-1
-0.9
-0.8
-0.7
-0.6
-0.5
-0.4
-0.3
-0.2
-0.1
0

(e)


Nyquist Diagram
Imaginary Axis
Real Axis
4
3
2
1
0
-1
-2
-3
-4
-3
-2
-1
0
1
2
3

(f)


Nyquist Diagram
Imaginary Axis
Real Axis
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4

(g)

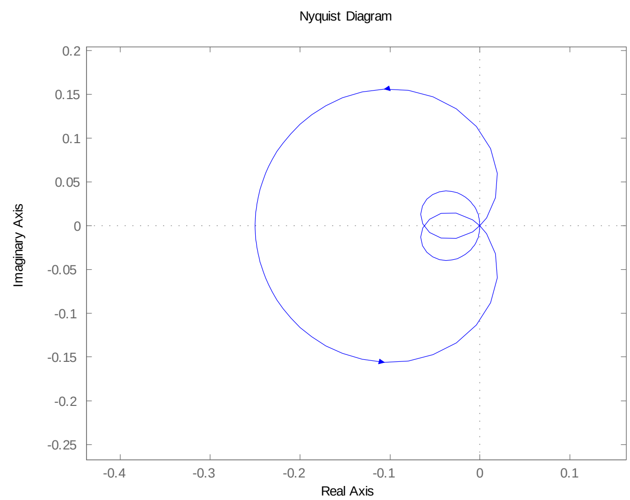
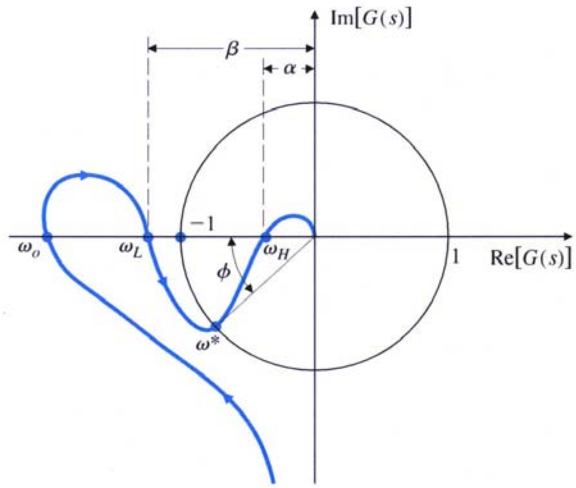

Nyquist Diagram
Imaginary Axis
Real Axis
0.2
0.15
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
-0.4
-0.3
-0.2
-0.1
0
0.1

Figure 6.90: Nyquist plot for Problem 24



(a) Problems and Solutions for Section 6.4

24. The Nyquist plot for some actual control systems resembles the one shown in Fig.6.90. What are the gain and phase margin(s) for the system of Fig. 6.90 given that $\alpha = 0.4$, $\beta = 1.3$, and $\phi = 40^o$. Describe what happens to the stability of the system as the gain goes from zero to a very large value. Sketch what the corresponding root locus must look like for such a system. Also sketch what the corresponding Bode plots would look like for the system.

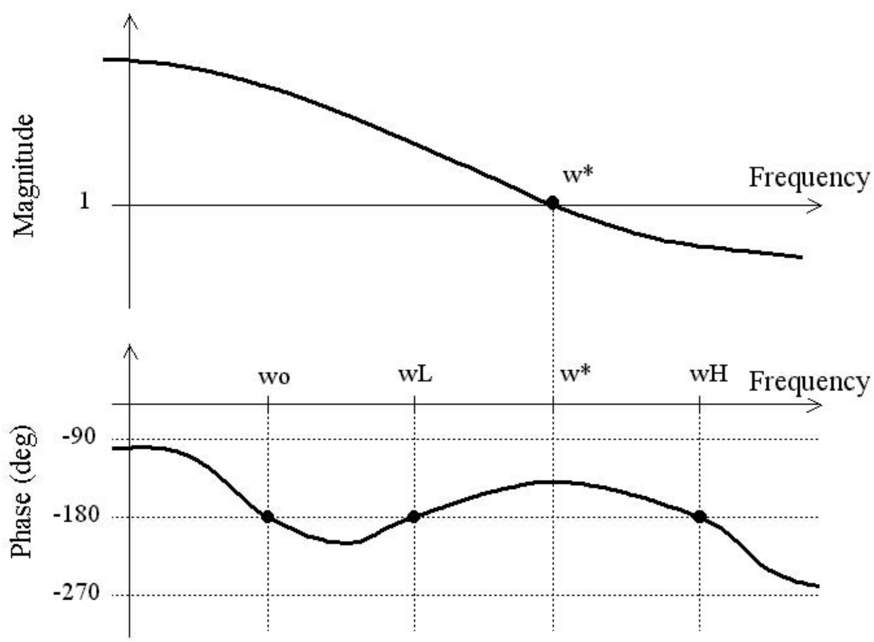
Solution :

The phase margin is defined as in Figure 6.34, $PM = \phi \ (\omega = \omega^*)$, but now there are several gain margins! If the system gain is increased (multiplied) by $\frac{1}{|\alpha|}$ or decreased (divided) by $|\beta|$, then the system will go unstable. This is a conditionally stable system. See Figure 6.40 for a typical root locus of a conditionally stable system.

$$\begin{aligned}\text{gain margin} &= \text{-20 log} \left|\alpha\right|_{dB} (\omega = \omega_H) \\ \text{gain margin} &= +20 \text{ log} \left|\beta\right|_{dB} (\omega = \omega_L)\end{aligned}$$

For a conditionally stable type of system as in Fig. 6.40, the Bode phase plot crosses -180° twice; however, for this problem we see from the Nyquist plot that it crosses 3 times! For very low values of gain, the entire

Nyquist plot would be shrunk, and the -1 point would occur to the left of the negative real axis crossing at ω_o, so there would be no encirclements and the system would be stable. As the gain increases, the -1 point occurs between ω_o and ω_L so there is an encirclement and the system is unstable. Further increase of the gain causes the -1 point to occur between ω_L and ω_H (as shown in Fig. 6.90) so there is no encirclement and the system is stable. Even more increase in the gain would cause the -1 point to occur between ω_H and the origin where there is an encirclement and the system is unstable. The root locus would look like Fig. 6.40 except that the very low gain portion of the loci would start in the LHP before they loop out into the RHP as in Fig. 6.40. The Bode plot would be vaguely like that drawn below:



25. The Bode plot for

$$G(s) = \frac{100[(s/10) + 1]}{s[(s/1) - 1][(s/100) + 1]}$$

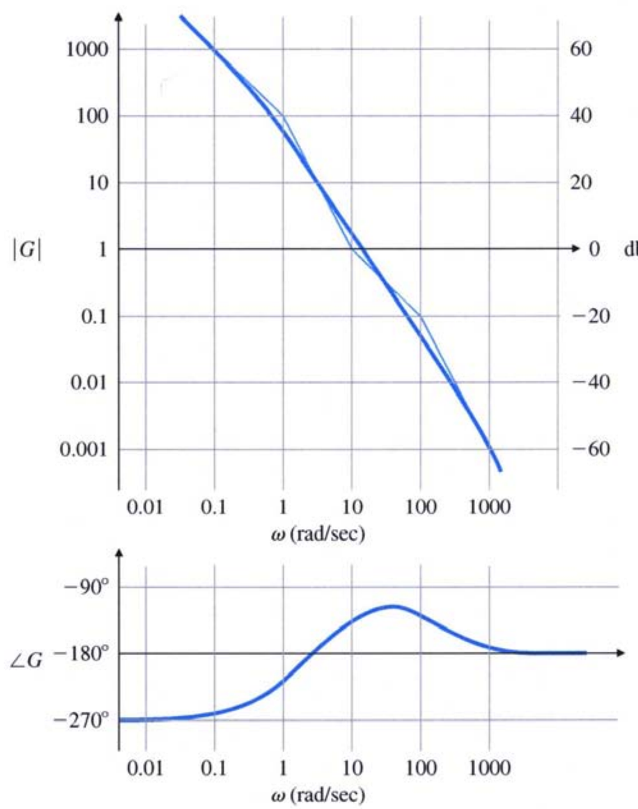
is shown in Fig. 6.91.

(a) Why does the phase start at -270^o at the low frequencies?

(b) Sketch the Nyquist plot for $G(s)$.

(c) Is the closed-loop system shown in Fig. 6.92 stable?

Figure 6.91: Bode plot for Problem 25



(d) Will the system be stable if the gain is lowered by a factor of 100? Make a rough sketch of a root locus for the system and qualitatively confirm your answer

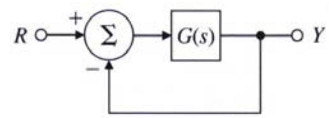
Solution :

(a) From the root locus, the phase at the low frequencies ($\omega = 0+$) is calculated as :

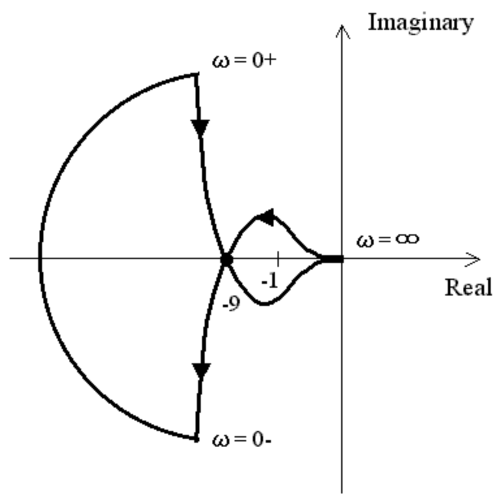
$$\begin{aligned} & \text{The phase at the point } \{s = j\omega(\omega = 0+)\} \\ = \; & -180^\circ(\text{pole : } s = 1) - 90^\circ(\text{pole : } s = 0) + 0^\circ(\text{zero : } s = -10) + 0^\circ(\text{pole : } s = -100) \\ = \; & -270^\circ \end{aligned}$$

Or, more simply, the RHP pole at $s = +1$ causes a -180^o shift from the -90^o that you would expect from a normal system with all the singularities in the LHP.

Figure 6.92: Control system for Problem 26



(b) The Nyquist plot for $G(s)$:



(c) As the Nyquist shows, there is one counter-clockwise encirclement of -1.

$$\Longrightarrow N = -1$$

We have one pole in RHP $\Longrightarrow P = 1$

$Z = N + P = -1 + 1 = 0 \Longrightarrow$ The closed-loop system is stable.

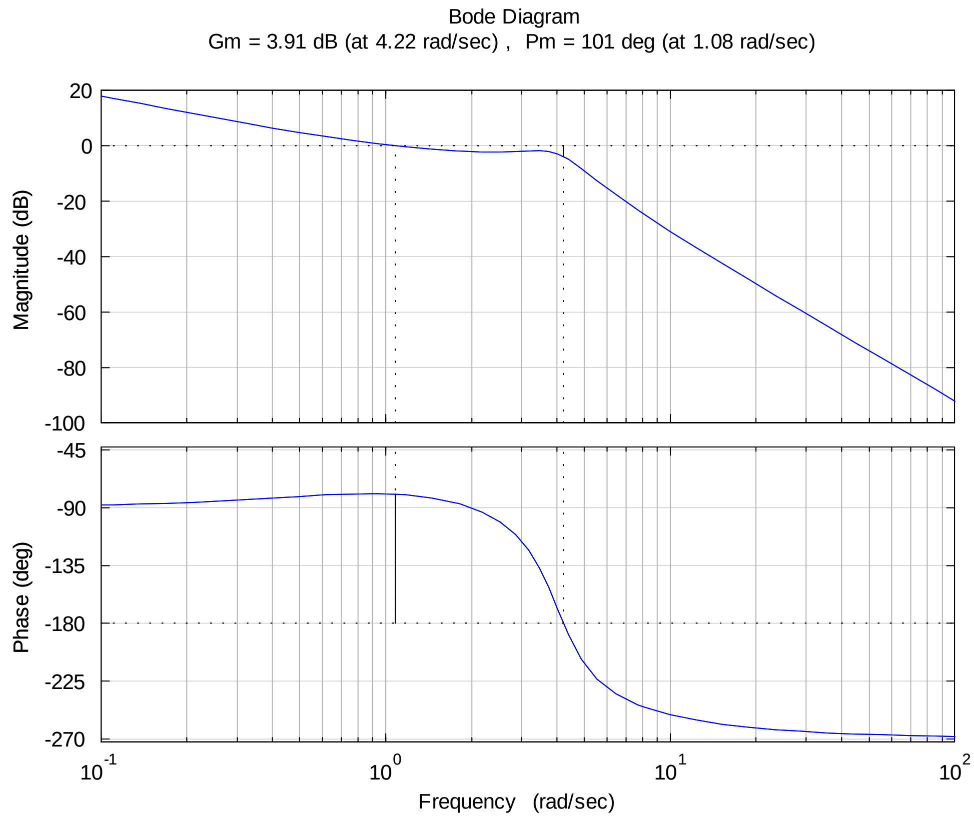
(d) The system goes unstable if the gain is lowered by a factor of 100.

26. Suppose that in Fig. 6.92,

$$G(s) = \frac{25(s+1)}{s(s+2)(s^2+2s+16)}.$$

Use MATLAB's `margin` to calculate the PM and GM for $G(s)$ and, based on the Bode plots, conclude which margin would provide more useful information to the control designer for this system.

Solution :



From the Bode plot,

$$PM = 101\,\mathrm{deg},\ GM = 3.9\mathrm{db} = 1.57$$

Since both PM and GM are positive, we can say that the closed-loop of this system is stable. But GM is so small that we must be careful not to increase the gain much, which leads the closed-loop system to be unstable. Clearly, the GM is the more important margin for this example.

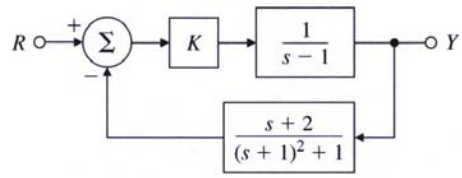
27. Consider the system given in Fig. 6.93.

(a) Use MATLAB to obtain Bode plots for $K = 1$ and use the plots to estimate the range of K for which the system will be stable.

(b) Verify the stable range of K by using `margin` to determine PM for selected values of K.

(c) Use `rlocus` and `rlocfind` to determine the values of K at the stability boundaries.

Figure 6.93: Control system for Problem 27

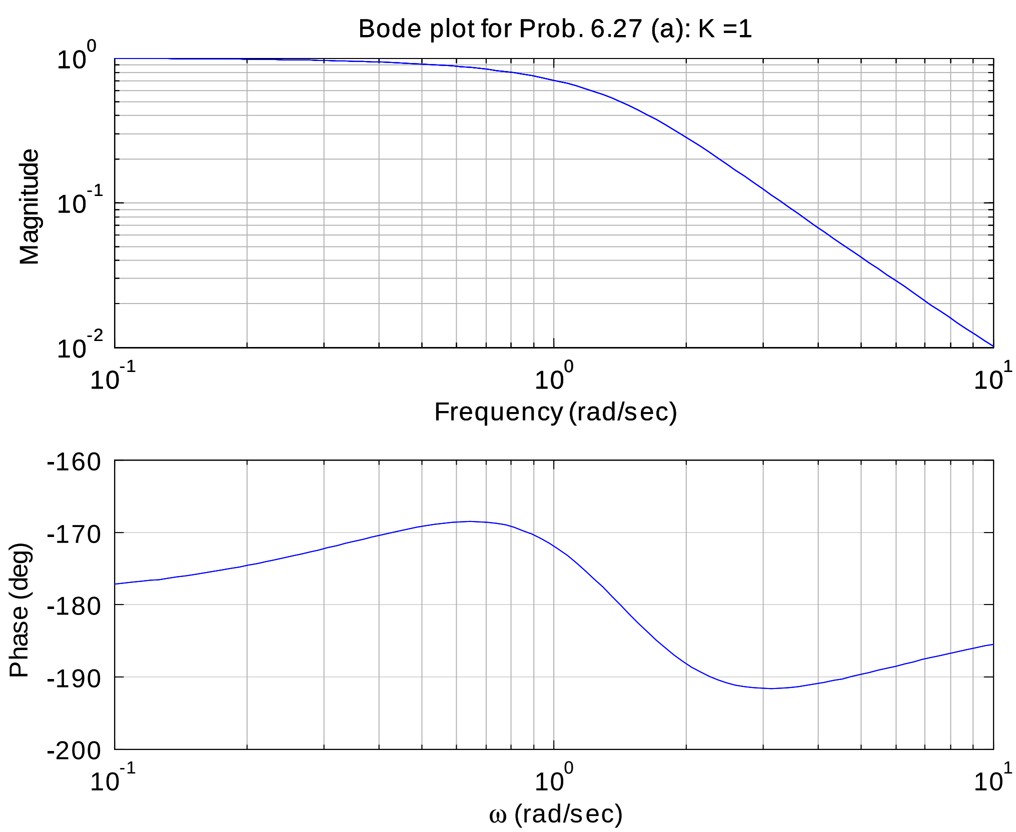


(d) Sketch the Nyquist plot of the system, and use it to verify the number of unstable roots for the unstable ranges of K.

(e) Using Routh's criterion, determine the ranges of K for closed-loop stability of this system.

Solution :

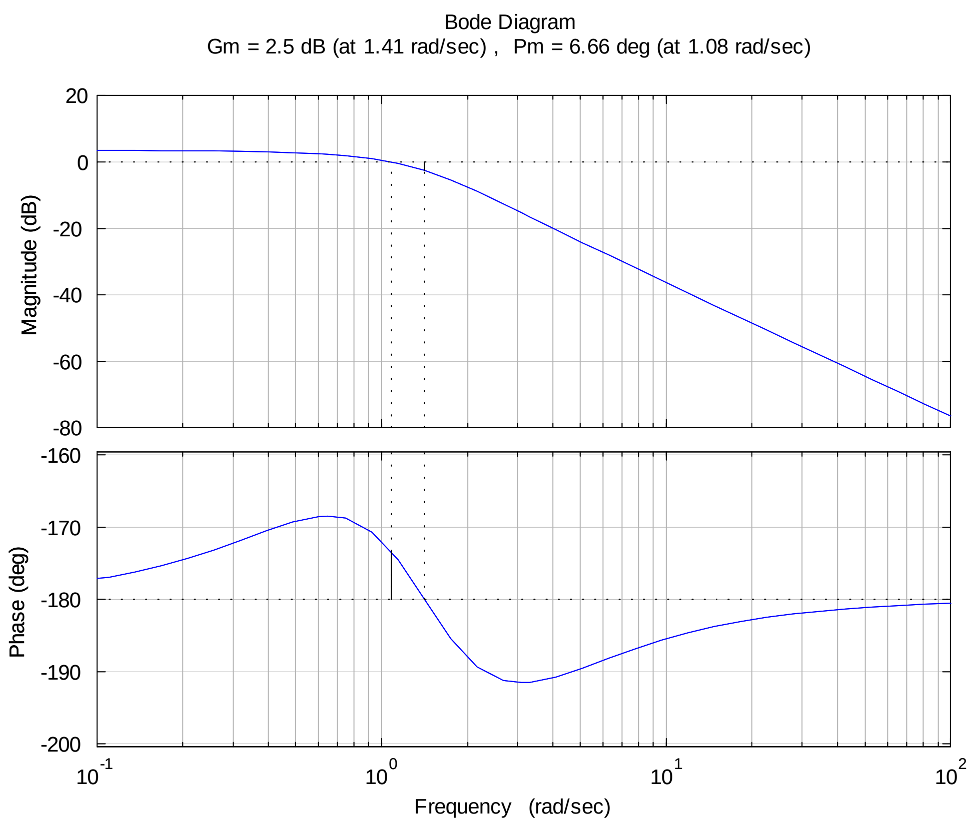
(a) The Bode plot for $K = 1$ is :



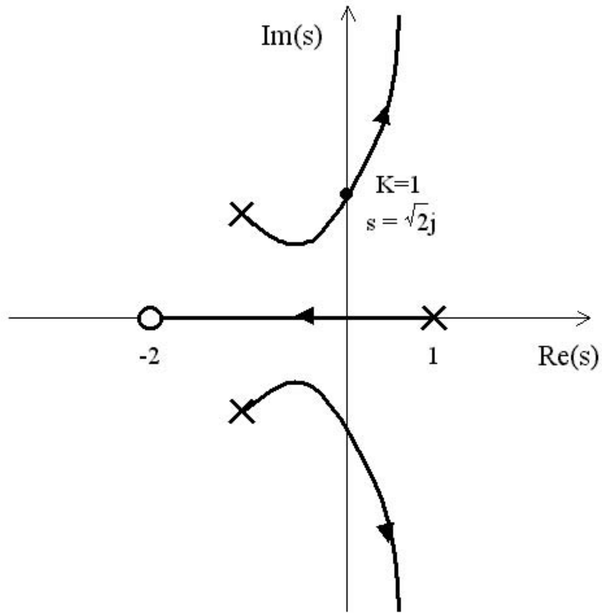
From the Bode plot, the closed-loop system is unstable for $K = 1$. But we can make the closed-system stable with positive GM by increasing the gain K up to the crossover frequency reaches at $\omega = 1.414$ rad/sec ($K = 2$), where the phase plot crosses the $-180°$ line. Therefore :

$$1 < K < 2 \implies \text{The closed-loop system is stable.}$$

(b) For example, $PM = 6.66\,\text{deg}$ for $K = 1.5$.



(c) Root locus is :



$j\omega$-crossing :

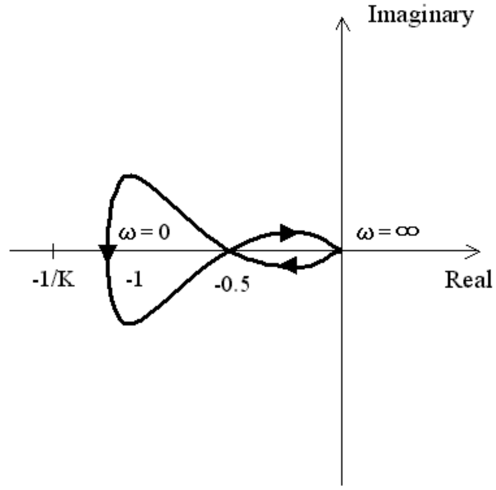
$$1 + K\frac{j\omega + 2}{(j\omega)^3 + (j\omega)^2 - 2} = 0$$

$$\begin{aligned} \omega^2 - 2K + 2 &= 0 \\ \omega(\omega^2 - K) &= 0 \end{aligned}$$

$$K = 2,\ \omega = \pm\sqrt{2},\ \text{or}\ K = 1,\ \omega = 0$$

Therefore,

$$1 < K < 2 \implies \text{The closed-loop system is stable.}$$



(d)

i. $0 < K < 1$
$N = 0,\ P = 1 \implies Z = 1$
One unstable closed-loop root.

ii. $1 < K < 2$
$N = -1,\ P = 1 \implies Z = 0$
Stable.

iii. $2 < K$
$N = 1,\ P = 1 \implies Z = 2$
Two unstable closed-loop roots.

(e) The closed-loop transfer function of this system is :

$$\begin{aligned}\frac{y(s)}{r(s)} &= \frac{k\dfrac{1}{s-1}}{1 + k\dfrac{1}{s-1} \times \dfrac{s+2}{(s+1)^2+1}} \\ &= \frac{K(s^2+2s+2)}{s^3+s^2+Ks+2K-2}\end{aligned}$$

So the characteristic equation is :

$$\implies s^3 + s^2 + Ks + 2K - 2 = 0$$

Using the Routh's criterion,

$$\begin{array}{lcc} s^3: & 1 & K \\ s^2: & 1 & 2K-2 \\ s^1: & 2-K & 0 \\ s^0: & 2k=2 & \end{array}$$

For stability,

$$\begin{aligned} 2-K &> 0 \\ 2K-2 &> 0 \end{aligned}$$

$$\Longrightarrow 2 > K > 1$$

$$\begin{array}{cc} 0 < K < 1 & \text{Unstable} \\ 1 < K < 2 & \text{Stable} \\ 2 < K & \text{Unstable} \end{array}$$

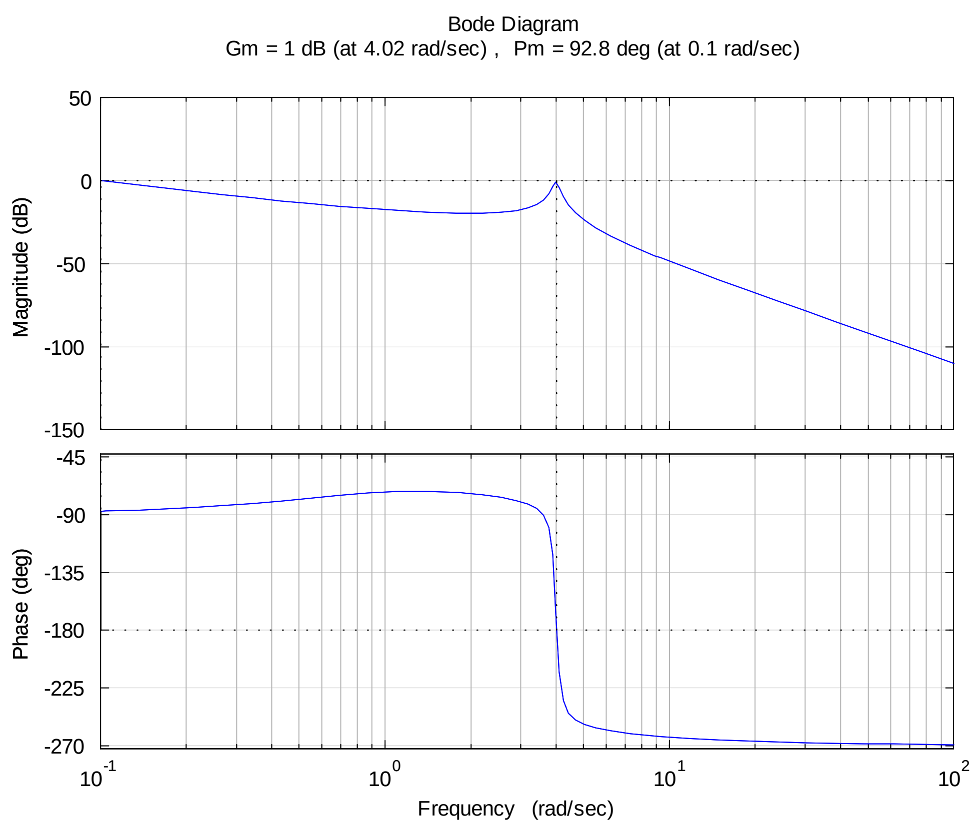
28. Suppose that in Fig. 6.92,

$$G(s) = \frac{3.2(s+1)}{s(s+2)(s^2+0.2s+16)}.$$

Use MATLAB's margin to calculate the PM and GM for $G(s)$ and comment on whether you think this system will have well damped closed-loop roots.

Solution :

MATLAB's margin plot for the given system is :



From the MATLAB margin routine, $PM = 92.8^o$. Based on this result, Fig. 6.36 suggests that the damping will be = 1; that is, the roots will be real. However, closer inspection shows that a very small increase in gain would result in an instability from the resonance leading one to believe that the damping of these roots is very small. Use of MATLAB's damp routine on the closed loop system confirms this where we see that there are two real poles ($\zeta = 1$) and two very lightly damped poles with $\zeta = 0.0027$. This is a good example where one needs to be careful to not use Matlab without thinking.

29. For a given system, show that the ultimate period P_u and the corresponding ultimate gain K_u for the Zeigler-Nichols method can be found using the following:

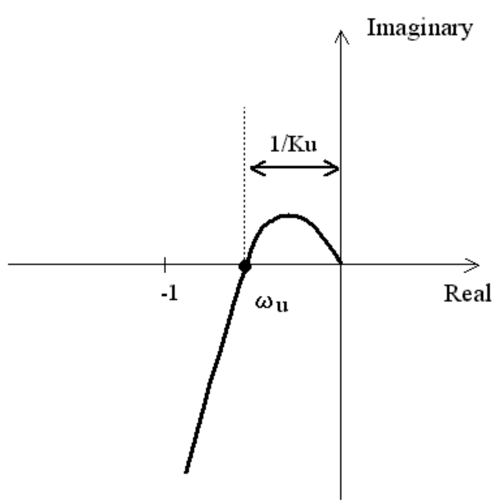
    (a) Nyquist diagram

    (b) Bode plot

    (c) root locus.

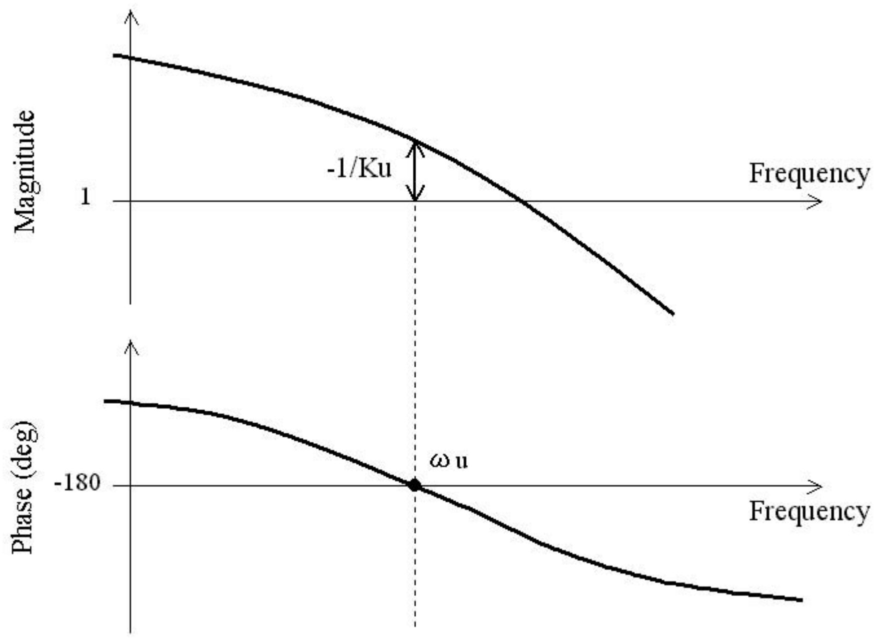
Solution :

(a) See sketch below.



$$P_u = \frac{2\pi}{\omega_u}$$

(b) See sketch below.



$$P_u = \frac{2\pi}{\omega_u}$$

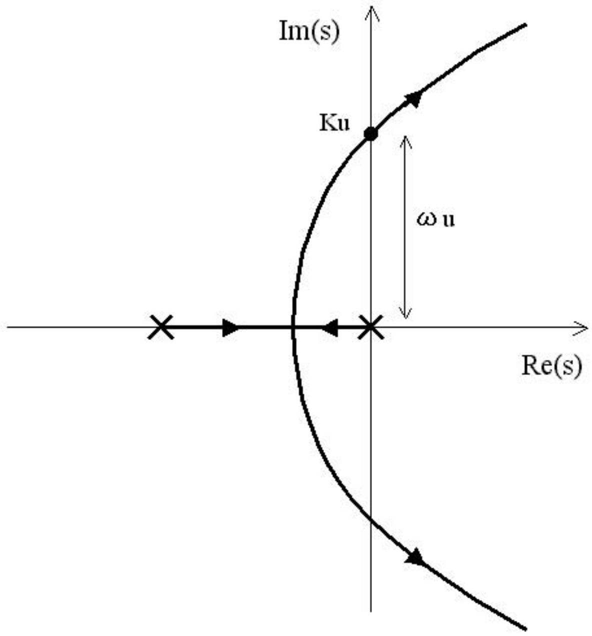
(c)

$$1 + K_u G(j\omega_u) = 0$$

$$1 + K_u \operatorname{Re}[G(j\omega_u)] + K_u j \operatorname{Im}[G(j\omega_u)] = 0$$

$$K_u = -\frac{1}{\operatorname{Re}[G(j\omega_u)]}$$

$$\operatorname{Im}[G(j\omega_u)] = 0; \text{ or } P_u = \frac{2\pi}{\omega_u}$$



30. If a system has the open-loop transfer function

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

with unity feedback, then the closed-loop transfer function is given by

$$T(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}.$$

Verify the values of the PM shown in Fig. 6.36 for $\zeta = 0.1$, 0.4, and 0.7.

Solution :

$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)},\ T(s) = \frac{G(s)}{1+G(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

ζ	PM from Eq. 6.32	PM from Fig. 6.36	PM from Bode plot
0.1	10°	10°	11.4° ($\omega = 0.99$ rad/sec)
0.4	40°	44°	43.1° ($\omega = 0.85$ rad/sec)
0.7	70°	65°	65.2° ($\omega = 0.65$ rad/sec)

31. Consider the unity feedback system with the open-loop transfer function

$$G(s) = \frac{K}{s(s+1)[(s^2/25)+0.4(s/5)+1]}.$$

(a) Use MATLAB to draw the Bode plots for $G(j\omega)$ assuming $K = 1$.

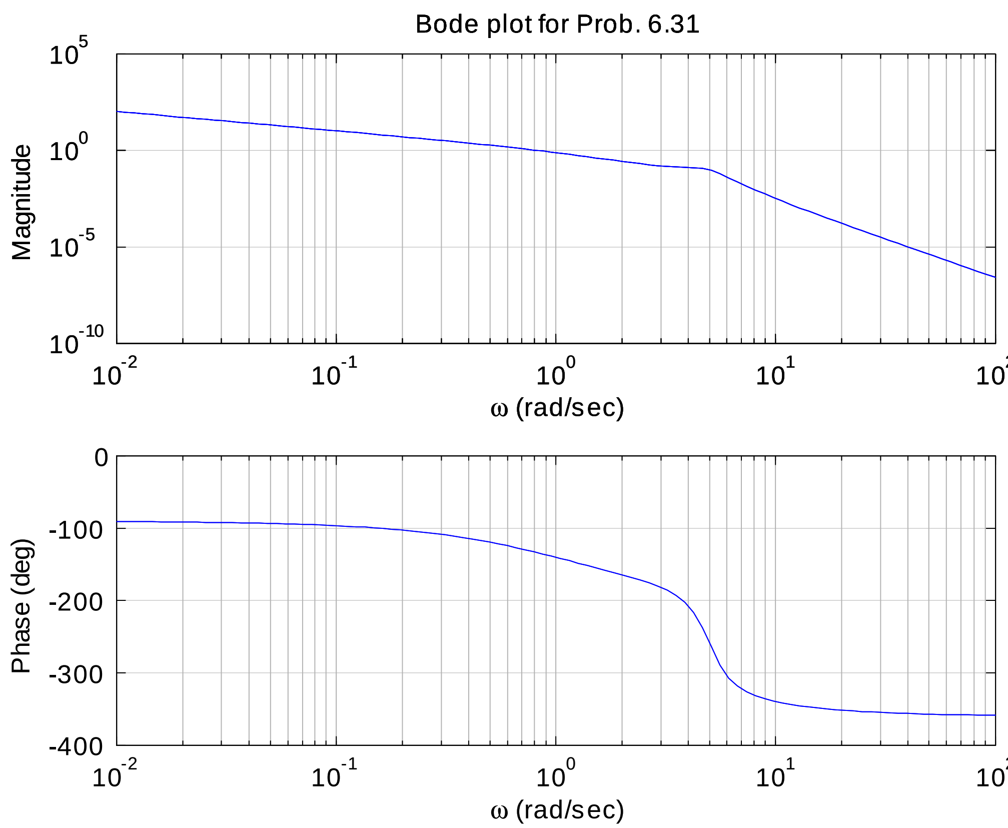
(b) What gain K is required for a PM of 45°? What is the GM for this value of K?

(c) What is K_v when the gain K is set for PM = 45°?

(d) Create a root locus with respect to K, and indicate the roots for a PM of 45°.

Solution :

(a) The Bode plot for $K = 1$ is shown below and we can see from margin that it results in a PM = 48^o.



(b) Although difficult to read the plot above, it is clear that a very slight increase in gain will lower the PM to 45^o, so try $K = 1.1$. The margin routine shows that this yields $PM = 45^o$ and $GM = 15$ db.

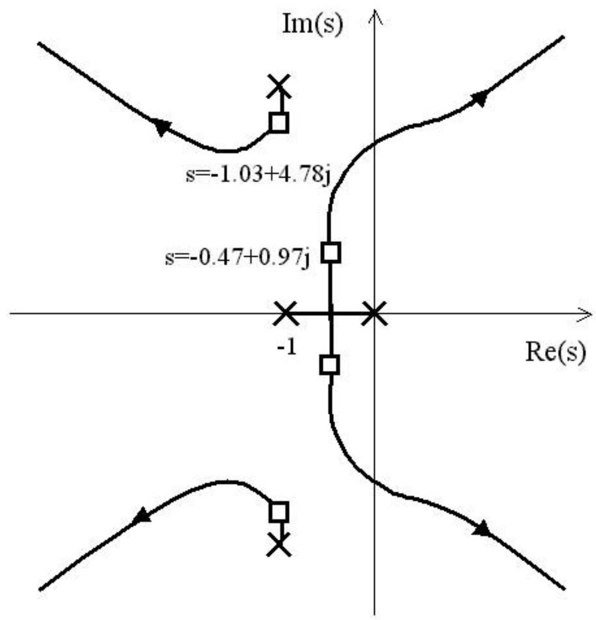
(c) $K_v = \lim_{s\to 0}\{sKG(s)\} = K = 1.1$ when K is set for PM=45°

$$K_v = 1.1$$

(d) The characteristic equation for PM of 45° :

$$1+\frac{1.1}{s(s+1)\left[\left(\frac{s}{5}\right)^2+0.4\left(\frac{s}{5}\right)+1\right]}=0$$

$$\Longrightarrow \quad s^4+3s^3+27s^2+25s+27.88=0$$
$$\Longrightarrow \quad s=-1.03\pm j4.78,\ -0.47\pm j0.97$$



32. For the system depicted in Fig. 6.94(a), the transfer-function blocks are defined by

$$G(s)=\frac{1}{(s+2)^2(s+4)} \quad \text{and} \quad H(s)=\frac{1}{s+1}.$$

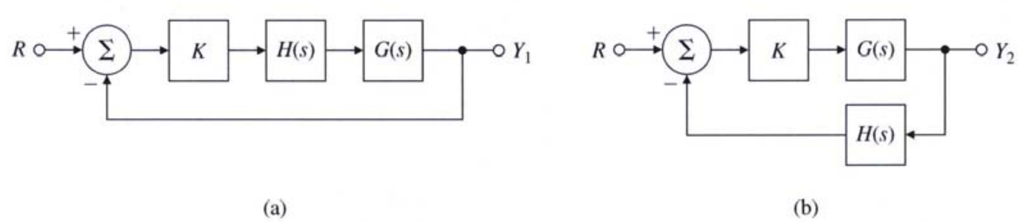
(a) Using rlocus and rlocfind, determine the value of K at the stability boundary.

(b) Using rlocus and rlocfind, determine the value of K that will produce roots with damping corresponding to $\zeta=0.707$.

(c) What is the gain margin of the system if the gain is set to the value determined in part (b)? Answer this question *without* using any frequency response methods.

(d) Create the Bode plots for the system, and determine the gain margin that results for PM = 65°. What damping ratio would you expect for this PM?

Figure 6.94: Block diagram for Problem 32: (a) unity feedback; (b) $H(s)$ in feedback



(e) Sketch a root locus for the system shown in Fig. 6.94(b).. How does it differ from the one in part (a)?

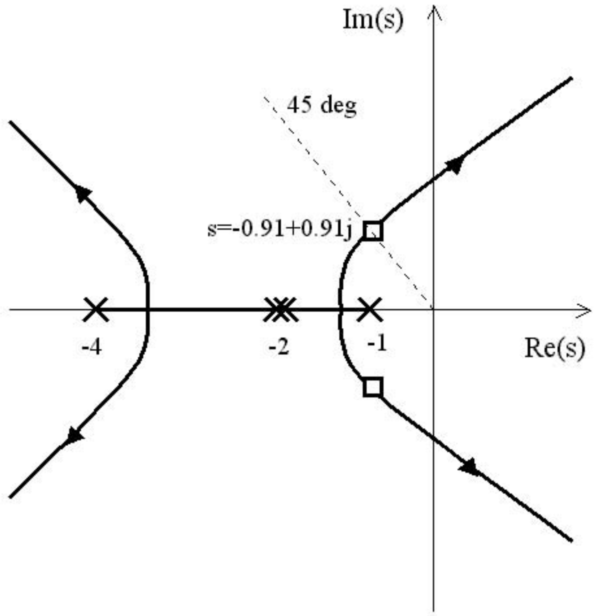
(f) For the systems in Figs. 6.94(a) and (b), how does the transfer function $Y_2(s)/R(s)$ differ from $Y_1(s)/R(s)$? Would you expect the step response to $r(t)$ be different for the two cases?

Solution :

(a) The root locus crosses $j\omega$ axis at $s_0 = j2$.

$$\begin{aligned} K &= \frac{1}{|H(s_0)G(s_0)|}|_{s_0=j2} \\ &= |j2+1|\,|j2+4|\,|j2+2|^2 \end{aligned}$$

$$\Longrightarrow K = 80$$



(b)

$$\zeta = 0.707 \implies 0.707 = \sin\theta \implies \theta = 45^\circ$$

From the root locus given,

$$\begin{aligned} s_1 &= -0.91 + j0.91 \\ K &= \frac{1}{|H(s_1)G(s_1)|}\Big|_{s_1=-0.91+j0.91} \\ &= |0.01 + j0.91|\,|3.09 + j0.91|\,|1.09 + j0.91|^2 \end{aligned}$$

$$\implies K = 5.9$$

(c)

$$GM = \frac{K_a}{K_b} = \frac{80}{5.9} = 13.5$$

(d) From the Root Locus :

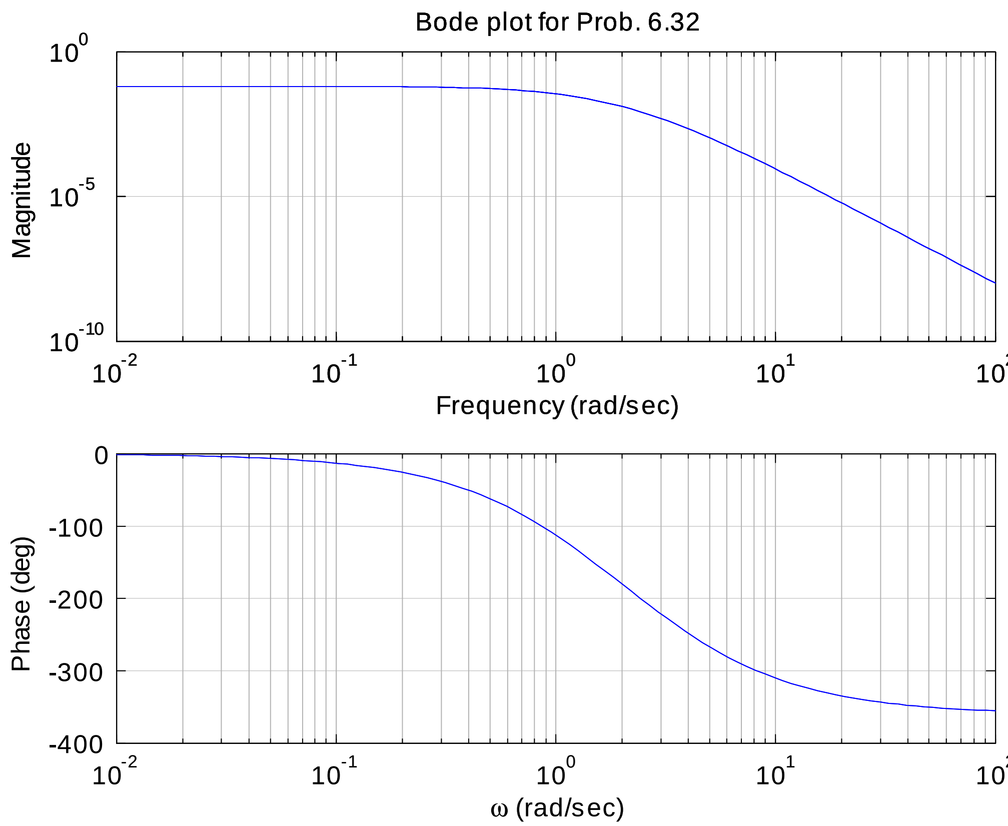
$$G(s)H(s) = \frac{1}{(s+1)(s+2)^2(s+4)}$$

PM=65° when $K = 30$. Instability occurs when $K = 80.0$.

$$\implies GM = 2.67$$

We approximate the damping ratio by $\zeta \simeq \frac{PM}{100}$

$$\zeta \simeq \frac{65}{100} = 0.65$$



(e) The root locus for Fig.6.94(a) is the same as that of Fig.6.94(b).

(f)

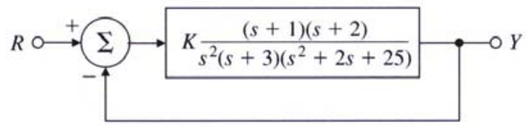
$$
\begin{aligned}
\frac{Y_1(s)}{R(s)} &= \frac{KG(s)H(s)}{1+KG(s)H(s)} = \frac{K}{(s+1)(s+2)^2(s+4)+K} \\
\frac{Y_2(s)}{R(s)} &= \frac{KG(s)}{1+KG(s)H(s)} = \frac{K(s+1)}{(s+1)(s+2)^2(s+4)+K}
\end{aligned}
$$

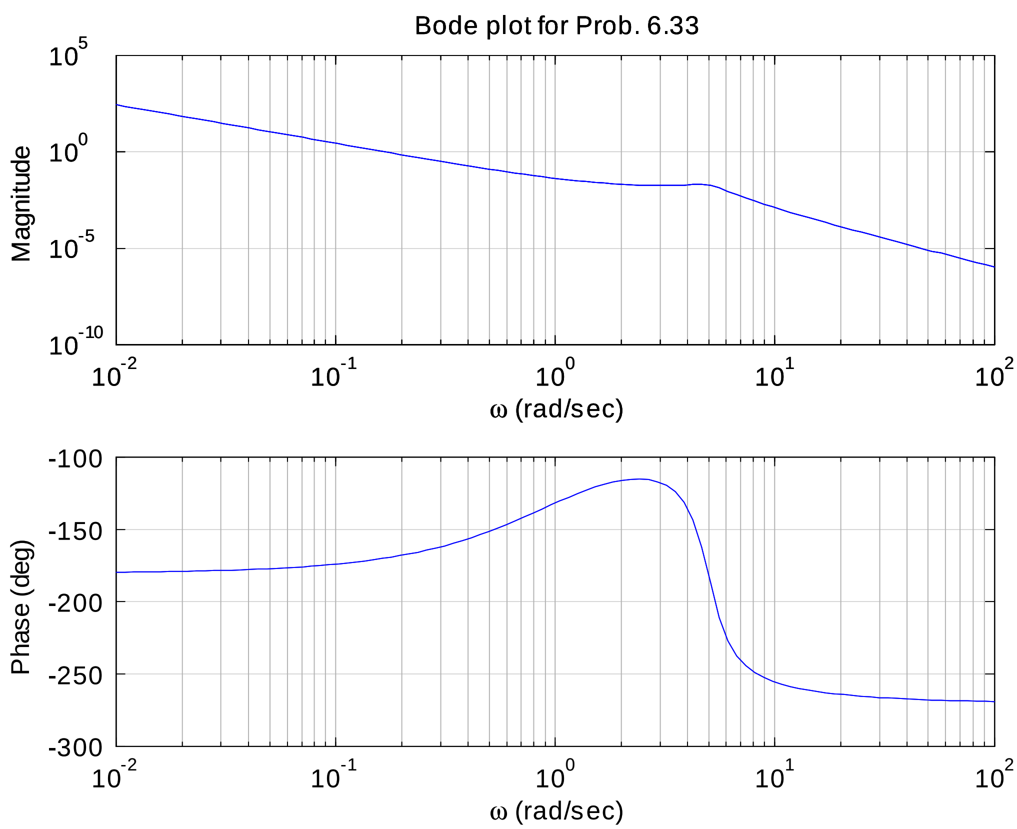
$\frac{Y_1(s)}{R(s)}$ and $\frac{Y_2(s)}{R(s)}$ have the same closed-loop poles. However, $\frac{Y_2(s)}{R(s)}$ has a zero, while $\frac{Y_1(s)}{R(s)}$ doesn't have a zero. We would therefore expect more overshoot from system (b).

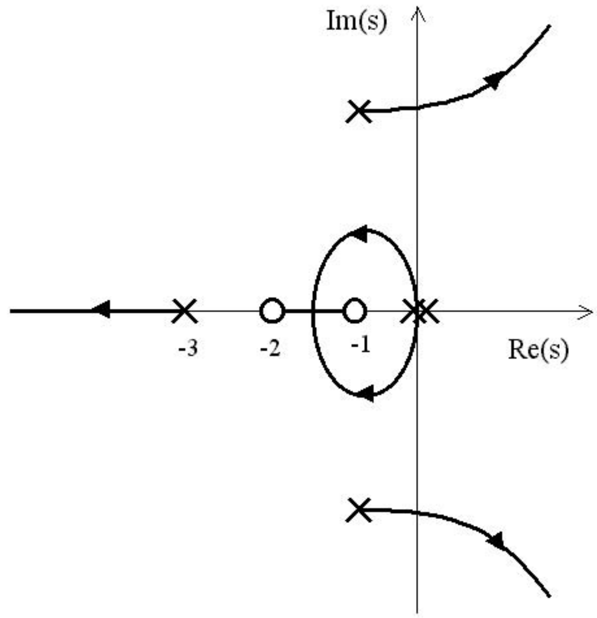
33. For the system shown in Fig. 6.95, use Bode and root-locus plots to determine the gain and frequency at which instability occurs. What gain (or gains) gives a PM of 20°? What is the gain margin when PM = 20°?

Solution :

Figure 6.95: Control system for Problem 33







The system with $K = 1$ gives,

$$\begin{aligned} GM &= 52\ (\omega = 5 \text{ rad/sec}) \\ PM &= 10^\circ\ (\omega = 0.165 \text{ rad/sec}) \end{aligned}$$

Therefore, instability occurs at $K_0 = 52$ and $\omega = 5$ rad/sec.

From the Bode plot, a PM of 20° is given by,

$$\begin{aligned} K_1 &= 3.9\ (\omega = 0.33 \text{ rad/sec}), \quad GM = \frac{52}{3.9} = 13 \\ K_2 &= 49\ (\omega = 4.6 \text{ rad/sec}), \quad GM = \frac{52}{49} = 1.06 \end{aligned}$$

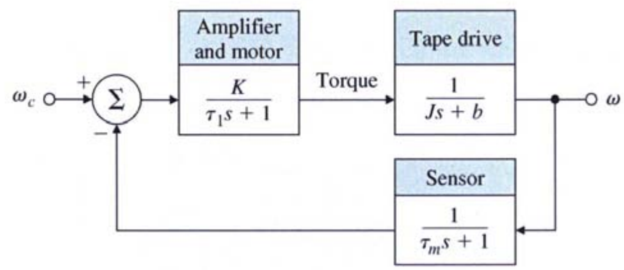
34. A magnetic tape-drive speed-control system is shown in Fig. 6.96. The speed sensor is slow enough that its dynamics must be included. The speed-measurement time constant is $\tau_m = 0.5$ sec; the reel time constant is $\tau_r = J/b = 4$ sec, where b = the output shaft damping constant = $1\ \text{N} \cdot \text{m} \cdot \text{sec}$; and the motor time constant is $\tau_1 = 1$ sec.

(a) Determine the gain K required to keep the steady-state speed error to less than 7% of the reference-speed setting.

(b) Determine the gain and phase margins of the system. Is this a good system design?

Solution :

Figure 6.96: Magnetic tape-drive speed control



(a) From Table 4.1, the error for this Type 1 system is

$$e_{ss} = \frac{1}{1+K}\,|\Omega_c|$$

Since the steady-state speed error is to be less than 7% of the reference speed,

$$\frac{1}{1+K_p} \leq 0.07$$

and for the system in Fig. 6.96 with the numbers plugged in, we see that $K_p = K$. Therefore, $K \geq 13$.

(b)

$$\begin{aligned} |G(s)| &= 0.79 \text{ at } \angle GH = -180^\circ \implies GM = \frac{1}{|GH|} = 1.3 \\ \angle GH &= -173^\circ \text{ at } |G(s)| = 1 \implies PM = \angle G + 180^\circ = 7^\circ \end{aligned}$$

GM is low $\implies$ The system is very close to instability.

PM is low $\implies$ The damping ratio is low. $\implies$ High overshoot.

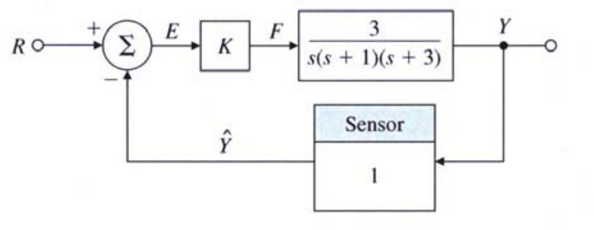
We see that to have a more stable system we have to lower the gain. With small gain, e_{ss} will be higher. Therefore, this is not a good design, and needs compensation.

35. For the system in Fig. 6.97, determine the Nyquist plot and apply the Nyquist criterion

(a) to determine the range of values of K (positive and negative) for which the system will be stable, and

(b) to determine the number of roots in the RHP for those values of K for which the system is unstable. Check your answer using a rough root-locus sketch.

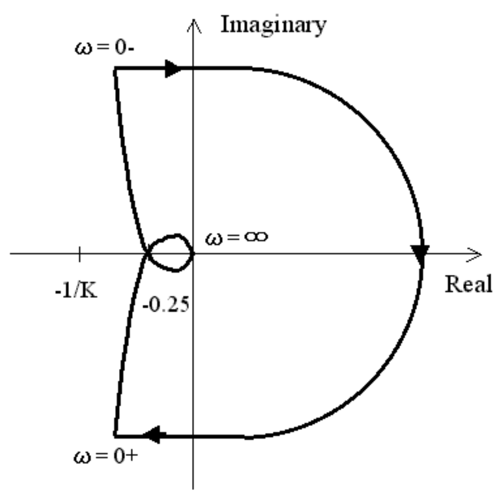
Figure 6.97: Control system for Problems 35, 69, and 70



Solution :

(a) & b.

$$KG(s) = K\frac{3}{s(s+1)(s+3)}$$



From the Nyquist plot above, we see that:

i.

$$-\infty < -\frac{1}{K} < -\frac{1}{4} \implies 0 < K < 4$$

There are no RHP open loop roots, hence $P = 0$ for all cases. For $0 < K < 4$, no encirclements of -1 so $N = 0$,

$$N = 0, \; P = 0 \implies Z = 0$$

The closed-loop system is stable. No roots in RHP.

ii.

$$-\frac{1}{4} < -\frac{1}{K} < 0 \implies 4 < K < \infty$$

Two encirclements of the -1 point, hence

$$N = 2,\ P = 0 \implies Z = 2$$

Two closed-loop roots in RHP.

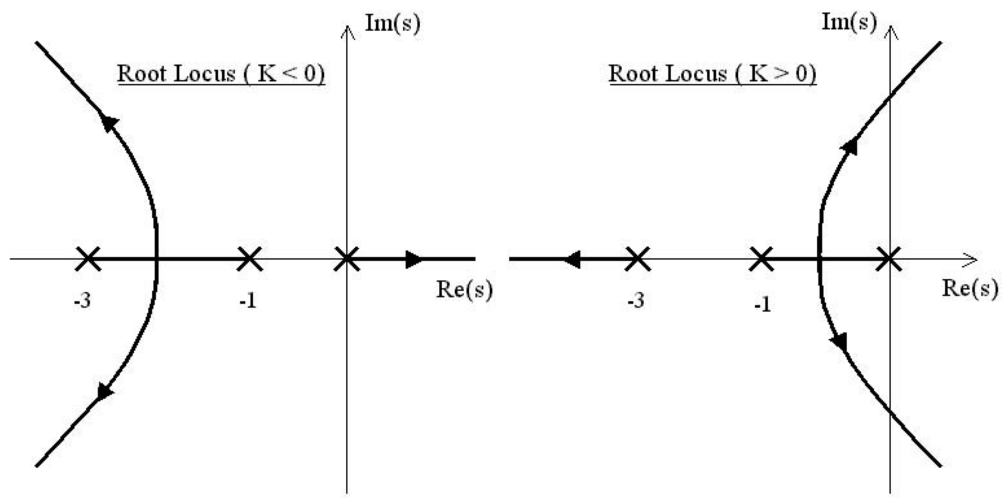
iii.

$$0 < -\frac{1}{K} \implies K < 0$$

$$N = 1,\ P = 0 \implies Z = 1$$

One closed-loop root in RHP.

The root loci below show the same results.



36. For the system shown in Fig. 6.98, determine the Nyquist plot and apply the Nyquist criterion.

    (a) to determine the range of values of K (positive and negative) for which the system will be stable, and

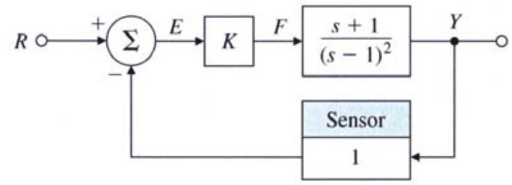
    (b) to determine the number of roots in the RHP for those values of K for which the system is unstable. Check your answer using a rough root-locus sketch.

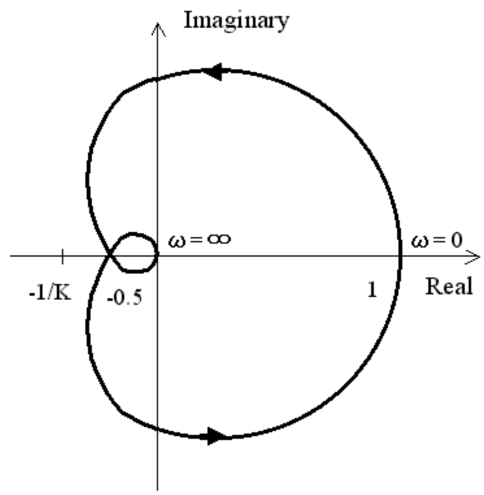
Solution :

(a) & b.

$$KG(s) = K\frac{s+1}{(s-1)^2}$$

Figure 6.98: Control system for Problem 36





From the Nyquist plot we see that:

i.

$$-\infty < -\frac{1}{K} < -\frac{1}{2} \implies 0 < K < 2$$

$$N = 0,\ P = 2 \implies Z = 2$$

Two closed-loop roots in RHP.

ii.

$$-\frac{1}{2} < -\frac{1}{K} < 0 \implies 2 < K$$

$$N = -2,\ P = 2 \implies Z = 0$$

The closed-loop system is stable.

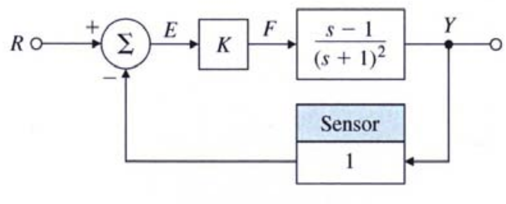
iii.

$$0 < -\frac{1}{K} < 1 \implies K < -1$$

$$N = -1,\ P = 2 \implies Z = 1$$

One closed-loop root in RHP.

Figure 6.99: Control system for Problem 37



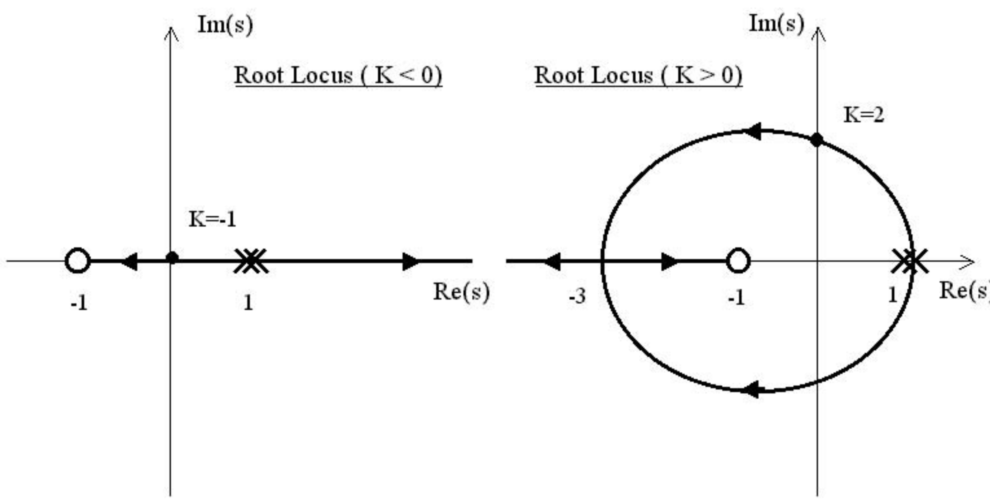
iv.

$$1 < -\frac{1}{K} < \infty \implies -1 < K < 0$$

$$N = 0,\ P = 2 \implies Z = 2$$

Two closed-loop roots in RHP.

These results are confirmed by looking at the root loci below:



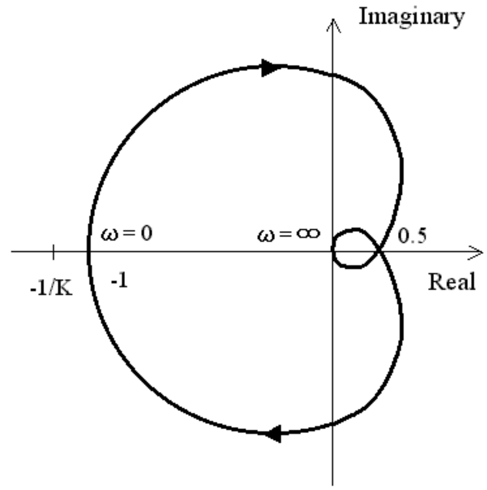
37. For the system shown in Fig. 6.99, determine the Nyquist plot and apply the Nyquist criterion.

   (a) to determine the range of values of K (positive and negative) for which the system will be stable, and

   (b) to determine the number of roots in the RHP for those values of K for which the system is unstable. Check your answer using a rough root-locus sketch.

Solution :

(a) & b.



$$KG(s) = K\frac{s-1}{(s+1)^2}$$

From the Nyquist plot we see that:

i.

$$-\infty < -\frac{1}{K} < -1 \implies 0 < K < 1$$

$$N = 0,\ P = 0 \implies Z = 0$$

The closed-loop system is stable.

ii.

$$-1 < -\frac{1}{K} < 0 \implies 1 < K$$

$$N = 1,\ P = 0 \implies Z = 1$$

One closed-loop root in RHP.

iii.

$$0 < -\frac{1}{K} < \frac{1}{2} \implies K < -2$$

$$N = 2,\ P = 0 \implies Z = 2$$

Two closed-loop roots in RHP.

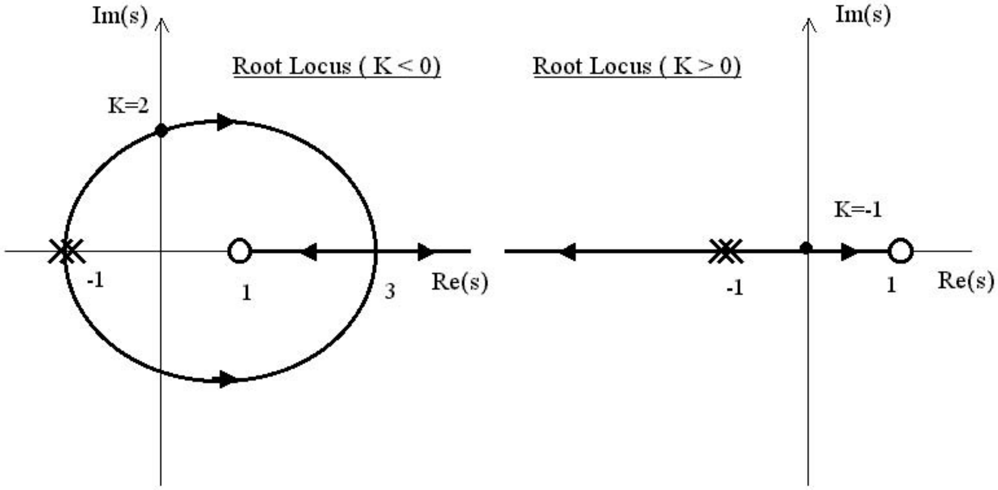
iv.

$$\frac{1}{2} < -\frac{1}{K} \implies -2 < K < 0$$

$$N = 0,\ P = 0 \implies Z = 0$$

The closed-loop system is stable.

These results are confirmed by looking at the root loci below:



38. The Nyquist diagrams for two stable, open-loop systems are sketched in Fig. 6.100. The proposed operating gain is indicated as K_0, and arrows indicate increasing frequency. In each case give a rough estimate of the following quantities for the closed-loop (unity feedback) system:

    (a) phase margin
    (b) damping ratio
    (c) range of gain for stability (if any)
    (d) system type (0, 1, or 2).

    Solution :

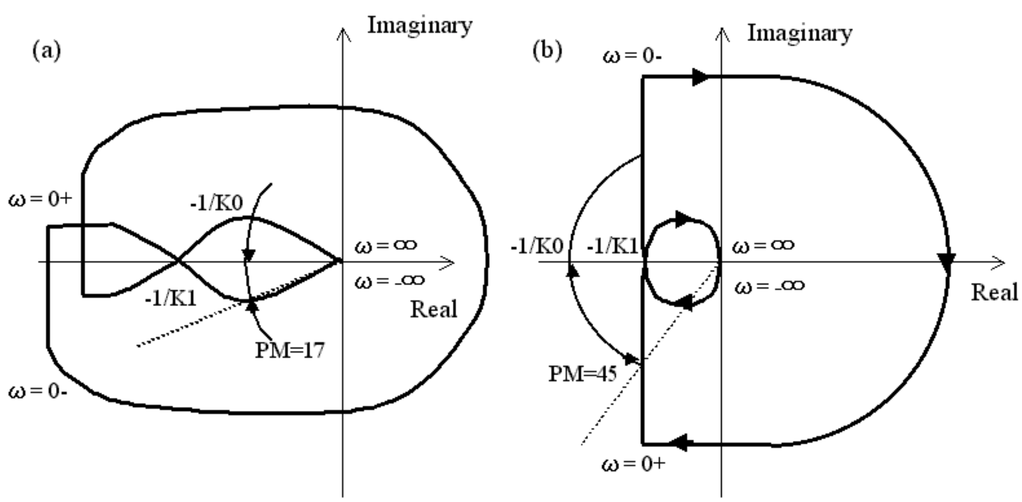
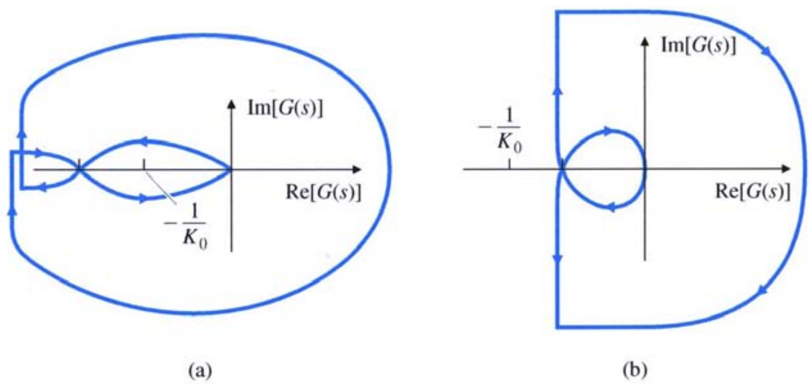


Figure 6.100: Nyquist plots for Problem 38



For both, with $K = K_0$:

$$N = 0,\ P = 0 \implies Z = 0$$

Therefore, the closed-loop system is stable.

	Fig.6.102(a)	Fig.6.102(b)
a. PM	$\simeq 17°$	$\simeq 45°$
b. Damping ratio	$0.17(\simeq \frac{17}{100})$	$0.45(\simeq \frac{45}{100})$

c. To determine the range of gain for stability, call the value of K where the plots cross the negative real axis as K_1. For case (a), $K > K_1$ for stability because gains lower than this amount will cause the -1 point to be encircled. For case (b), $K < K_1$ for stability because gains greater than this amount will cause the -1 point to be encircled.

d. For case (a), the 360^o loop indicates two poles at the origin, hence the system is Type 2. For case (b), the 180^o loop indicates one pole at the origin, hence the system is Type 1.

39. The steering dynamics of a ship are represented by the transfer function

$$\frac{V(s)}{\delta_r(s)} = G(s) = \frac{K[-(s/0.142)+1]}{s(s/0.325+1)(s/0.0362)+1)},$$

where v is the ship's lateral velocity in meters per second, and δ_r is the rudder angle in radians.

(a) Use the MATLAB command `bode` to plot the log magnitude and phase of $G(j\omega)$ for $K = 0.2$

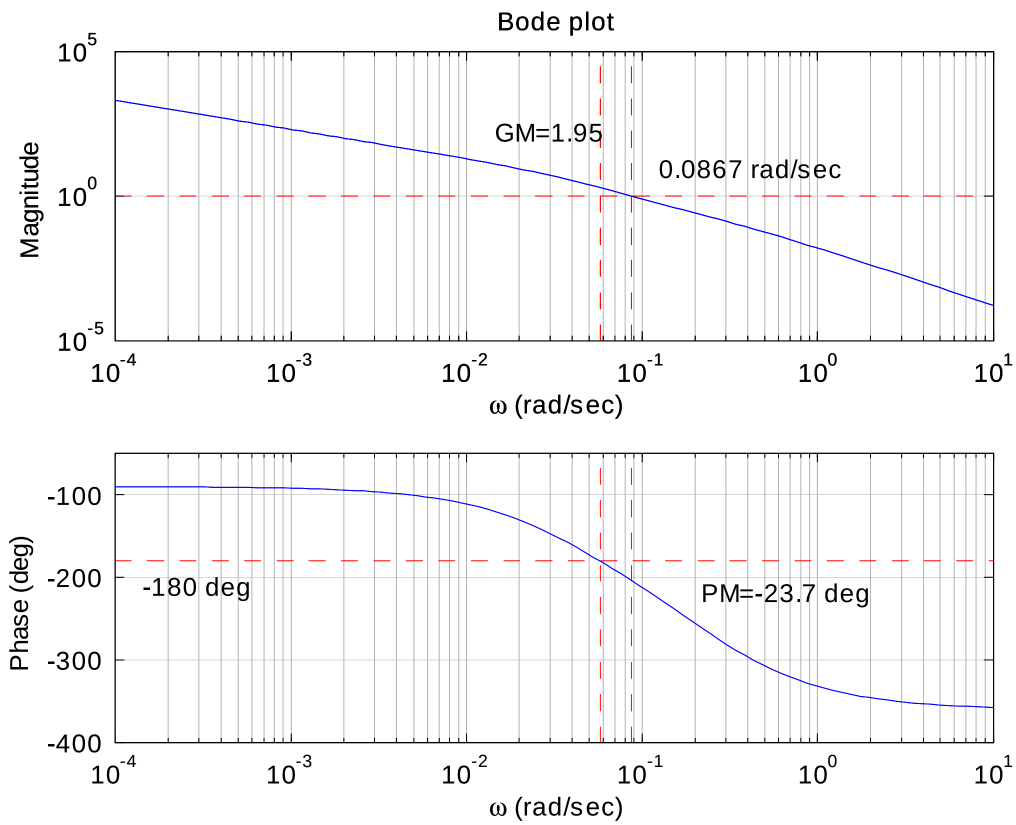
(b) On your plot, indicate the crossover frequency, PM, and GM,

(c) Is the ship steering system stable with $K = 0.2$?

(d) What value of K would yield a PM of 30^o and what would the crossover frequency be?

Solution :

(a) The Bode plot for $K = 0.2$ is :

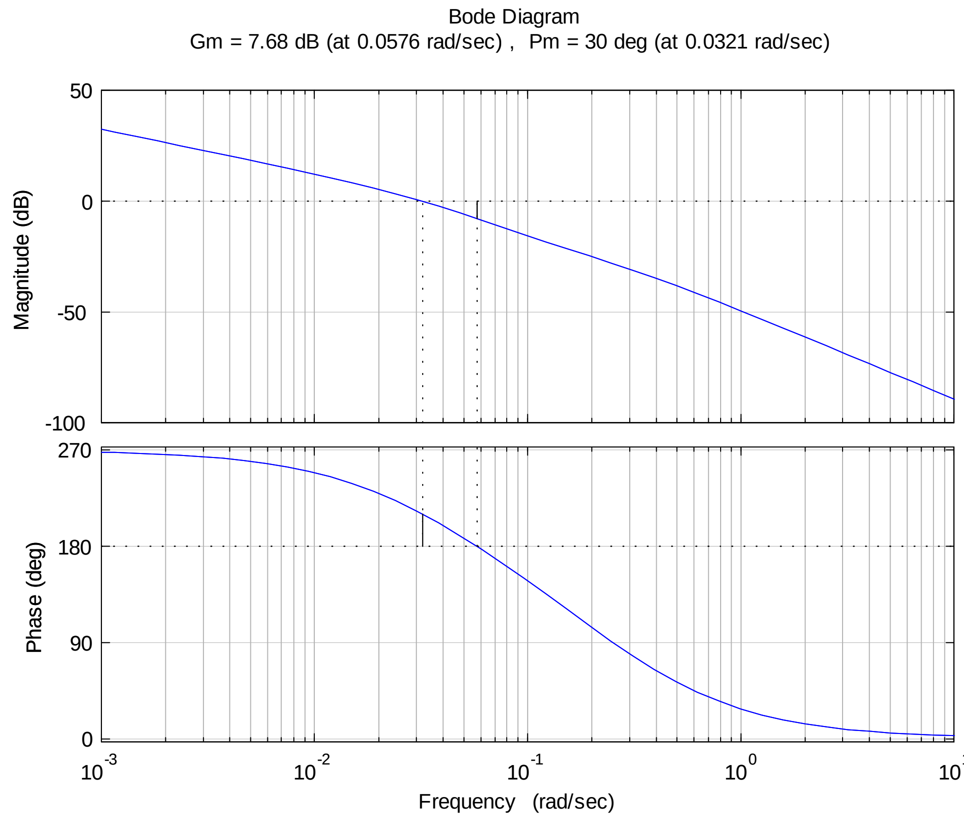


(b) From the Bode plot above :

$$\begin{aligned} \omega_c &= 0.0867 \text{ rad/sec} \\ PM &= -23.7 \text{ deg} \\ GM &= 1.95 \end{aligned}$$

(c) Since $PM < 0$, the closed-loop system with $K = 0.2$ is unstable.

(d) From the Bode plot above, we can get better PM by decreasing the gain K. Then we will find that $K = 0.0421$ yields $PM = 30°$ at the crossover frequency $\omega_c = 0.032$ rad/sec. The Bode plot with $K = 0.0421$ is :



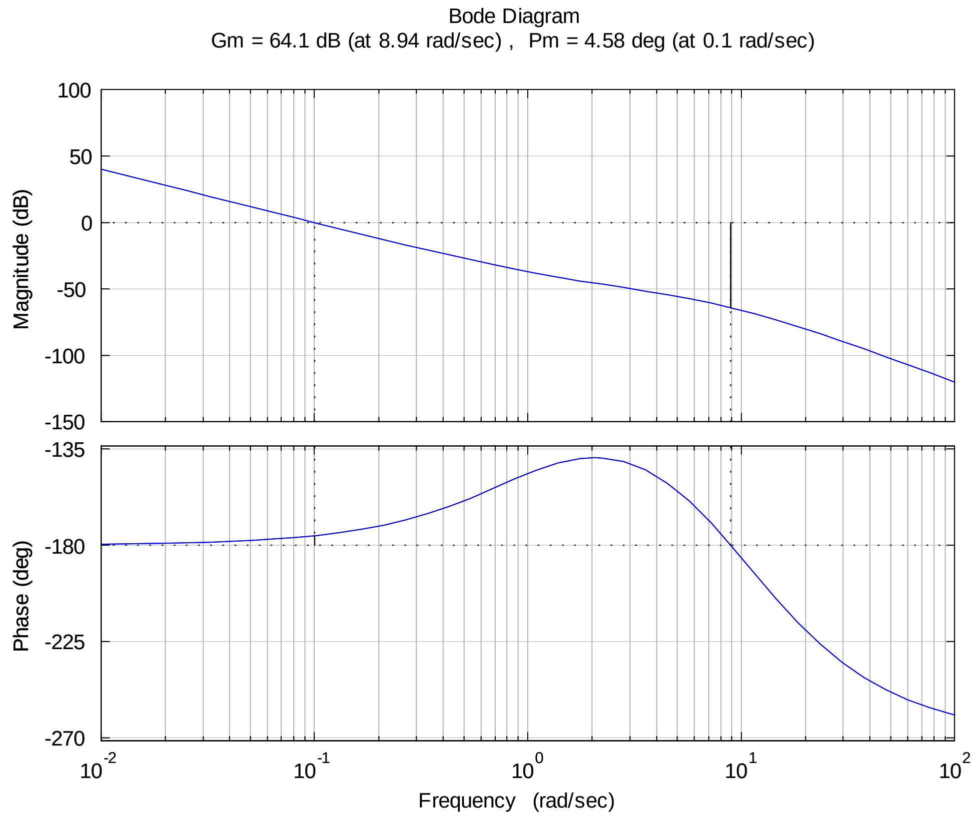
40. For the open-loop system

$$KG(s) = \frac{K(s+1)}{s^2(s+10)^2}.$$

Determine the value for K at the stability boundary and the values of K at the points where $PM = 30°$.

Solution :

The bode plot of this system with $K = 1$ is :



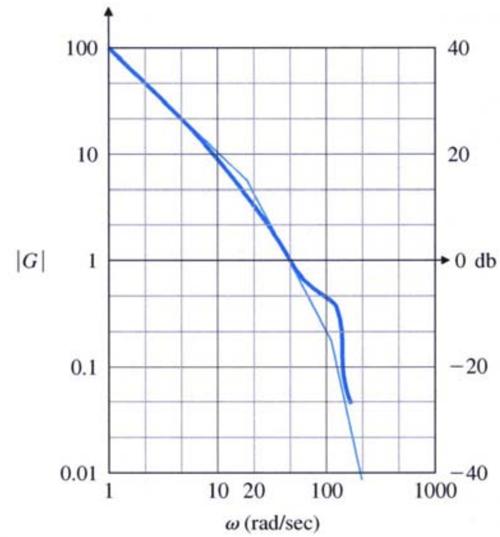
Since $GM = 64.1$ db ($\simeq 1600$), the range of K for stability is :

$$K < 1600$$

From the Bode plot, the magnitude at the frequency with $-150°$ phase is 0.0188 (-34.5 dB) at 0.8282 rad/sec and 0.00198 (-54.1 db) at 4.44 rad/sec. Therefore, the values of K at the points where $PM = 30°$ is :

$$\begin{aligned} K &= \frac{1}{0.0188} = 53.2, \\ K &= \frac{1}{0.00198} = 505 \end{aligned}$$

Figure 6.101: Magnitude frequency response for Problem 41



(a) Problems and Solutions for Section 6.5

41. The frequency response of a plant in a unity feedback configuration is sketched in Fig. 6.101. Assume the plant is open-loop stable and minimum phase.

    (a) What is the velocity constant K_v for the system as drawn?

    (b) What is the damping ratio of the complex poles at $\omega = 100$?

    (c) What is the PM of the system as drawn? (Estimate to within $\pm 10^o$.)

Solution :

(a) From Fig. 6.101,

$$K_v = \lim_{s \to 0} sG = |\text{Low frequency asymptote of } G(j\omega)|_{\omega=1} = 100)$$

(b) Let

$$G_1(s) = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\left(\frac{s}{\omega_n}\right) + 1}$$

For the second order system $G_1(s)$,

$$|G_1(j\omega)|_{\omega=1} = \frac{1}{2\zeta} \qquad (1)$$

From Fig. 6.101 :

$$|G_1(j\omega)|_{\omega=100} = \frac{|G(j\omega)|_{\omega=100}}{|\text{Asymptote of } G(j\omega)|_{\omega=100}} \cong \frac{0.4}{0.2} = 2 \qquad (2)$$

From (1) and (2) we have :

$$\frac{1}{2\zeta} = 2 \implies \zeta = 0.25$$

(c) Since the plant is a minimum phase system, we can apply the Bode's approximate gain-phase relationship.

When $|G| = 1$, the slope of $|G|$ curve is $\cong$ -2.

$$\implies \angle G(j\omega) \cong -2 \times 90^\circ \ = -180^\circ$$

$$PM \cong \angle G(j\omega) + 180^\circ \ = 0^\circ$$

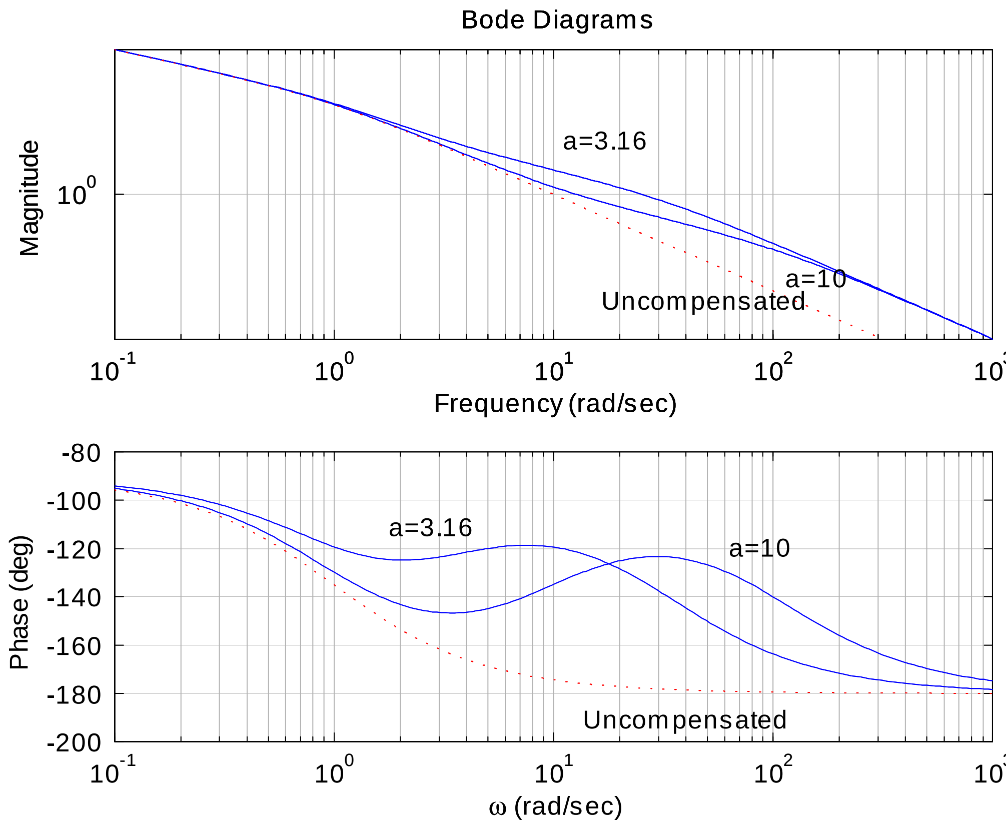
Note : Actual PM by Matlab calculation is 6.4°, so this approximation is within the desired accuracy.

42. For the system

$$G(s) = \frac{100(s/a+1)}{s(s+1)(s/b+1)},$$

where $b = 10a$, find the approximate value of a that will yield the best PM by sketching only candidate values of the frequency response magnitude.

Solution :



Without the zero and pole that contain the a & b terms, the plot of $|G|$ shows a slope of -2 at the $|G| = 1$ crossover at 10 rad/sec. We clearly need to install the zero and pole with the a & b terms somewhere at frequencies greater 1 rad/sec. This will increase the slope from -2 to -1 between the zero and pole. So the problem simplifies to selecting a so that the -1 slope region between the zero and pole brackets the crossover frequency. That scenario will maximize the PM. Referring to the plots above, we see that $3.16 < a < 10$, makes the slope of the asymptote of $|G|$ be -1 at the crossover and represent the two extremes of possibilities for a -1 slope. The maximum PM will occur half way between these extremes on a log scale, or

$$\Longrightarrow \; a = \sqrt{3.16 \times 10} = 5.6$$

Note : Actual PM is as follows :

$$
\begin{aligned}
PM &= 46.8^\circ \text{ for } a = 3.16\ (\omega_c = 25.0 \text{ rad/sec}) \\
PM &= 58.1^\circ \text{ for } a = 5.6\ \ (\omega_c = 17.8 \text{ rad/sec}) \\
PM &= 49.0^\circ \text{ for } a = 10\ (\omega_c = 12.6 \text{ rad/sec})
\end{aligned}
$$

Problem and Solution for Section 6.6

43. For the open-loop system

$$KG(s) = \frac{K(s+1)}{s^2(s+10)^2}.$$

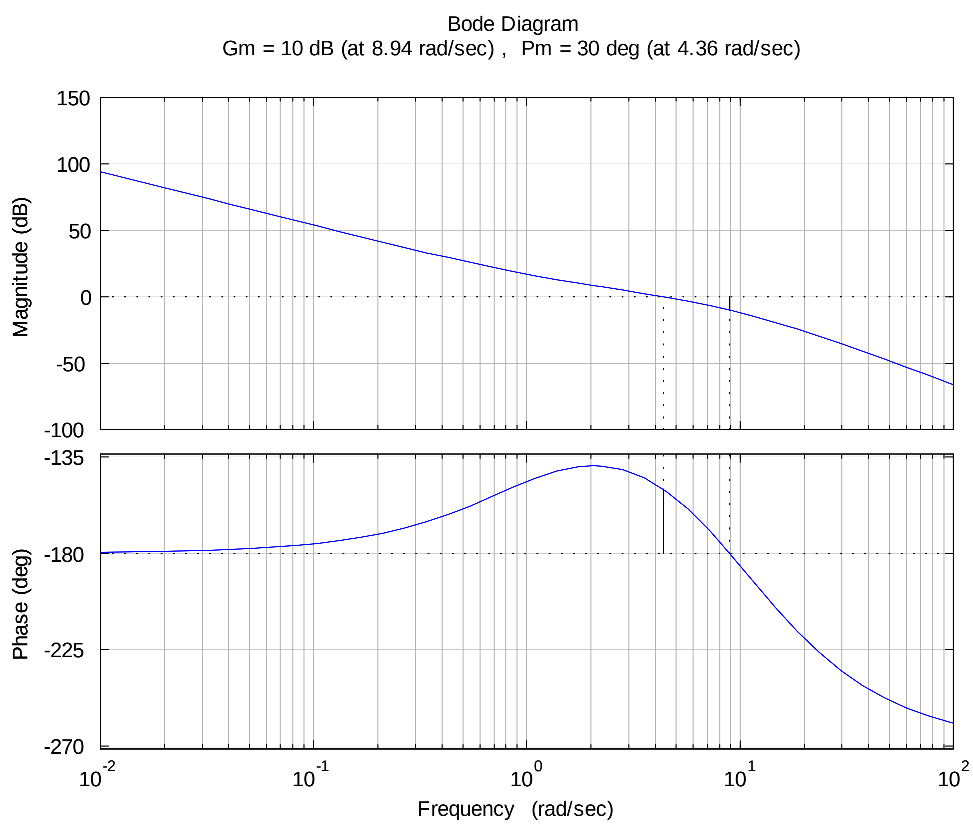
Determine the value for K that will yield PM $\geq 30°$ and the maximum possible closed-loop bandwidth. Use MATLAB to find the bandwidth.

Solution :

From the result of Problem 6.39., the value of K that will yield $PM \geq 30°$ is :

$$53.2 \leq K \leq 505$$

The maximum closed-loop bandwidth will occur with the maximum gain K within the allowable region; therefore, the maximum bandwidth will occur with $K = 505$. The Bode plot of the closed loop system with $K = 505$ is :



Looking at the point with Magnitude 0.707(-3 db), the maximum possible closed-loop bandwidth is :

$$\omega_{BW,\ \max} \simeq 7.7 \text{ rad/sec.}$$

Problems and Solutions for Section 6.7

44. For the lead compensator

$$D(s) = \frac{Ts+1}{\alpha Ts+1},$$

where $\alpha < 1$.

(a) Show that the phase of the lead compensator is given by

$$\phi = \tan^{-1}(T\omega) - \tan^{-1}(\alpha T\omega).$$

(b) Show that the frequency where the phase is maximum is given by

$$\omega_{\max} = \frac{1}{T\sqrt{\alpha}},$$

and that the maximum phase corresponds to

$$\sin\phi_{\max} = \frac{1-\alpha}{1+\alpha}.$$

(c) Rewrite your expression for $\omega_{\max}$ to show that the maximum-phase frequency occurs at the geometric mean of the two corner frequencies on a logarithmic scale:

$$\log\omega_{\max} = \frac{1}{2}\left(\log\frac{1}{T} + \log\frac{1}{\alpha T}\right).$$

(d) To derive the same results in terms of the pole-zero locations, rewrite $D(s)$ as

$$D(s) = \frac{s+z}{s+p},$$

and then show that the phase is given by

$$\phi = \tan^{-1}\left(\frac{\omega}{|z|}\right) - \tan^{-1}\left(\frac{\omega}{|p|}\right),$$

such that

$$\omega_{\max} = \sqrt{|z||p|}.$$

Hence the frequency at which the phase is maximum is the square root of the product of the pole and zero locations.

Solution :

(a) The frequency response is obtained by letting $s = j\omega$,

$$D(j\omega) = K\frac{Tj\omega + 1}{\alpha Tj\omega + 1}$$

The phase is given by,$\phi = \tan^{-1}(T\omega) - \tan^{-1}(\alpha T\omega)$

(b) Using the trigonometric relationship,

$$\tan(A - B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}$$

then

$$\tan(\phi) = \frac{T\omega - \alpha T\omega}{1 - \alpha T^2\omega^2}$$

and since,

$$\sin^2(\phi) = \frac{\tan^2(\phi)}{1 + \tan^2(\phi)}$$

then

$$\sin(\phi) = \sqrt{\frac{\omega^2T^2(1-\alpha)^2}{1 + \alpha^2\omega^4T^4 + (1+\alpha^2)\omega^2T^2}}$$

To determine the frequency at which the phase is a maximum, let us set the derivative with respect to ω equal to zero,

$$\frac{d\sin(\phi)}{d\omega} = 0$$

which leads to

$$2\omega T^2(1-\alpha)^2(1 - \alpha\omega^4T^4) = 0$$

The value $\omega = 0$ gives the maximum of the function and setting the second part of the above equation to zero then,

$$\omega^4 = \frac{1}{\alpha^2T^4}$$

or

$$\omega_{\max} = \frac{1}{\sqrt{\alpha}T}$$

The maximum phase contribution, that is, the peak of the $\angle D(s)$ curve corresponds to,

$$\sin\phi_{\max} = \frac{1-\alpha}{1+\alpha}$$

or

$$\alpha = \frac{1 - \sin\phi_{\max}}{1 + \sin\phi_{\max}}$$

$$\tan\phi_{\max} = \frac{\omega_{\max}T - \alpha\omega_{\max}T}{1 + \omega_{\max}^2T^2} = \frac{1-\alpha}{2\sqrt{\alpha}}$$

(c) The maximum frequency occurs midway between the two break frequencies on a logarithmic scale,

$$\begin{aligned}\log \omega_{\max} &= \log \frac{\frac{1}{\sqrt{T}}}{\sqrt{\alpha T}} \\ &= \log \frac{1}{\sqrt{T}} + \log \frac{1}{\sqrt{\alpha T}} \\ &= \frac{1}{2}\left(\log \frac{1}{T} + \log \frac{1}{\alpha T}\right)\end{aligned}$$

as shown in Fig. 6.53.

(d) Alternatively, we may state these results in terms of the pole-zero locations. Rewrite $D(s)$ as,

$$D(s) = K\frac{(s+z)}{(s+p)}$$

then

$$D(j\omega) = K\frac{(j\omega+z)}{(j\omega+p)}$$

and

$$\phi = \tan^{-1}\left(\frac{\omega}{|z|}\right) - \tan^{-1}\left(\frac{\omega}{|p|}\right)$$

or

$$\tan\phi = \frac{\frac{\omega}{|z|} - \frac{\omega}{|p|}}{1 + \frac{\omega}{|z|}\frac{\omega}{|p|}}$$

Setting the derivative of the above equation to zero we find,

$$\left(\frac{\omega}{|z|} - \frac{\omega}{|p|}\right)\left(1 + \frac{\omega^2}{|z|\,|p|}\right) - \frac{2\omega}{|z|\,|p|}\left(\frac{\omega}{|z|} - \frac{\omega}{|p|}\right) = 0$$

and

$$\omega_{\max} = \sqrt{|z|\,|p|}$$

and

$$\log \omega_{\max} = \frac{1}{2}\left(\log|z| + \log|p|\right)$$

Hence the frequency at which the phase is maximum is the square root of the product of the pole and zero locations.

45. For the third-order servo system

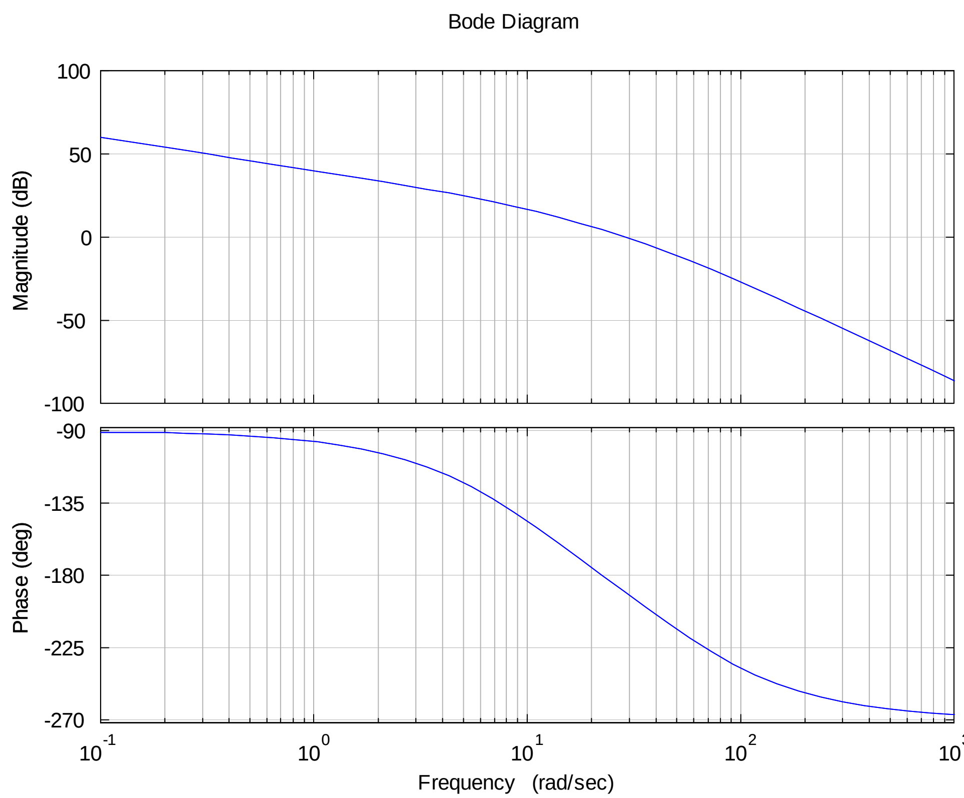
$$G(s) = \frac{50,000}{s(s+10)(s+50)}.$$

Design a lead compensator so that PM $\geq 50°$ and $\omega_{BW} \geq 20$ rad/sec using Bode plot sketches, then verify and refine your design using MATLAB.

Solution :

Let's design the lead compensator so that the system has $PM \geq 50°$ & $\omega_{WB} \simeq \omega_c \geq 20$ rad/sec.

The Bode plot of the given system is :



Start with a lead compensator design with :

$$D(s) = \frac{Ts+1}{\alpha Ts+1} \ (\alpha < 1)$$

Since the open-loop crossover frequency $\omega_c (\simeq \omega_{BW})$ is already above 20 rad/sec, we are going to just add extra phase around $\omega = \omega_c$ in order to satisfy $PM = 50°$.

Let's add phase lead $\geq 60°$. From Fig. 6.54,

$$\frac{1}{\alpha} \simeq 20 \Longrightarrow \text{choose } \alpha = 0.05$$

To apply maximum phase lead at $\omega = 20$ rad/sec,

$$\omega = \frac{1}{\sqrt{\alpha}T} = 20 \Longrightarrow \frac{1}{T} = 4.48,\ \frac{1}{\alpha T} = 89.4$$

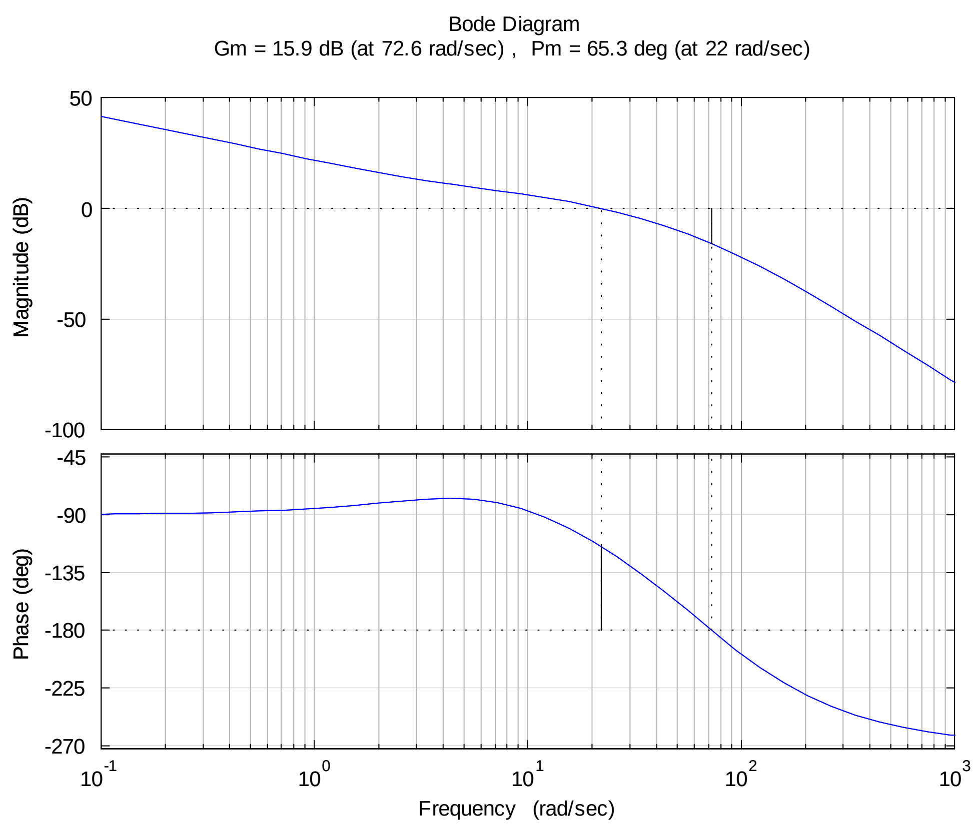
Therefore by applying the lead compensator with some gain adjustments :

$$D(s) = 0.12 \times \frac{\dfrac{s}{4.5}+1}{\dfrac{s}{90}+1}$$

we get the compensated system with :

$$PM = 65^\circ,\ \omega_c = 22 \text{ rad/sec, so that } \omega_{BW} \gtrsim 25 \text{ rad/sec.}$$

The Bode plot with designed compensator is :



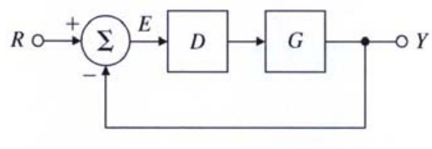


Figure 6.102: Control system for Problem 46

46. For the system shown in Fig. 6.102, suppose that

$$G(s) = \frac{5}{s(s+1)(s/5+1)}.$$

Design a lead compensation $D(s)$ with unity DC gain so that PM $\geq 40°$ using Bode plot sketches, then verify and refine your design using MATLAB. What is the approximate bandwidth of the system?

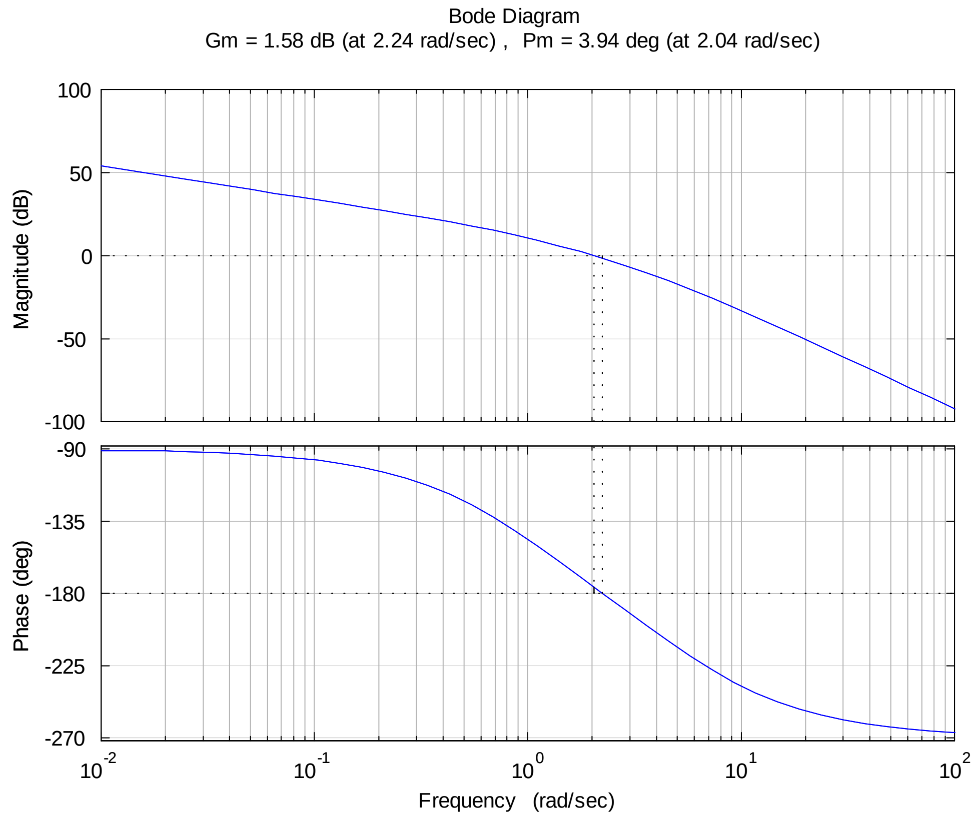
Solution :

Start with a lead compensator design with :

$$D(s) = \frac{Ts+1}{\alpha Ts+1}$$

which has unity DC gain with $\alpha < 1$.

The Bode plot of the given system is :



Since $PM = 3.9°$, let's add phase lead $\geq 60°$. From Fig. 6.53,

$$\frac{1}{\alpha} \simeq 20 \Longrightarrow \text{choose } \alpha = 0.05$$

To apply maximum phase lead at $\omega = 10$ rad/sec,

$$\omega = \frac{1}{\sqrt{\alpha}T} = 10 \Longrightarrow \frac{1}{T} = 2.2, \ \frac{1}{\alpha T} = 45$$

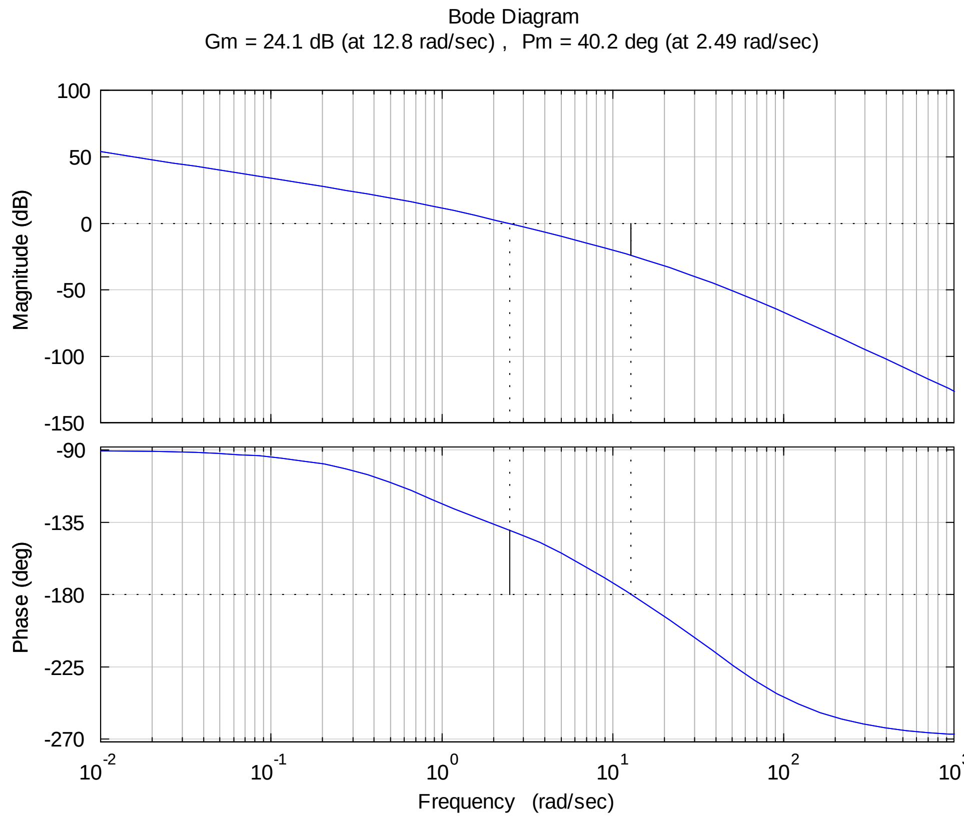
Therefore by applying the lead compensator :

$$D(s) = \frac{\frac{s}{2.2} + 1}{\frac{s}{45} + 1}$$

we get the compensated system with :

$$PM = 40°, \ \omega_c = 2.5$$

The Bode plot with designed compensator is :



From Fig. 6.51, we see that $\omega_{BW} \simeq 2 \times \omega_c \simeq 5$ rad/sec.

47. Derive the transfer function from T_d to θ for the system in Fig. 6.70. Then apply the Final Value Theorem (assuming T_d = constant) to determine whether $\theta(\infty)$ is nonzero for the following two cases:

(a) When $D(s)$ has no integral term: $\lim_{s\to 0} D(s) = \text{constant}$;

(b) When $D(s)$ has an integral term:

$$D(s) = \frac{D'(s)}{s},$$

where $\lim_{s\to 0} D'(s) = \text{constant}$.

Solution :

The transfer function from T_d to θ :

$$\frac{\Theta(s)}{T_d(s)} = \frac{\frac{0.9}{s^2}}{1 + \frac{0.9}{s^2}\frac{2}{s+2}D(s)}$$

where $T_d(s) = |T_d|/s$.

(a) Using the final value theorem :

$$\begin{aligned}\theta(\infty) &= \lim_{t\to\infty}\theta(t) = \lim_{s\to 0} s\Theta(t) = \lim_{s\to 0}\frac{\frac{0.9}{s^2}}{\frac{s^2(s+2)+1.8D(s)}{s^2(s+2)}}\frac{|T_d|}{s} \\ &= \frac{|T_d|}{\lim\limits_{s\to 0} D(s)} = \frac{|T_d|}{\text{constant}} \neq 0\end{aligned}$$

Therefore, there will be a steady state error in θ for a constant T_d input if there is no integral term in D(s).

(b)

$$\begin{aligned}\theta(\infty) &= \lim_{t\to\infty}\theta(t) = \lim_{s\to 0} s\Theta(t) = \lim_{s\to 0}\frac{\frac{0.9}{s^2}}{\frac{s^3(s+2)+1.8D'(s)}{s^3(s+2)}}\frac{|T_d|}{s} \\ &= \frac{0}{1.8\lim\limits_{s\to 0} D'(s)} = 0\end{aligned}$$

So when D(s) contains an integral term, a constant T_d input will result in a zero steady state error in θ.

48. The inverted pendulum has a transfer function given by Eq. (2.31), which is similar to

$$G(s) = \frac{1}{s^2 - 1}.$$

(a) Design a lead compensator to achieve a PM of 30° using Bode plot sketches, then verify and refine your design using MATLAB.

(b) Sketch a root locus and correlate it with the Bode plot of the system.

(c) Could you obtain the frequency response of this system experimentally?

Solution :

(a) Design the lead compensator :

$$D(s) = K\frac{Ts+1}{\alpha Ts+1}$$

such that the compensated system has $PM \simeq 30°$ & $\omega_c \simeq 1$ rad/sec. (Actually, the bandwidth or speed of response was not specified, so any crossover frequency would satisfy the problem statement.)

$$\alpha = \frac{1-\sin(30°)}{1+\sin(30°)} = 0.32$$

To apply maximum phase lead at $\omega = 1$ rad/sec,

$$\omega = \frac{1}{\sqrt{\alpha}T} = 1 \Longrightarrow \frac{1}{T} = 0.57, \ \frac{1}{\alpha T} = 1.77$$

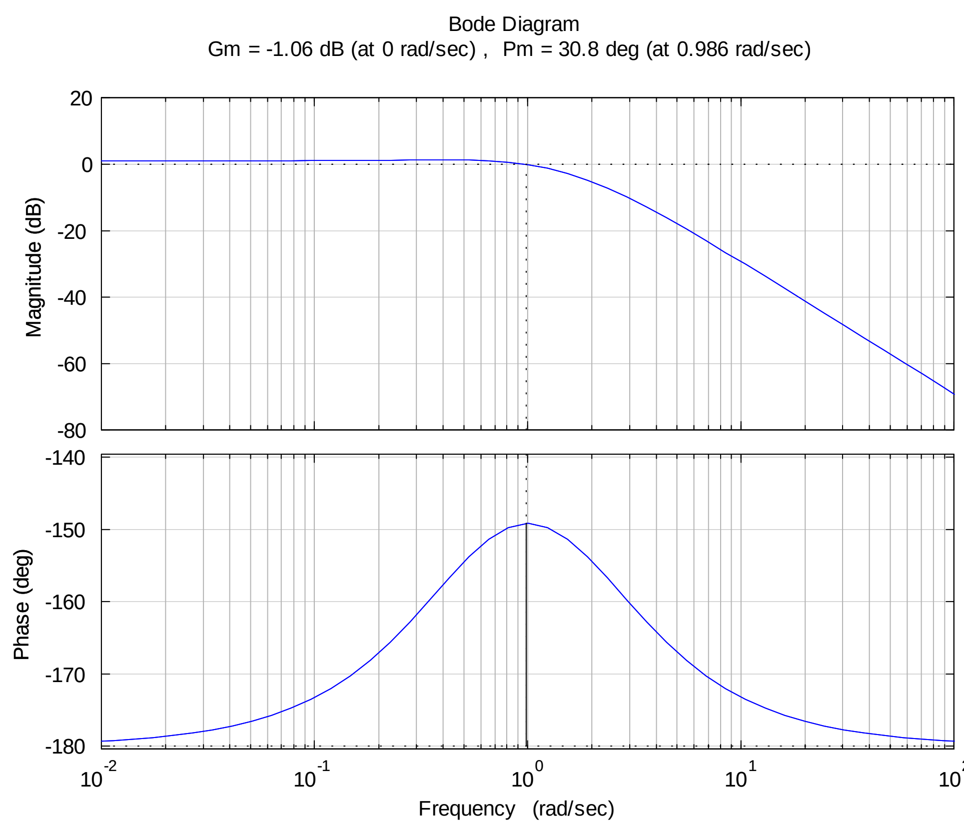
Therefore by applying the lead compensator :

$$D(s) = K\frac{\frac{s}{0.57}+1}{\frac{s}{1.77}+1}$$

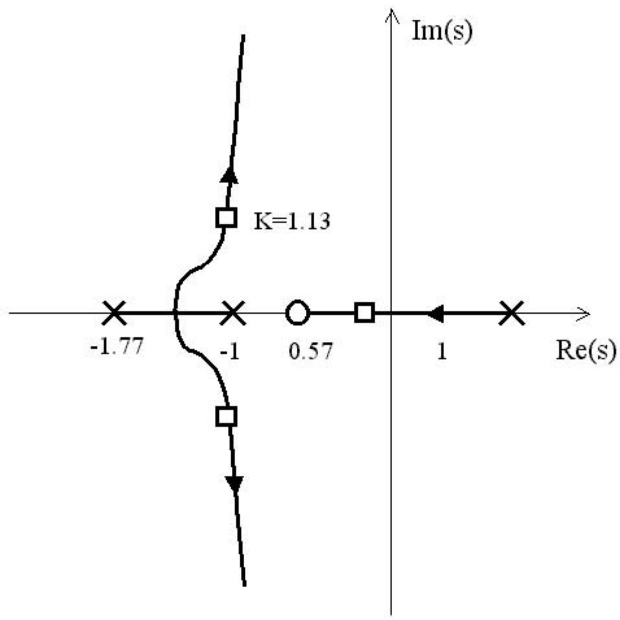
By adjusting the gain K so that the crossover frequency is around 1 rad/sec, $K = 1.13$ results in :

$$PM = 30.8^\circ$$

The Bode plot of compensated system is :



(b) Root Locus of the compensated system is :



and confirms that the system yields all stable roots with reasonable damping. However, it would be a better design if the gain was raised some in order to increase the speed of response of the slow real root. A small decrease in the damping of the complex roots will result.

(c) No, because the sinusoid input will cause the system to blow up because the open loop system is unstable. In fact, the system will "blow up" even without the sinusoid applied. Or, a better description would be that the pendulum will fall over until it hits the table.

49. The open-loop transfer function of a unity feedback system is

$$G(s) = \frac{K}{s(s/5+1)(s/50+1)}.$$

(a) Design a lag compensator for $G(s)$ using Bode plot sketches so that the closed-loop system satisfies the following specifications:

i. The steady-state error to a unit ramp reference input is less than 0.01.

ii. PM $\geq = 40°$

(b) Verify and refine your design using MATLAB.

Solution :

Let's design the lag compensator :

$$D(s) = \frac{Ts+1}{\alpha Ts+1},\ \alpha > 1$$

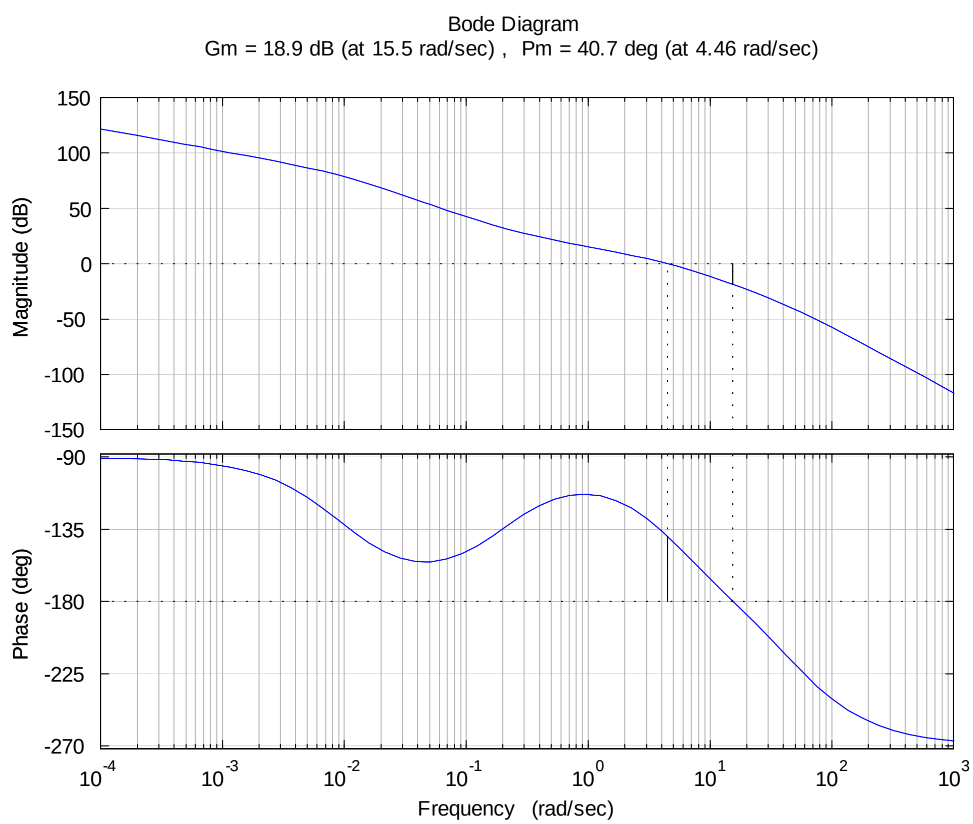
From the first specification,

$$\begin{aligned}
\text{Steady-state error to unit ramp} &= \lim_{s\to 0} s\left|\frac{D(s)G(s)}{1+D(s)G(s)}\frac{1}{s^2}-\frac{1}{s^2}\right| < 0.01 \\
&\Longrightarrow \frac{1}{K} < 0.01 \\
&\Longrightarrow \text{Choose } K = 150
\end{aligned}$$

Uncompensated, the crossover frequency with $K = 150$ is too high for a good PM. With some trial and error, we find that the lag compensator,

$$D(s) = \frac{\frac{s}{0.2}+1}{\frac{s}{0.01}+1}$$

will lower the crossover frequency to $\omega_c \simeq 4.46$ rad/sec where the $PM = 40.7°$.



50. The open-loop transfer function of a unity feedback system is

$$G(s) = \frac{K}{s(s/5+1)(s/200+1)}.$$

(a) Design a lead compensator for $G(s)$ using Bode plot sketches so that the closed-loop system satisfies the following specifications:

i. The steady-state error to a unit ramp reference input is less than 0.01.

ii. For the dominant closed-loop poles the damping ratio $\zeta \geq 0.4$.

(b) Verify and refine your design using MATLAB including a direct computation of the damping of the dominant closed-loop poles.

Solution :

Let's design the lead compensator :

$$D(s) = \frac{Ts+1}{\alpha Ts+1},\ \alpha < 1$$

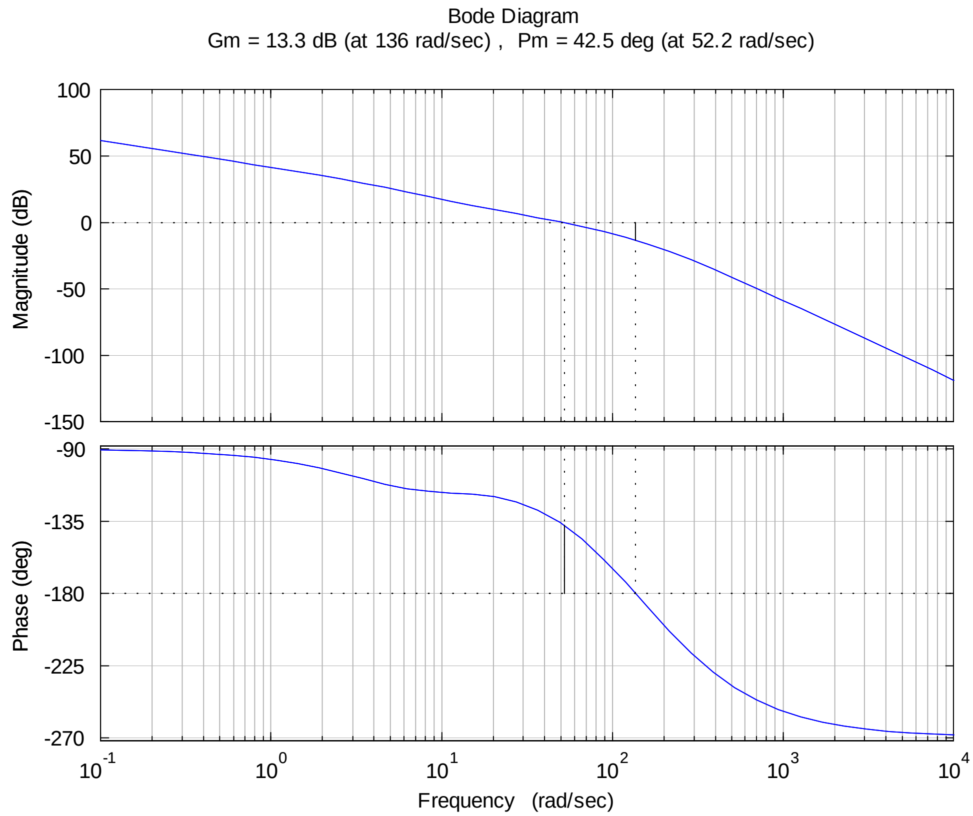
From the first specification,

$$\begin{aligned} \text{Steady-state error to unit ramp} &= \lim_{s\to 0} s \left| \frac{D(s)G(s)}{1+D(s)G(s)}\frac{1}{s^2} - \frac{1}{s^2} \right| < 0.01 \\ &\Longrightarrow \frac{1}{K} < 0.01 \\ &\Longrightarrow \text{Choose } K = 150 \end{aligned}$$

From the approximation $\zeta \simeq \frac{PM}{100}$, the second specification implies $PM \geq 40$. After trial and error, we find that the compensator,

$$D(s) = \frac{\frac{s}{10}+1}{\frac{s}{100}+1}$$

results in a $PM = 42.5°$ and a crossover frequency $\omega_c \simeq 51.2$ rad/sec as shown by the `margin` output:



and the use of `damp` verifies the damping to be $\zeta = 0.42$ for the complex closed-loop roots which exceeds the requirement.

51. A DC motor with negligible armature inductance is to be used in a position control system. Its open-loop transfer function is given by

$$G(s) = \frac{50}{s(s/5+1)}.$$

   (a) Design a compensator for the motor using Bode plot sketches so that the closed-loop system satisfies the following specifications:

      i. The steady-state error to a unit ramp input is less than 1/200.

      ii. The unit step response has an overshoot of less than 20%.

      iii. The bandwidth of the compensated system is no less than that of the uncompensated system.

   (b) Verify and/or refine your design using MATLAB including a direct computation of the step response overshoot.

Solution :

The first specification implies that a loop gain greater than 200 is required. Since the open loop gain of the plant is 50, a gain from the compensator, K, is required where

$$K > 4 \Longrightarrow \text{ so choose } K = 5$$

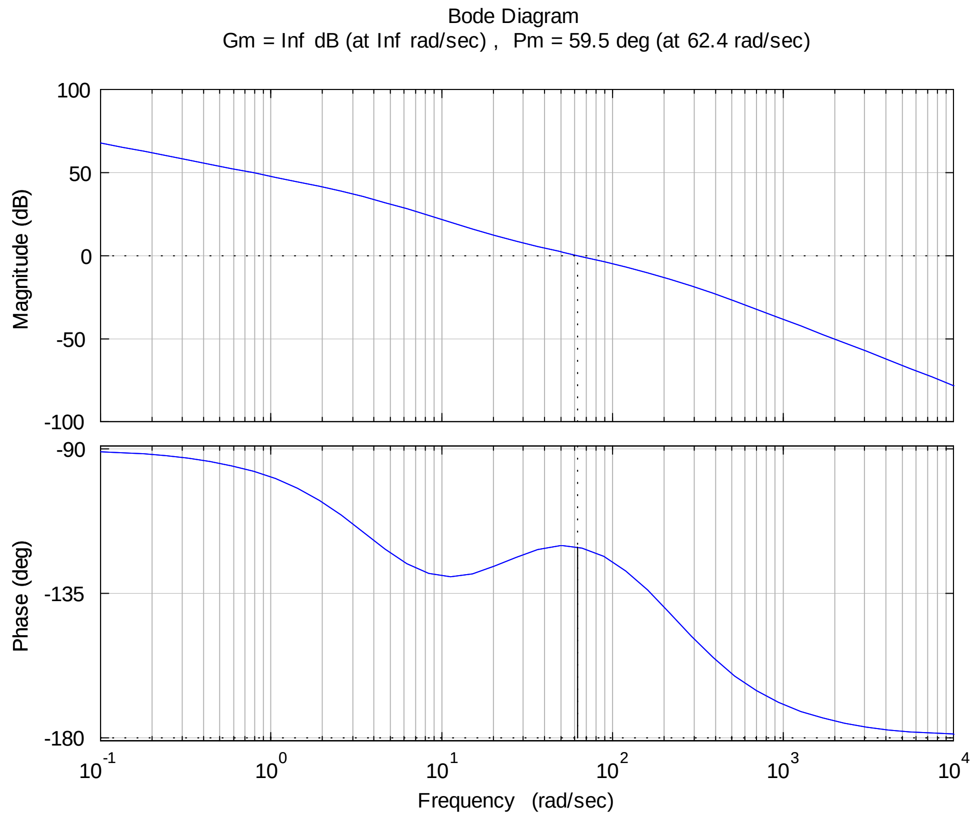
From Figure 3.23, we see that the second specification implies that :

$$\text{Overshoot} < 20\% \Longrightarrow \zeta > 0.5 \Longrightarrow PM > 50^\circ$$

A sketch of the Bode asymptotes of the open loop system with the required loop gain shows a crossover frequency of about 30 rad/sec at a slope of -2; hence, the PM will be quite low. To add phase with no decrease in the crossover frequency, a lead compensator is required. Figure 6.53 shows that a lead ratio of 10:1 will provide about 55^o of phase increase and the asymptote sketch shows that this increase will be centered at the crossover frequency if we select the break points at

$$D(s) = \frac{\frac{s}{20} + 1}{\frac{s}{200} + 1}.$$

Use of Matlab's `margin` routine shows that this compensation results in a $PM = 59^\circ$ and a crossover frequency $\omega_c \simeq 60$ rad/sec.



and using the `step` routine on the closed loop system shows the step response to be less than the maximum allowed 20%.

52. The open-loop transfer function of a unity feedback system is

$$G(s) = \frac{K}{s(1+s/5)(1+s/20)}.$$

(a) Sketch the system block diagram including input reference commands and sensor noise.

(b) Design a compensator for $G(s)$ using Bode plot sketches so that the closed-loop system satisfies the following specifications:

i. The steady-state error to a unit ramp input is less than 0.01.

ii. PM $\geq 45°$

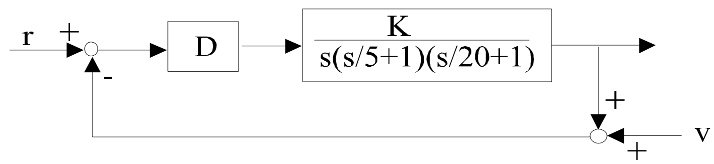
iii. The steady-state error for sinusoidal inputs with $\omega < 0.2$ rad/sec is less than 1/250.

iv. Noise components introduced with the sensor signal at frequencies greater than 200 rad/sec are to be attenuated at the output by at least a factor of 100,.

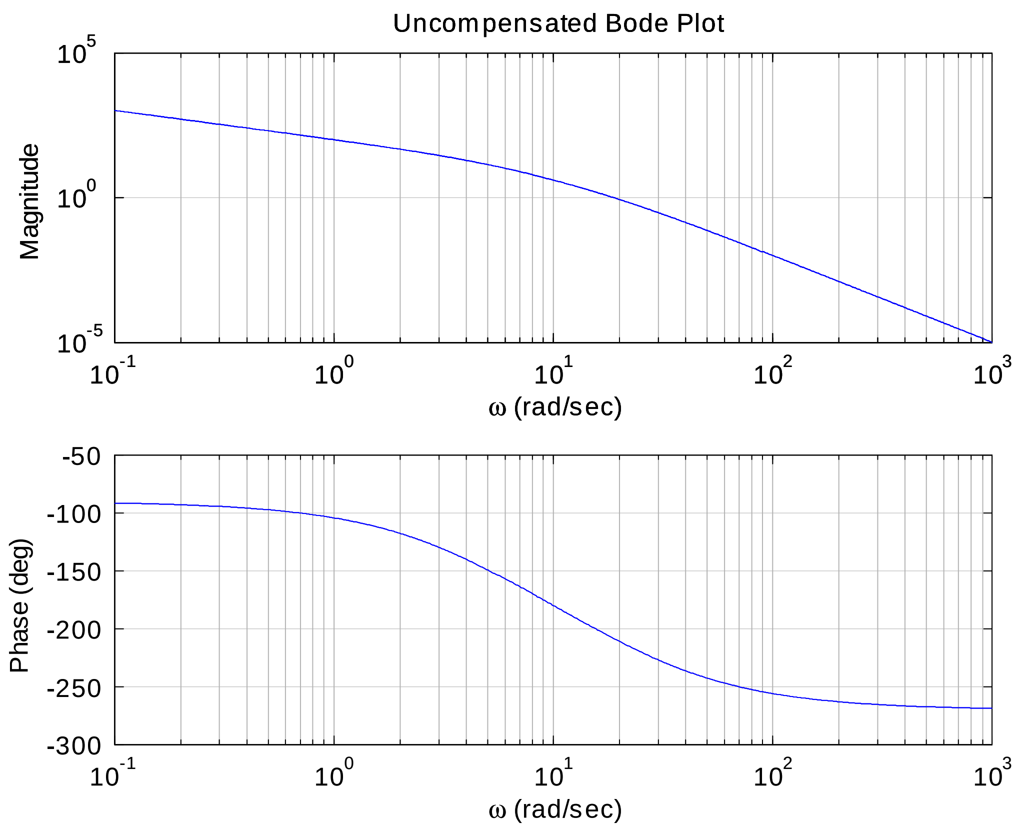
(c) Verify and/or refine your design using MATLAB including a computation of the closed-loop frequency response to verify (iv).

Solution :

a. The block diagram shows the noise, v, entering where the sensor would be:



b. The first specification implies $K_v \geq 100$ and thus $K \geq 100$. The bode plot with $K = 1$ and $D = 1$ below shows that there is a negative PM but all the other specs are met. The easiest way to see this is to hand plot the asymptotes and mark the constraints that the gain must be ≥ 250 at $\omega \leq 0.2$ rad/sec and the gain must be ≤ 0.01 for $\omega \geq 200$ rad/sec.



In fact, the specs are exceeded at the low frequency side, and slightly exceeded on the high frequency side. But it will be difficult to increase the phase at crossover without violating the specs. From a hand plot of the asymptotes, we see that a combination of lead and lag will do the trick. Placing the lag according to

$$D_{lag}(s) = \frac{(s/2+1)}{(s/0.2+1)}$$

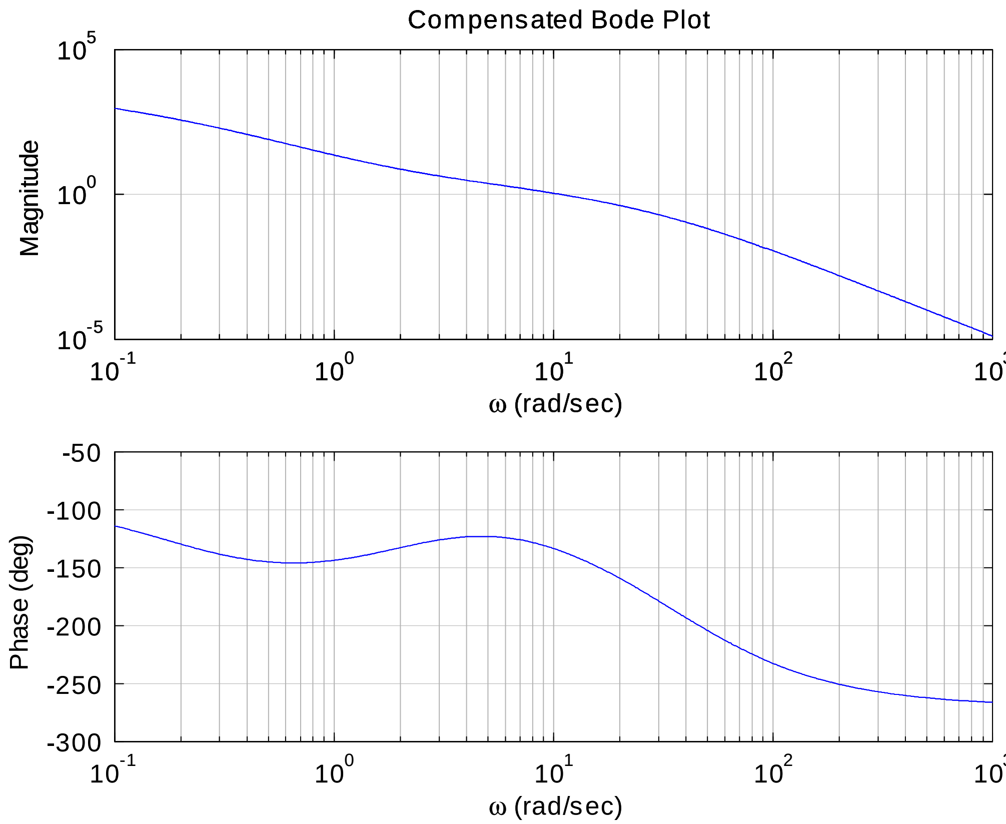
will lower the gain curve at frequencies just prior to crossover so that a -1 slope is more easily achieved at crossover without violating the high frequency constraint. In addition, in order to obtain as much phase at crossover as possible, a lead according to

$$D_{lead}(s) = \frac{(s/5+1)}{(s/50+1)}$$

will preserve the -1 slope from $\omega = 5$ rad/sec to $\omega = 20$ rad/sec which will bracket the crossover frequency and should result in a healthy PM. A look at the Bode plot shows that all specs are met except the $PM = 44$. Perhaps close enough, but a slight increase in lead should do the trick. So our final compensation is

$$D(s) = \frac{(s/2+1)}{(s/0.2+1)}\frac{(s/4+1)}{(s/50+1)}$$

with $K = 100$. This does meet all specs with $PM = 45^o$ exactly, as can be seen by examining the Bode plot below.



53. Consider a type I unity feedback system with

$$G(s) = \frac{K}{s(s+1)}.$$

Design a lead compensator using Bode plot sketches so that $K_v = 20\,\text{sec}^{-1}$ and PM $> 40^\circ$. Use MATLAB to verify and/or refine your design so that it meets the specifications.

Solution :

Use a lead compensation :

$$D(s) = \frac{Ts+1}{\alpha Ts+1},\ \alpha > 1$$

From the specification, $K_v = 20\ \text{sec}^{-1}$,

$$\Longrightarrow \quad K_v = \lim_{s\to 0} sD(s)G(s) = K = 20$$
$$\Longrightarrow \quad K = 20$$

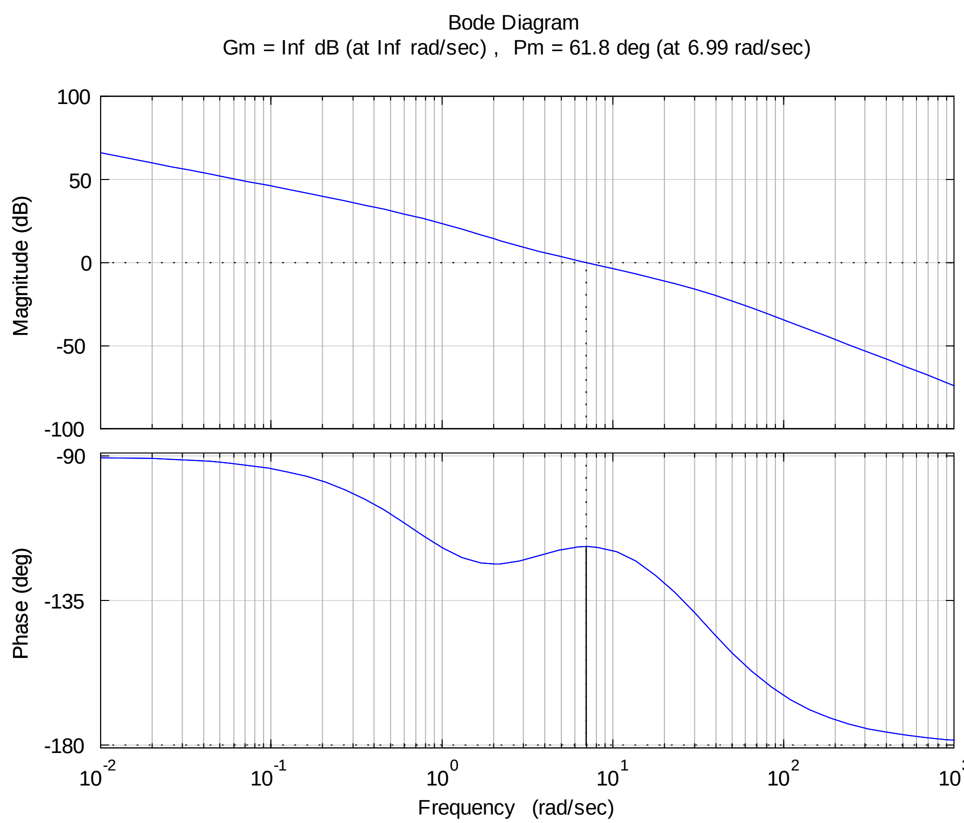
From a hand sketch of the uncompensated Bode plot asymptotes, we see that the slope at crossover is -2, hence the PM will be poor. In fact, an exact computation shows that

$$PM = 12.75 \text{ (at } \omega_c = 4.42 \text{ rad/sec)}$$

Adding a lead compensation

$$D(s) = \frac{\frac{s}{3}+1}{\frac{s}{30}+1}$$

will provide a -1 slope in the vicinity of crossover and should provide plenty of PM. The Bode plot below verifies that indeed it did and shows that the $PM = 62^\circ$ at a crossover frequency $\cong 7$ rad/sec thus meeting all specs.



54. Consider a satellite-attitude control system with the transfer function

$$G(s) = \frac{0.05(s+25)}{s^2(s^2+0.1s+4)}.$$

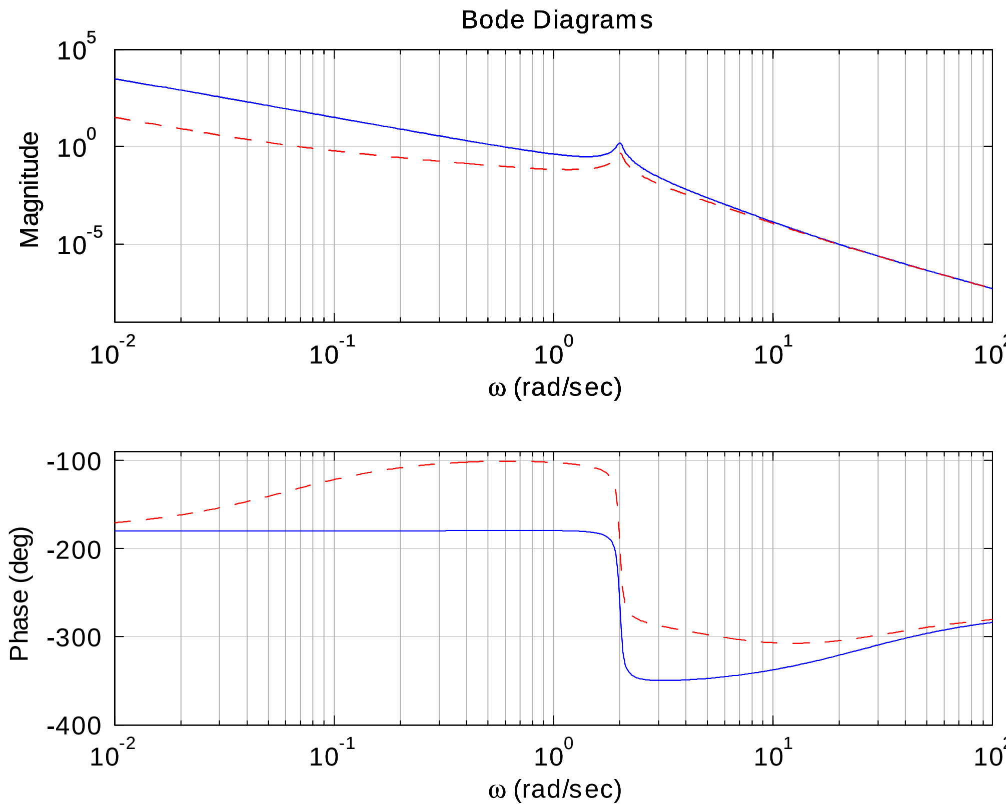
Amplitude-stabilize the system using lead compensation so that GM $\geq$ 2 (6 db), and PM $\geq 45°$, keeping the bandwidth as high as possible with a single lead.

Solution :

The sketch of the uncompensated Bode plot asymptotes shows that the slope at crossover is -2; therefore, a lead compensator will be required in order to have a hope of meeting the PM requirement. Furthermore, the resonant peak needs to be kept below magnitude 1 so that it has no chance of causing an instability (this is amplitude stabilization). This latter requirement means we must lower gain at the resonance. Using the single lead compensator,

$$D(s) = \frac{(s+0.06)}{(s+6)}$$

will lower the low frequency gain by a factor of 100, provide a -1 slope at crossover, and will lower the gain some at the resonance. Thus it is a good first cut at a compensation. The MATLAB Bode plot shows the uncompensated and compensated and verifies our intent. Note especially that the resonant peak never crosses magnitude 1 for the compensated (dashed) case.



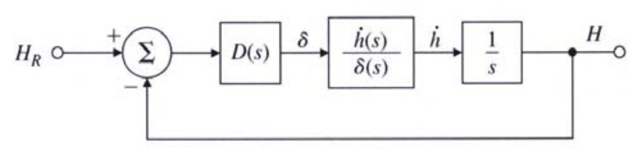
The MATLAB margin routine shows a $GM = 6.3$ db and $PM = 48°$ thus meeting all specs.

55. In one mode of operation the autopilot of a jet transport is used to control altitude. For the purpose of designing the altitude portion of the autopilot loop, only the long-period airplane dynamics are important. The linearized relationship between altitude and elevator angle for the long-period dynamics is

$$G(s) = \frac{h(s)}{\delta(s)} = \frac{20(s + 0.01)}{s(s^2 + 0.01s + 0.0025)} \frac{\text{ft}}{\text{deg}}.$$

The autopilot receives from the altimeter an electrical signal proportional to altitude. This signal is compared with a command signal (proportional to the altitude selected by the pilot), and the difference provides an error signal. The error signal is processed through compensation, and the result is used to command the elevator actuators. A block diagram of this system is shown in Fig. 6.103. You have been given the task of designing the compensation. Begin by considering a proportional control law $D(s) = K$.

Figure 6.103: Control system for Problem 55



(a) Use MATLAB to draw a Bode plot of the open-loop system for $D(s) = K = 1$.

(b) What value of K would provide a crossover frequency (i.e., where $|G| = 1$) of 0.16 rad/sec?

(c) For this value of K, would the system be stable if the loop were closed?

(d) What is the PM for this value of K?

(e) Sketch the Nyquist plot of the system, and locate carefully any points where the phase angle is 180° or the magnitude is unity.

(f) Use MATLAB to plot the root locus with respect to K, and locate the roots for your value of K from part (b).

(g) What steady-state error would result if the command was a step change in altitude of 1000 ft?

For parts (h)and (i), assume a compensator of the form

$$D(s) = K\frac{Ts+1}{\alpha Ts+1}.$$

(h) Choose the parameters K, T, and α so that the crossover frequency is 0.16 rad/sec and the PM is greater that 50°. Verify your design by superimposing a Bode plot of $D(s)G(s)/K$ on top of the Bode plot you obtained for part (a), and measure the PM directly.

(i) Use MATLAB to plot the root locus with respect to K for the system including the compensator you designed in part (h). Locate the roots for your value of K from part (h).

(j) Altitude autopilots also have a mode where the rate of climb is sensed directly and commanded by the pilot.

   i. Sketch the block diagram for this mode,

   ii. define the pertinent $G(s)$,

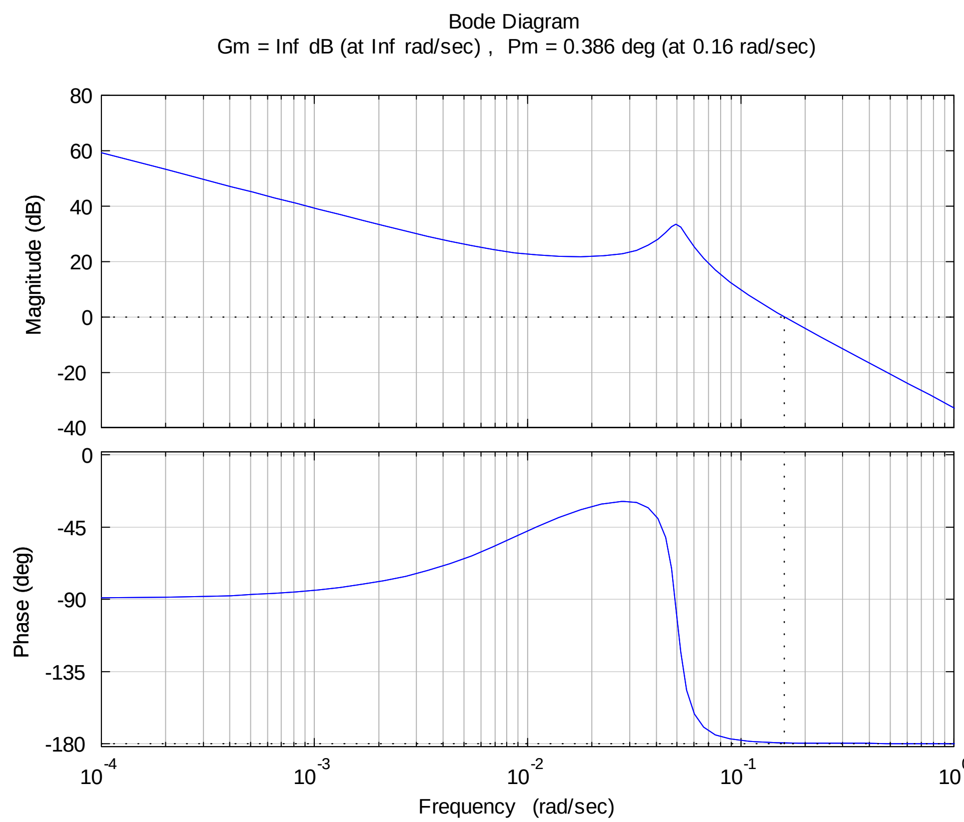
   iii. design $D(s)$ so that the system has the same crossover frequency as the altitude hold mode and the PM is greater than 50°

Solution :

The plant transfer function :

$$\frac{h(s)}{\delta(s)} = \frac{80\left(\frac{s}{0.01}+1\right)}{s\left\{\left(\frac{s}{0.05}\right)^2 + 2\frac{0.1}{0.05}s + 1\right\}}$$

(a) See the Bode plot :



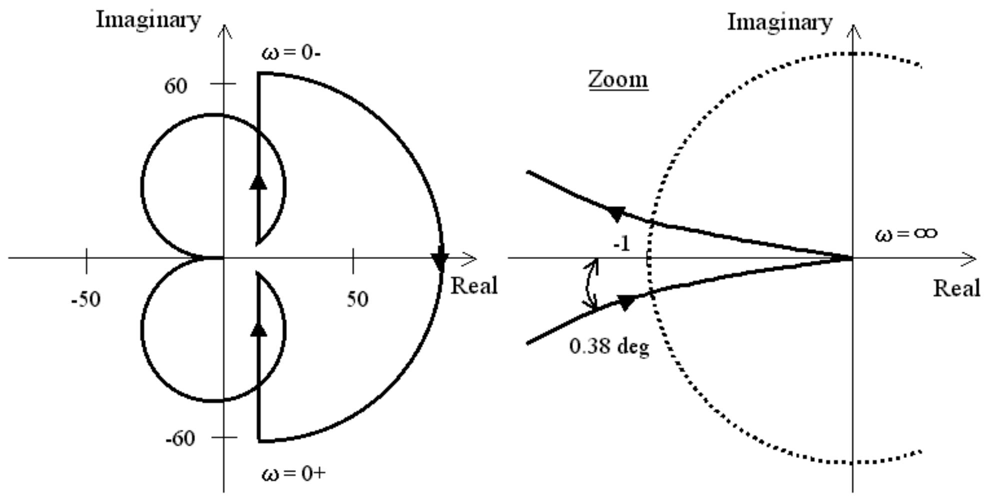
(b) Since $|G| = 865$ at $\omega = 0.16$,

$$K = \frac{1}{|G|}|_{\omega=0.16} = 0.0012$$

(c) The system would be stable, but poorly damped.

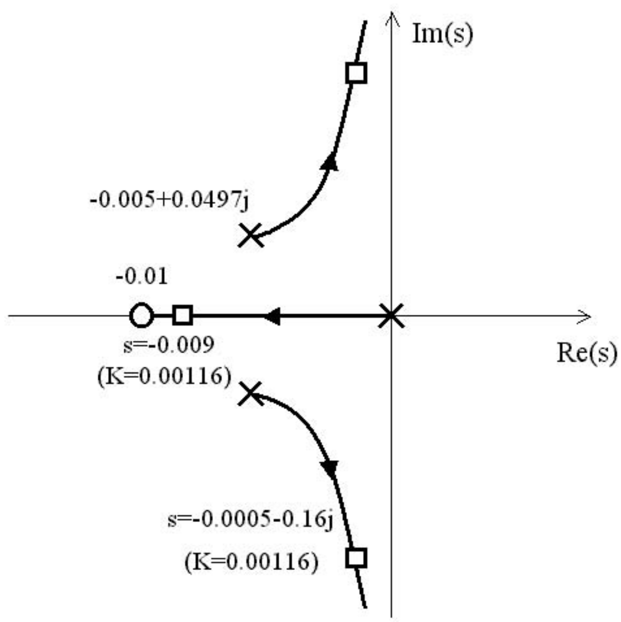
(d) $PM = 0.39°$

(e) The Nyquist plot for $D(j\omega)G(j\omega)$:



The phase angle never quite reaches $-180°$.

(f) See the Root locus :



The closed-loop roots for $K = 0.0012$ are :

$$s = -0.009,\ -0.005 \pm j0.16$$

(g) The steady-state error e_∞ :

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} s \frac{1}{1+K\dfrac{h(s)}{\delta(s)}} \frac{1000}{s} \\ &= 0 \end{aligned}$$

as it should be for this Type 1 system.

(h) Phase margin of the plant :

$$PM = 0.39^\circ \ (\omega_c = 0.16 \text{ rad/sec})$$

Necessary phase lead and $\dfrac{1}{\alpha}$:

$$\text{necessary phase lead} = 50^\circ - 0.39^\circ \ \simeq 50^\circ$$

From Fig. 6.54 :

$$\Longrightarrow \frac{1}{\alpha} = 8$$

Set the maximum phase lead frequency at ω_c :

$$\omega = \frac{1}{\sqrt{\alpha}T} = \omega_c = 0.16 \Longrightarrow T = 18$$

so the compensation is

$$D(s) = K\frac{18s+1}{2.2s+1}$$

For a gain K, we want $|D(j\omega_c)G(j\omega_c)| = 1$ at $\omega = \omega_c = 0.16$. So evaluate via Matlab

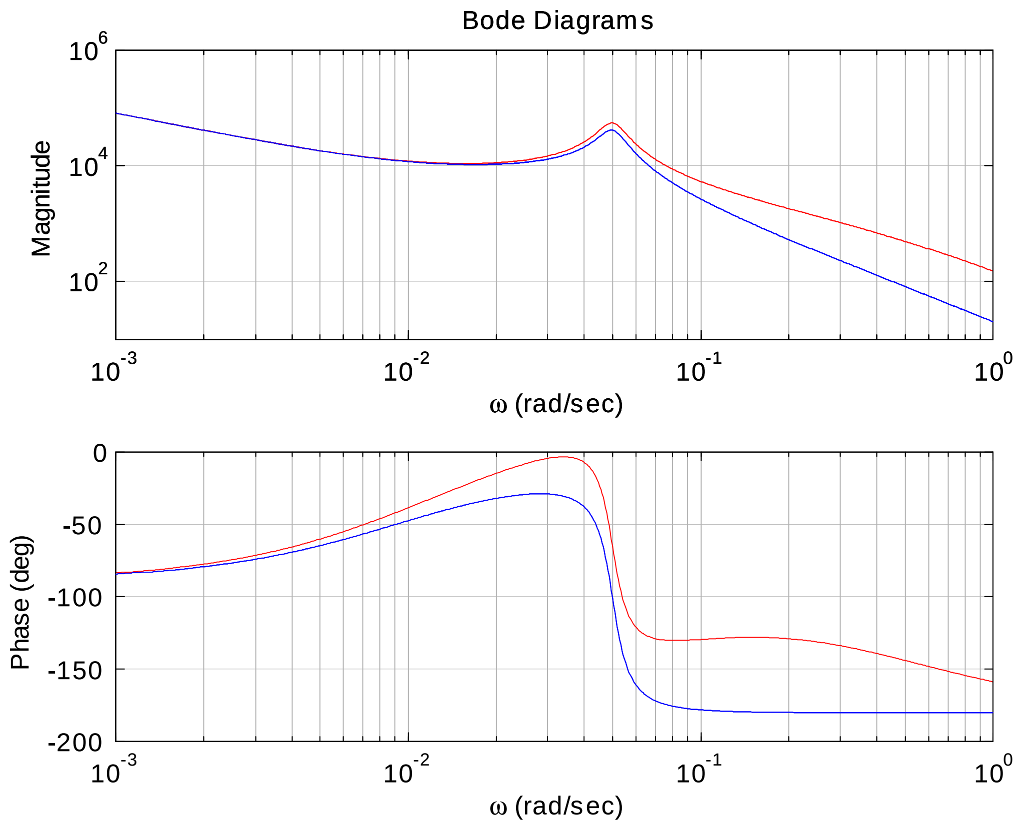
$$\begin{aligned} \left|\frac{D(j\omega_c)G(j\omega_c)}{K}\right|_{\omega_c=0.16} \text{ and find it} &= 2.5\times 10^3 \\ \Longrightarrow \quad K &= \frac{1}{2.5\times 10^3} = 4.0\times 10^{-4} \end{aligned}$$

Therefore the compensation is :

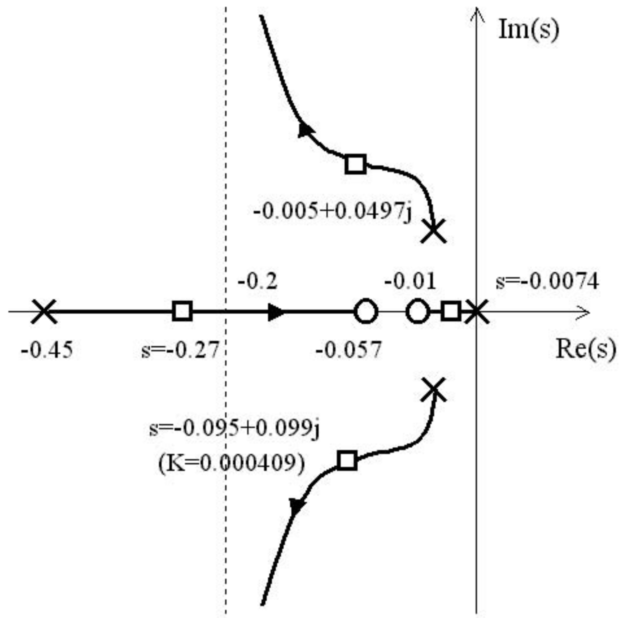
$$D(s) = 4.0\times 10^{-4}\frac{18s+1}{2.2s+1}$$

which results in the Phase margin :

$$PM = 52^\circ \ (\omega_c = 0.16 \text{ rad/sec})$$



(i) See the Root locus :

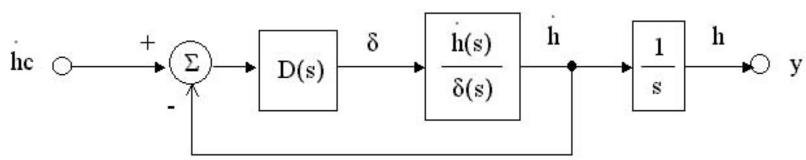


The closed-loop roots for $K = 4.0 \times 10^{-4}$ are :

$$s = -0.27,\ -0.0074,\ -0.095 \pm j0.099$$

(j) In this case, the reference input and the feedback parameter are the rate of climb.

i. The block diagram for this mode is :



ii. Define $G(s)$ as :

$$G(s) = \frac{\dot{h}(s)}{\delta(s)} = \frac{80\left(\dfrac{s}{0.01}+1\right)}{s\left(\dfrac{s}{0.05}\right)^2 + 2\dfrac{0.1}{0.05}s + 1}$$

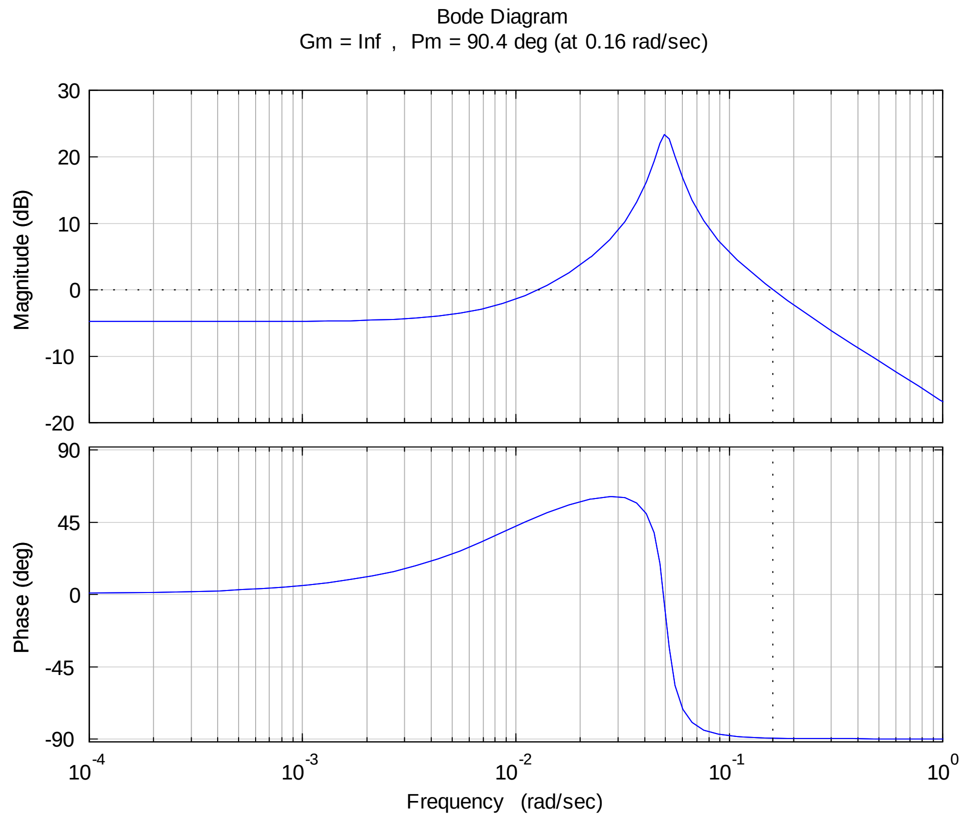
iii. By evaluating the gain of $G(s)$ at $\omega = \omega_c = 0.16$, and setting K equal to its inverse, we see that proportional feedback :

$$D(s) = K = 0.0072$$

satisfies the given specifications by providing:

$$PM = 90^\circ\ (\omega_c = 0.16 \text{ rad/sec})$$

The Bode plot of the compensated system is :



56. For a system with open-loop transfer function $_c$

$$G(s) = \frac{10}{s[(s/1.4)+1][(s/3)+1]},$$

design a lag compensator with unity DC gain so that PM $\geq$40°. What is the approximate bandwidth of this system?

Solution :

Lag compensation design :

Use

$$D(s) = \frac{Ts+1}{\alpha Ts+1}$$

K=1 so that DC gain of $D(s) = 1$.

(a) Find the stability margins of the plant without compensation by plotting the Bode, find that:

$$\begin{aligned} PM &= -20^\circ \ (\omega_c = 3.0 \text{ rad/sec}) \\ GM &= 0.44 \ (\omega = 2.05 \text{ rad/sec}) \end{aligned}$$

(b) The lag compensation needs to lower the crossover frequency so that a $PM \simeq 40°$ will result, so we see from the uncompensated Bode that we need the crossover at about

$$\Longrightarrow \omega_{c,new} = 0.81$$

where

$$|G(j\omega_c)| = 10.4$$

so the lag needs to lower the gain at $\omega_{c,new}$ from 10.4 to 1.

(c) Pick the zero breakpoint of the lag to avoid influencing the phase at $\omega = \omega_{c,new}$ by picking it a factor of 20 below the crossover, so

$$\frac{1}{T} = \frac{\omega_{c,new}}{20}$$

$$\Longrightarrow T = 25$$

(d) Choose α :
Since $D(j\omega) \cong \frac{1}{\alpha}$ for $\omega >> \frac{1}{T}$, let

$$\frac{1}{\alpha} = \frac{1}{|G(j\omega_{c,new})|}$$

$$\alpha = |G(j\omega_{c,new})| = 10.4$$

(e) Compensation :

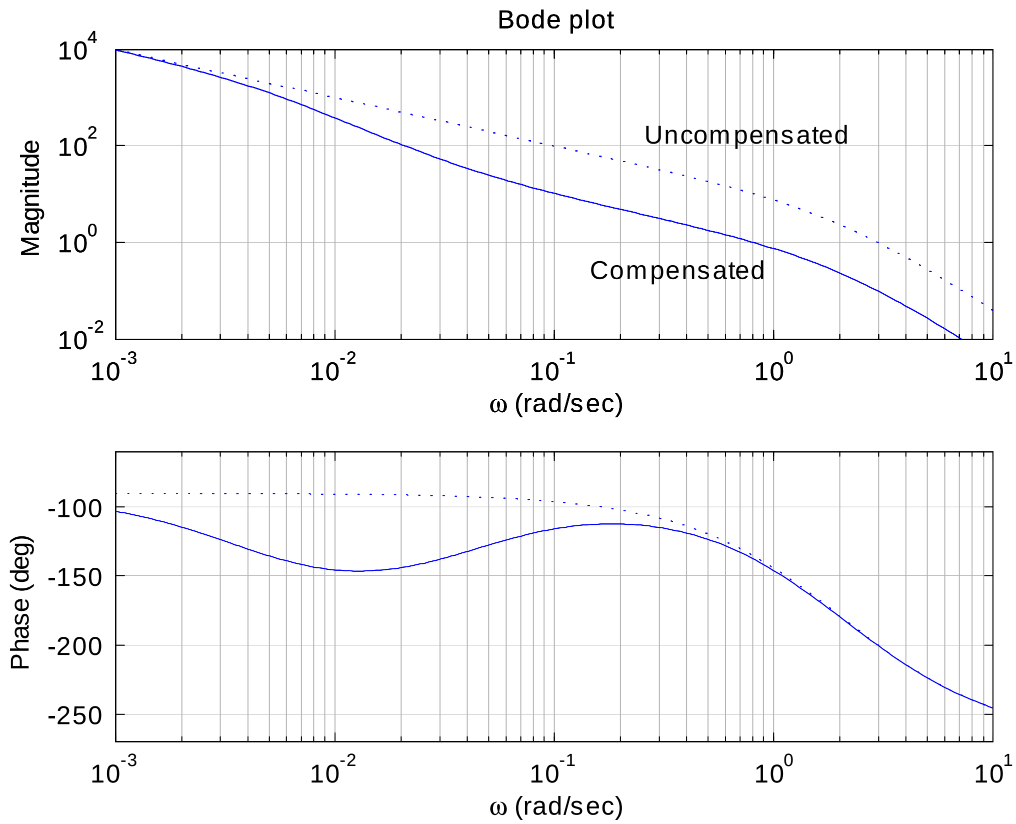
$$D(s) = \frac{\frac{s}{0.04} + 1}{\frac{s}{0.0038} + 1}$$

(f) Stability margins of the compensated system :

$$\begin{aligned} PM &= 42° \ (\omega_c = 0.8 \text{ rad/sec}) \\ GM &= 4.4 \ (\omega = 2.0 \text{ rad/sec}) \end{aligned}$$

Approximate bandwidth ω_{BW}:

$$PM \cong 42° \Longrightarrow \omega_{BW} \cong 2\omega_c = 1.6 \text{ (rad/sec)}$$



57. For the ship-steering system in Problem 39,

(a) Design a compensator that meets the following specifications:

i. velocity constant $K_v = 2$,

ii. PM $\geq 50°$,

iii. unconditional stability (PM > 0 for all $\omega \leq \omega_c$, the crossover frequency).

(b) For your final design, draw a root locus with respect to K, and indicate the location of the closed-loop poles.

Solution :

The transfer function of the ship steering is

$$\frac{V(s)}{\delta_r(s)} = G(s) = \frac{K[-(s/0.142) + 1]}{s(s/0.325 + 1)(s/0.0362) + 1)}.$$

(a) Since the velocity constant, K_v must be 2, we require that $K = 2$.

i. The phase margin of the uncompensated ship is

$$PM = -111^\circ \ (\omega_c = 0.363 \text{ rad/sec})$$

which means it would be impossible to stabilize this system with one lead compensation, since the maximum phase increase would be 90^o. There is no specification leading to maintaining a high bandwidth, so the use of lag compensation appears to be the best choice. So we use a lag compensation:

$$D(s) = \frac{Ts+1}{\alpha Ts+1}$$

ii. The crossover frequency which provides $PM \simeq 50^\circ$ is obtained by looking at the uncompensated Bode plot below, where we see that the crossover frequency needs to be lowered to

$$\omega_{c,new} = 0.017,$$

where the uncompensated gain is

$$|G(j\omega_{c,new})| = 107$$

iii. Keep the zero of the lag a factor of 20 below the crossover to keep the phase lag from the compensation from fouling up the PM, so we find:

$$\begin{aligned} \frac{1}{T} &= \frac{\omega_{c,new}}{20} \\ &\implies T = 1.2 \times 10^3 \end{aligned}$$

iv. Choose α so that the gain reduction is achieved at crossover :

$$\alpha = |G(j\omega_{c,new})| = 107$$

$(D(j\omega) \simeq \dfrac{1}{\alpha}$ for $\omega \gg \dfrac{1}{T})$

v. So the compensation is :

$$D(s) = \frac{1200s+1}{12.6s+1} = \frac{\frac{s}{0.0008}+1}{\frac{s}{0.08}+1}$$

vi. Stability margins of the compensated system :

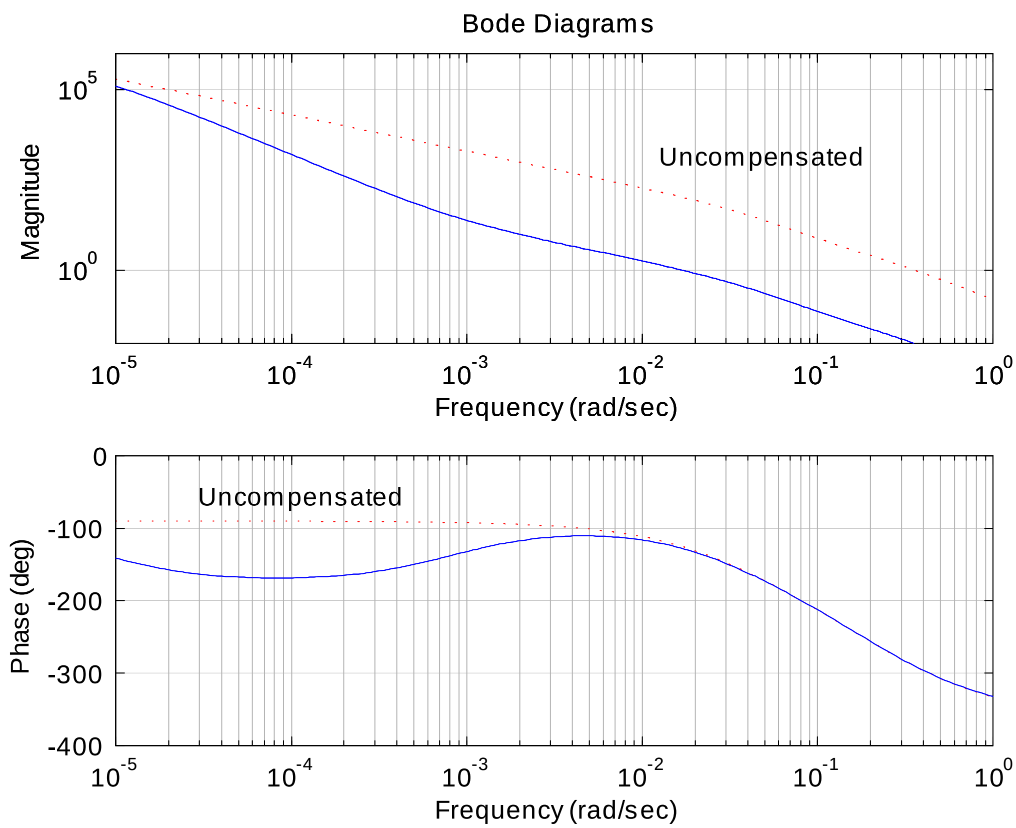
$$\begin{aligned} PM &= 52.1^\circ \ (\omega_c = 0.017 \text{ rad/sec}) \\ GM &= 5.32 \ (\omega = 0.057 \text{ rad/sec}) \end{aligned}$$

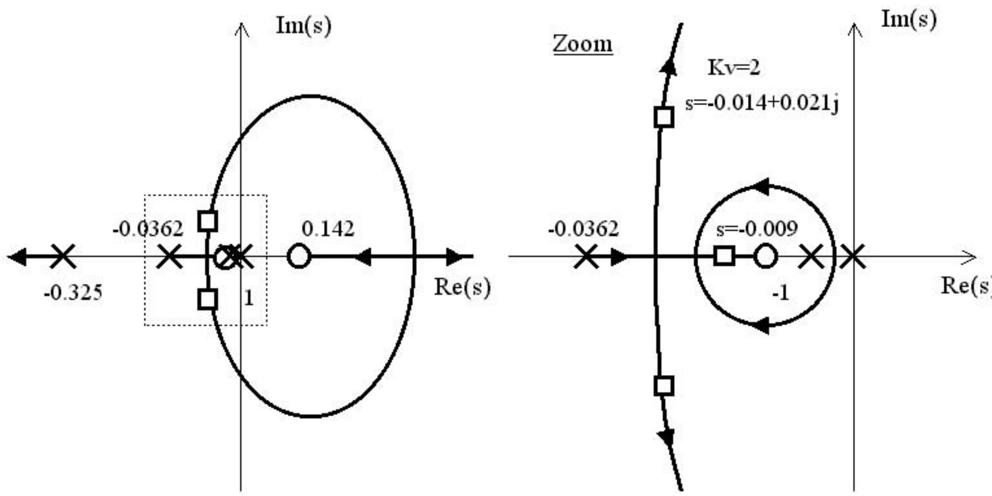
and the system is unconditionally stable since the phase > 0 for all $\omega < \omega_c$.as can be seen by the plot below.

(b) See the root locus. (Note that this is a zero degree root locus.)

The closed-loop roots for $K = 2$ are :

$$s = -0.33,\ -0.0009,\ -0.014 \pm j0.021$$





58. For a unity feedback system with

$$G(s) = \frac{1}{s(\frac{s}{20}+1)(\frac{s^2}{100^2}+0.5\frac{s}{100}+1)} \tag{2}$$

(a) A lead compensator is introduced with $\alpha = 1/5$ and a zero at $1/T = 20$. How must the gain be changed to obtain crossover at $\omega_c = 31.6$ rad/sec, and what is the resulting value of K_v?

(b) With the lead compensator in place, what is the required value of K for a lag compensator that will readjust the gain to a K_v value of 100?

(c) Place the pole of the lag compensator at 3.16 rad/sec, and determine the zero location that will maintain the crossover frequency at $\omega_c = 31.6$ rad/sec. Plot the compensated frequency response on the same graph.

(d) Determine the PM of the compensated design.

Solution :

(a) From a sketch of the asymptotes with the lead compensation (with $K_1 = 1$) :

$$D_1(s) = K_1 \frac{\frac{s}{20}+1}{\frac{s}{100}+1}$$

in place, we see that the slope is -1 from zero frequency to $\omega = 100$ rad/sec. Therefore, to obtain crossover at $\omega_c = 31.6$ rad/sec, the gain $K_1 = 31.6$ is required. Therefore,

$$K_v = 31.6$$

(b) To increase K_v to be 100, we need an additional gain of 3.16 from the lag compensator at very low frequencies to yield $K_v = 100$.

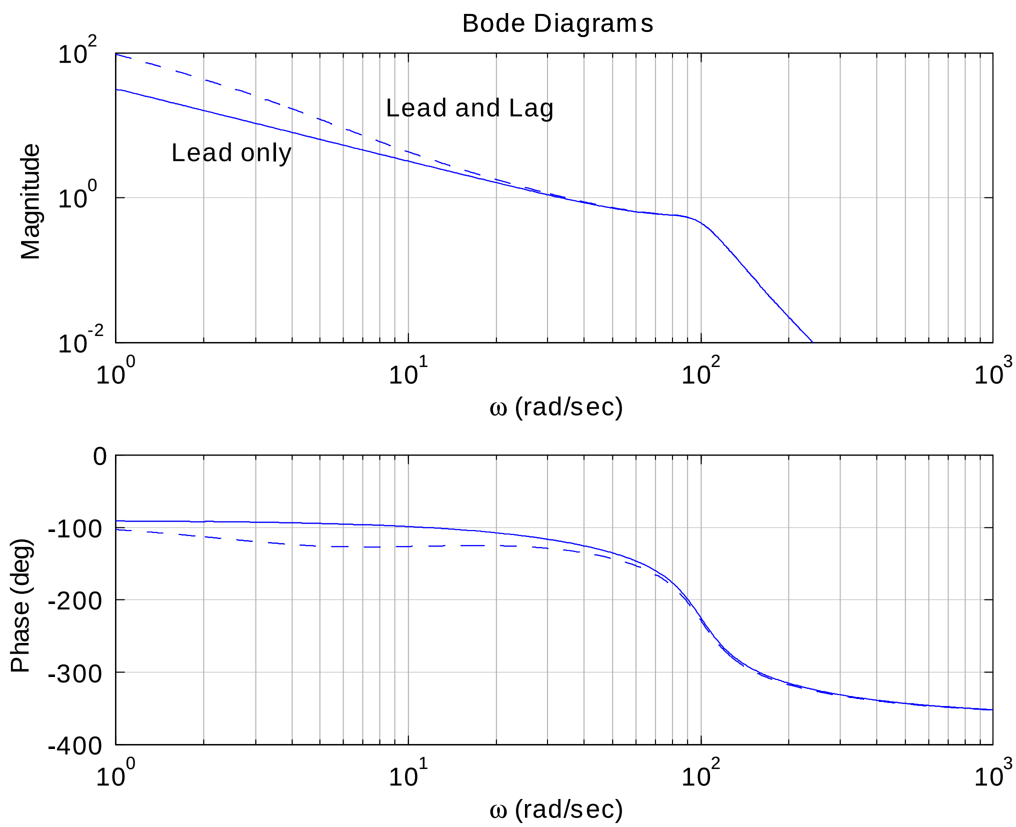
(c) For a low frequency gain increase of 3.16, and the pole at 3.16 rad/sec, the zero needs to be at 10 in order to maintain the crossover at $\omega_c = 31.6$ rad/sec. So the lag compensator is

$$D_2(s) = 3.16 \frac{\frac{s}{10}+1}{\frac{s}{3.16}+1}$$

and

$$D_1(s)D_2(s) = 100 \frac{\frac{s}{20}+1}{\frac{s}{100}+1} \frac{\frac{s}{10}+1}{\frac{s}{3.16}+1}$$

The Bode plots of the system before and after adding the lag compensation are



(d) By using the `margin` routine from MATLAB, we see that

$$PM = 49^\circ \ (\omega_c = 34.5 \text{ deg/sec})$$

59. Golden Nugget Airlines had great success with their free bar near the tail of the airplane. (See Problem 5.39) However, when they purchased a much larger airplane to handle the passenger demand, they discovered that there was some flexibility in the fuselage that caused a lot of unpleasant yawing motion at the rear of the airplane when in turbulence and was causing the revelers to spill their drinks. The approximate transfer function for the dutch roll mode (See Section 10.3.1) is

$$\frac{r(s)}{\delta_r(s)} = \frac{8.75(4s^2 + 0.4s + 1)}{(s/0.01 + 1)(s^2 + 0.24s + 1)}$$

where r is the airplane's yaw rate and δ_r is the rudder angle. In performing a Finite Element Analysis (FEA) of the fuselage structure and adding those dynamics to the dutch roll motion, they found that the transfer function needed additional terms that reflected the fuselage lateral bending that occurred due to excitation from the rudder and turbulence. The revised transfer function is

$$\frac{r(s)}{\delta_r(s)} = \frac{8.75(4s^2 + 0.4s + 1)}{(s/0.01 + 1)(s^2 + 0.24s + 1)} \cdot \frac{1}{(\frac{s^2}{\omega_b^2} + 2\zeta\frac{s}{\omega_b} + 1)}$$

where ω_b is the frequency of the bending mode ($= 10$ rad/sec) and ζ is the bending mode damping ratio ($= 0.02$). Most swept wing airplanes have a "yaw damper" which essentially feeds back yaw rate measured by a rate gyro to the rudder with a simple proportional control law. For the new Golden Nugget airplane, the proportional feedback gain, $K = 1$, where

$$\delta_r(s) = -Kr(s). \tag{3}$$

(a) Make a Bode plot of the open-loop system, determine the PM and GM for the nominal design, and plot the step response and Bode magnitude of the closed-loop system. What is the frequency of the lightly damped mode that is causing the difficulty?

(b) Investigate remedies to quiet down the oscillations, but maintain the same low frequency gain in order not to affect the quality of the dutch roll damping provided by the yaw rate feedback. Specifically, investigate one at a time:

i. increasing the damping of the bending mode from $\zeta = 0.02$ to $\zeta = 0.04$. (Would require adding energy absorbing material in the fuselage structure)

ii. increasing the frequency of the bending mode from $\omega_b = 10$ rad/sec to $\omega_b = 20$ rad/sec. (Would require stronger and heavier structural elements)

iii. adding a low pass filter in the feedback, that is, replace K in Eq. (3) with $KD(s)$ where

$$D(s) = \frac{1}{s/\tau_p + 1}. \tag{4}$$

Pick τ_p so that the objectionable features of the bending mode are reduced while maintaining the PM $\geq 60^o$.

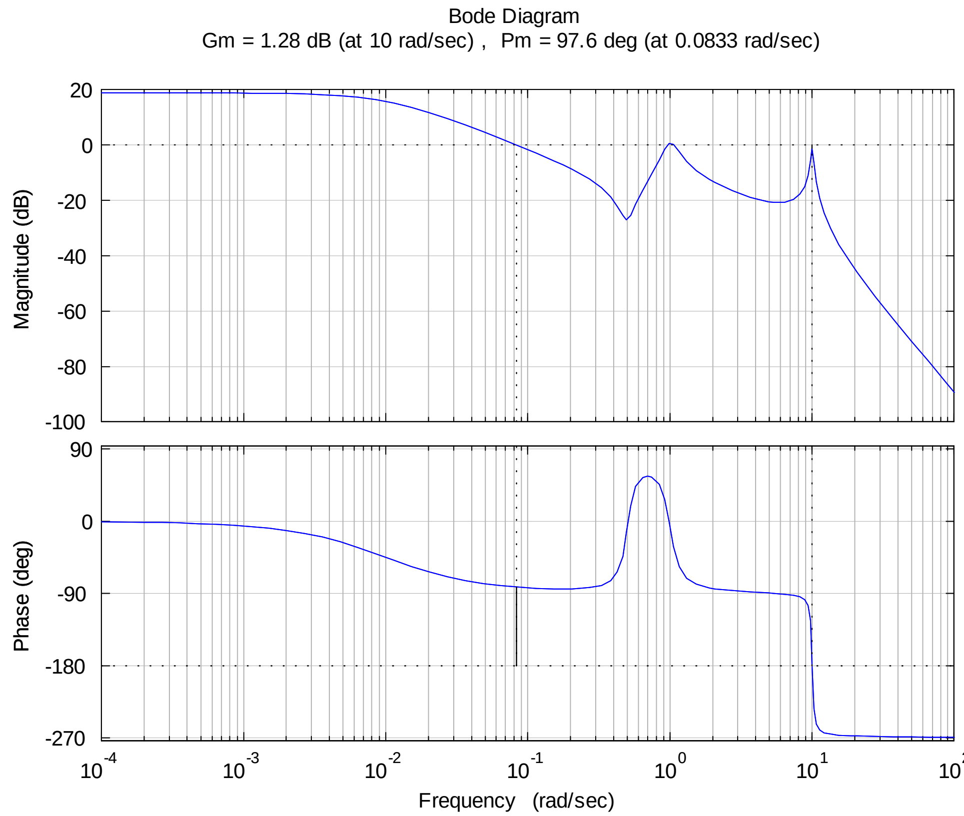
iv. adding a notch filter as described in Section 5.4.3. Pick the frequency of the notch zero to be at ω_b with a damping of $\zeta = 0.04$ and pick the denominator poles to be $(s/100 + 1)^2$ keeping the DC gain of the filter $= 1$.

(c) Investigate the sensitivity of the two compensated designs above (iii and iv) by determining the effect of a reduction in the bending mode frequency of -10%. Specifically, re-examine the two designs by tabulating the GM, PM, closed loop bending mode damping ratio and resonant peak amplitude, and qualitatively describe the differences in the step response.

(d) What do you recommend to Golden Nugget to help their customers quit spilling their drinks? (Telling them to get back in their seats is not an acceptable answer for this problem! Make the recommendation in terms of improvements to the yaw damper.)

Solution :

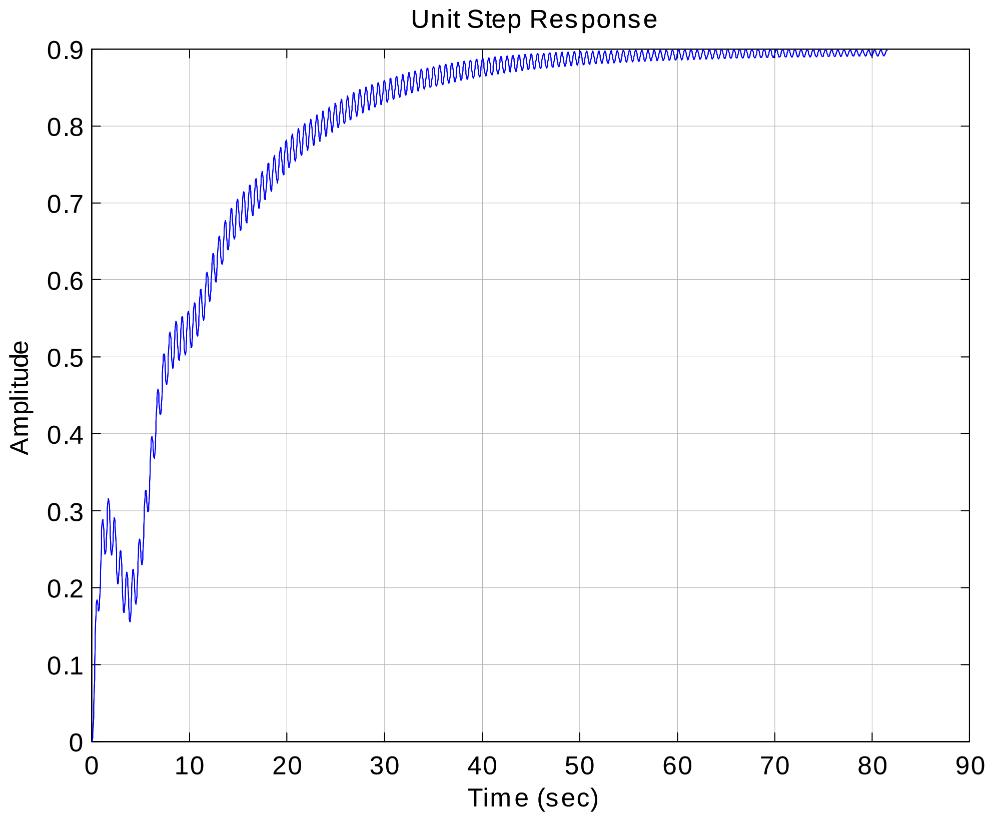
(a) The Bode plot of the open-loop system is :



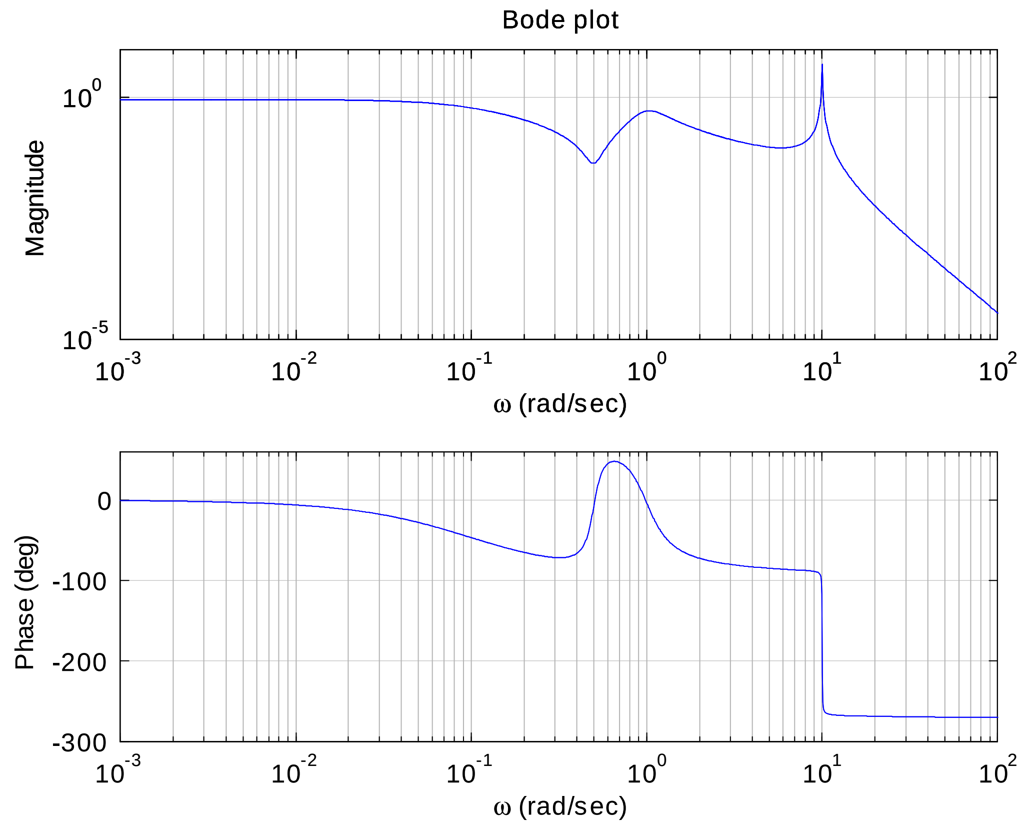
$$
\begin{aligned}
PM &= 97.6^\circ \ (\omega = 0.0833 \text{ rad/sec}) \\
GM &= 1.28 \ (\omega = 10.0 \text{ rad/sec})
\end{aligned}
$$

The low GM is caused by the resonance being close to instability.

The closed-loop system unit step response is :



The Bode plot of the closed-loop system is :

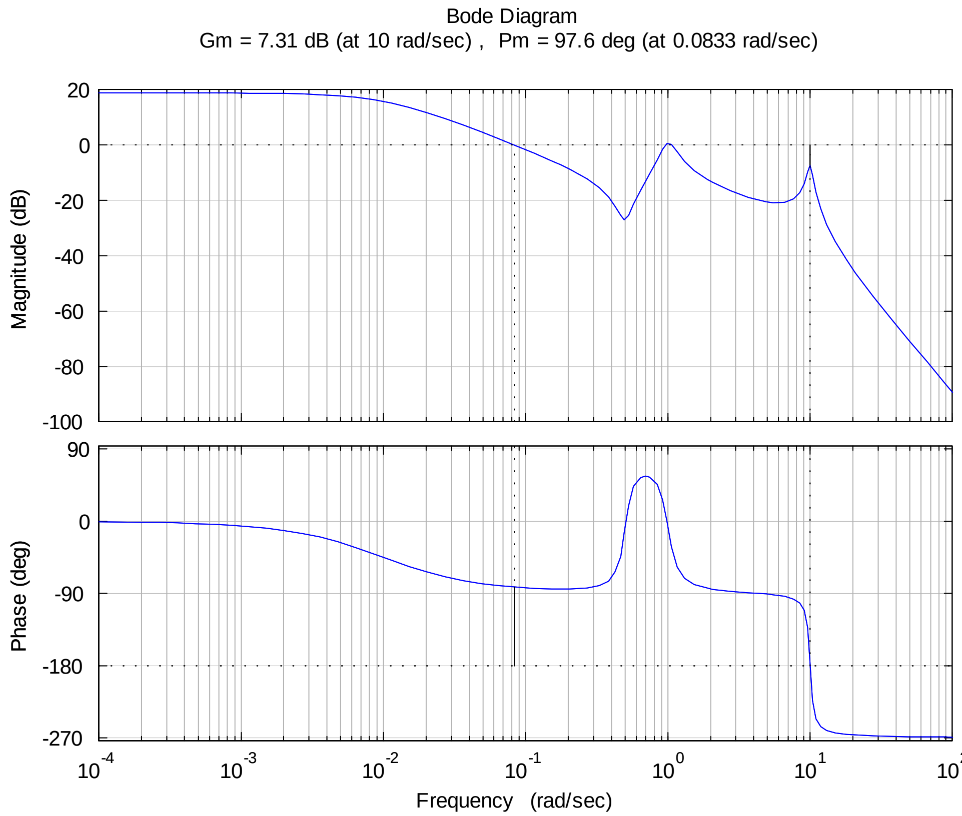


From the Bode plot of the closed -loop system, the frequency of the lightly damped mode is :

$$\omega \simeq 10 \text{ rad/sec}$$

and this is borne out by the step response that shows a lightly damped oscillation at 1.6 Hz or 10 rad/sec.

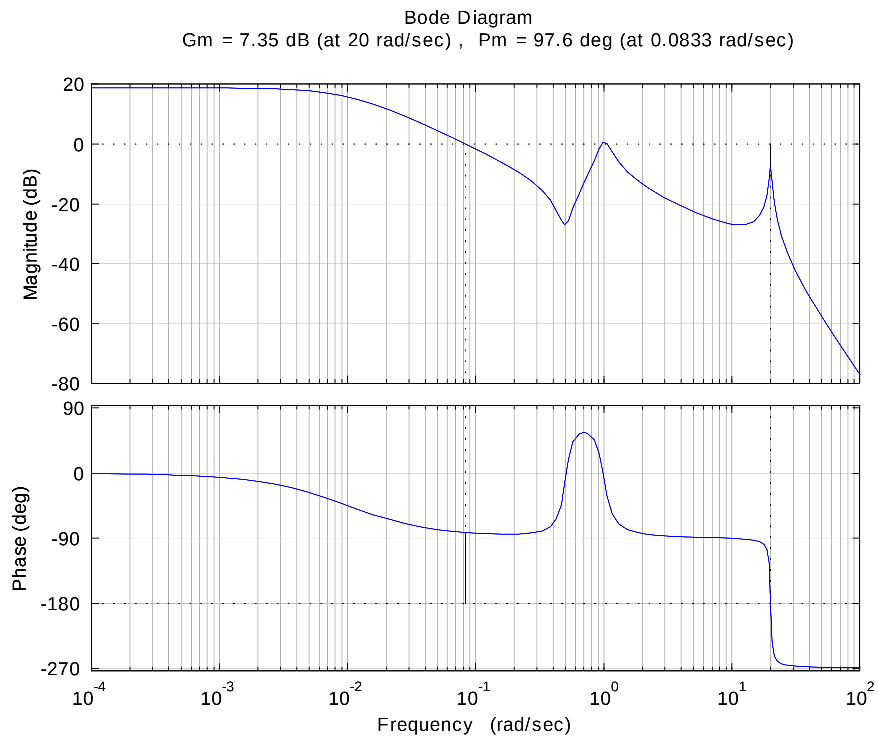
i. The Bode plot of the system with the bending mode damping increased from $\zeta = 0.02$ to $\zeta = 0.04$ is :



$$
\begin{aligned}
PM &= 97.6^\circ \ (\omega = 0.0833 \text{ rad/sec}) \\
GM &= 7.31 \ (\omega = 10.0 \text{ rad/sec})
\end{aligned}
$$

and we see that the GM has increased considerably because the resonant peak is well below magnitude 1; thus the system will be much better behaved.

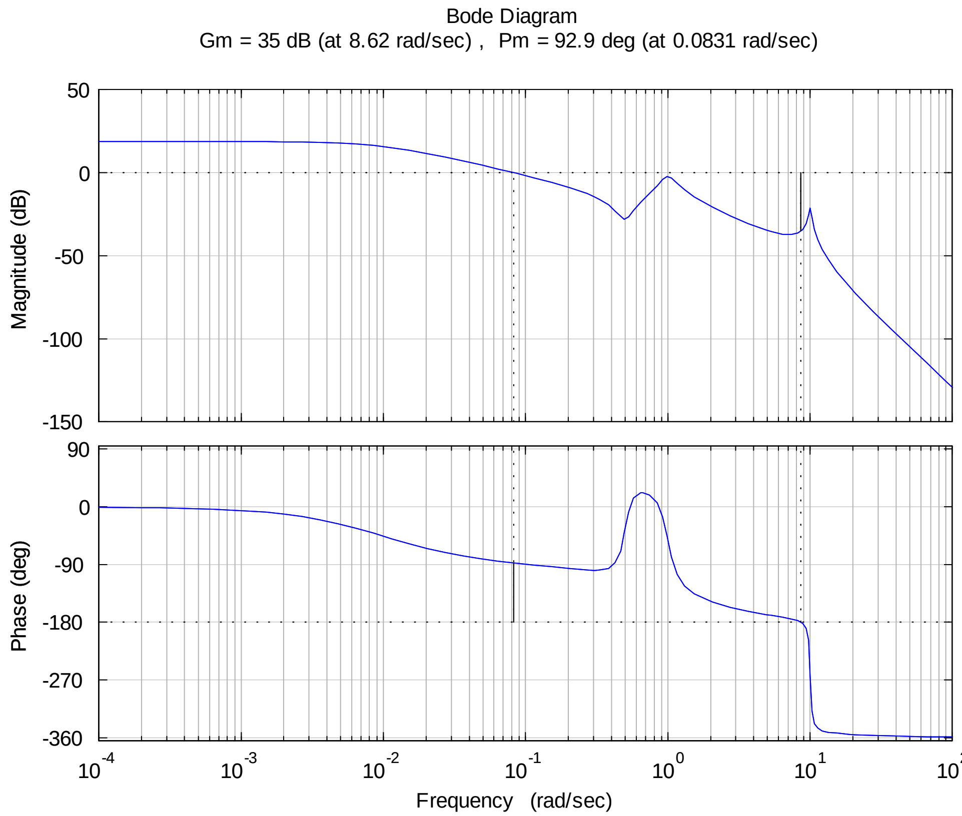
ii. The Bode plot of this system ($\omega_b = 10$ rad/sec $\Longrightarrow \omega_b = 20$ rad/sec) is :



$$
\begin{aligned}
PM &= 97.6^\circ \ (\omega = 0.0833 \text{ rad/sec}) \\
GM &= 7.34 \ (\omega = 20.0 \text{ rad/sec})
\end{aligned}
$$

and again, the GM is much improved and the resonant peak is significantly reduced from magnitude 1.

iii. By picking up $\tau_p = 1$, the Bode plot of the system with the low pass filter is :



$$
\begin{aligned}
PM &= 92.9^\circ\ (\omega = 0.0831 \text{ rad/sec}) \\
GM &= 34.97\ (\omega = 8.62 \text{ rad/sec})
\end{aligned}
$$

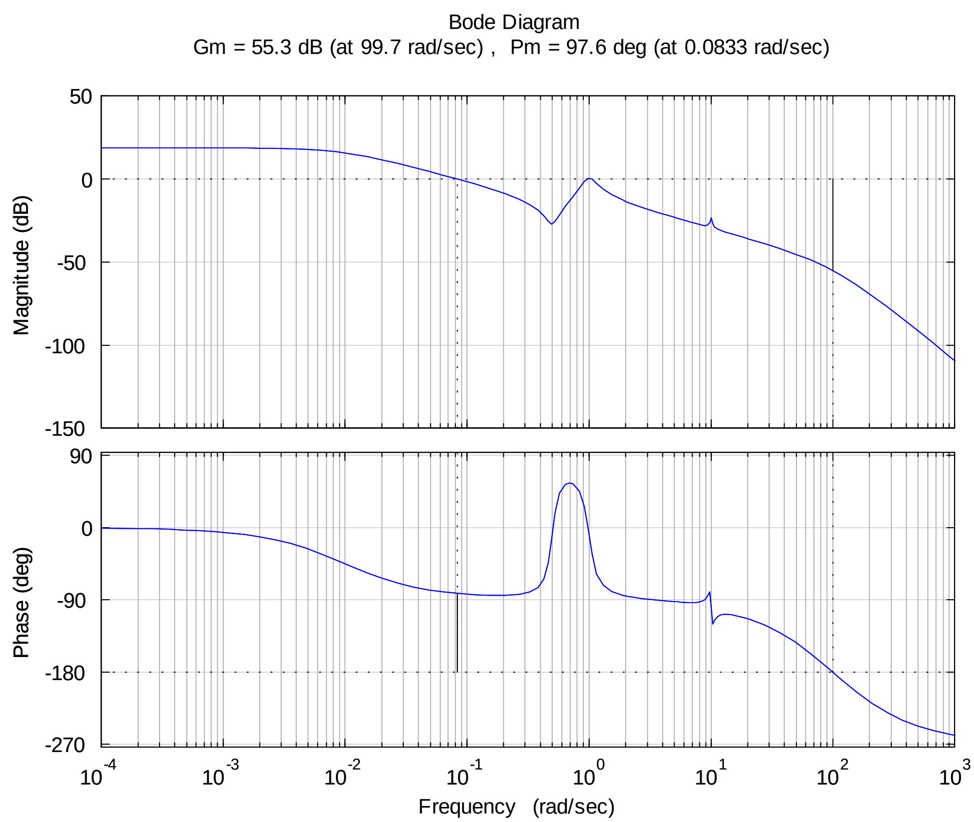
which are healthy margins and the resonant peak is, again, well below magnitude 1.

iv. The Bode plot of the system with the given notch filter is :

$$
\begin{aligned}
PM &= 97.6^\circ\ (\omega = 0.0833 \text{ rad/sec}) \\
GM &= 55.3\ (\omega = 99.7 \text{ rad/sec})
\end{aligned}
$$

which are the healthiest margins of all the designs since the notch filter has essentially canceled the bending mode resonant peak.

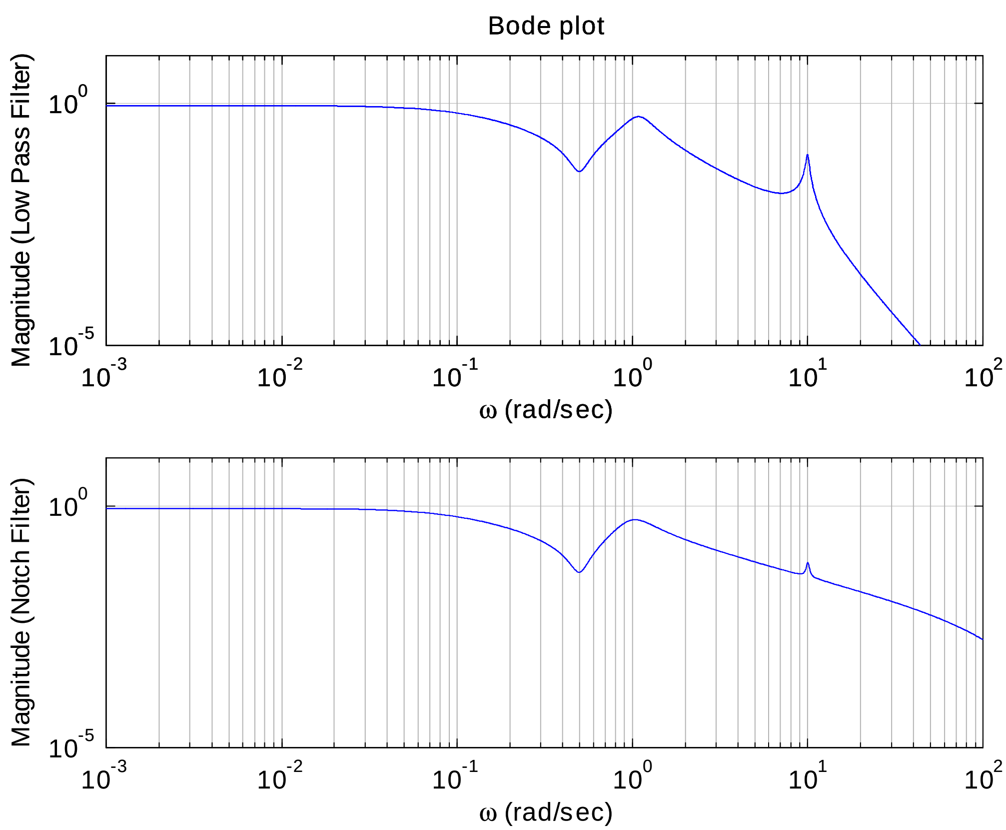
(b) Generally, the notch filter is very sensitive to where to place the notch zeros in order to reduce the lightly damped resonant peak. So if you want to use the notch filter, you must have a good estimation of the location of the bending mode poles and the poles must remain at that location for all aircraft conditions. On the other hand, the low pass filter is relatively robust to where to place its break point.


Bode Diagram
Gm = 55.3 dB (at 99.7 rad/sec) , Pm = 97.6 deg (at 0.0833 rad/sec)
Magnitude (dB)
50
0
-50
-100
-150
Phase (deg)
90
0
-90
-180
-270
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)

Evaluation of the margins with the bending mode frequency lowered by 10% will show a drastic reduction in the margins for the notch filter and very little reduction for the low pass filter.

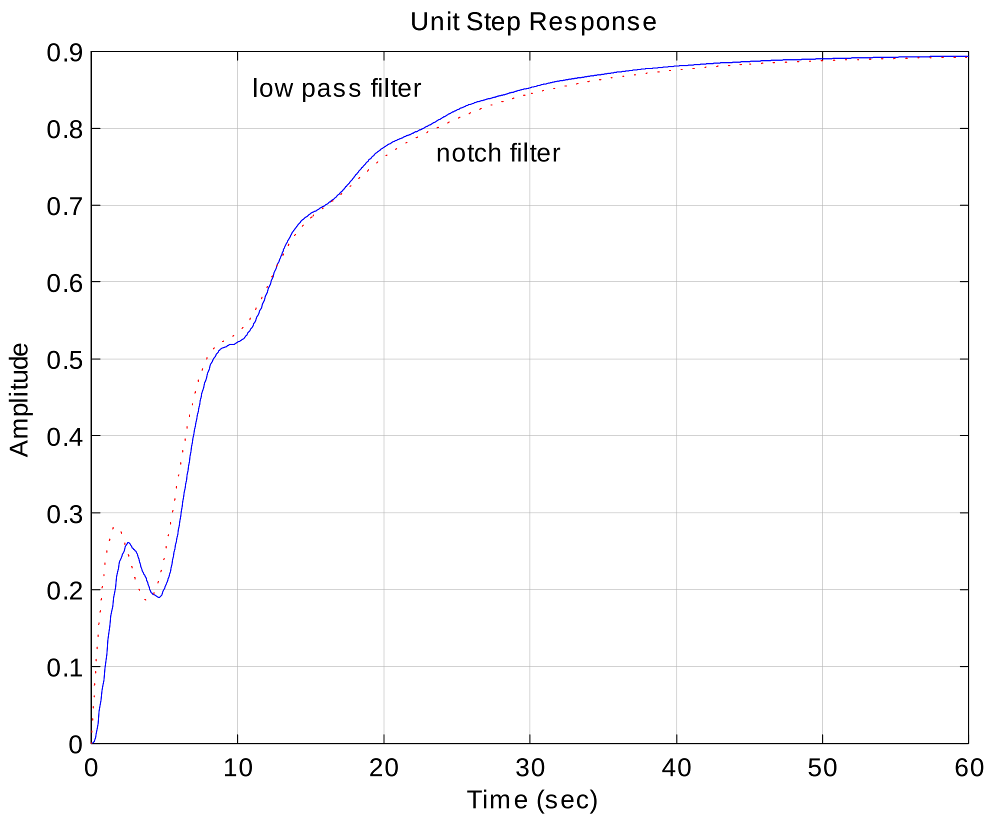
	Low Pass Filter	Notch Filter
GM	34.97 ($\omega = 8.62$ rad/sec)	55.3 ($\omega = 99.7$ rad/sec)
PM	92.9° ($\omega = 0.0831$ rad/sec)	97.6° ($\omega = 0.0833$ rad/sec)
Closed-loop bending mode damping ratio	$\simeq 0.02$	$\simeq 0.04$
Resonant peak	0.087	0.068

The magnitude plots of the closed-loop systems are :



The closed-loop step responses are :

(c) While increasing the natural damping of the system would be the best solution, it might be difficult and expensive to carry out. Likewise, increasing the frequency typically is expensive and makes the structure heavier, not a good idea in an aircraft. Of the remaining


Unit Step Response
low pass filter
notch filter
Amplitude
Time (sec)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
0
10
20
30
40
50
60

two options, it is a better design to use a low pass filter because of its reduced sensitivity to mismatches in the bending mode frequency. Therefore, the best recommendation would be to use the low pass filter.

60. Consider a system with the open-loop transfer function (loop gain)

$$G(s) = \frac{1}{s(s+1)(s/10+1)}.$$

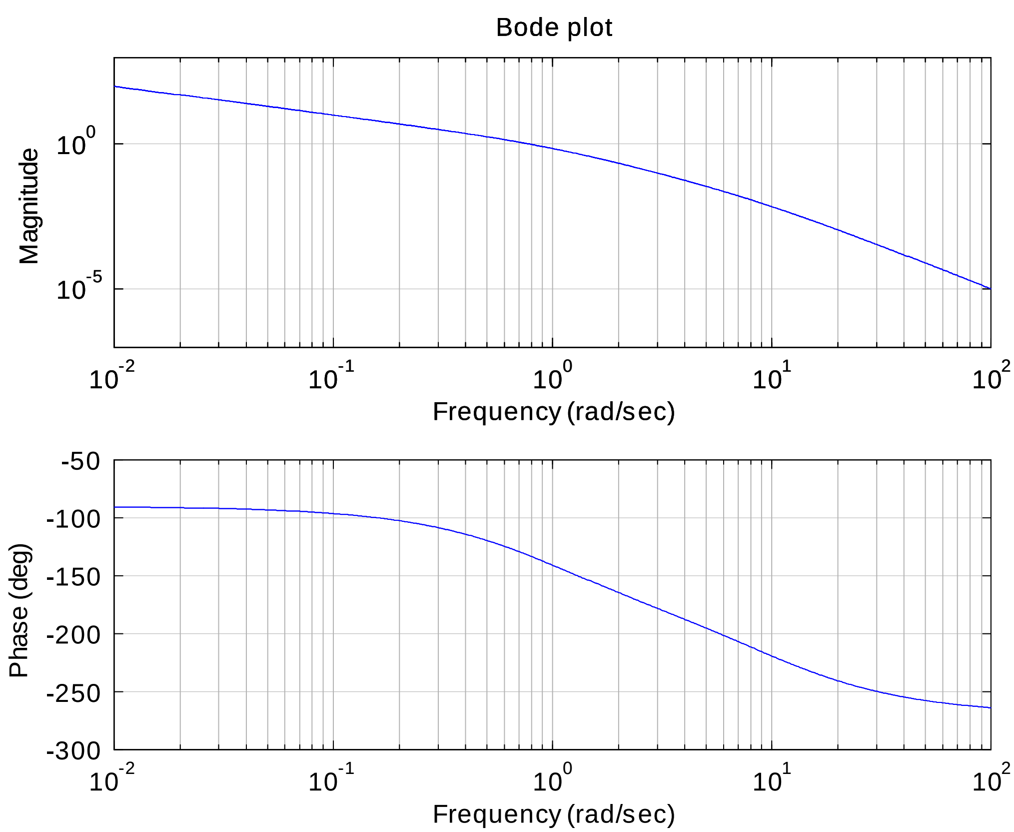
(a) Create the Bode plot for the system, and find GM and PM.

(b) Compute the sensitivity function and plot its magnitude frequency response.

(c) Compute the Vector Margin (VM).

Solution :

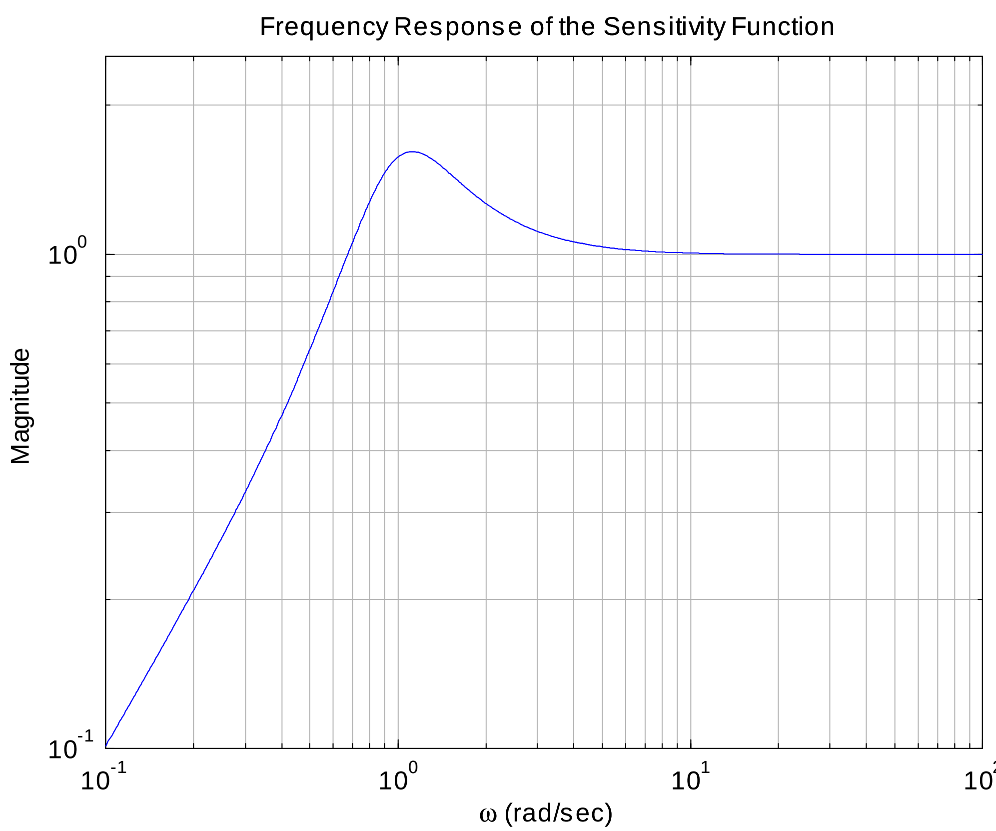
(a) The Bode plot is :



(b) Sensitivity function is :

$$S(s) = \frac{1}{1+G(s)} = \frac{1}{1+\dfrac{1}{s(s+1)(\frac{s}{10}+1)}}$$

The magnitude frequency response of this sensitivity function is :



(c) Vector Margin is defined as :

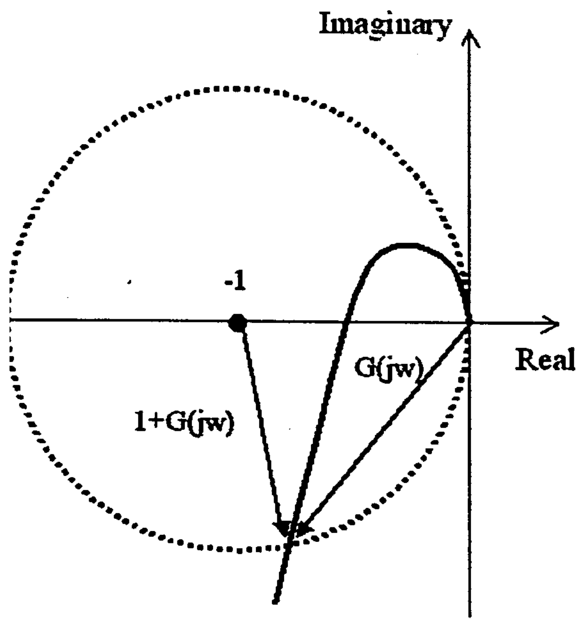
$$VM = \min_{\omega} \frac{1}{|s(j\omega)|} = \frac{1}{1.61} = 0.62$$

61. Prove that the sensitivity function $S(s)$ has magnitude greater than 1 inside a circle with a radius of 1 centered at the -1 point. What does

this imply about the shape of the Nyquist plot if closed-loop control is to outperform open-loop control at all frequencies?

Solution :

$$S(s) = \frac{1}{1+G(s)}$$



Inside the unit circle, $|1 + G(s)| < 1$ which implies $|S(s)| > 1$.

Outside the unit circle, $|1 + G(s)| > 1$ which implies $|S(s)| < 1$.

On the unit circle, $|1 + G(s)| = 1$ which means $|S(s)| = 1$.

If the closed-loop control is going to outperform open-loop control then $|S(s)| \leq 1$ for all s. This means that the Nyquist plot must lie outside the circle of radius one centered at -1.

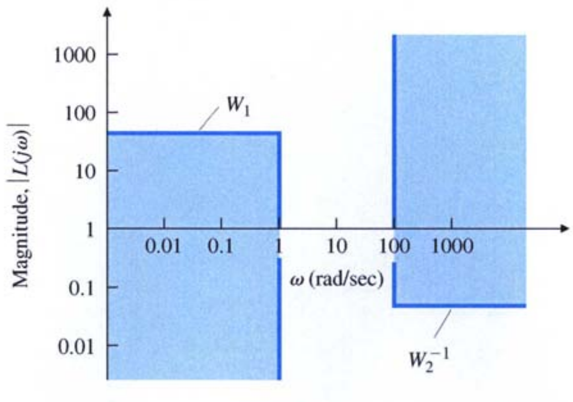
62. Consider the system in Fig. 6.102 with the plant transfer function

$$G(s) = \frac{10}{s(s/10+1)}.$$

We wish to design a compensator $D(s)$ that satisfies the following design specifications:

(a) i. $K_v = 100$,
  ii. PM $\geq 45°$,
  iii. sinusoidal inputs of up to 1 rad/sec to be reproduced with $\leq 2\%$ error,

Figure 6.104: Control system constraints for Problem 62



iv. sinusoidal inputs with a frequency of greater than 100 rad/sec to be attenuated at the output to $\leq 5\%$ of their input value.

(b) Create the Bode plot of $G(s)$, choosing the open-loop gain so that $K_v = 100$.

(c) Show that a *sufficient* condition for meeting the specification on sinusoidal inputs is that the magnitude plot lies outside the shaded regions in Fig. 6.104. Recall that

$$\frac{Y}{R} = \frac{KG}{1+KG} \quad \text{and} \quad \frac{E}{R} = \frac{1}{1+KG}.$$

(d) Explain why introducing a lead network alone cannot meet the design specifications.

(e) Explain why a lag network alone cannot meet the design specifications.

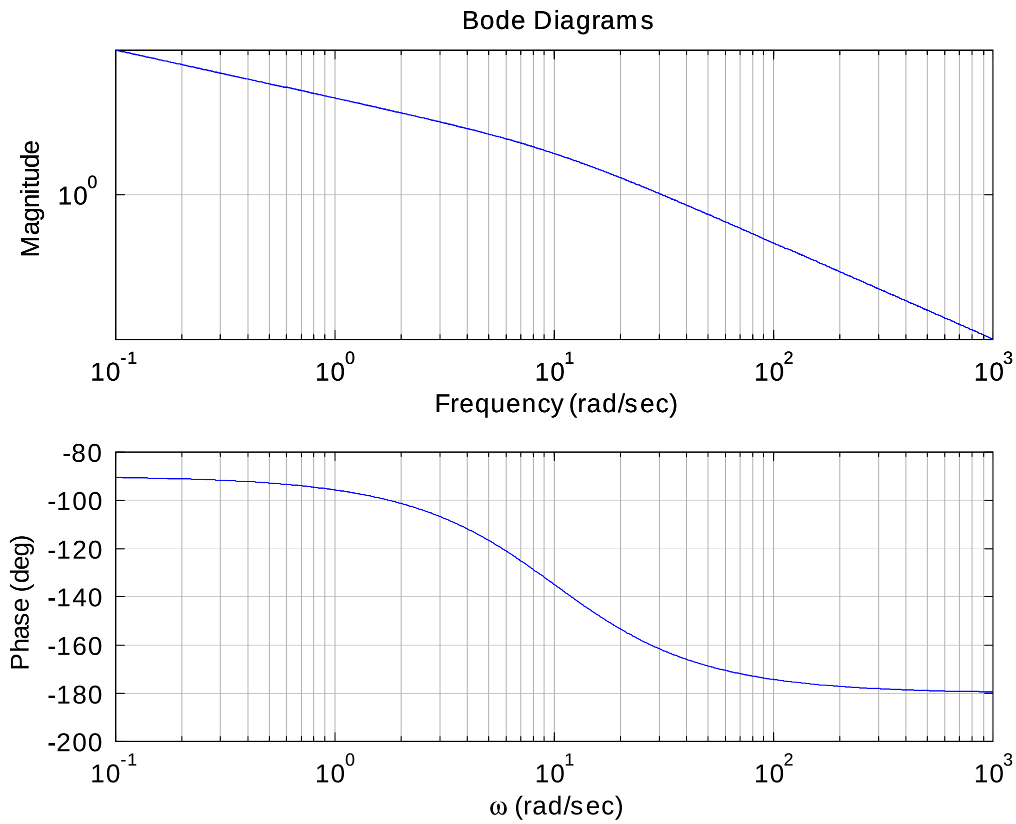
(f) Develop a full design using a lead-lag compensator that meets all the design specifications, without altering the previously chosen low frequency open-loop gain.

Solution :

(a) To satisfy the given velocity constant K_v,

$$\begin{aligned} K_v &= \lim_{s \to 0} sKG(s) = 10K = 100 \\ &\Longrightarrow \quad K = 10 \end{aligned}$$

(b) The Bode plot of $G(s)$ with the open-loop gain $K = 10$ is :



(c) From the 3rd specification,

$$\begin{aligned}\left|\frac{E}{R}\right| &= \left|\frac{1}{1+KG}\right| < 0.02\ (2\%)\\ &\Longrightarrow \ |KG| > 49 \text{ (at } \omega < 1 \text{ rad/sec)}\end{aligned}$$

From the 4th specification,

$$\begin{aligned}\left|\frac{Y}{R}\right| &= \left|\frac{KG}{1+KG}\right| < 0.05\ (5\%)\\ &\Longrightarrow \ |KG| < 0.0526 \text{ (at } \omega > 100 \text{ rad/sec)}\end{aligned}$$

which agree with the figure.

(d) A lead compensator may provide a sufficient PM, but it increases the gain at high frequency so that it violates the specification above.

(e) A lag compensator could satisfy the PM specification by lowering the crossover frequency, but it would violate the low frequency specification, W_1.

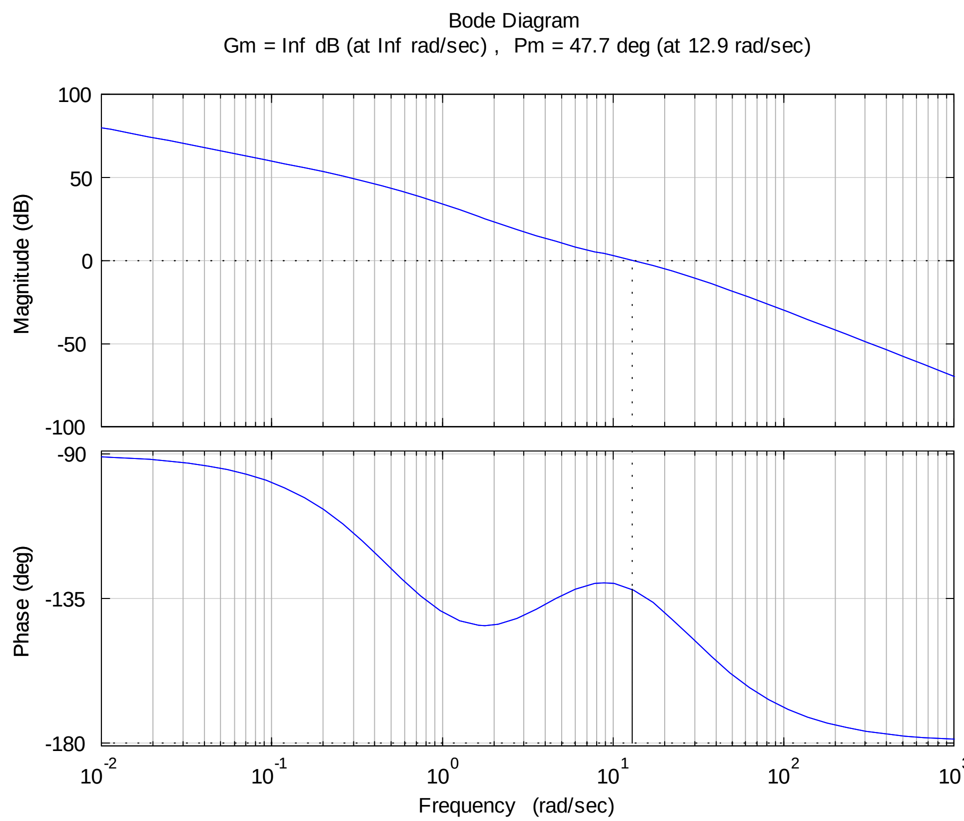
(f) One possible lead-lag compensator is :

$$D(s) = 100 \frac{\frac{s}{8.52}+1}{\frac{s}{22.36}+1} \frac{\frac{s}{4.47}+1}{\frac{s}{0.568}+1}$$

which meets all the specification :

$$\begin{aligned} K_v &= 100 \\ PM &= 47.7^\circ \text{ (at } \omega_c = 12.9 \text{ rad/sec)} \\ |KG| &= 50.45 \text{ (at } \omega = 1 \text{ rad/sec)} > 49 \\ |KG| &= 0.032 \text{ (at } \omega = 100 \text{ rad/sec)} < 0.0526 \end{aligned}$$

The Bode plot of the compensated open-loop system $D(s)G(s)$ is :



Problems and Solutions for Section 6.8

63. Assume that the system

$$G(s) = \frac{e^{-T_d s}}{s+10},$$

has a 0.2-sec time delay ($T_d = 0.2$ sec). While maintaining a phase margin $\geq 40^\circ$, find the maximum possible bandwidth using the following:

(a) One lead-compensator section

$$D(s) = K\frac{s+a}{s+b},$$

where $b/a = 100$;

(b) Two lead-compensator sections

$$D(s) = K\left(\frac{s+a}{s+b}\right)^2,$$

where $b/a = 10$.

(c) Comment on the statement in the text about the limitations on the bandwidth imposed by a delay.

Solution :

(a) One lead section :

With $b/a = 100$, the lead compensator can add the maximum phase lead :

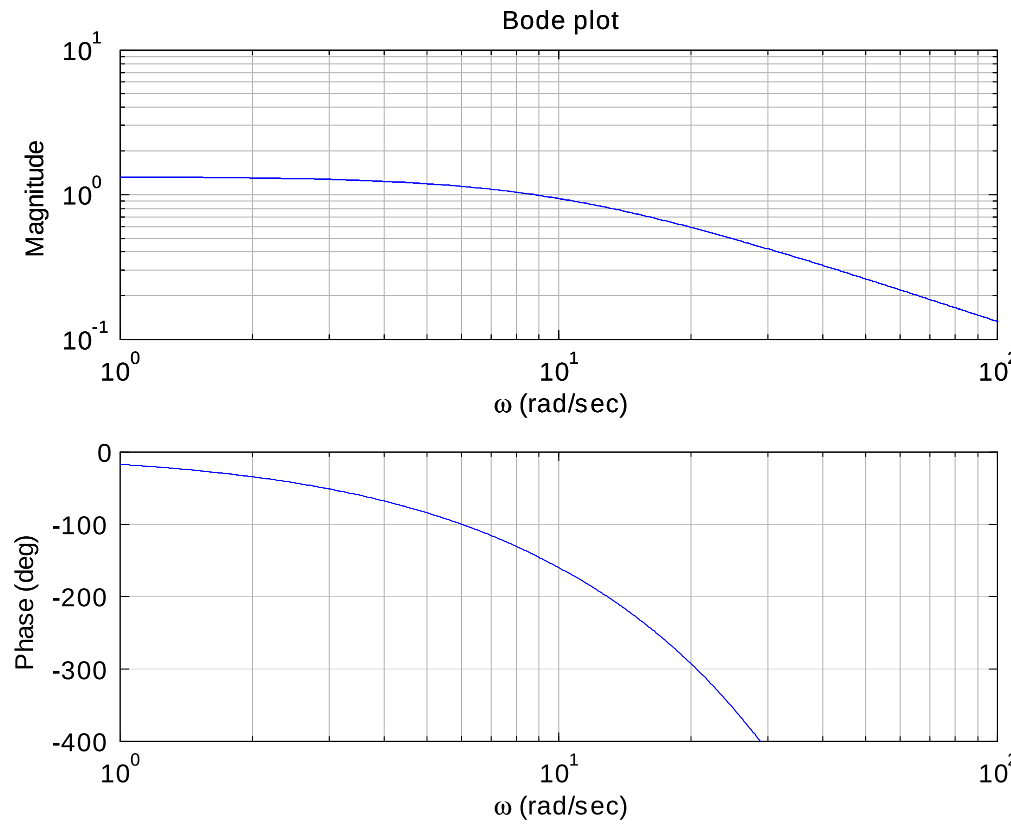
$$\begin{aligned} \phi_{\max} &= \sin^{-1}\frac{1-\frac{a}{b}}{1+\frac{a}{b}} \\ &= 78.6 \text{ deg (at } \omega = 10a \text{ rad/sec)} \end{aligned}$$

By trial and error, a good compensator is :

$$\begin{aligned} K &= 1202,\ a = 15 \Longrightarrow D_a(s) = 1202\frac{s+15}{s+1500} \\ PM &= 40^\circ \text{ (at } \omega_c = 11.1 \text{ rad/sec)} \end{aligned}$$

The Bode plot is shown below. Note that the phase is adjusted for the time delay by subtracting ωT_d at each frequency point while there is no effect on the magnitude. For reference, the figures also include the case of proportional control, which results in :

$$K = 13.3,\ PM = 40^\circ \text{ (at } \omega_c = 8.6 \text{ rad/sec)}$$



(b) Two lead sections :

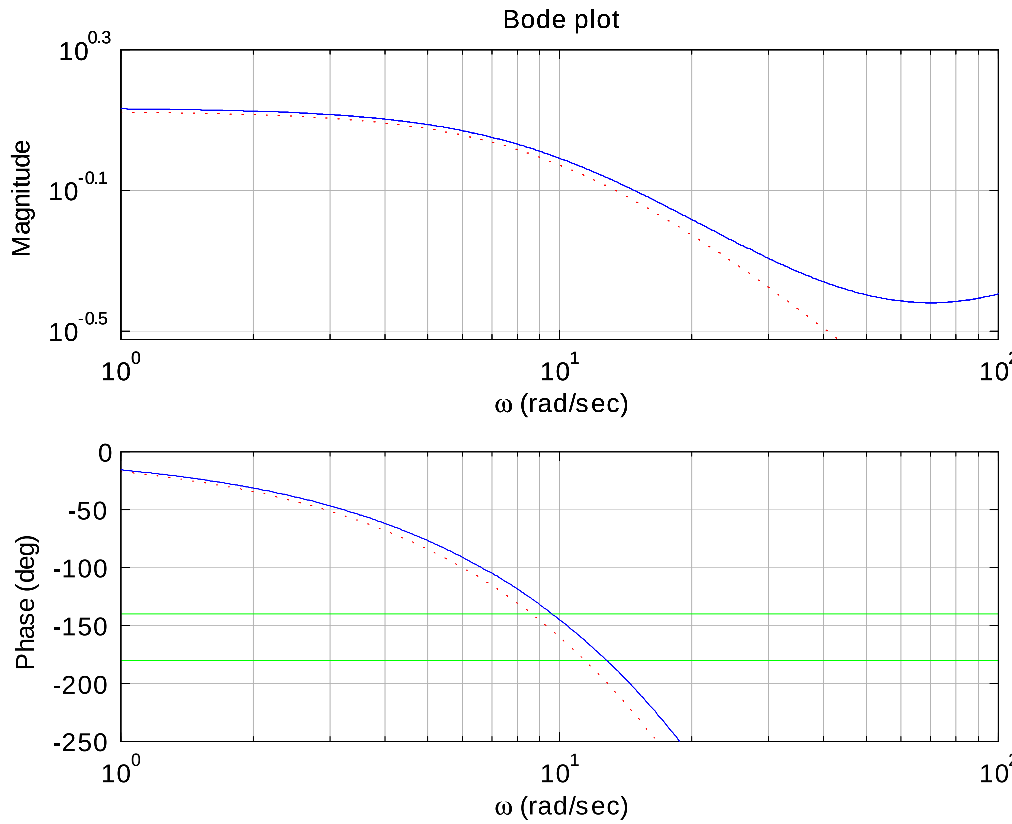
With $b/a = 10$, the lead compensator can add the maximum phase lead :

$$
\begin{aligned}
\phi_{\max} &= \sin^{-1}\frac{1-\frac{a}{b}}{1+\frac{a}{b}} \\
&= 54.9 \text{ deg (at } \omega = \sqrt{10}a \text{ rad/sec)}
\end{aligned}
$$

By trial and error, one of the possible compensators is :

$$
\begin{aligned}
K &= 1359,\ a = 70 \Longrightarrow D_b(s) = 1359\frac{(s+70)^2}{(s+700)^2} \\
PM &= 40^\circ \text{ (at } \omega_c = 9.6 \text{ rad/sec)}
\end{aligned}
$$

The Bode plot is shown below.



(c) The statement in the text is that it should be difficult to stabilize a system with time delay at crossover frequencies, $\omega_c \gtrsim 3/T_d$. This problem confirms this limit, as the best crossover frequency achieved was $\omega_c = 9.6$ rad/sec whereas $3/T_d = 15$ rad/sec. Since the bandwidth is approximately twice the crossover frequency, the limitations imposed on the bandwidth by the time delay is verified.

64. Determine the range of K for which the following systems are stable:

(a) $G(s) = K\frac{e^{-4s}}{s}$

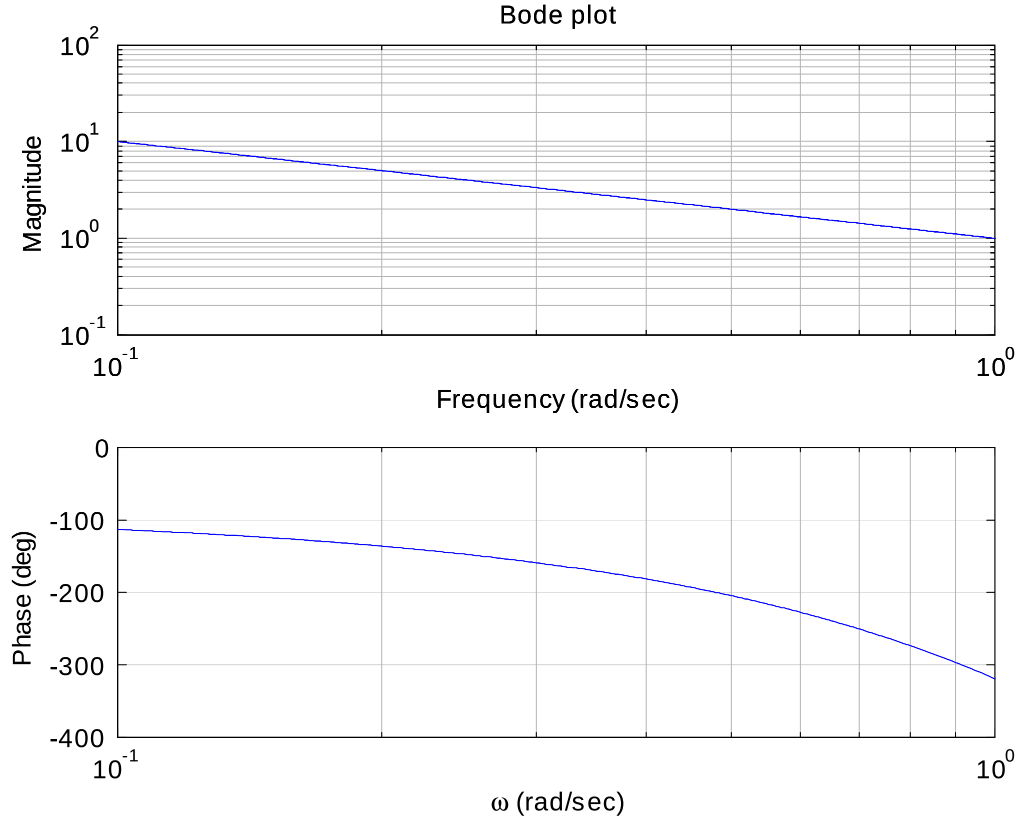
(b) $G(s) = K\frac{e^{-s}}{s(s+2)}$

Solution :

(a)

$$\left|\frac{G(j\omega)}{K}\right| = 2.54, \text{ when } \angle\frac{G(j\omega)}{K} = -180^\circ$$

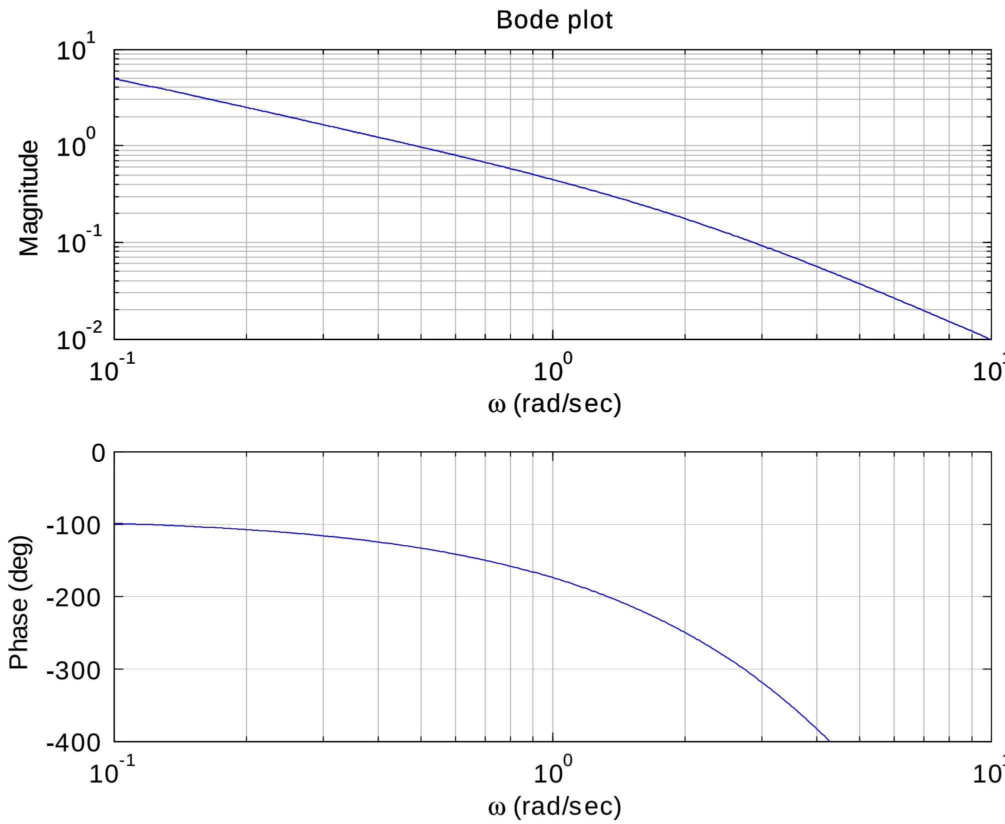
$$\text{range of stability : } 0 < K < \frac{1}{2.54}$$



(b)

$$\left|\frac{G(j\omega)}{K}\right| = 0.409 = \frac{1}{2.45}, \text{ when } \angle\frac{G(j\omega)}{K} = -180^\circ$$

$$\text{range of stability}: \; 0 < K < 2.45$$



65. In Chapter 5, we used various approximations for the time delay, one of which is the first order Padé

$$e^{-T_d s} \cong H_1(s) = \frac{1 - T_d s/2}{1 + T_d s/2}.$$

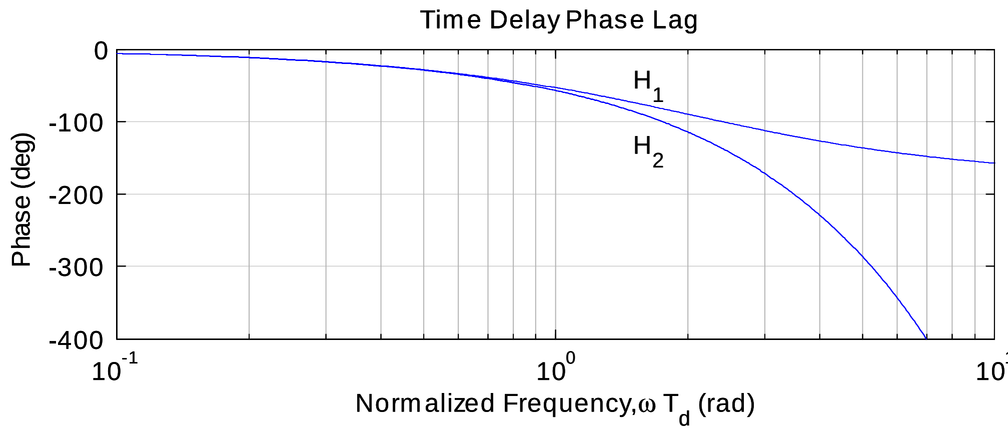
Using frequency response methods, the exact time delay

$$H_2(s) = e^{-T_d s}.$$

can be used. Plot the phase of $H_1(s)$ and $H_2(s)$ and discuss the implications.

Solution :

The approximation $H_1(j\omega)$ and the true phase $H_2(j\omega)$ are compared in the plot below:



$H_1(j\omega)$ closely approximates the correct phase of the delay (phase of $H_2(s)$) for $\omega T_d \lesssim \frac{\pi}{2}$ and progressively worsens above that frequency. The implication is that the $H_1(s)$ approximation should not be trusted for crossover frequencies $\omega_c \gtrsim \frac{\pi}{2T_d}$. Instead, one should use the exact phase for the time delay given by $H_2(s)$.

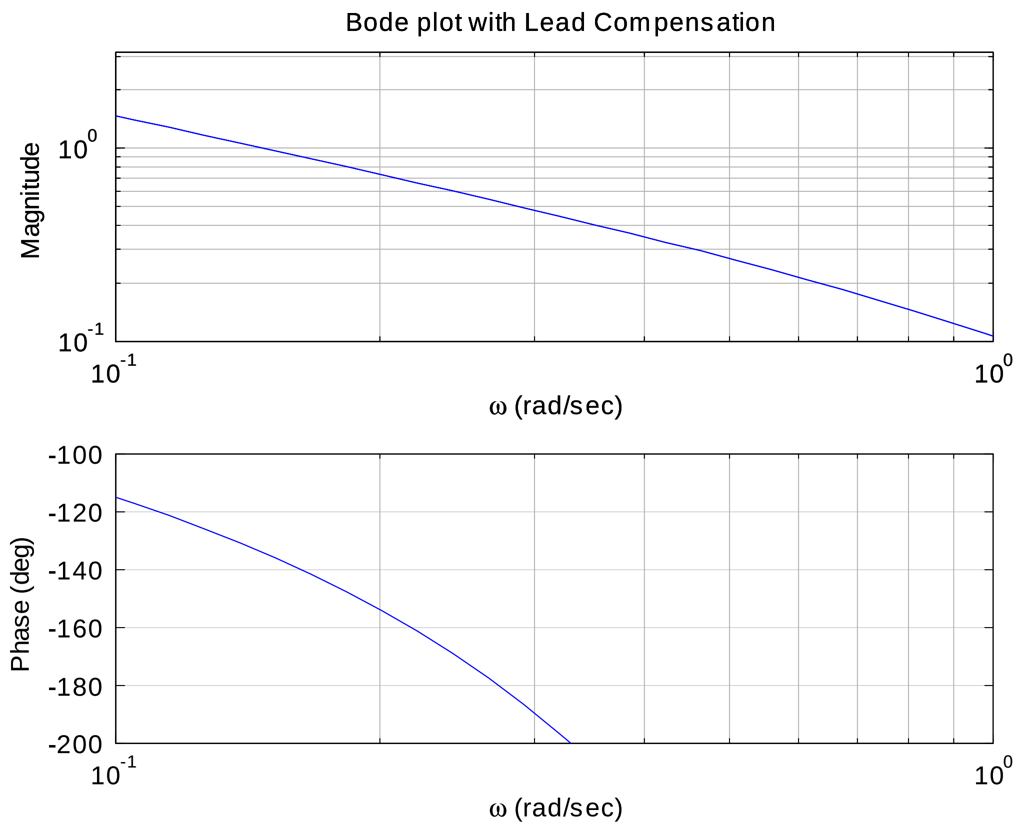
66. Consider the heat exchanger of Example 2.15 with the open-loop transfer function

$$G(s) = \frac{e^{-5s}}{(10s+1)(60s+1)}.$$

(a) Design a lead compensator that yields PM $\geq 45°$ and the maximum possible closed-loop bandwidth.

(b) Design a PI compensator that yields PM $\geq 45°$ and the maximum possible closed-loop bandwidth.

Solution :



(a) First, make sure that the phase calculation includes the time delay lag of $-T_d\omega = -5\omega$. A convenient placement of the lead zero is at $\omega = 0.1$ because that will preserve the -1 slope until the lead pole. We then raise the gain until the specified PM is obtained in order to maximize the crossover frequency. The resulting lead compensator,

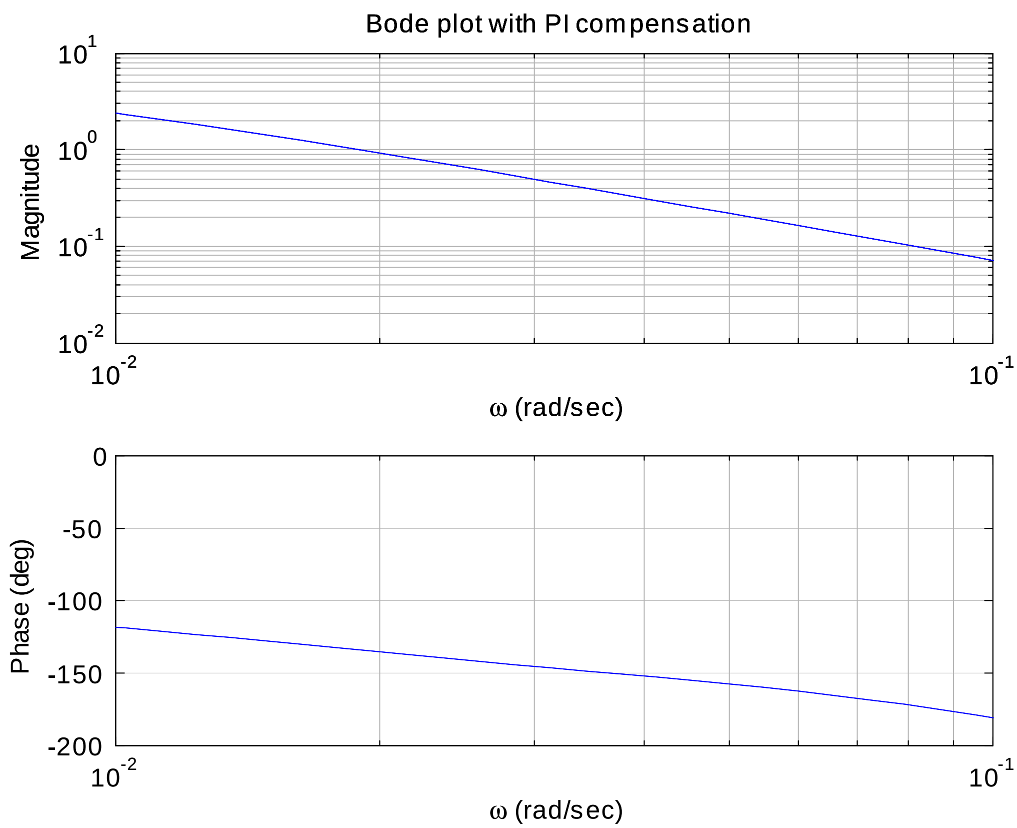
$$D(s) = \frac{90(s+0.1)}{(s+1)}$$

yields $PM = 46°$ as seen by the Bode below.Also note that the crossover frequency, $\omega_c = 0.15$ rad/sec, which can be read approximately from the plot above, and verified by using the `margin` command in MATLAB with the phase adjusted by the time delay lag.

(b) The brealpoint of the PI compensator needs to be kept well below 0.1 in order to maintain a positive phase margin at as high a crossover frequency as possible. In Table 4.1, Zeigler-Nichols suggest a breakpoint at $\omega = 1/17$, so we will select a PI of the form :

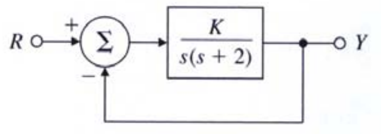
$$D(s) = K\left(1 + \frac{1}{20s}\right)$$

and select the gain so that the PM specification is met. For $K = 0.55$ the phase margin is $46°$ as shown by the Bode below:



Note with this compensation that $\omega_c = 0.02$ rad/sec, which is considerably lower than that yielded by the lead compensation.

Figure 6.105: Control system for Problem 67



Problems and Solutions for Section 6.9

67. A feedback control system is shown in Fig.6.105. The closed-loop system is specified to have an overshoot of less than 30% to a step input.

    (a) Determine the corresponding PM specification in the frequency domain and the corresponding closed-loop resonant peak value M_r. (See Fig. 6.38)

    (b) From Bode plots of the system, determine the maximum value of K that satisfies the PM specification.

    (c) Plot the data from the Bode plots (adjusted by the K obtained in part (b)) on a copy of the Nichols chart in Fig. 6.84 and determine the resonant peak magnitude M_r. Compare that with the approximate value obtained in part (a).

    (d) Use the Nichols chart to determine the resonant peak frequency ω_r and the closed-loop bandwidth.

    Solution :

    (a) From Fig. 6.38 :

    $$M_p \leq 0.3 \Longrightarrow PM \geq 40^o \Longrightarrow M_r \leq 1.5$$

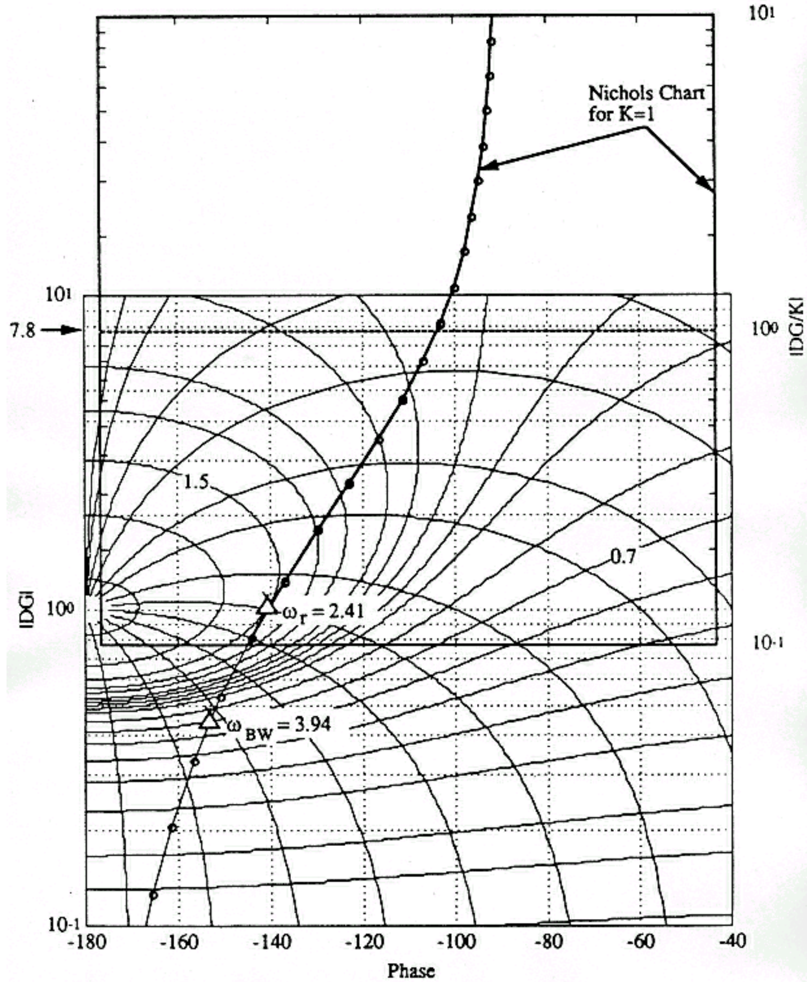
    $$\text{resonant peak : } M_r \leq 1.5$$

    (b) A sketch of the asymptotes of the open loop Bode shows that a PM of $\cong 40^o$ is obtained when $K = 8$. A Matlab plot of the Bode can be used to refine this and yields

    $$K = 7.81$$

    for $PM = 40^o$.

    (c) The Nichols chart below shows that $M_r = 1.5$ which agrees exactly with the prediction from Fig. 6.38:



(d) The corresponding frequency where the curve is tangent to $M_r = 1.5$ is:

$$\omega_r = 2.41 \text{ rad/sec}$$

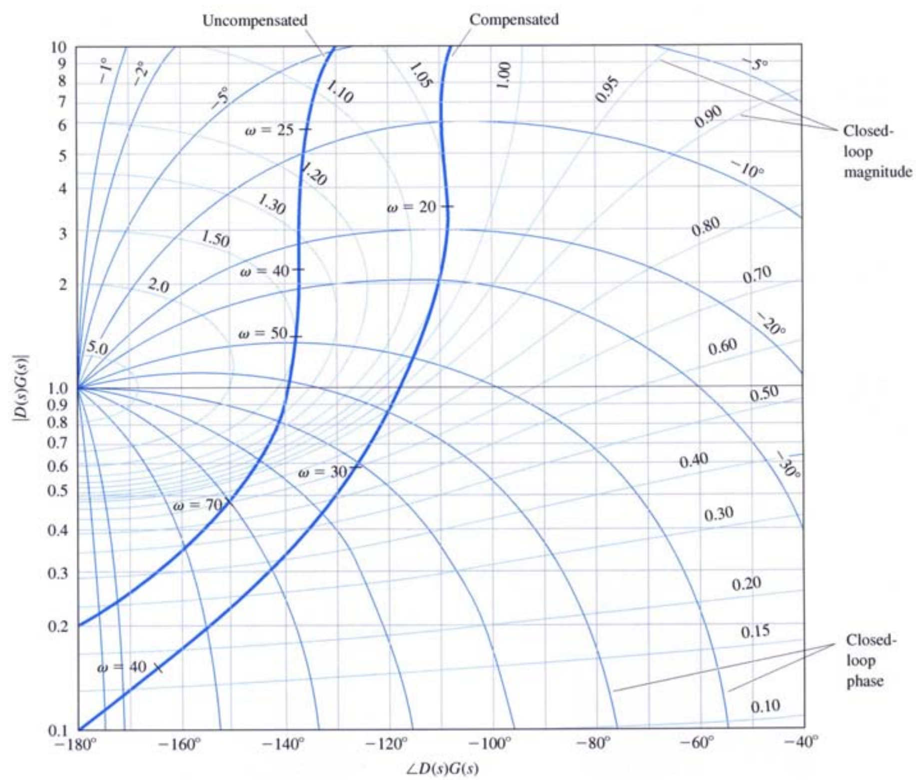
as can be determined by noting the frequency from the Bode plot that corresponds to the point on the Nichols chart.

The bandwidth ω_{BW} is determined by where the curve crosses the closed-loop magnitude of 0.7 and noting the frequency from the Bode plot that corresponds to the point on the Nichols chart

$$\omega_{BW} = 3.94 \text{ rad/sec}$$

68. The Nichols plot of an uncompensated and a compensated system are shown in Fig. 6.106.

Figure 6.106: Nichols plot for Problem 68



(a) What are the resonance peaks of each system?
(b) What are the PM and GM of each system?
(c) What are the bandwidths of each system?
(d) What type of compensation is used?

Solution :

(a) Resonant peak :

Uncompensated system : Resonant peak = 1.5 ($\omega_r = 50$ rad/sec)
Compensated system : Resonant peak = 1..05 ($\omega_r = 20$ rad/sec)

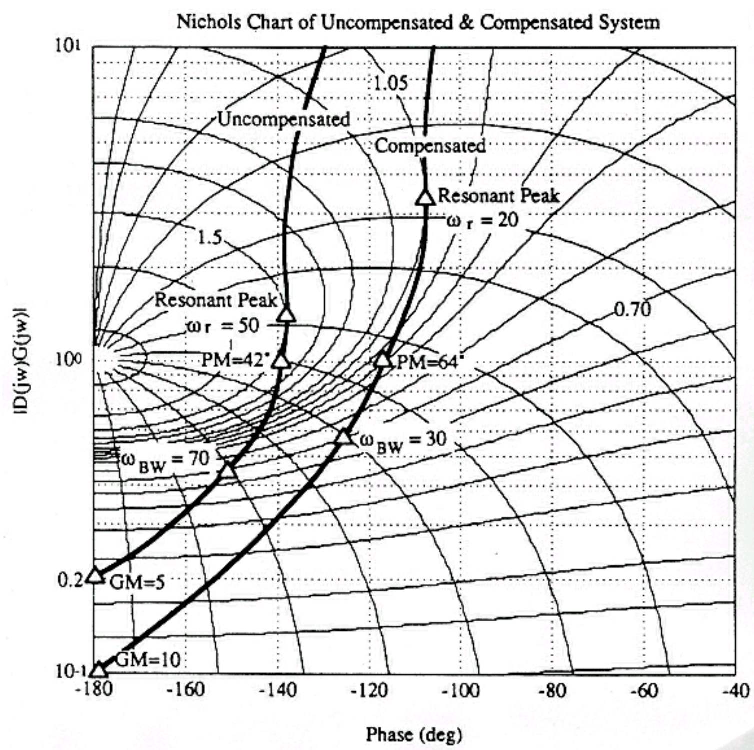
(b) PM, GM :

$$\text{Uncompensated system} \quad : \quad PM = 42^\circ,\ GM = \frac{1}{0.2} = 5$$
$$\text{Compensated system} \quad : \quad PM = 64^\circ,\ GM = \frac{1}{0.1} = 10$$

(c) Bandwidth :

Uncompensated system : Bandwidth = 70 rad/sec
Compensated system : Bandwidth = 30 rad/sec

(d) Lag compensation is used, since the bandwidth is reduced.



69. Consider the system shown in Fig. 6.97.

(a) Construct an inverse Nyquist plot of $[Y(j\omega)/E(j\omega)]^{-1}$.

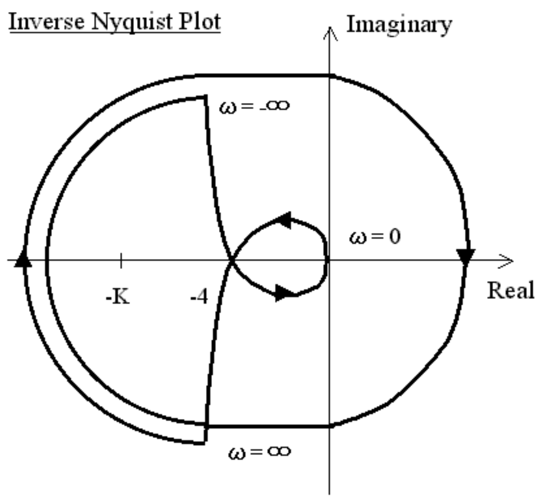
(b) Show how the value of K for neutral stability can be read directly from the inverse Nyquist plot.

(c) For $K = 4$, 2, and 1, determine the gain and phase margins.

(d) Construct a root-locus plot for the system, and identify corresponding points in the two plots. To what damping ratios ζ do the GM and PM of part (c) correspond?

Solution :

(a) See the inverse Nyquist plot.



(b) Let

$$G(j\omega) = \frac{Y(j\omega)}{E(j\omega)}$$

The characteristic equation with $s = j\omega$:

$$1 + K_u G(j\omega) = 0$$

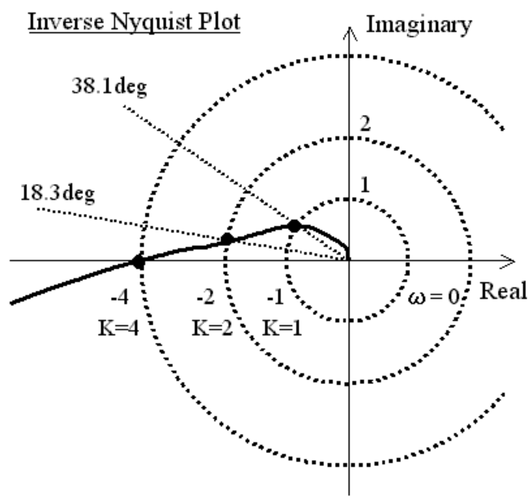
$$\Longrightarrow G^{-1} = -K_u$$

From the inverse Nyquist plot,

$$-K_u = -4 \Longrightarrow K_u = 4$$

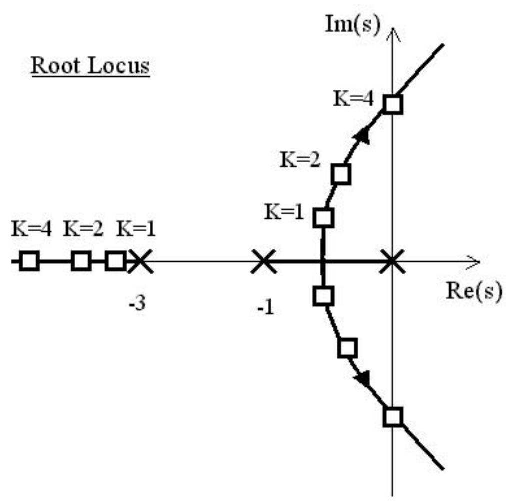
(c)

K	GM	PM
4	$\frac{-4}{-4} = 1$	$0°$
2	$\frac{-4}{-2} = 2$	$18.3°$
1	$\frac{-4}{-1} = 4$	$38.1°$



(d)

K	closed-loop poles	ζ
4	-4 $\pm j1.73$	0
2	-3.63 $-0.19 \pm j1.27$	0.14
1	-3.37 $-0.31 \pm j0.89$	0.33



70. An unstable plant has the transfer function

$$\frac{Y(s)}{F(s)} = \frac{s+1}{(s-1)^2}.$$

A simple control loop is to be closed around it, in the same manner as the block diagram in Fig. 6.97.

(a) Construct an inverse Nyquist plot of Y/F.

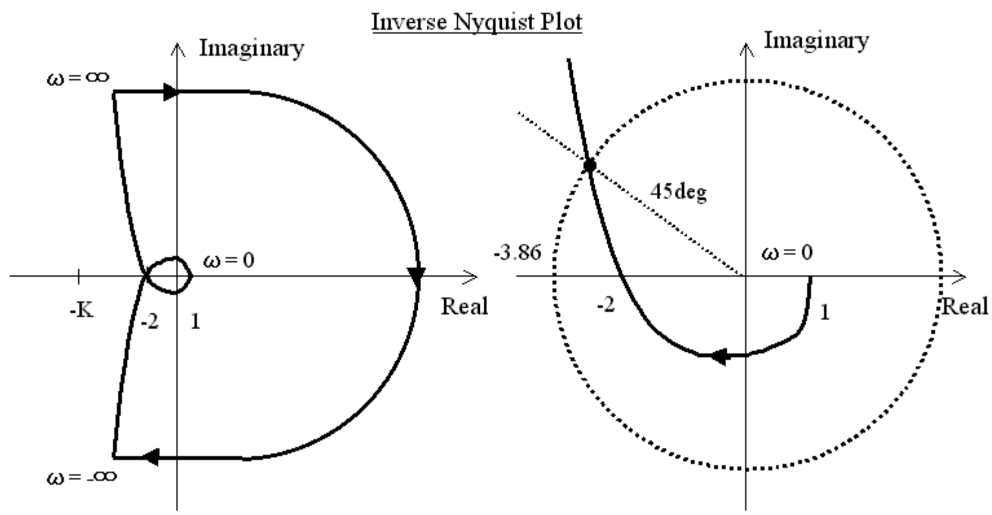
(b) Choose a value of K to provide a PM of 45°. What is the corresponding GM?

(c) What can you infer from your plot about the stability of the system when $K < 0$?

(d) Construct a root-locus plot for the system, and identity corresponding points in the two plots. In this case, to what value of ζ does PM = 45° correspond?

Solution :

(a) The plots are :



(b) From the inverse Nyquist plot, $K = 3.86$ provides a phase margin of 45°.
Since $K = 2$ gives $\angle G(j\omega)^{-1} = 180°$,

$$GM = \frac{2}{3.86} = 0.518$$

Note that GM is less than 1, but the system with $K = 3.86$ is stable.

$$K = 3.86, \; GM = 0.518$$

(c) We can apply stability criteria to the inverse Nyquist plot as follows :

Let

$$
\begin{aligned}
N &= \text{Net number of clockwise encirclement of } -K \\
P &= \text{Number of poles of } G^{-1} \text{ in RHP} \\
(&= \text{Number of zeros of } G \text{ in RHP}) \\
Z &= \text{Number of closed-loop system roots in RHP}
\end{aligned}
$$

Then,

$-K < -2$	$K > 2 \Longrightarrow N = 0, P = 0$ $\Longrightarrow Z = 0 \Longrightarrow$ Stable
$-2 < -K < 1$	$-1 > K > 2 \Longrightarrow N = 2, P = 0$ $\Longrightarrow Z = 2 \Longrightarrow$ Two unstable closed-loop roots
$-K > 1$	$K < -1 \Longrightarrow N = 1, P = 0$ $\Longrightarrow Z = 1 \Longrightarrow$ One unstable closed-loop root

Then, we can infer from the inverse Nyquist plot the stability situation when K is negative. In summary, when K is negative, there are either one or two unstable roots, and the system is always unstable.

(d) The stability situation seen in the root locus plot agrees with that obtained from the inverse Nyquist plot.
They show :

$K > 2$	Stable
$-1 < K < 2$	Two unstable closed-loop roots
$K < -1$	One unstable closed-loop root

For the phase margin 45°,

$$
\begin{aligned}
\text{closed-loop roots} &= -0.932 \pm 1.999j \\
\zeta &= 0.423
\end{aligned}
$$

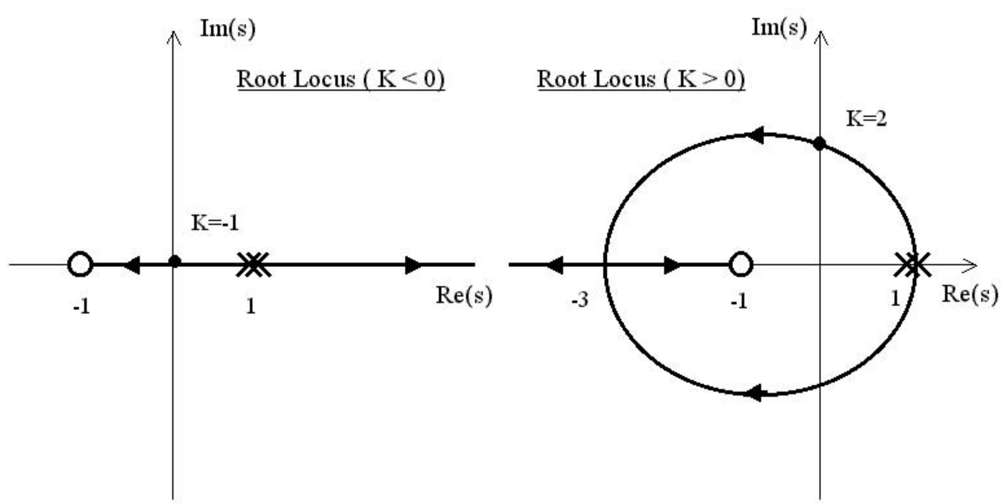
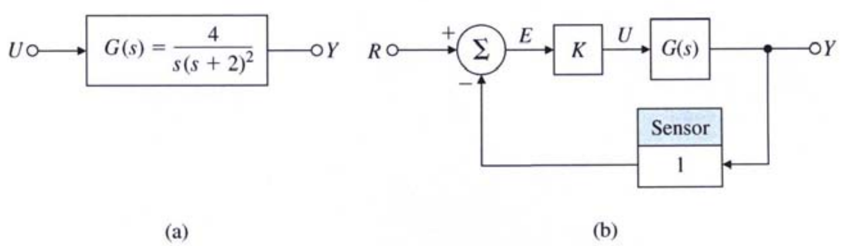


Figure 6.107: Control system for Problem 71



71. Consider the system shown in Fig. 6.107(a).

(a) Construct a Bode plot for the system.

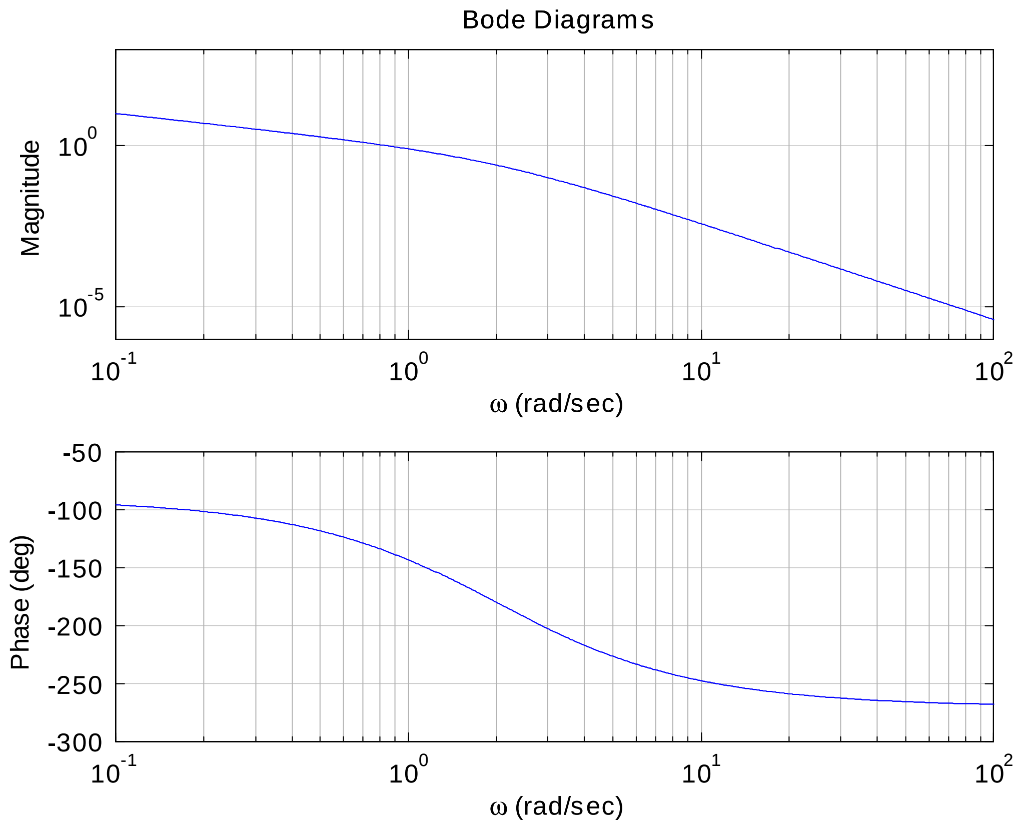
(b) Use your Bode plot to sketch an inverse Nyquist plot.

(c) Consider closing a control loop around $G(s)$, as shown in Fig. 6.107(b). Using the inverse Nyquist plot as a guide, read from your Bode plot the values of GM and PM when $K = 0.7$, 1.0, 1.4, and 2. What value of K yields PM $= 30°$?

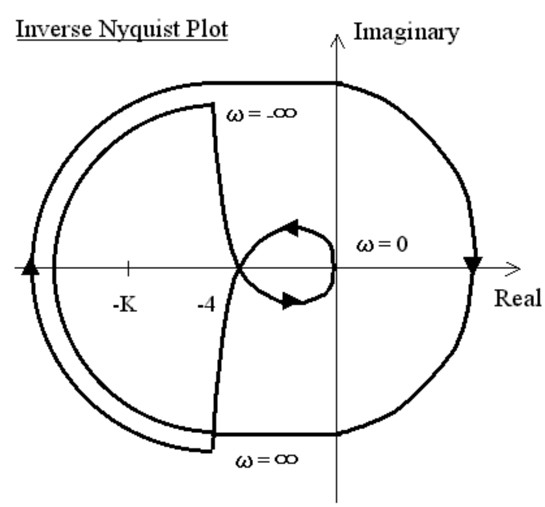
(d) Construct a root-locus plot, and label the same values of K on the locus. To what value of ζ does each pair of PM/GM values correspond? Compare the ζ vs PM with the rough approximation in Fig. 6.37

Solution :

(a) The figure follows :

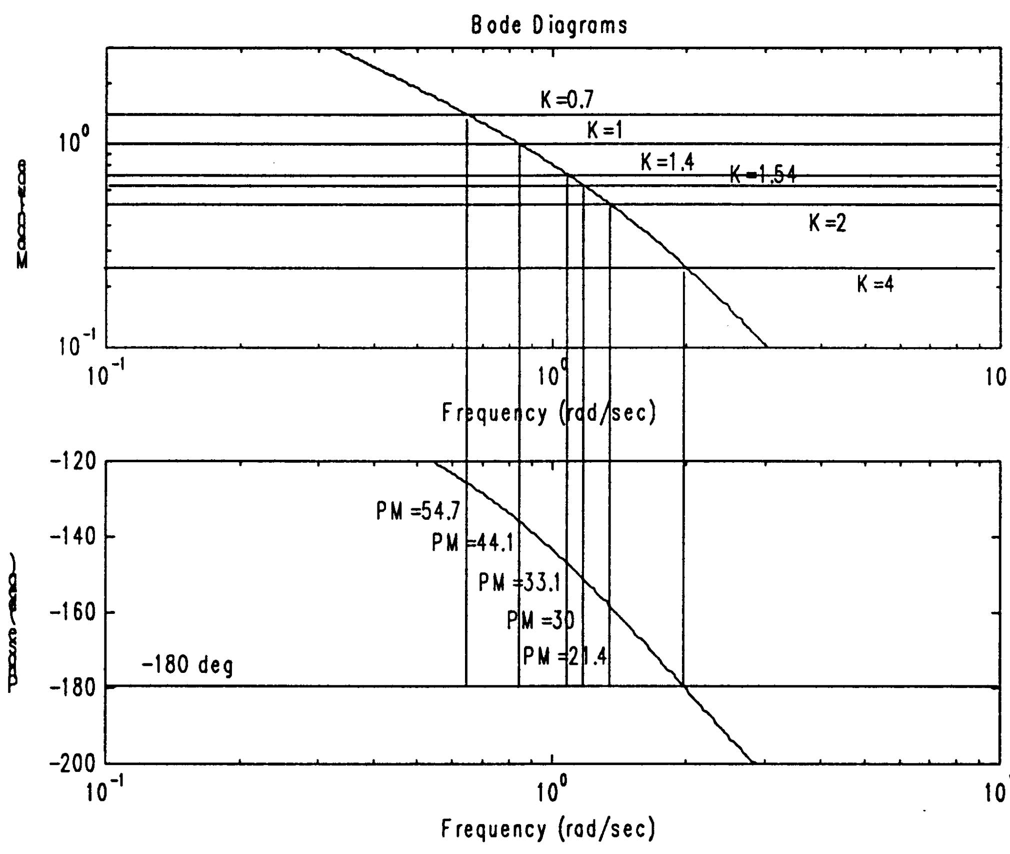


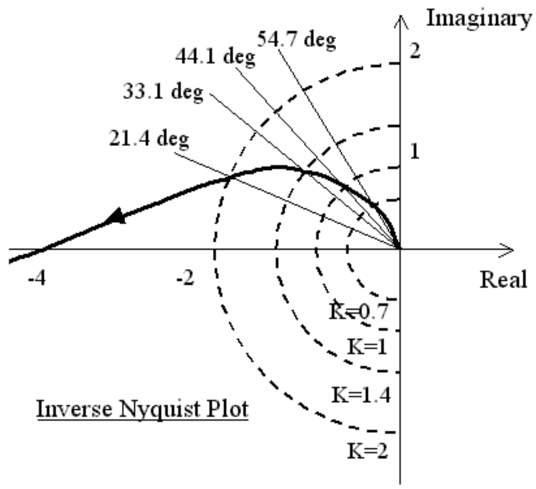
(b) The figure follows :



(c)

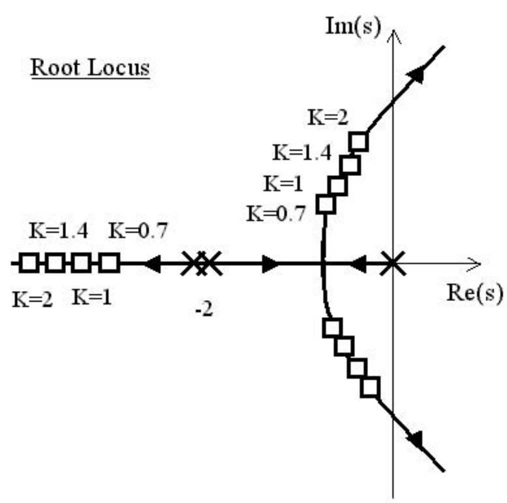
K	GM	PM
$K = 0.7$	$5.71\ (\omega = 2.00)$	$54.7^\circ\ (\omega_c = 0.64)$
$K = 1$	$4.00\ (\omega = 2.00)$	$44.1^\circ\ (\omega_c = 0.85)$
$K = 1.4$	$2.86\ (\omega = 2.00)$	$33.1^\circ\ (\omega_c = 1.08)$
$K = 2$	$2.00\ (\omega = 2.00)$	$21.4^\circ\ (\omega_c = 1.36)$
For $PM = 30^\circ$ $K = 1.54$	$2.60\ (\omega = 2.00)$	$30.0^\circ\ (\omega_c = 1.15)$





(d)

K	closed-loop roots	ζ
$K = 0.7$	-2.97 $-0.51 \pm 0.82j$	0.53
$K = 1$	-3.13 $-0.43 \pm 1.04j$	0.38
$K = 1.4$	-3.30 $-0.35 \pm 1.25j$	0.27
$K = 2$	-3.51 $-0.25 \pm 1.49j$	0.16



Chapter 7

State-Space Design

Problems and Solutions for Section 7.3: Block diagrams and State Space

1. Write the dynamic equations describing the circuit in Fig. 7.82. Write the equations as a second-order differential equation in $y(t)$. Assuming a zero input, solve the differential equation for $y(t)$ using Laplace-transform methods for the parameter values and initial conditions shown in the figure. Verify your answer using the `initial` command in MATLAB.

   Solution:

$$i = C\frac{dy}{dt} \qquad (1)$$

$$v = L\frac{di}{dt} \qquad (2)$$

$$u(t) - L\frac{di}{dt} - Ri(t) - y(t) = 0$$

$$\frac{di}{dt} = \frac{u}{L} - \frac{R}{L}i - \frac{1}{C}y \qquad (3)$$

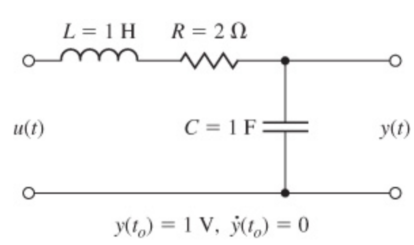


Figure 7.82: Circuit for Problem 7.1

Substituting the given values for L, R, and C we have for equation (3):

$$\frac{di}{dt} = u - 2\frac{dy}{dt} - y$$
$$\ddot{y} + 2\dot{y} + y = u$$

Characteristic equation:

$$\begin{aligned} s^2 + 2s + 1 &= 0 \\ (s+1)^2 &= 0 \end{aligned}$$

So:

$$y(t) = A_1 e^{-t} + A_2 t e^{-t}$$

Solving for the coefficients:

$$\begin{aligned} y(t) &= A_1 e^{-t} + A_2 t e^{-t} \\ y(t_0) &= A_1 e^{-t_0} + A_2 t_0 e^{t_0} = 1 \\ \dot{y}(t) &= -A_1 e^{-t} + A_2 e^{-t} - A_2 t e^{-t} \\ \dot{y}(t_0) &= -A_1 e^{-t_0} + A_2 e^{-t_0} - A_2 t e^{-t_0} = 0 \end{aligned}$$

$$\Rightarrow A_2 = e^{t_0} \quad \text{and} \quad A_1 = (1-t_0)e^{t_0}$$
$$y(t) = (1-t_0)e^{t_0 - t} + t e^{t_0 - t}.$$

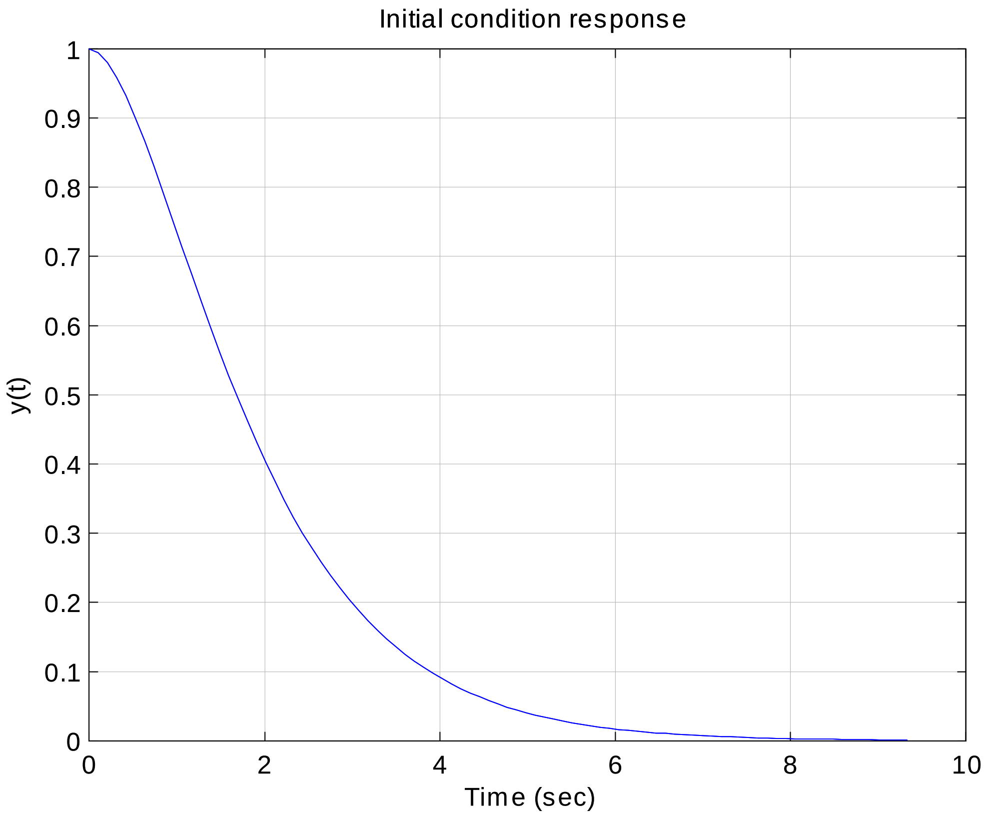
To verify the solution using MATLAB, re-write the differential equation in state space form,

$$\begin{aligned} \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u = \mathsf{a}\mathbf{x} + \mathsf{b}u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \mathsf{c}\mathbf{x}, \end{aligned}$$

where $\mathbf{x} = [y\ \dot{y}]^T$. Then the following MATLAB statements,

```
a=[0,1;-1,-2];
b=[0;1];
c=[1,0];
d=[0];
sys=ss(a,b,c,d);
xo=[1;0];
[y,t,x]=initial(sys,xo);
plot(t,y);
grid;
xlabel('Time (sec)');
ylabel('y(t)');
title('Initial condition response');
```

generate the initial condition response shown below that agrees with the analytical solution above.



Problem 7.1: Initial condition response.

2. A schematic for the satellite and scientific probe for the Gravity Probe-B (GP-B) experiment that was launched on April 30, 2004 is sketched in Fig. 7.83. Assume that the mass of the spacecraft plus helium tank, m_1, is 2000 kg and the mass of the probe, m_2, is 1000 kg. A rotor will float inside the probe and will be forced to follow the probe with a capacitive forcing mechanism. The spring constant of the coupling, k, is 3.2×10^6. The viscous damping b is 4.6×10^3.

   (a) Write the equations of motion for the system consisting of masses m_1 and m_2 using the inertial position variables, y_1 and y_2.

   (b) The actual disturbance u is a micrometeorite, and the resulting motion is very small Therefore, rewrite your equations with the scaled variables $z_1 = 10^6 y_1$, $z_2 = 10^6 y_2$, and $v = 1000u$.

   (c) Put the equations in state-variable form using the state $\mathbf{x} = [z_1 \quad \dot{z}_1 \quad z_2 \quad \dot{z}_2]^T$, the output $y = z_2$, and the input an impulse, $u = 10^{-3}\delta(t)$ N·sec on mass m_1.

   (d) Using the numerical values, enter the equations of motion into Matlab in the form

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}v \quad (1)$$

$$y = \mathbf{H}\mathbf{x} + Jv \quad (2)$$

and define the MATLAB system: sysGPB = ss(F,G,H,J). Plot the response of y caused by the impulse with the MATLAB command impulse(sysGPB). This is the signal the rotor must follow.

(e) Use the MATLAB commands p = eig(F) to find the poles (or roots) of the system and z = tzero(F,G,H, J) to find the zeros of the system.

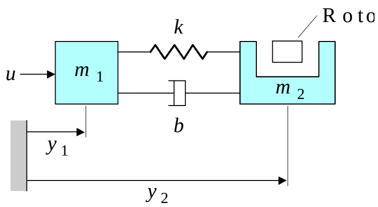


Figure 7.83: Schematic diagram of the GP-B satellite and probe.

Solution:

(a) The rotor is not part of the problem and can be ignored in writing the equations of motion

$$\begin{aligned} m_1\ddot{y}_1 &= u - k\,(y_1 - y_2) - b\,(\dot{y}_1 - \dot{y}_2) \\ m_2\ddot{y}_2 &= -k\,(y_2 - y_1) - b\,(\dot{y}_2 - \dot{y}_1) \end{aligned}$$

(b) Let's put in the values for the parameters as well as scale the variables as requested.

$$\begin{aligned} 2000\left(10^{-6}\ddot{z}_1\right) &= \frac{1}{1000}v - 10^{-6}(3.2\times 10^{6})\,(z_1 - z_2) - 10^{-6}(4.6\times 10^{3})\,(\dot{z}_1 - \dot{z}_2) \\ 1000\left(10^{-6}\ddot{z}_2\right) &= -10^{-6}(3.2\times 10^{6})\,(z_2 - z_1) - 10^{-6}(4.6\times 10^{3})\,(\dot{z}_2 - \dot{z}_1) \end{aligned}$$

which becomes

$$\begin{aligned} \ddot{z}_1 &= -(1.6\times 10^{3})\,(z_1 - z_2) - (2.3)\,(\dot{z}_1 - \dot{z}_2) + \frac{1}{2}v \\ \ddot{z}_2 &= -(3.2\times 10^{3})\,(z_2 - z_1) - (4.6)\,(\dot{z}_2 - \dot{z}_1) \end{aligned}$$

(c) The state-variable form for $\mathbf{x} = [z_1 \quad \dot{z}_1 \quad z_2 \quad \dot{z}_2]^T$ is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(1.6\times 10^{3})\,(x_1 - x_3) - (2.3)\,(x_2 - x_4) + \frac{1}{2}v \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -(3.2\times 10^{3})\,(x_3 - x_1) - (4.6)\,(x_4 - x_2) \end{aligned}$$

or, in matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1.6\times 10^{3} & -2.3 & 1.6\times 10^{3} & 2.3 \\ 0 & 0 & 0 & 1 \\ 3.2\times 10^{3} & 4.6 & -3.2\times 10^{3} & -4.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} v$$

and the output equation is

$$y = [0 \quad 0 \quad 1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + 0$$

(d) The system matrices

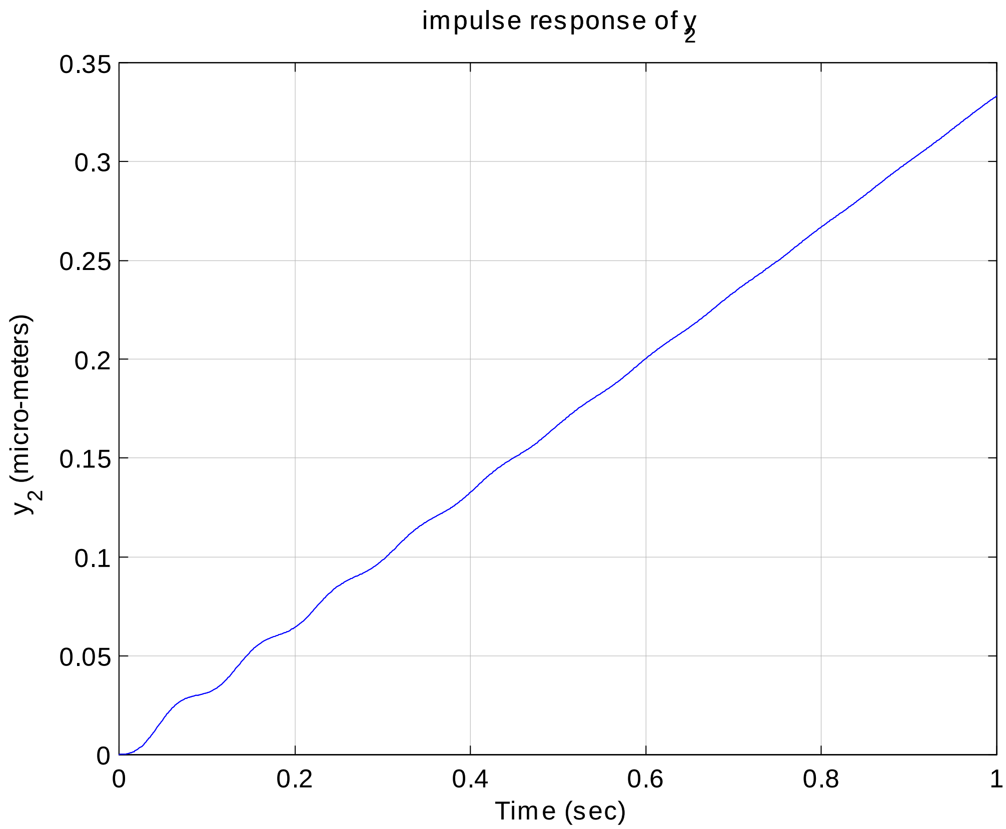
$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1.6 \times 10^3 & -2.3 & 1.6 \times 10^3 & 2.3 \\ 0 & 0 & 0 & 1 \\ 3.2 \times 10^3 & 4.6 & -3.2 \times 10^3 & -4.6 \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{H} = [0 \quad 0 \quad 1 \quad 0] \qquad \text{and} \qquad J = 0$$

plus the MATLAB statements:

```
sysGPB = ss(F,G,H,J);

t=0:0.001:1;

y=impulse(sysGPB,t);    % u = 10^-3 implies that v = 1

plot(t,y)
```

produce the plot below.



Impulse response for Problem 7.2.

The micro meteorite hits the first mass and imparts a velocity of 0.33μm/sec to the two mass system. It also excites the resonant mode of relative motion between the masses that dies out in less than a second.

Problems and Solutions for Section 7.4: Analysis of the State Equations

3. Give the state description matrices in control-canonical form for the following transfer functions:

(a) $\dfrac{1}{4s+1}$

(b) $\dfrac{5(s/2+1)}{(s/10+1)}$

(c) $\dfrac{2s+1}{s^2+3s+2}$

(d) $\dfrac{s+3}{s(s^2+2s+2)}$

(e) $\dfrac{(s+10)(s^2+s+25)}{s^2(s+3)(s^2+s+36)}$

Solution:

(a) $F = -0.25,\ G = 1,\ H = 0.25,\ J = 0.$

(b) $F = -10,\ G = 1,\ H = -200,\ J = 25.$

Hint: Do a partial fraction expansion to find the J term first.

(c)

$$\mathbf{F} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix},\ \mathbf{G} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \mathbf{H} = \begin{bmatrix} 2 & 1 \end{bmatrix},\ J = [0].$$

(d)

$$\mathbf{F} = \begin{bmatrix} -2 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ \mathbf{G} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{H} = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix},\ J = [0].$$

(e)

$$\mathbf{F} = \begin{bmatrix} -4 & -39 & -108 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},\ \mathbf{G} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},\ \mathbf{H} = \begin{bmatrix} 0 & 1 & 11 & 35 & 250 \end{bmatrix},\ J = [0].$$

4. Use the MATLAB function `tf2ss` to obtain the state matrices called for Problem 7.3.

Solution:

In all cases, simply form *num* and *den* given below and then use the MATLAB command `[F,G,H,J] = tf2ss(num,den)`.

(a) $num = \begin{bmatrix} 0 & 1 \end{bmatrix}$, $den = \begin{bmatrix} 4 & 1 \end{bmatrix}$.

(b) $num = \begin{bmatrix} 5/2 & 5 \end{bmatrix}$, $den = \begin{bmatrix} 1/10 & 1 \end{bmatrix}$.

(c) $num = \begin{bmatrix} 0 & 2 & 1 \end{bmatrix}$, $den = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$.

(d) $num = \begin{bmatrix} 0 & 0 & 1 & 3 \end{bmatrix}$, $den = \begin{bmatrix} 1 & 2 & 2 & 0 \end{bmatrix}$.

(e) $num = \begin{bmatrix} 0 & 0 & 1 & 11 & 35 & 250 \end{bmatrix}$, $den = \begin{bmatrix} 1 & 4 & 1 & 39 & 108 & 0 & 0 \end{bmatrix}$.

Note that the answers are the same as for Problem 7.2.

Hint: The MATLAB function `conv` will save time when forming the numerator and denominator for part (e).

5. Give the state description matrices in normal-mode form for the transfer functions of Problem 7.3. Make sure that all entries in the state matrices are realvalued by keeping any pairs of complex conjugate poles together, and realize them as a separate subblock in control canonical form.

Solution:

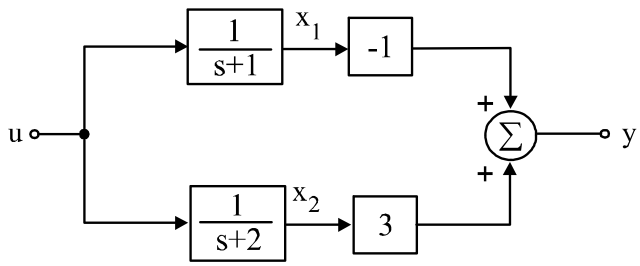
(a) $F = -0.25,\ G = 1,\ H = 0.25,\ J = 0.$

(b) $F = -10,\ G = 1,\ H = -200,\ J = 25.$

(c)

$$\frac{2s+1}{s^2+3s+2} = \frac{2s+1}{(s+1)(s+2)} = \frac{-1}{s+1} + \frac{3}{s+2},$$

The computation can also be done using the `residue` command in MATLAB.



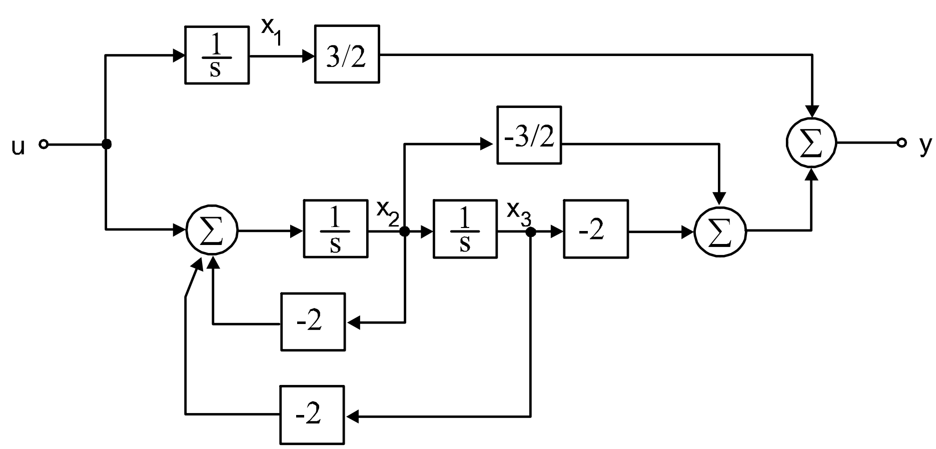
Block diagram for Problem 7.5 (c).

$$\mathbf{F} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{H} = \begin{bmatrix} -1 & 3 \end{bmatrix}, \ J = [0].$$

(d)

$$\frac{s+3}{s(s^2+2s+2)} = \frac{3/2}{s} - \frac{3/2s+2}{s^2+2s+2},$$

The computation can also be done using the residue command in Matlab.



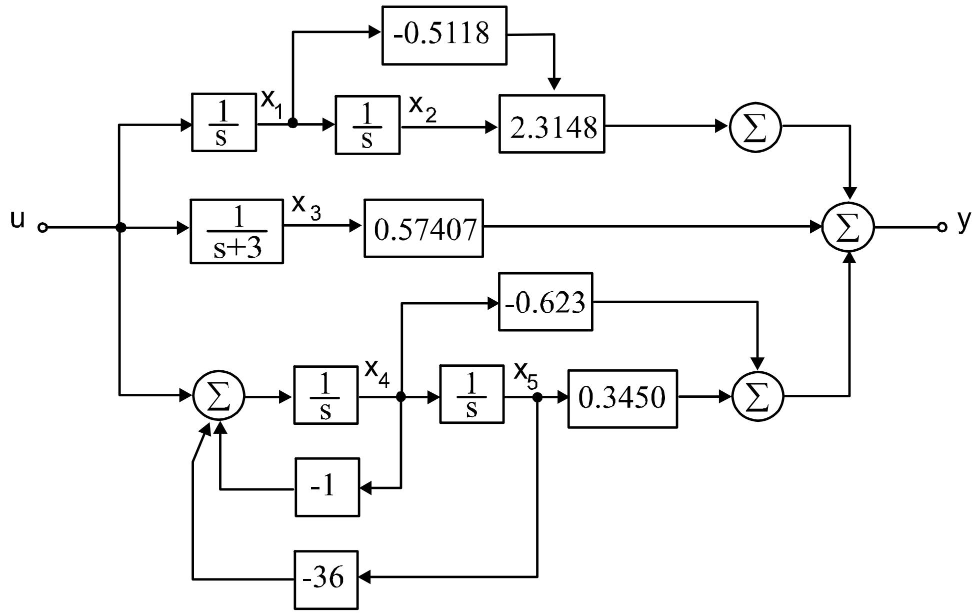
Block diagram for Problem 7.5 (d).

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & 1 & 0 \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{H} = \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} & -2 \end{bmatrix}, \ J = [0].$$

(e) The hard part is getting the expansion,

$$\frac{(s+10)(s^2+s+25)}{s^2(s+3)(s^2+s+36)} = \frac{-0.5118s+2.3148}{s^2} + \frac{0.57407}{s+3} + \frac{-0.0622s+0.3452}{s^2+s+36}$$

You can use the Matlab function residue to obtain this. From the figure, we have,



Block diagram for Problem 7.5 (e).

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -1 & -36 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{H} = \begin{bmatrix} -0.5118 & 2.3148 & 0.57407 & -0.0622 & 0.3452 \end{bmatrix}, J = [0].$$

6. A certain system with state $\mathbf{x}$ is described by the state matrices,

$$\mathbf{F} = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$
$$\mathbf{H} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad J = 0.$$

Find the transformation $\mathbf{T}$ so that if $\mathbf{x} = \mathbf{T}\mathbf{z}$, the state matrices describing the dynamics of $\mathbf{z}$ are in control canonical form. Compute the new matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and D.

Solution:

Following the procedure outlined in the chapter, we have,

$$\mathcal{C} = \begin{bmatrix} \mathbf{G} & \mathbf{FG} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}.$$

$$\begin{aligned} t_2 &= \begin{bmatrix} 0 & 1 \end{bmatrix} \mathcal{C}^{-1} = \frac{1}{5} \begin{bmatrix} 3 & -1 \end{bmatrix}, \\ t_1 &= t_2\mathbf{F} = \frac{1}{5} \begin{bmatrix} -4 & 3 \end{bmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned}
\mathbf{T}^{-1} &= \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & -1/5 \end{bmatrix} \Longrightarrow \mathbf{T} = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}, \\
\mathbf{A} &= \mathbf{T}^{-1}\mathbf{F}\mathbf{T} = \begin{bmatrix} -2 & -2 \\ 1 & 0 \end{bmatrix}, \ \mathbf{B} = \mathbf{T}^{-1}\mathbf{G} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
\mathbf{C} = \mathbf{H}\mathbf{T} &= \begin{bmatrix} 1 & 3 \end{bmatrix}, \ D = J = 0.
\end{aligned}$$

7. Show that the transfer function is not changed by a linear transformation of state.

Solution:

Assume the original system is,

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{F}\mathbf{x} + \mathbf{G}u, \\
y &= \mathbf{H}\mathbf{x} + Ju, \\
G(s) &= \mathbf{H}(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G} + J.
\end{aligned}$$

Assume a change of state from $\mathbf{x}$ to $\mathbf{z}$ using the nonsingular transformation $\mathbf{T}$,

$$\mathbf{x} = \mathbf{T}\mathbf{z}.$$

The new system matrices are,

$$\mathbf{A} = \mathbf{T}^{-1}\mathbf{F}\mathbf{T}, \ \mathbf{B} = \mathbf{T}^{-1}\mathbf{G}, \ \mathbf{C} = \mathbf{H}\mathbf{T}, \ D = J.$$

The transfer function is,

$$\begin{aligned}
G_z(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D \\
&= \mathbf{H}\mathbf{T}(s\mathbf{I} - \mathbf{T}^{-1}\mathbf{F}\mathbf{T})^{-1}\mathbf{T}^{-1}\mathbf{G} + J.
\end{aligned}$$

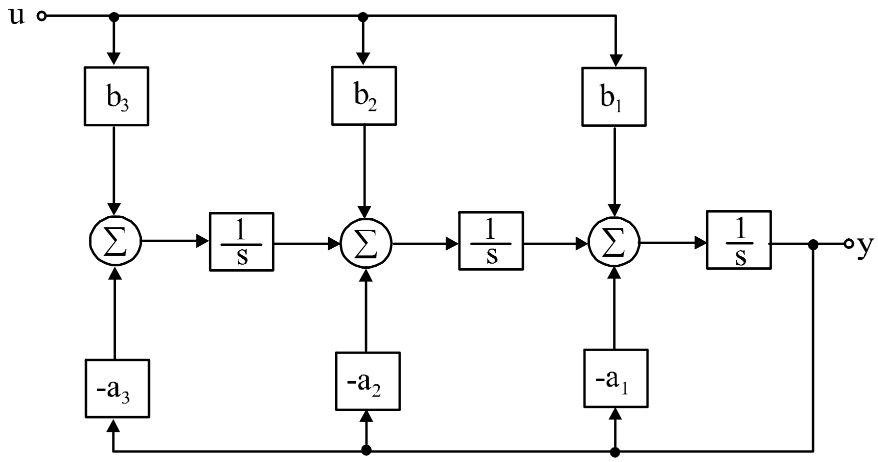
If we factor $\mathbf{T}$ on the left and $\mathbf{T}^{-1}$ on the right of the $(s\mathbf{I} - \mathbf{T}^{-1}\mathbf{F}\mathbf{T})^{-1}$ term, we obtain,

$$\begin{aligned}
G_z(s) &= \mathbf{H}\mathbf{T}(s\mathbf{T}\mathbf{T}^{-1} - \mathbf{T}^{-1}\mathbf{F}\mathbf{T})^{-1}\mathbf{T}^{-1}\mathbf{G} + J \\
&= \mathbf{H}\mathbf{T}\mathbf{T}^{-1}(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{T}\mathbf{T}^{-1}\mathbf{G} + J = \mathbf{H}(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G} + J = G(s).
\end{aligned}$$

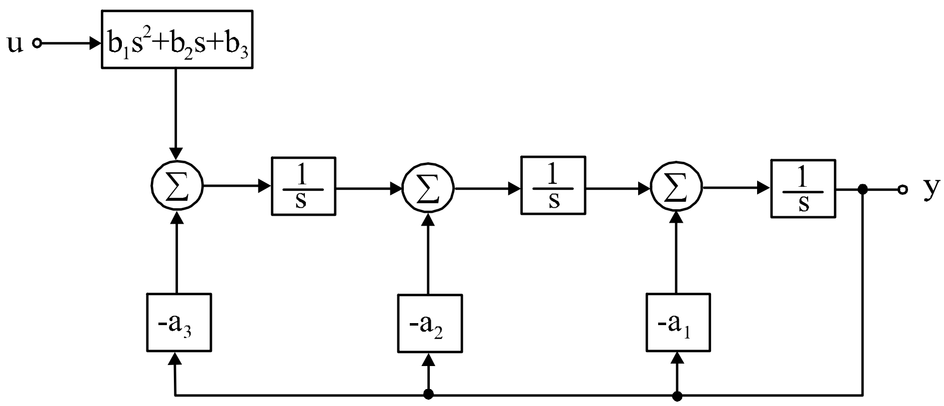
8. Use block-diagram reduction or Mason's rule to find the transfer function for the system in observer canonical form depicted by Fig. 7.31.

Solution:

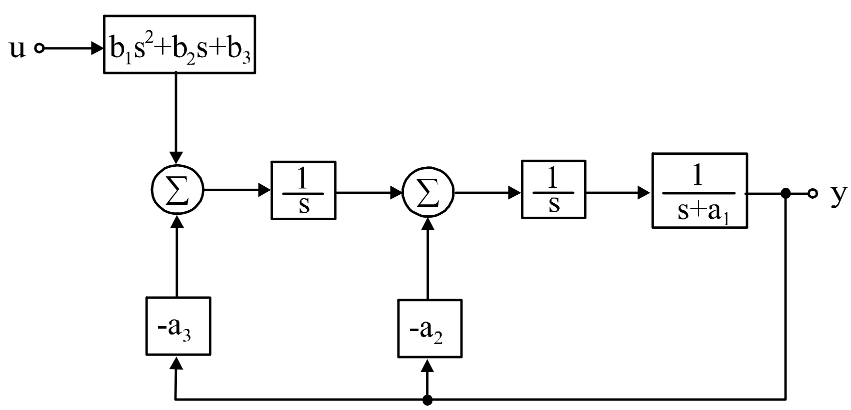
(a) We will show the process for the general third-order case shown below. Combine the feed-forward terms to produce the top figure on the next page. Then, reduce the last loop to get the figure on the bottom.



Problem 7.8: Observer canonical form.



Observer canonical form: feedforward terms combined.



Observer canonical form: one loop reduced.

(a) Using Mason's rule:

Forward path gains:

$$p_1 = \frac{b_3}{s^3}, \quad p_2 = \frac{b_2}{s^2}, \quad p_3 = \frac{b_1}{s}$$

Loop path gains:

$$\ell_1 = -\frac{a_3}{s^3}, \quad \ell_2 = -\frac{a_2}{s^2}, \quad \ell_3 = -\frac{a_1}{s}$$

$$\frac{Y}{R} = \frac{p_1 + p_2 + p_3}{1 - \ell_1 - \ell_2 - \ell_3} = \frac{\frac{b_3}{s^3} + \frac{b_2}{s^2} + \frac{b_1}{s}}{1 + \frac{a_3}{s^3} + \frac{a_2}{s^2} + \frac{a_1}{s}} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}.$$

9. Suppose we are given a system with state matrices $\mathbf{F}, \mathbf{G}, \mathbf{H}$ ($J = 0$ in this case). Find the transformation $\mathbf{T}$ so that, under Eqs. (7.24) and (7.25), the new state description matrices will be in observer canonical form.

   Solution:

   Express the transformation matrix in terms of its column vectors,

$$\mathbf{T} = [t_1 \;\; t_2 \;\; t_3]$$

   Then if $\mathbf{A}$ is in observer canonical form,

$$\begin{aligned} \mathbf{TA}_o &= [* \;\; t_1 \;\; t_2] = \mathbf{FT} = [\mathbf{F}t_1 \;\; \mathbf{F}t_2 \;\; \mathbf{F}t_3] \\ \mathbf{C}_o &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = \mathbf{HT} = [\mathbf{H}t_1 \;\; \mathbf{H}t_2 \;\; \mathbf{H}t_3]. \end{aligned}$$

   From these,

$$\begin{bmatrix} \mathbf{H} \\ \mathbf{HF} \\ \mathbf{HF}^2 \end{bmatrix} t_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow t_3 = \mathcal{O}^{-1} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T, \; t_2 = \mathbf{F}t_3, \; t_1 = \mathbf{F}t_2.$$

10. Use the transformation matrix in Eq. (7.41) to explicitly multiply out the equations at the end of Example 7.10.

    Solution:

$$\mathbf{A}_m = \mathbf{T}^{-1}\mathbf{A}_c\mathbf{T} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix}.$$

$$\begin{aligned} \mathbf{B}_m &= \mathbf{T}^{-1}\mathbf{B}_c = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \\ \mathbf{C}_m &= \mathbf{C}_c\mathbf{T} = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \end{bmatrix}. \end{aligned}$$

11. Find the state transformation that takes the observer canonical form of Eq. (7.35) to the modal canonical form.

    Solution:

    We wish to find the transformation $\mathbf{T}$ such that,

$$\mathbf{A}_m = \mathbf{T}^{-1}\mathbf{A}_o\mathbf{T} = diag(\lambda_1, \lambda_2, ..., \lambda_n).$$

$$\begin{aligned}\mathbf{B}_m &= \mathbf{T}^{-1}\mathbf{B}_o,\\ \mathbf{C}_m &= \mathbf{C}_o\mathbf{T}.\end{aligned}$$

The columns of $\mathbf{T}$ are the eigenvectors of $\mathbf{A}_o$. The eigenvectors of $\mathbf{A}_o$ are all of the form (which can be proved by induction):

$$\begin{aligned} t_i &= \begin{bmatrix} 1 \\ \lambda_i + a_1 \\ \lambda_i^2 + a_1\lambda_i + a_2 \\ \lambda_i^3 + a_1\lambda_i^2 + a_2\lambda_i + a_3 \\ \vdots \\ \lambda_i^{n-1} + a_1\lambda_i^{n-2} + a_2\lambda_i^{n-3} + \ldots + a_{n-1} \end{bmatrix}, \; i = 1, 2, ..., n. \\ \mathbf{T} &= \begin{bmatrix} t_1 & t_2 & ... & t_n \end{bmatrix}. \end{aligned}$$

For the cases where there are *repeated* eigenvalues, and a full set of linearly *independent* eigenvectors do not exist, then the generalized eigenvectors need to be computed to transform the system to Jordan form (see Strang, 1988).

12. a) Find the transformation $\mathbf{T}$ that will keep the description of the tape-drive system of Example 7.11 in modal canonical form but will convert each element of the input matrix $\mathbf{B}_m$ to unity.

    b) Use MATLAB to verify that your transformation does the job.

    Solution:

    (a) We would like to find a transformation matrix $\mathbf{T}_1$such that,

$$\mathbf{B}_m = \mathbf{T}_1^{-1}\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T.$$

Since $\mathbf{T}_1$ is full rank, this is equivalent to solving $\mathbf{G} = \mathbf{T}_1\mathbf{B}_m$. Recall that the magnitude of the eigenvectors are can be scaled by any arbitrary constant, so long as the direction in state space is preserved. Thus we can scale each of the eigenvectors of $\mathbf{T}$ found in Example 7.3 and keep the solution in modal form. Let,

$$\mathbf{T}_1 = \mathbf{TN} = [n_1t_1 \; n_2t_2 \cdots n_5t_5],$$

where t_i are the eigenvectors, n_i are scalars and N=$\mathsf{diag}(\mathsf{n}_1,\cdots,\mathsf{n}_5)$. Now,

$$\mathbf{G} = \mathbf{T}_1\mathbf{B}_m = \mathbf{TNB}_m = \mathbf{Tn}, \Rightarrow \mathbf{n} = \mathbf{T}^{-1}\mathbf{G},$$

and $\mathbf{n} = [n_1 \; n_2 \cdots n_5]^T$. Thus, to find the transformation $\mathbf{T}_1$, we compute $\mathbf{n}$ and multiply to get $\mathbf{T}_1 = \mathbf{TN} = \mathbf{T}\mathsf{diag}(n_1 \; n_2 \cdots n_5)$.

(b) In MATLAB,

$$
\begin{aligned}
T &= \begin{bmatrix} 1.4708 & -0.17221 & -1.4708 & -1.5432 & -3.3237 \\ 0.7377 & 2.8933 & -0.7377 & -2.9037 & -0.1282 \\ -0.7376 & -5.4133 & 0.6767 & -1.3533 & -4.0599 \\ 0.9227 & -6.5022 & -0.9227 & -2.7962 & -9.9016 \\ -0.0014 & 0.1663 & 0.0014 & 0.0449 & 3.9341 \end{bmatrix}, \\
\mathsf{n} &= \mathsf{T\backslash G}, \\
\mathsf{N} &= \mathsf{diag(n)}, \\
\mathsf{T1} &= \mathsf{T * N}, \\
T1 &= \begin{bmatrix} -0.3883 & 0.0247 & 2.8708 & -4.8234 & 2.3162 \\ -0.0897 & -0.0129 & 0.0000 & 1.2239 & -1.1214 \\ 2.9239 & 0.0000 & 2.8708 & -8.2970 & 2.5023 \\ -0.9314 & 0.0376 & 0.0000 & 2.1053 & -1.2115 \\ 0.0133 & 0.0001 & -0.0000 & -0.0745 & 1.0612 \end{bmatrix}, \\
B_m &= T1\backslash G = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T, \\
A_m &= T1\backslash F * T1 = \begin{bmatrix} -0.6371 & 0.0257 & -0.0000 & 0.0000 & -0.0000 \\ -17.2941 & -0.6371 & 0.0000 & -0.0000 & -0.0000 \\ -0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.0000 \\ -0.0000 & 0.0000 & 0.0000 & -0.5075 & -0.0000 \\ -0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.9683 \end{bmatrix}.
\end{aligned}
$$

13. a) Find the state transformation that will keep the description of the tape-drive system of Example 7.11 in modal canonical form but will cause the poles to be displayed in $\mathbf{A}_m$ in order of increasing magnitude.
b) Use MATLAB to verify your result in part (a), and give the complete new set of state matrices as $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and D.

Solution:

(a) To change the order of the eigenvalues (poles), p_i, in A_m, all we need to do is re-order the eigenvectors in T. In this case, $|p_5| > |p_1| = |p_2| > |p_4| > |p_3|$. Thus, $T_2 = [t_5\ t_1\ t_2\ t_4\ t_3]$.

(b) Our solution uses the MATLAB `sort` command to re-order eigenvectors. Note that this approach is independent of the size of the system matrix A_m. T is the same as in Problem 7.10. Because of the equality of the magnitudes of the complex eigenvectors, we can switch two of the columns of the matrix T_2. In MATLAB,

```
p = eig(F);

[f, indices] = sort(abs(p));

T2 = T(:,indices);

n = T2\G;

T3 = T2*diag(n);
```

Am2 = T3\F*T3;

$$Am2 = \begin{bmatrix} -0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.0000 \\ 0.0000 & -0.5075 & -0.0000 & 0.0000 & -0.0000 \\ -0.0000 & 0.0000 & -0.6371 & 0.0257 & -0.0000 \\ 0.0000 & 0.0000 & -17.2941 & -0.6371 & -0.0000 \\ 0.0000 & 0.0000 & -0.0000 & 0.0000 & -0.9683 \end{bmatrix},$$

Bm2 = T3\G;

Bm2 = $[1\ 1\ 1\ 1\ 1]^T$

Cm2 = h3*T3;

$Cm2 = \begin{bmatrix} 2.8708 & -6.5602 & 1.2678 & 0.0123 & 2.4092 \end{bmatrix}$,

Dm2 = 0;

14. Find the characteristic equation for the modal-form matrix $\mathbf{A}_m$ of Eq. (7.17a) using Eq. (7.58).

Solution:

$$\det(s\mathbf{I} - \mathbf{F}) = \det \begin{bmatrix} s+4 & 0 \\ 0 & s+3 \end{bmatrix} = (s+4)(s+3)$$

Since $\mathbf{A}_m$ was already in modal form, your solution is easily checked by inspection.

15. Given the system,

$$\dot{\mathbf{x}} = \begin{bmatrix} -4 & 1 \\ -2 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u,$$

with zero initial conditions, find the steady-state value of $\mathbf{x}$ for a step input u.

Solution:

We are given $\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u$. Steady-state means that $\dot{\mathbf{x}} = 0$ and a step input (or unit step) means $u = 1(t)$. Thus, assuming that $\mathbf{F}$ is invertible (which you can check), we have,

$$\mathbf{0} = \mathbf{F}\mathbf{x}_{ss} + \mathbf{G} \Longrightarrow \mathbf{x}_{ss} = -\mathbf{F}^{-1}\mathbf{G} = \begin{bmatrix} -4 & 1 \\ -2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 2/3 \end{bmatrix}.$$

This can be verified in MATLAB with step(F,G,H,J,1) where H=eye(size(F)) and J=[0;0].

16. Consider the system shown in Fig. 7.83.

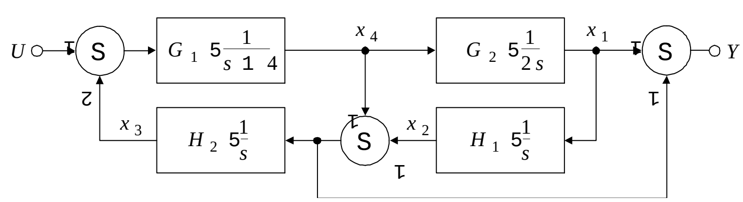


Figure 7.84: A block diagram for Problem 7.16.

a) Find the transfer function from U to Y.
b) Write state equations for the system using the state variables indicated.

Solution:

(a) The system is equivalent to the block diagram shown. Following the block diagram back to known state variables,

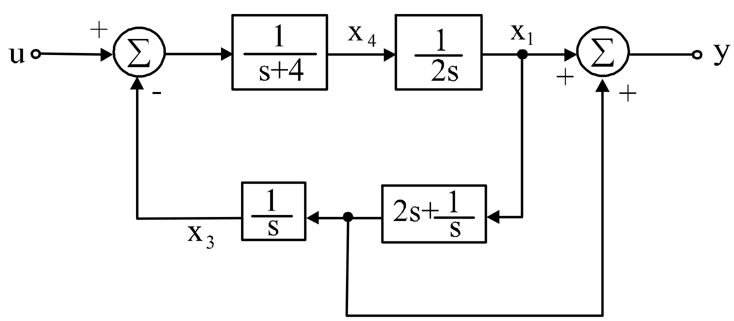
$$y = x_1 \left[1 + 2s + \frac{1}{s}\right],$$

and,

$$x_1 = \frac{1}{2s(s+4)} \left[u - \frac{1}{s}\left(2s + \frac{1}{s}\right) x_1\right],$$

resulting in,

$$\frac{Y(s)}{U(s)} = \frac{2s^3 + s^2 + s}{2s^4 + 8s^3 + 2s^2 + 1}.$$



Block diagram for solution of Problem 7.16 (a).

Another possible solution is in terms of Mason's rule.

(b)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix} \mathbf{x}.$$

With practice, you should be able to see quickly that $\dot{x}_3$ is simply the input to block H_2, which is the sum of x_2 and x_4. But this is nothing more than the third row of the matrix equation above. Your results can be checked for consistency using MATLAB's command ss2tf.

17. Using the indicated state variables, write the state equations for each of the systems shown in Fig. 7.85. Find the transfer function for each system using both block-diagram manipulation and matrix algebra [as in Eq. (7.48)].

Solution:

(a) Performing a partial fraction expansion on $(s+2)/(s+4)$, Fig. 7.85(a) can be redrawn as shown below.

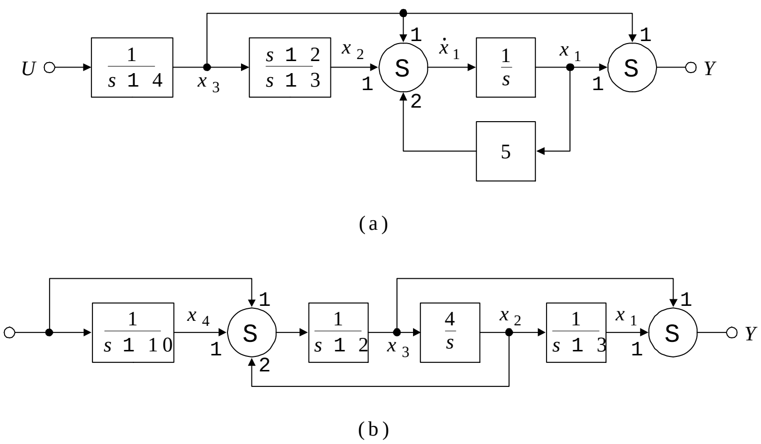
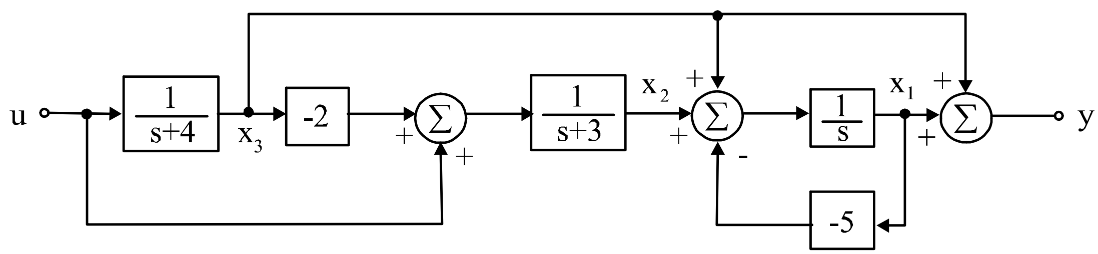


Figure 7.85: Block diagrams for Problem 7.17.



Redrawn block diagram for solution to Problem 7.17(a).

By inspection of the block diagram, the state equations are,

$$\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} -5 & 1 & 1 \\ 0 & -3 & -2 \\ 0 & 0 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u, \\
y &= \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \mathbf{x}.
\end{aligned}$$

Computing $G(s) = \mathbf{H}(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G}$, the following transfer function is obtained:

$$\frac{Y(s)}{U(s)} = \frac{s^2 + 10s + 20}{s^3 + 12s^2 + 47s + 60}.$$

We can use the MATLAB ss2tf command to verify this result.

(b) Using the second block diagram given in Fig. 7.85(b), we can write,

$$\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} -3 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & -10 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} u, \\
y &= \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}.
\end{aligned}$$

Computing $\mathbf{H}(s\mathbf{I}-\mathbf{F})^{-1}\mathbf{G}$ (using MATLAB's ss2tf), we obtain the following transfer function,

$$\frac{Y(s)}{U(s)} = \frac{s^3+14s^2+37s+44}{s^4+15s^3+60s^2+112s+120}.$$

18. For each of the listed transfer functions, write the state equations in both control and observer canonical form. In each case draw a block diagram and give the appropriate expressions for $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$.
a) $\dfrac{s^2-2}{s^2(s^2-1)}$ (control of an inverted pendulum by a force on the cart)
b) $\dfrac{3s+4}{s^2+2s+2}$

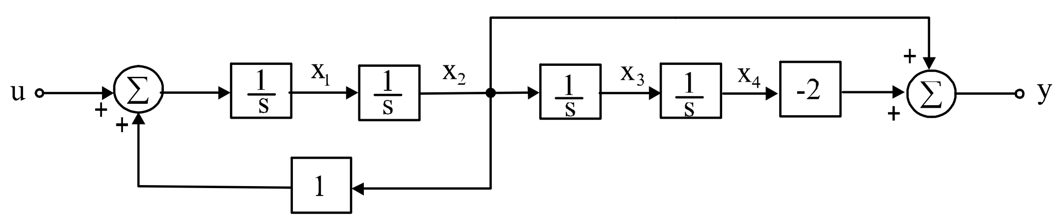
Solution:

(a)

$$\frac{Y(s)}{U(s)} = \frac{s^2-2}{s^2(s^2-1)}.$$

This transfer function can be realized in controller canonical form as shown below. From the figure, we have,

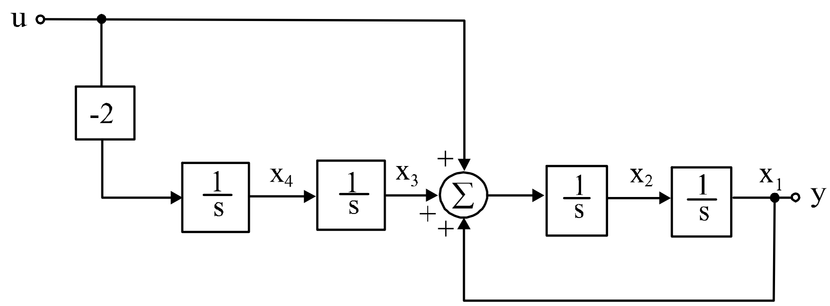
$$\begin{aligned}
\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} &= \begin{bmatrix}0&1&0&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} + \begin{bmatrix}1\\0\\0\\0\end{bmatrix}u,\\
y &= \begin{bmatrix}0&1&0&-2\end{bmatrix}\mathbf{x}.
\end{aligned}$$



Controller canonical form for the transfer function of Problem 7.18(a).

The block diagram for observer canonical form is shown below. From the figure, we have,

$$\begin{aligned}
\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} &= \begin{bmatrix}0&1&0&0\\1&0&1&0\\0&0&0&1\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} + \begin{bmatrix}0\\1\\0\\-2\end{bmatrix}u,\\
y &= \begin{bmatrix}1&0&0&0\end{bmatrix}\mathbf{x}.
\end{aligned}$$



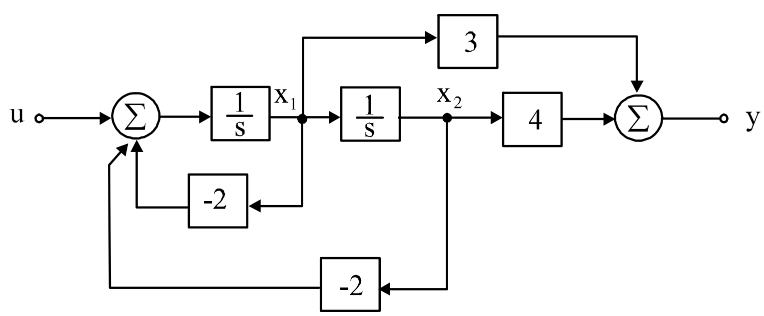
Observer canonical form for the transfer function of Problem 7.18(a).

(b)

$$\frac{Y(s)}{U(s)} = \frac{3s+4}{s^2+2s+2}.$$

This transfer function can be realized in Controller canonical form as shown below. From the figure, we have,

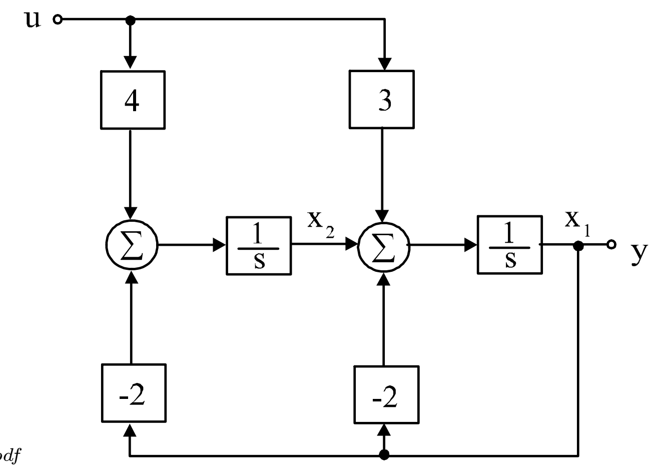
$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}-2 & -2\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}u,\\ y &= \begin{bmatrix}3 & 4\end{bmatrix}\mathbf{x}.\end{aligned}$$



Controller canonical form for the transfer function of Problem 7.18(b).

The block diagram for observer canonical form is shown below. From the figure, we have:

$$\begin{aligned}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}-2 & 1\\ -2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}3\\ 4\end{bmatrix}u,\\ y &= \begin{bmatrix}1 & 0\end{bmatrix}\mathbf{x}.\end{aligned}$$



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Observer canonical form for the transfer function of Problem 7.18(b).

19. Consider the transfer function, $G(s) = \frac{Y(s)}{U(s)} = \frac{s+1}{s^2+5s+6}$. (7.263)
a) By rewriting Eq. (7.263) in the form,

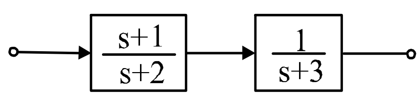
$$G(s) = \frac{1}{s+3}\left(\frac{s+1}{s+2}\right),$$

find a **series realization** of $G(s)$ as a cascade of two first-order systems.
b) Using a partial-fraction expansion of $G(s)$, find a **parallel realization** of $G(s)$.
c) Realize $G(s)$ in control canonical form.

Solution:

(a) The series realization shown below is given by:

$$G(s) = \left(\frac{1}{s+3}\right)\left(\frac{s+1}{s+2}\right) = \hat{g}_2(s)\hat{g}_1(s).$$



Series connection of $G(s)$ for Problem 7.19(a).

$$\begin{aligned}
\text{For } \hat{g}_1(s),\ \dot{x}_1 &= -2x_1 + u_1,\ y_1 = -x_1 + u_1. \\
\text{For } \hat{g}_2(s),\ \dot{x}_2 &= -3x_2 + u_2,\ y_2 = x_2.
\end{aligned}$$

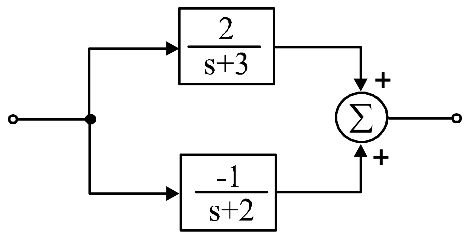
The series interconnections result in $u = u_1,\ y = y_2,\ u_2 = y_1$. Therefore,

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -2 & 0 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u, \\
y &= \begin{bmatrix} 0 & 1 \end{bmatrix}\mathbf{x}.
\end{aligned}$$

The MATLAB command series can also be used.

(b) The parallel realization, shown below, is given by:

$$G(s) = \frac{2}{s+3} + \frac{-1}{s+2} = \hat{g}_1(s) + \hat{g}_2(s).$$



Parallel connection of $G(s)$ for Problem 7.19(b).

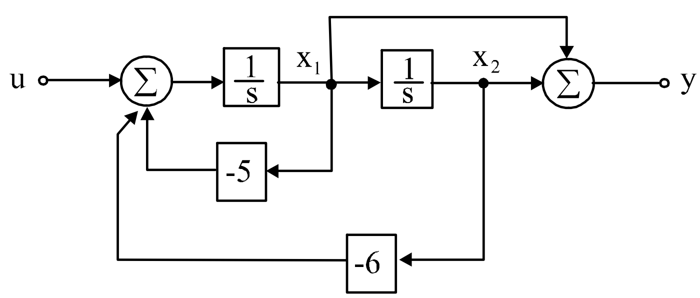
$$\begin{aligned} \text{For } \hat{g}_1(s), \quad \dot{x}_1 &= -3x_1 + u_1,\ y_1 = 2x_1. \\ \text{For } \hat{g}_2(s), \quad \dot{x}_2 &= -2x_2 + u_2,\ y_2 = -x_2. \end{aligned}$$

The interconnections are $u_1 = u_2 = u,\ y = y_1 + y_2$. Therefore,

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 2 & -1 \end{bmatrix} \mathbf{x}. \end{aligned}$$

The MATLAB command parallel can also be used.

(c) Control canonical form, shown in Fig. 7.17, is realized by simply picking off the appropriate coefficients of the original (strictly proper) transfer function. If the original function is not strictly proper, then it should be reduced to a feedthrough term plus a strictly proper term.



Controller canonical form of $G(s)$ for Problem 7.19(c).

For $G(s)$, we have,

$$\mathbf{F} = \begin{bmatrix} -5 & -6 \\ 1 & 0 \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{H} = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

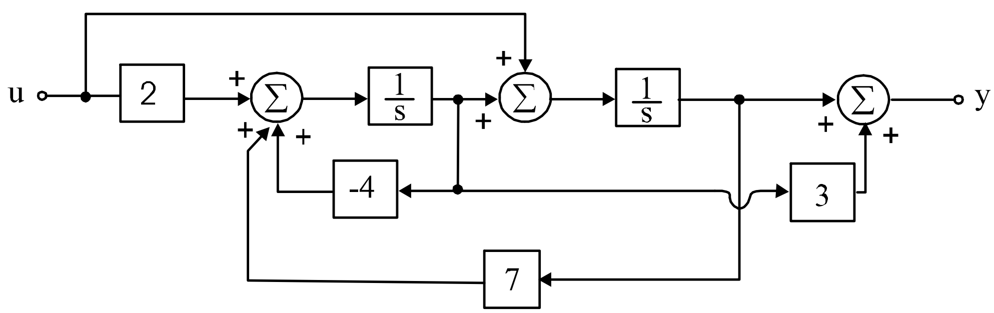
Problems and Solutions for Section 7.5: Control-Law Design for Full-State Feedback

20. Consider the plant described by,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 7 & -4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u,$$
$$y = [\ 1 \quad 3\ ]\mathbf{x}.$$

a) Draw a block diagram for the plant with one integrator for each state variable.
b) Find the transfer function using matrix algebra.
c) Find the closed-loop characteristic equation if the feedback is
(1) $u = -[\ K_1 \quad K_2\ ]\mathbf{x}$; (2) $u = -Ky$.

Solution:



State realization showing integrators explicitly.

(a) See figure.

(b) Using the formula $G(s) = \mathbf{H}(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G}$, we obtain,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{7s + 27}{s^2 + 4s - 7}.$$

The MATLAB command ss2tf can also be used.

(c)

(i) State feedback, $u = -[K_1\ K_2]\mathbf{x}$.

$$\begin{aligned} \det(\lambda\mathbf{I} - \mathbf{F} + \mathbf{GK}) &= \det \begin{bmatrix} \lambda + K_1 & -1 + K_2 \\ -7 + 2K_2 & \lambda + 4 + 2K_2 \end{bmatrix} \\ &= \lambda^2 + \lambda(4 + 2K_2 + K_1) + (6K_1 + 7K_2 - 7) = 0. \end{aligned}$$

(ii) Output feedback,

$$u = -Ky = -K \begin{bmatrix} 1 & 3 \end{bmatrix} \mathbf{x} = -\begin{bmatrix} K & 3K \end{bmatrix} \mathbf{x}.$$

This yields the following closed-loop characteristic equation:

$$\lambda^2 + \lambda(7K + 4) + (27K - 7) = 0.$$

Hints: If you have already solved the case for state feedback, simply plug $K_1 = K$ and $K_2 = 3K$ into the characteristic equation for state feedback and find the characteristic equation for output feedback. The output vector $\mathbf{H}$ fixes the ratio among the state variables. Secondly, although there were products of K_1 and K_2 when we were forming the determinant, they should all cancel in your final answer. The reason for this is that the characteristic equation $\dot{\mathbf{x}} = (\mathbf{F} - \mathbf{GK})\mathbf{x}$ is *linear* in $\mathbf{K}$.

21. For the system,

$$\begin{aligned}\dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x},\end{aligned}$$

design a state feedback controller that satisfies the following specifications:

- Closed-loop poles have a damping coefficient $\zeta = 0.707$.
- Step-response peak time is under 3.14 sec.

Verify your design with MATLAB.

Solution:

For a second-order system, the specification on rise time can be translated into a value of $\omega_n$by the equation $\omega_d = \pi$. Then determine ω_n from $\omega_d = \omega_n\sqrt{1-\zeta^2}$. This yields $\omega_n = 1.414$. Using full state feedback, we would like the a characteristic equation to be,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2s + 2 = 0.$$

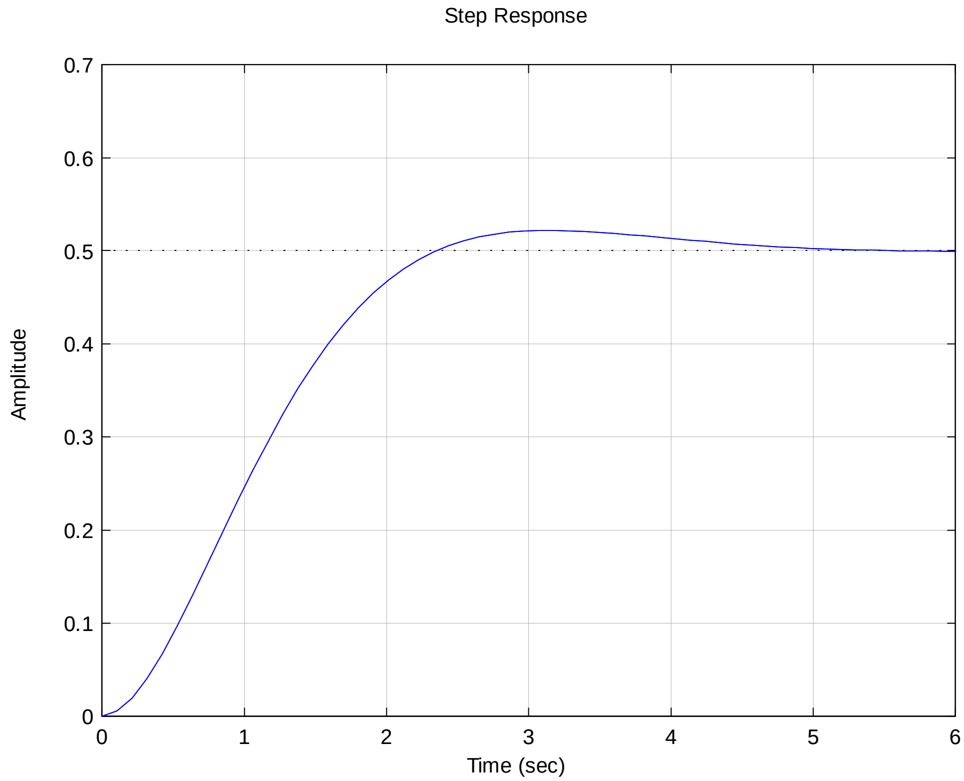
Using state feedback $u = -\mathbf{Kx}$, we get,

$$\dot{\mathbf{x}} = (\mathbf{F} - \mathbf{GK})\mathbf{x} = \begin{bmatrix} 0 & 1 \\ -6 - k_1 & -5 - k_2 \end{bmatrix}\mathbf{x}.$$

Hence the closed-loop characteristic equation is,

$$s^2 + (5 + k_2)s + (6 + k_1) = 0.$$

Comparing coefficients, $k_1 = -4$ and $k_2 = -3$. The MATLAB command `place` can also be used. The reference step can be simulated in MATLAB with $u = -\mathbf{Kx} + r$, and the MATLAB command `step`, as shown below.



Step response for Problem 7.21.

22. a) Design a state feedback controller for the following system so that the closed-loop step response has an overshoot of less than 25% and a 1% settling time under 0.115 sec.:

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ 0 & -10 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}. \end{aligned}$$

b) Use the `step` command in MATLAB to verify that your design meets the specifications. If it does not, modify your feedback gains accordingly.

Solution:

(a) For the overshoot specification,

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} < 25\% \Longrightarrow \zeta \cong 0.4.$$

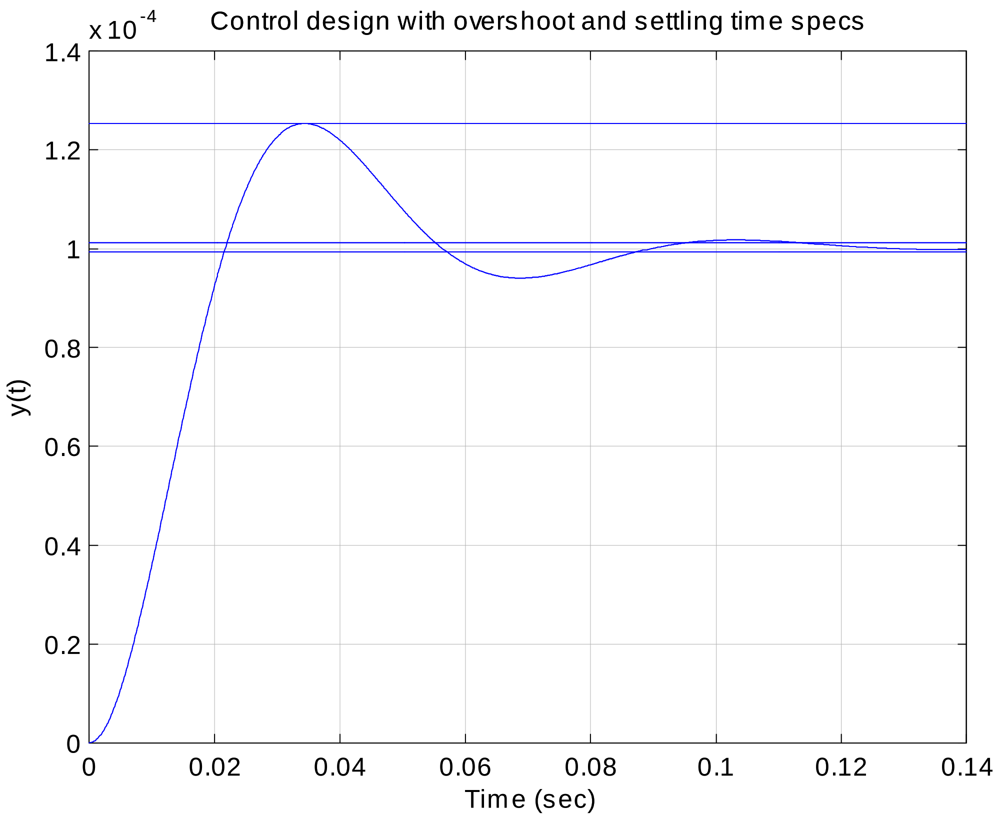
For the 1% settling time specification, we use,

$$e^{-\zeta\omega_n t_s} = 0.01 \Longrightarrow \omega_n = \frac{4.6}{\zeta t_s}.$$

(b) This can be implemented in MATLAB with the following code:

```
F = [0,1;0,-10];
G = [0;1];
H = [1,0];
J = 0;
zeta = 0.404; % Tweak values slightly so that specs are met.
ts = 0.114;
wn = 4.6/(ts*zeta);
p = roots([1, 2*zeta*wn, wn^2]);
k = place(F,G,p);
sysCL=ss(F-G*k,G,H,J)
step(sysCL);
```

The step response is shown next.



Step response for Problem 7.22.

23. Consider the system,

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & -2 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}.$$

a) Design a state feedback controller for the system so that the closed-loop step response has an overshoot of less than 5% and a 1% settling time under 4.6 sec.
b) Use the step command in MATLAB to verify that your design meets the specifications. If it does not, modify your feedback gains accordingly.

Solution:

(a) There are many different approaches to designing the control law. We will attack the problem using a symmetric root locus. We assume the output is x_1. Although the system is third-order, we can still use the second-order order rules of thumb in order to get an estimate of where we would like the closed loop poles.

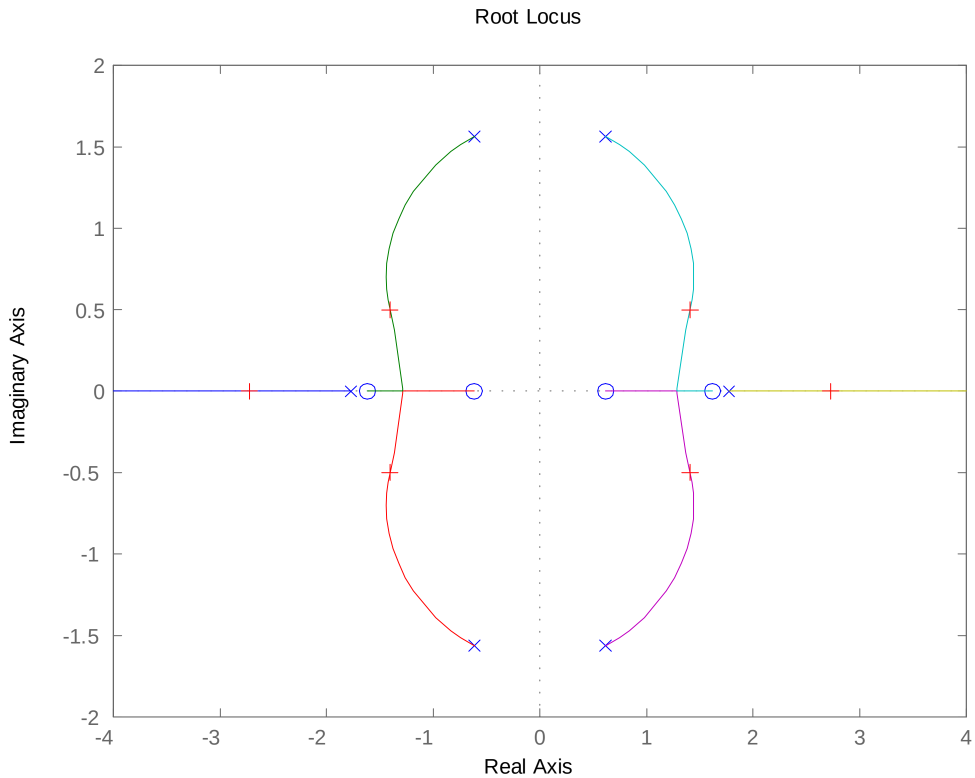
$$\sigma = \frac{4.6}{t_s} \Longrightarrow \zeta\omega_n = 1,$$
$$M_p \leq 5\% \Longrightarrow \zeta > 0.7.$$

The open-loop poles are at -1.45 and $-0.77 \pm j1.47$ and the open-loop zeros are at -1.37 and 0.37. The symmetric root locus is shown on the next page and was generated using the following MATLAB code:

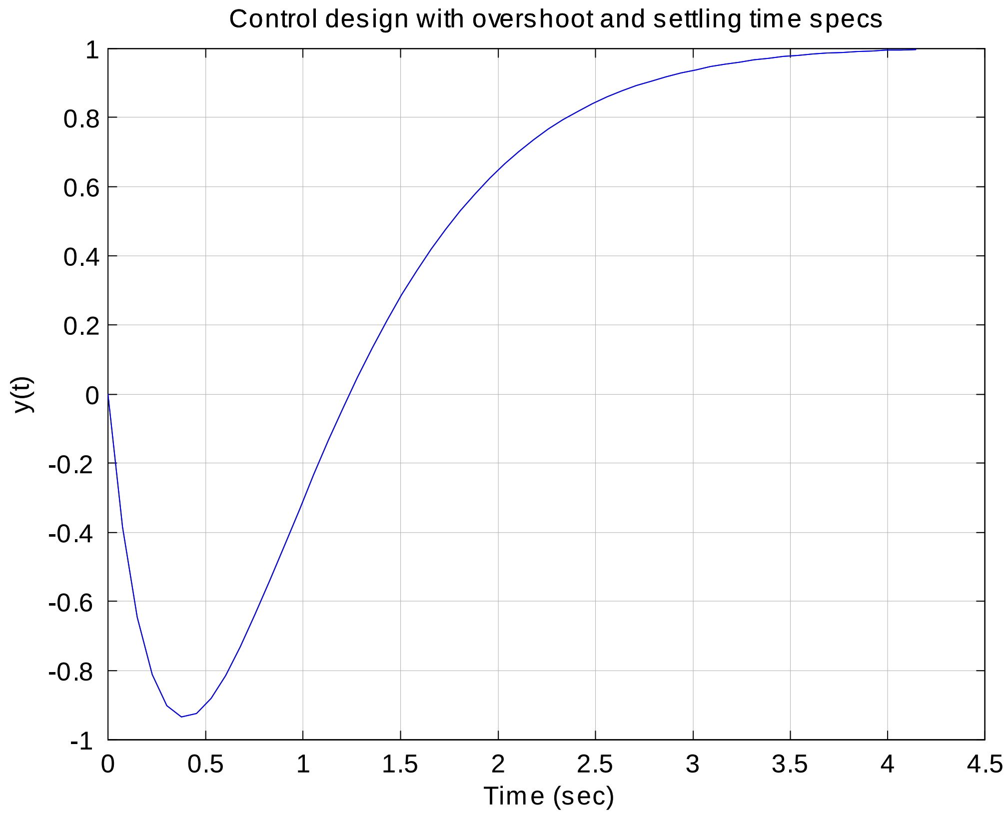
```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% the function srl is used to compute the roots of the symmetric root locus
function [k,p]=srl(f,g,h)
a=[f 0*f;-h'*h -f'];
b=[g;0*g];
c=[0*h g'];
rlocus(a,b,c,0);
[k,p]=rlocfind(a,b,c,0)
```

Note that crosses indicate where the closed-loop pole locations have been selected, which roughly correspond to the ζ and ω_n suggested by the rules of thumb for a second-order system with no zeros. The control gains $\mathbf{K} = \begin{bmatrix} 0.78 & 0.07 & 0.28 \end{bmatrix}$ correspond to these closed loop pole locations. The MATLAB command place can be used to verify this computation. The step response is shown next using the MATLAB step command.

Technically, this closed-loop step response meets the 4.6 sec 1% settling time and 5% overshoot. However, the right half plane zero close to the origin gives catastrophic results in terms of undershoot. This should alert the reader to the importance of paying attention to the zeros of the system, especially in the right half plane.



Symmetric root locus for Problem 7.23.



Closed-loop step response for Problem 7.23.

24. Consider the system in Fig. 7.83.

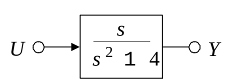


Figure 7.86: System for Problem 7.24.

a) Write a set of equations that describes this system in the control canonical form as $\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u$ and $y = \mathbf{H}\mathbf{x}$.
b) Design a control law of the form,

$$u = -\left[\begin{array}{cc} K_1 & K_2 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \end{array}\right],$$

that will place the closed-loop poles at $s = -2 \pm 2j$.

Solution:

(a) Let's write this system in the control canonical form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u,$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}.$$

(b) If $u = -[K_1 \; K_2]\mathbf{x}$, the poles of the closed-loop system satisfy $\det(s\mathbf{I} - \mathbf{F} + \mathbf{GK}) = 0$. Thus,

$$\det \begin{bmatrix} s + K_1 & -1 + K_2 \\ 4 & s \end{bmatrix} = 0 \Longrightarrow s^2 + K_1 s + 4 + K_2 = 0.$$

The closed-loop characteristic equation is,

$$(s + 2 - 2j)(s + 2 + 2j) = s^2 + 4s + 8 = 0.$$

Comparing coefficients, we have $K_1 = 4$ and $K_2 = 4$. The MATLAB command `place` can also be used to verify this result.

25. *Output Controllability*: In many situations a control engineer may be interested in controlling the output y rather than the state $\mathbf{x}$. A system is said to be **output controllable** if at any time you are able to transfer the output from zero to any desired output y^* in a finite time using an appropriate control signal u^*. Derive necessary and sufficient conditions for a continuous system ($\mathbf{F}$,$\mathbf{G}$,$\mathbf{H}$) to be output controllable. Are output and state controllability related? If so, how?

Solution:

Because we are considering *linear* systems, if you can take the state from some initial state to some final condition in a finite time with a finite input, then you can also take it to the same state in infinitesimal time using impulsive inputs. To express this mathematically, let u be,

$$u(t) = g_1\delta(t) + g_2\delta^{(1)}(t) + \cdots + g_n\delta^{(n-1)}(t),$$

where $\delta(t)$ represents a delta function, $\delta^{(1)}(t)$ represents the first derivative of a delta function (a unit doublet), etc., and the g_i are scalars to be determined. Let,

$$u^* = [g_1 \;\; g_2 \cdots g_n]^T,$$

then

$$\mathbf{x}(0+) - \mathbf{x}(0-) = \mathcal{C}\; u^*.$$

Hence, we have found a control signal that will drive the state to arbitrary values given the non-singularity of the controllability matrix, $\mathcal{C}$.

In fact, the invertibility of $\mathcal{C}$ is a necessary and sufficient condition for state controllability. For output controllability,

$$\mathbf{Hx}(0+) - \mathbf{Hx}(0-) = \mathbf{H}\mathcal{C}\; u^*,$$
$$y(0+) - y(0-) = \mathbf{H}\mathcal{C}\; u^*.$$

Assuming (without loss of generality) that $y(0-) = 0$, we have,

$$y(0+) = [\mathbf{HG} \;\; \mathbf{HFG} \cdots \mathbf{HF}^{n-1}\mathbf{G}]u^*.$$

Therefore, a system is output controllable if and only if,

$$[\mathbf{HG}\ \ \mathbf{HFG}\cdots\mathbf{HF}^{n-1}\mathbf{G}] \text{ is full rank.}$$

This is always true (for a single-input single-output system) unless the transfer function is zero. Of course, state controllability implies output controllability, but output controllability does *not* imply state controllability.

26. Consider the system,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 4 & 0 & 0 \\ -1 & -4 & 0 & 0 \\ 5 & 7 & 1 & 15 \\ 0 & 0 & 3 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}u.$$

a) Find the eigenvalues of this system. (*Hint:* Note the block-triangular structure.)
b) Find the controllable and uncontrollable modes of this system.
c) For each of the uncontrollable modes, find a vector $\mathbf{v}$ such that,

$$\mathbf{v}^T\mathbf{G} = 0, \quad \mathbf{v}^T\mathbf{F} = \lambda\mathbf{v}^T.$$

d) Show that there are an infinite number of feedback gains $\mathbf{K}$ that will relocate the modes of the system to -5, -3, -2, and -2.
e) Find the unique matrix $\mathbf{K}$ that achieves these pole locations and prevents initial conditions on the uncontrollable part of the system from ever affecting the controllable part.

Solution:

(a) Because the system is block lower triangular, we can determine the eigenvalues of the system by taking the union of the eigenvalues of each of the blocks along the main (block)diagonal.

$$\begin{aligned} s^2+4s+4 &= 0 \Longrightarrow -2,-2. \\ s^2+2s-48 &= 0 \Longrightarrow 6,-8 \end{aligned}$$

Thus, the eigenvalues of the system are $-2,-2,6,$ and -8. (Easily checked with MATLAB's `eig` command).

(b) To find the controllable or uncontrollable modes of the system, we follow method learned in Problem 7.28. Specifically, we find an orthogonal similarity transformation which transforms ($\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$) to ($\bar{\mathbf{F}}$, $\bar{\mathbf{G}}$, $\bar{\mathbf{H}}$) where $\bar{\mathbf{F}}$ is an upper-Hessenberg matrix. (See Problem 7.28 for details). Observe that this system is almost in the desired form already! Simply by interchanging the state variables x_3 and x_4, we can transform the system into the proper form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_4 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 & 0 \\ -1 & -4 & 0 & 0 \\ 0 & 0 & -3 & 3 \\ 5 & 7 & 15 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_4 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}u.$$

Now the controllable and uncontrollable modes can be determined by inspection. The uncontrollable modes correspond to the eigenvalues of the $\mathbf{F}_{11}$ block, so -2 and -2 are both uncontrollable modes. Similarly, the controllable modes from the $\mathbf{F}_{22}$ block are -8 and 6.

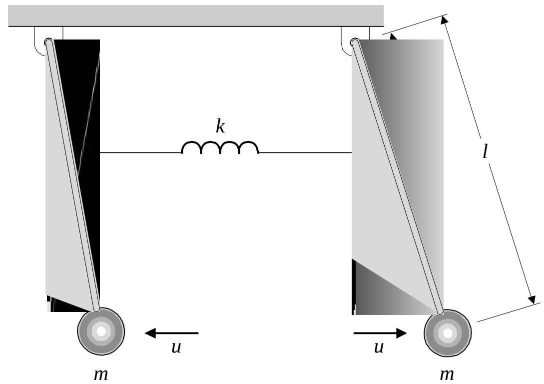


Figure 7.87: Coupled pendulums for Problem 7.27.

MATLAB function ctrbf will give a similar result, although the order of the state variables may be switched.

(c) Notice that we need the left eigenvectors of $\mathbf{F}$ that is orthogonal to $\mathbf{G}$. The only left eigenvector of $\mathbf{F}$ that is orthogonal to $\mathbf{G}$ is $\begin{bmatrix} 1 & 2 & 0 & 0 \end{bmatrix}^T$.

(d) Because the modes at -2 and -2 are uncontrollable, we expect that state feedback will not have any affect on these modes. Writing an expression for the feedback we have,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 & 0 \\ -1 & -4 & 0 & 0 \\ 5-k_1 & 7-k_2 & 1-k_3 & 15-k_4 \\ 0 & 0 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

Notice that the system matrix is still block diagonal. The characteristic equation of the $\mathbf{F}_{22}$ block gives,

$$\det(s\mathbf{I} - \mathbf{F}_{22}) = s^2 + (2+k_3)s + (3k_3 + 3k_4 - 48) = 0.$$

Picking $k_3 = 6$ and $k_4 = 15$ will place the controllable roots at -3 and -5. Since k_1 and k_2 are arbitrary, there are an infinite number of feedback gains that will relocate the modes of the system to the desired locations.

(e) To completely decouple the controllable and uncontrollable portions of the system, we make the $\mathbf{F}_{21}$ block identically zero by setting $k_1 = 5$ and $k_2 = 7$.

27. Two pendulums, coupled by a spring, are to be controlled by two equal and opposite forces u, which are applied to the pendulum bobs as shown in Fig. 7.86. The equations of motion are

$$ml^2\ddot{\theta}_1 = -ka^2(\theta_1 - \theta_2) - mgl\theta_1 - lu,$$
$$ml^2\ddot{\theta}_2 = -ka^2(\theta_2 - \theta_1) - mgl\theta_2 + lu.$$

a) Show that the system is uncontrollable. Can you associate a physical meaning with the controllable and uncontrollable modes?
b) Is there any way that the system can be made controllable?

Solution:

(a) Using the state vector $\mathbf{x} = [\theta_1\ \dot{\theta}_1\ \theta_2\ \ \dot{\theta}_2]^T$,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\left(\frac{ka^2}{ml^2} + \frac{g}{l}\right) & 0 & \frac{ka^2}{ml^2} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{ka^2}{ml^2} & 0 & -\left(\frac{ka^2}{ml^2} + \frac{g}{l}\right) & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ -\frac{1}{ml} \\ 0 \\ \frac{1}{ml} \end{bmatrix} u.$$

The controllability matrix is determined as,

$$\begin{aligned} \mathcal{C} &= \begin{bmatrix} \mathbf{G} & \mathbf{FG} & \mathbf{F}^2\mathbf{G} & \mathbf{F}^3\mathbf{G} \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\frac{1}{ml} & 0 & \frac{1}{ml}\left(\frac{ka^2}{ml^2} + \frac{g}{l}\right) + \frac{ka^2}{m^3l^3} \\ -\frac{1}{ml} & 0 & \frac{1}{ml}\left(\frac{ka^2}{ml^2} + \frac{g}{l}\right) + \frac{ka^2}{m^3l^3} & 0 \\ 0 & \frac{1}{ml} & 0 & -\frac{ka^2}{m^3l^3} - \frac{1}{ml}\left(\frac{ka^2}{ml^2} + \frac{g}{l}\right) \\ \frac{1}{ml} & 0 & -\frac{ka^2}{m^3l^3} - \frac{1}{ml}\left(\frac{ka^2}{ml^2} + \frac{g}{l}\right) & 0 \end{bmatrix} \end{aligned}$$

Then ($\mathbf{F}$,$\mathbf{G}$) is uncontrollable since det($\mathcal{C}$)=0. If we re-write the state equations in terms of the state vector,

$$\mathbf{z} = \begin{bmatrix} \alpha & \dot{\alpha} & \beta & \dot{\beta} \end{bmatrix}^T,$$

where, $\alpha = \theta_1 + \theta_2$, and, $\beta = \theta_1 - \theta_2$, then the resulting equations of motion are,

$$\begin{aligned} ml^2\ddot{\alpha} &= -mgl\alpha \\ ml^2\ddot{\beta} &= -2ka^2\beta - mgl\beta - 2lu. \end{aligned}$$

Clearly, α, the "pendulum mode" (or the symmetric mode, i.e., the pendulums swinging together), is uncontrollable and, β, the "spring mode"(i.e., the unsymmetric mode) is controllable.

(b) Yes, make the forces unequal, i.e., let $u_1 \neq u_2$, or eliminate one of the forces, i.e., let $u_1 = 0$, or let $u_2 = 0$.

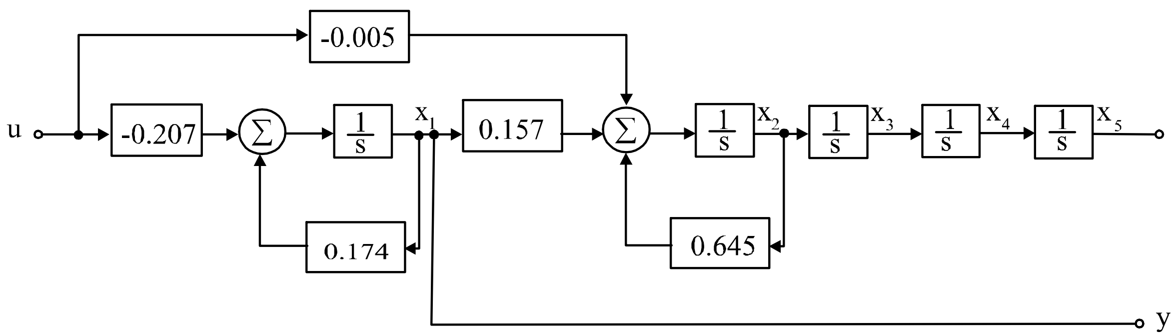
28. The state-space model for a certain application has been given to us with the following state description matrices:

$$\mathbf{F} = \begin{bmatrix} 0.174 & 0 & 0 & 0 & 0 \\ 0.157 & 0.645 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} -0.207 \\ -0.005 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

a) Draw a block diagram of the realization with an integrator for each state variable.
b) A student has computed det $\mathcal{C} = 2.3 \times 10^{-7}$ and claims that the system is uncontrollable. Is the student right or wrong? Why?
c) Is the realization observable?

Solution:

(a) The block diagram is shown below.



Block diagram for Problem 7.28.

(b) The system is controllable because a control signal u (command) reaches all the state variables of the system through the integrators in Fig. 7.26. The determinant of the controllability matrix is small, $\det(\mathcal{C}) = -2.3 \times 10^{-7}$, due to poor scaling of system variables. For example, if the control signal is scaled by 100, then $\det(\mathcal{C}) = -2.3 \times 10^{3}$.

(c) The realization is unobservable. You can check $\det(\mathcal{O})$ or just observe from the block diagram that there is no path from the state variables x_2, x_3, x_4, or x_5 to the output y.

29. *Staircase Algorithm (Van Dooren et al., 1978)*: Any realization (**F**,**G**,**H**) can be transformed by an **orthogonal similarity transformation** to ($\bar{\mathbf{F}}$,$\bar{\mathbf{G}}$,$\bar{\mathbf{H}}$), where $\bar{\mathbf{F}}$ is an **upper Hessenberg matrix** (having one nonzero diagonal above the main diagonal):

$$\bar{\mathbf{F}} = \mathbf{T}^T\mathbf{F}\mathbf{T} = \begin{bmatrix} * & \alpha_1 & \mathbf{0} & 0 \\ * & * & \ddots & 0 \\ * & * & \ddots & \alpha_{n-1} \\ * & * & \cdots & * \end{bmatrix}, \quad \bar{\mathbf{G}} = \mathbf{T}^T\mathbf{G} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g_1 \end{bmatrix},$$

where $g_1 \neq 0$, and,

$$\bar{\mathbf{H}} = \mathbf{H}\mathbf{T} = [h_1 \cdots h_n], \quad \mathbf{T}^{-1} = \mathbf{T}^T.$$

Orthogonal transformations correspond to a **rotation** of the vectors (represented by the matrix columns) being transformed with no change in length.

a) Prove that if $\alpha_i = 0$ and $\alpha_{i+1}, \ldots, \alpha_{n-1} \neq 0$ for some i, then the controllable and uncontrollable modes of the system can be identified after this transformation has been done.

b) How would you use this technique to identify the observable and unobservable modes of (**F, G, H**)?

c) What advantage does this approach for determining the controllable and uncontrollable modes have over transforming the system to any other form?

d) How can we use this approach to determine a basis for the controllable and uncontrollable subspaces, as in Problem 7.43?

This algorithm can be used to design a numerically stable algorithm for pole placement [see Minimis and Paige (1982)]. The name of the algorithm comes from the multi-input version in which the α_i are the blocks that make $\bar{\mathbf{F}}$ resemble a staircase.

Solution:

(a) If $\alpha_i = 0$,

$$\bar{\mathbf{F}} = \mathbf{T}^T\mathbf{F}\mathbf{T}\begin{bmatrix} * & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & \alpha_2 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & \alpha_{i+1} & 0 & 0 & 0 \\ * & * & * & * & * & \ddots & 0 & 0 \\ * & * & * & * & * & * & * & \alpha_{n-1} \\ * & * & * & * & * & * & * & * \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{11} & \mathbf{0} \\ \mathbf{F}_{21} & \mathbf{F}_{22} \end{bmatrix}, \quad \bar{\mathbf{G}} = \mathbf{T}^T\mathbf{G} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ g_1 \end{bmatrix},$$

This suggests naturally splitting up the state vector into two parts $\mathbf{x} = [x_1 \; x_2]^T$ where x_1 and x_2 are vectors of the appropriate size (depending upon which $\alpha_i = 0$). Then recognize that the equations are,

$$\begin{aligned} \dot{x}_1 &= \mathbf{F}_{11}x_1, \\ \dot{x}_2 &= \mathbf{F}_{21}x_1 + \mathbf{F}_{22}x_2 + g_1 u. \end{aligned}$$

Notice that the control signal u and the state x_2 do *not* effect the state x_1. Thus, all of the modes associated with the block $\mathbf{F}_{11}$ are uncontrollable. All of the states in x_2 are controllable. This is easily checked by forming the controllability matrix associated with the pair ($\mathbf{F}_{22}$,g_1). Hence, the system has been split into its controllable and uncontrollable parts.

(b) Use duality, i.e., transform $[\mathbf{F}^T\mathbf{H}^T]$ into Hessenberg form.

(c) Because $\mathbf{T}^{-1} = \mathbf{T}^T$, three advantages are recognized:

(i) Better numerical accuracy.

(ii) The controllable-uncontrollable decomposition is immediate.

(iii) Repeated roots are handled.

(d) Simply split $\mathbf{T}$ and extract the controllable and uncontrollable subspaces,

$$\mathbf{T} = [\underbrace{\mathbf{T}_1}_{i} \; \underbrace{\mathbf{T}_2}_{n-i}], \tag{3}$$

$$\mathbf{T}_1 = \mathcal{N}(\mathcal{C}^T), \quad \mathbf{T}_2 = \mathcal{R}(\mathcal{C}). \tag{4}$$

See the MATLAB ctrbf (and obsvf) functions.

Problems and Solutions for Section 7.6: Selection of Pole Locations for Good Design

30. The normalized equations of motion for an inverted pendulum at angle θ on a cart are,

$$\ddot{\theta} = \theta + u, \quad \ddot{x} = -\beta\theta - u,$$

where x is the cart position, and the control input u is a force acting on the cart.
a) With the state defined as $\mathbf{x} = [\theta, \dot{\theta}, x, \dot{x}]^T$, find the feedback gain $\mathbf{K}$ that places the closed-loop poles at $s = -1, -1, -1 \pm 1j$.
For parts (b) through (d), assume that $\beta = 0.5$.
b) Use the symmetric root locus to select poles with a bandwidth as close as possible to those of

part (a), and find the control law that will place the closed-loop poles at the points you selected.
c) Compare the responses of the closed-loop systems in parts (a) and (b) to an initial condition of $\theta = 10°$. You may wish to use the initial command in MATLAB.
d) Compute $\mathbf{N_x}$ and N_u for zero steady-state error to a constant command input on the cart position, and compare the step responses of each of the two closed-loop systems.

Solution:

(a) The state space equations of motion are,

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} u.$$

We require the closed-loop characteristic equation to be,

$$\alpha_c(s) = (s+1)^2(s^2+2s+2) = s^4 + 4s^3 + 7s^2 + 6s + 2.$$

From the above state equations,

$$\det(s\mathbf{I} - \mathbf{F} + \mathbf{GK}) = s^4 + (k_2 - k_4)s^3 + (k_1 - k_3 - 1)s^2 + k_4(1-\beta)s + k_3(1-\beta) \equiv \alpha_c(s)$$

Comparing coefficients yields:

$$\begin{aligned} k_1 &= \frac{10-8\beta}{1-\beta},\ k_2 = \frac{10-4\beta}{1-\beta},\ k_3 = \frac{2}{1-\beta},\ k_4 = \frac{6}{1-\beta}, \\ \mathbf{K} &= \begin{bmatrix} 12 & 16 & 4 & 12 \end{bmatrix}. \end{aligned}$$

(b) The symmetric root locus is shown below, where we have chosen $\mathbf{H} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$. The following MATLAB commands can be used to generate the symmetric root locus,

```
% Symmetric root locus
a=[F, 0*F;-H'*H, -F'];
b=[G;0*G];
c=[0*H, G'];
d=0;
rlocus(a,b,c,d);
```

The chosen pole locations, shown on the symmetric root locus, result in a feedback gain of (using MATLAB's place command),

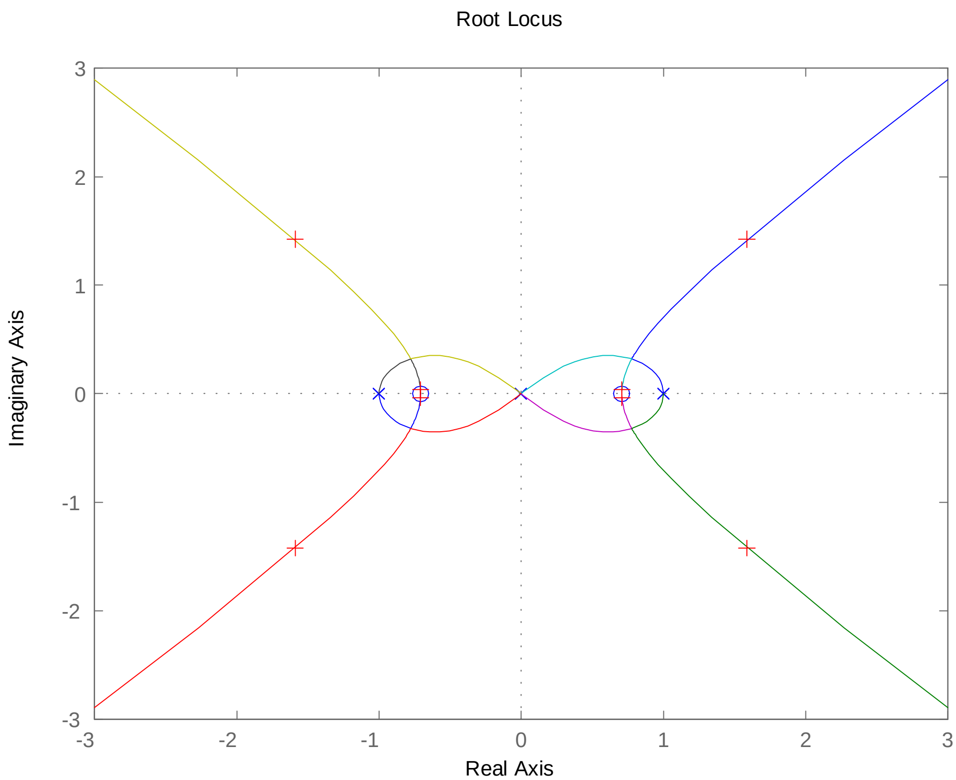
$$\mathbf{K} = \begin{bmatrix} 13.5 & 18.36 & 3.9 & 13.98 \end{bmatrix}.$$

(c) The initial condition response to $\theta(0) = 10°$ for both control designs in (a) and (b) is shown on the next page.

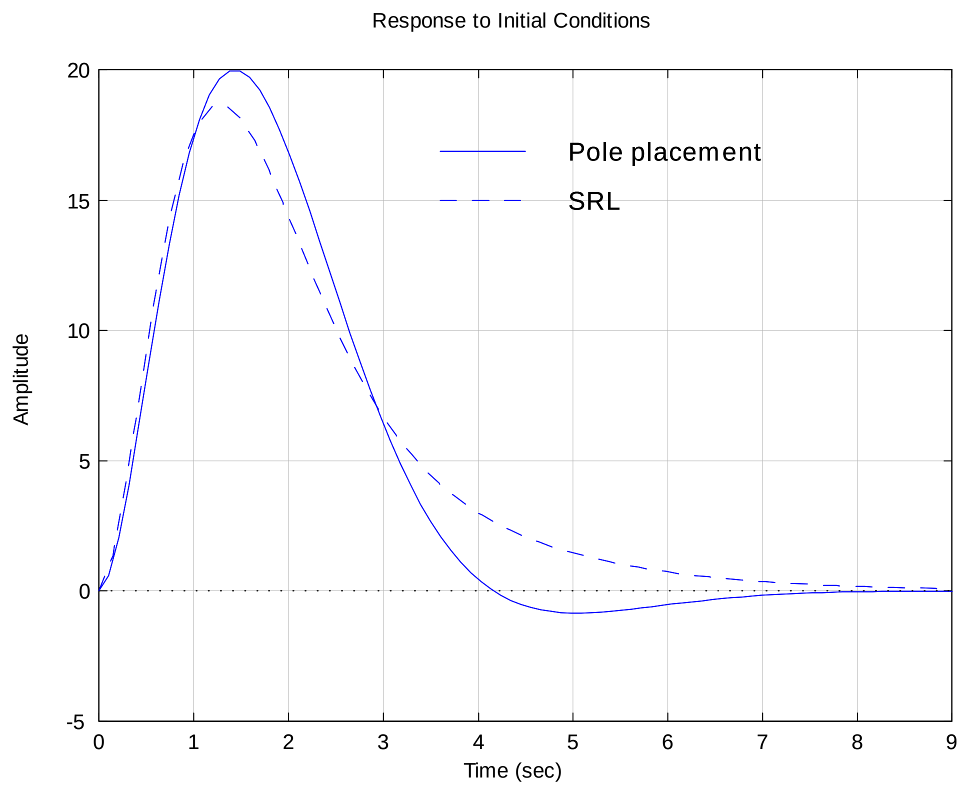
(d) To compute $N_{\mathbf{x}}$ and N_u for zero steady-state error to a constant command input on cart position, x, we solve the equations,

$$\begin{bmatrix} \mathbf{F} & \mathbf{G} \\ \mathbf{H}_2 & J \end{bmatrix} \begin{bmatrix} N_{\mathbf{x}} \\ N_u \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix}.$$

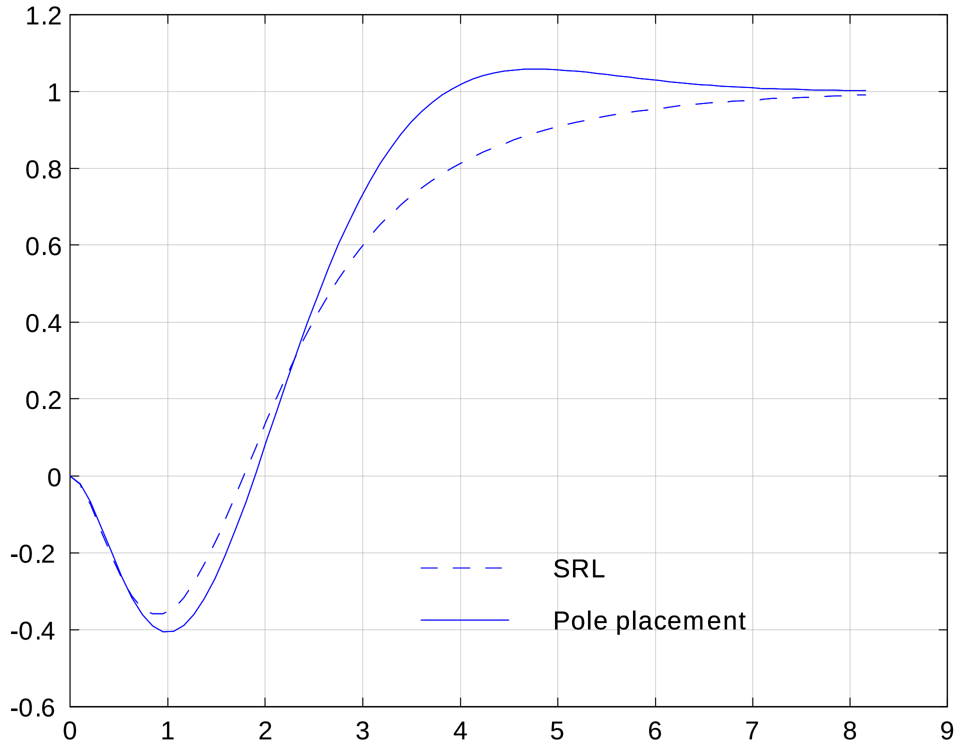
This yields $N_{\mathbf{x}} = [0\ 0\ 1\ 0]^T$ and $N_u = 0$. The step responses for each of the closed-loop systems (using the MATLAB `step` command) are shown next.



Symmetric root locus for Problem for Problem 7.30.



Initial condition response with $\theta(0) = 10^{\circ}$ for Problem 7.30.



Step response using $N_{\mathbf{x}}$ and N_u for Problem 7.30.

31. Consider the feedback system in Fig. 7.88. Find the relationship between K, T, and ξ such that the closed-loop transfer function minimizes the integral of the time multiplied by the absolute value of the error (ITAE) criterion,

$$\mathcal{J} = \int_0^{\infty} t|e|dt,$$

for a step input. Assume $\omega_0 = 1$.

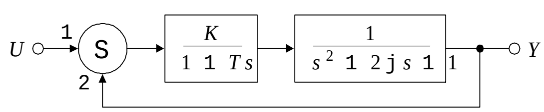


Figure 7.88: Control system for Problem 31.

Solution:

From the diagram:

$$\frac{Y(s)}{U(s)} = \frac{K/T}{s^3 + (\frac{1}{T} + 2\zeta)s^2 + (2 + \frac{2\zeta}{T})s + \frac{2+K}{T}}.$$

From the ITAE requirements [see Franklin, Powell, Emami-Naeini 3rd. Edition, pp. 508], we need to have,

$$\alpha_c(s) = s^3 + 1.75s^2 + 2.15s + 1.$$

Comparing the coefficients,

$$\frac{1}{T} + 2\zeta = 1.75, \qquad 2 + \frac{2\zeta}{T} = 2.15, \qquad \frac{2+K}{T} = 1.$$

32. Prove that the Nyquist plot for LQR design avoids a circle of radius one centered at the -1 point as shown in Fig. 7.89. Show that this implies that $\frac{1}{2}$ <GM< ∞ the "upward" gain margin is GM= ∞, and there is a "downward" GM= $\frac{1}{2}$, and the phase margin is at least PM= $\pm 60°$. Hence the LQR gain matrix, $\mathbf{K}$, can be multiplied by a large scalar or reduced by half with guaranteed closed-loop system stability.

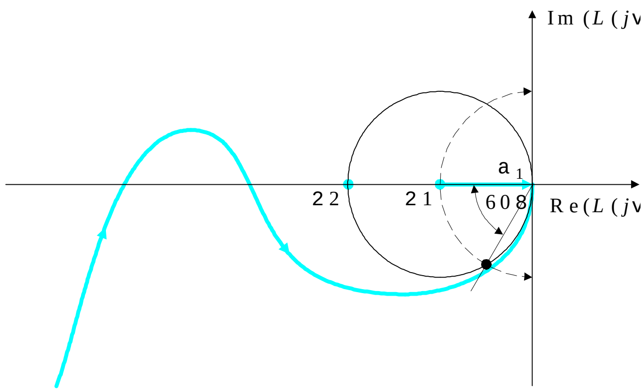


Figure 7.89: Nyquist plot for an optimal regulator.

Solution:

It has been proved (Anderson and Moore, 1990) that the Nyquist plot for LQR design avoids a circle of radius one centered at the -1 point as shown in Fig. 7.89. This leads to extraordinary phase and gain margin properties as shown below. First note that the state-feedback system can be re-drawn in the usual feedback configuration as shown on the next page. Using Eq. (7.60) and factoring $(s\mathbf{I} - \mathbf{F})$, we have

$$\begin{aligned}
\alpha_c(s) &= \det[s\mathbf{I} - (\mathbf{F} - \mathbf{G}\mathbf{K})] \qquad (5a)\\
&= \det\{(s\mathbf{I} - \mathbf{F})[\mathbf{I} + (s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G}\mathbf{K}]\}\\
&= \det(s\mathbf{I} - \mathbf{F}) \det\left([\mathbf{I} + (s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G}\mathbf{K}]\right)\\
&= D(s)[1 + \mathbf{K}(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G}].
\end{aligned}$$

Now[1], using the above equation and Eq. (7.95) we can write,

$$\alpha_c(s)\alpha_c(-s) = D(s)D(-s)[1+\mathbf{K}(s\mathbf{I}-\mathbf{F})^{-1}\mathbf{G}][1+\mathbf{K}(-s\mathbf{I}-\mathbf{F})^{-1}\mathbf{G}], \qquad (6a)$$
$$= 1+\rho G_0(s)G_0(-s).$$

Setting $s = j\omega$ we obtain,

$$\begin{aligned}\alpha_c(j\omega)\alpha_c(-j\omega) &= |D(j\omega)|^2|[1+\mathbf{K}(j\omega\mathbf{I}-\mathbf{F})^{-1}\mathbf{G}]|^2 \\ &= |D(j\omega)|^2|[1+\rho|G(j\omega)|^2|.\end{aligned} \qquad (7a)$$

But since $\rho|G(j\omega)|^2 \geq 0$, we can conclude that the return difference must satisfy,

$$|1+\mathbf{K}(j\omega\mathbf{I}-\mathbf{F})^{-1}\mathbf{G}| \geq 1. \qquad (8)$$

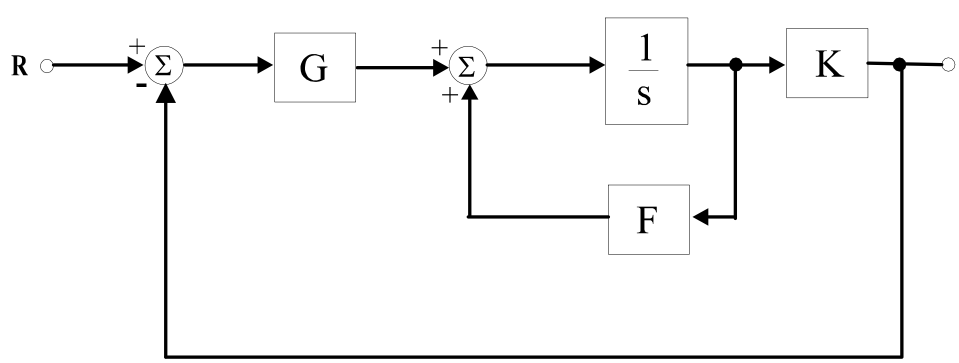
Let us re-write the loop gain as sum of its real and imaginary parts,

$$L(j\omega) = \mathbf{K}(j\omega\mathbf{I}-\mathbf{F})^{-1}\mathbf{G} = \mathrm{Re}\,[L(j\omega))+j\mathrm{Im}(L(j\omega)]\,. \qquad (9)$$

Finally, Eq. (8) implies that,

$$([\mathrm{Re}(L(j\omega)]+1)^2+[\mathrm{Im}(L(j\omega)]^2 \geq 1, \qquad (10)$$

which means that the Nyquist plot must indeed avoid a circle centered at -1 with unit radius. The Nyquist plot approaches the origin for large frequencies, and we find that the "upward" gain margin GM $= \infty$. The only other point on the negative real axis, in the proximity of the same circle, that the Nyquist plot can possibly cross is close to -2. This implies a "downward" gain margin of GM $= \frac{1}{2}$ (see also Problem 6.24). Hence the LQR gain matrix can be multiplied by a large scalar or reduced by half with guaranteed closed-loop system stability. As far as the determination of the phase margin PM is concerned, the closest possible approach point of the Nyquist plot to the -1 point is shown in Fig. 7.89. From this figure we conclude that the PM is at least $60°$. These margins are remarkable and it is not realistic to assume that they can be achieved in practice because of the presence of modeling errors and lack of sensors!



Optimal regulator in a feedback configuration.

[1]We have used the following result from matrix theory: if $\mathbf{A}$ is n×m matrix and $\mathbf{B}$ is m×n then $\det[\mathbf{I}_n-\mathbf{AB}] = \det[\mathbf{I}_m-\mathbf{BA}]$. See Appendix C.

Problems and Solutions for Section 7.7: Estimator Design

33. Consider the system

$$\mathbf{F} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{H} = [1 \ \ 2],$$

and assume that you are using feedback of the form $u = -\mathbf{K}\mathbf{x} + r$, where r is a reference input signal.
a) Show that $(\mathbf{F},\mathbf{H})$ is observable.
b) Show that there exists a $\mathbf{K}$ such that $(\mathbf{F} - \mathbf{GK},\mathbf{H})$ is unobservable.

c) Compute a $\mathbf{K}$ of the form $\mathbf{K} = [1, K_2]$ that will make the system unobservable as in part (b); that is, find K_2 so that the closed-loop system is not observable.
d) Compare the open-loop transfer function with the transfer function of the closed-loop system of part (c). What is the unobservability due to?

Solution:

(a)

$$\mathcal{O} = \begin{bmatrix} \mathbf{H} \\ \mathbf{HF} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

is nonsingular. Therefore, $(\mathbf{F},\mathbf{H})$ is observable.

(b) Let,

$$\mathcal{O}_{unobs} = \begin{bmatrix} \mathbf{H} \\ \mathbf{H}(\mathbf{F} - \mathbf{GK}) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -K_1 & 1 - K_2 \end{bmatrix}.$$

So if $\det(\mathcal{O}_{unobs}) = 1 - K_2 + 2K_1 = 0$, then $(\mathbf{F} - \mathbf{GK},\mathbf{H})$ is unobservable.

(c) $K_1 = 1 \Longrightarrow 1 - K_2 + 2 = 0 \Longrightarrow K_2 = 3$. The result can be verified using MATLAB's `place` command.

(d)

$$\begin{aligned} G_{ol}(s) &= \mathbf{H}(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G} = \frac{s+2}{s^2+2s-1} = \frac{s+2}{(s-0.414)(s+2.414)}. \\ G_{cl}(s) &= \mathbf{H}(s\mathbf{I} - \mathbf{F} + \mathbf{GK})^{-1}\mathbf{G} = \frac{s+2}{s^2+3s+2} = \frac{s+2}{(s+2)(s+1)} = \frac{1}{(s+1)}. \end{aligned}$$

The computations can be carried out using MATLAB's `ss2tf` command. So the unobservability is due to a cancellation of one of the closed-loop poles with the zero of the system. In other words, this closed-loop mode is unobservable from the output.

34. Consider a system with the transfer function,

$$G(s) = \frac{9}{s^2 - 9}.$$

a) Find $(\mathbf{F}_0, \mathbf{G}_0, \mathbf{H}_0)$ for this system in observer canonical form.
b) Is $(\mathbf{F}_0, \mathbf{G}_0)$ controllable?
c) Compute $\mathbf{K}$ so that the closed-loop poles are assigned to $s = -3 \pm 3j$.
d) Is the closed-loop system of part (c) observable?

e) Design a full-order estimator with estimator-error poles at $s = -12 \pm 12j$.
f) Suppose the system is modified to have a zero:

$$G_1(s) = \frac{9(s+1)}{s^2 - 9}.$$

Prove that if $u = -\mathbf{K}\mathbf{x} + r$, there is a feedback gain $\mathbf{K}$ that makes the closed-loop system unobservable. [Again assume an observer canonical realization for $G_1(s)$.]

Solution:

(a) For a transfer function,

$$G(s) = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2},$$

the observer canonical form becomes,

$$\mathbf{F}_o = \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix}, \ \mathbf{G}_o = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \end{bmatrix}, \ \mathbf{H}_o = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

(b) To check whether $(\mathbf{F}_o, \mathbf{G}_o)$ is controllable we form the controllability matrix,

$$\mathcal{C} = \begin{bmatrix} \mathbf{G}_o & \mathbf{F}_o\mathbf{G}_o \end{bmatrix} = \begin{bmatrix} 0 & 9 \\ 9 & 0 \end{bmatrix} \Rightarrow \det(\mathcal{C}) = -81 \neq 0.$$

Thus, the system is controllable.

(c) $\mathbf{K} = \begin{bmatrix} 3 & 2/3 \end{bmatrix}$. The result can be verified using MATLAB's place command.

(d) The system is in observer canonical form. Hence, it is guaranteed to be observable. To check,

$$\mathcal{O} = \begin{bmatrix} \mathbf{H}_o \\ \mathbf{H}_o\mathbf{F}_o \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Longrightarrow \det(\mathcal{O}) = 1 \neq 0 \neq 0.$$

(e) Solving $\det(s\mathbf{I} - \mathbf{F}_o + \mathbf{L}\mathbf{H}_o) = (s+12)^2 + 144$ for $\mathbf{L}$ yields $\mathbf{L} = \begin{bmatrix} 24 & 297 \end{bmatrix}^T$. The result can be verified using MATLAB's place command.

(f) The realization for,

$$G(s) = \frac{9(s+1)}{s^2 - 9},$$

in observer canonical form yields,

$$\mathbf{F}_o = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix}, \ \mathbf{G}_o = \begin{bmatrix} 9 \\ 9 \end{bmatrix}, \ \mathbf{H}_o = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Thus, for the system with feedback,

$$\mathcal{O} = \begin{bmatrix} \mathbf{H}_o \\ \mathbf{H}_o(\mathbf{F}_o - \mathbf{G}_o\mathbf{K}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -9k_1 & 1 - 9k_2 \end{bmatrix} \Longrightarrow \det(\mathcal{O}) = 1 - 9k_2.$$

Thus,

$$\{(k_1, k_2) \ | k_2 = 1/9\},$$

is the set of all k_1 and k_2 that make the system unobservable. So we have shown that there exists a feedback gain $\mathbf{K}$ which makes the closed-loop system unobservable. Note that for $k_2 = 1/9$,

$$\det(s\mathbf{I} - \mathbf{F}_o + \mathbf{G}_o\mathbf{K}) = \det \begin{bmatrix} s + 9k_1 & -1 + 9k_2 \\ -9 + 9k_1 & s + 9k_2 \end{bmatrix} = (s + 9k_1)(s + 1).$$

Thus, the reason why the system becomes unobservable is that the pole at $s = -1$ is cancelled by a zero.

35. Explain how the controllability, observability, and stability properties of a linear system are related.

Solution:

controllability $\Longrightarrow \det\,[\mathbf{G}\ \mathbf{FG}\ \mathbf{F}^2\mathbf{G} \cdots \mathbf{F}^{n-1}\mathbf{G}] \neq 0.$

observability $\Longrightarrow \det \begin{bmatrix} \mathbf{H} \\ \mathbf{HF} \\ \mathbf{HF}^2 \\ \vdots \\ \mathbf{HF}^{n-1} \end{bmatrix} \neq 0.$

stability $\Longrightarrow$ Re{eigenvalues($\mathbf{F}$)} < 0.

So,in general, there is no connection between these three properties. However, for a minimal realization (controllable, observable), internal and external stabilities are the same.

Note that the mathematical relations given above are idealizations, much like a frictionless plane in physics. In practice, it is important to consider the singular values of the controllability or observability matrices and their proximity to the $j\omega$ axis. For example, if two of the eigenvectors of the controllability matrix are nearly parallel, then the system is nearly uncontrollable and large actuator signals may be required to get the system to a particular state in state space.

36. Consider the electric circuit shown in Fig. 7.90.

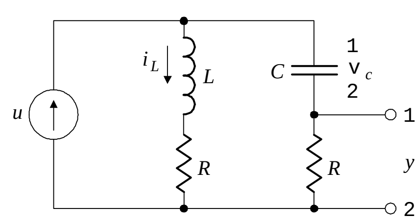


Figure 7.90: Electric circuit for Problem 35.

a) Write the internal (state) equations for the circuit. The input $u(t)$ is a current, and the output y is a voltage. Let $x_1 = i_L$ and $x_2 = v_c$.
b) What condition(s) on R, L, and C will guarantee that the system is controllable?
c) What condition(s) on R, L, and C will guarantee that the system is observable?

Solution:

(a) Apply Kirchhoff's voltage and current laws, with $x_1 = i_L$ and $x_2 = v_c$, we obtain,

$$\begin{aligned} L\dot{x}_1 + Rx_1 &= x_2 + RC\dot{x}_2, \\ \dot{x}_2 &= u - x_1, \\ y &= (u - x_1)R \end{aligned}$$

Thus,

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -2R/L & 1/L \\ -1/C & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} R/L \\ 1/C \end{bmatrix} u, \\ y &= \begin{bmatrix} -R & 0 \end{bmatrix} \mathbf{x} + Ru. \end{aligned}$$

(b) The condition for the system to be uncontrollable is $\det(\mathcal{C}) = 0$.

$$\begin{aligned} \mathcal{C} &= \begin{bmatrix} \mathbf{G} & \mathbf{FG} \end{bmatrix} = \begin{bmatrix} R/L & -2R^2/L^2 + 1/LC \\ 1/C & -R/LC \end{bmatrix}. \\ \det(\mathcal{C}) &= R^2/L^2C - 1/LC^2. \end{aligned}$$

Thus, the system is controllable if $R^2 \neq L/C$.

(c) The condition for the system to be unobservable is,

$$\begin{aligned} \mathcal{O} &= \begin{bmatrix} \mathbf{H} \\ \mathbf{HF} \end{bmatrix} = \begin{bmatrix} -R & 0 \\ 2R^2/L & -R/L \end{bmatrix}. \\ \det(\mathcal{O}) &= R^2/L. \end{aligned}$$

Since $\det(\mathcal{O}) \neq 0$ for any R, L, C except $R = 0$ or $L = \infty$, the system is observable.

37. The block diagram of a feedback system is shown in Fig. 7.91. The system state is,

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_p \\ \mathbf{x}_f \end{bmatrix},$$

and the dimensions of the matrices are as follows:

$$\begin{aligned} &\mathbf{F} = n \times n, \quad \mathbf{L} = n \times 1, \\ &\mathbf{G} = n \times 1, \quad \mathbf{x} = 2n \times 1, \\ &\mathbf{H} = 1 \times n, \quad r = 1 \times 1, \\ &\mathbf{K} = 1 \times n, \quad y = 1 \times 1, \end{aligned}$$

a) Write state equations for the system.

b) Let $\mathbf{x} = \mathbf{Tz}$, where

$$\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix}.$$

Show that the system is not controllable.

c) Find the transfer function of the system from r to y.

Solution:

(a) We have,

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_f \end{bmatrix} = \begin{bmatrix} \mathbf{F} & -\mathbf{GK} \\ \mathbf{LH} & \mathbf{F}-\mathbf{LH}-\mathbf{GK} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_f \end{bmatrix} + \begin{bmatrix} \mathbf{G} \\ \mathbf{G} \end{bmatrix} r.$$

(b) In order to apply our transformation of coordinates, we need $\mathbf{T}^{-1}$,

$$\mathbf{T} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \Longrightarrow \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix}.$$

Thus,

$$\begin{aligned}
\mathbf{F}_{cl} &= \mathbf{T}^{-1} \begin{bmatrix} \mathbf{F} & -\mathbf{GK} \\ \mathbf{LH} & \mathbf{F}-\mathbf{LH}-\mathbf{GK} \end{bmatrix} \mathbf{T} = \begin{bmatrix} \mathbf{F}-\mathbf{GK} & -\mathbf{GK} \\ \mathbf{0} & \mathbf{F}-\mathbf{LH} \end{bmatrix}, \\
\mathbf{G}_{cl} &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{G} \\ \mathbf{G} \end{bmatrix} = \begin{bmatrix} \mathbf{G} \\ \mathbf{0} \end{bmatrix}, \\
\mathbf{H}_{cl} &= \begin{bmatrix} \mathbf{H} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{H} & \mathbf{0} \end{bmatrix}.
\end{aligned}$$

In the new coordinate system, we have,

$$\begin{aligned}
\dot{\mathbf{z}} &= \begin{bmatrix} \mathbf{F}-\mathbf{GK} & -\mathbf{GK} \\ \mathbf{0} & \mathbf{F}-\mathbf{LH} \end{bmatrix} \mathbf{z} + \begin{bmatrix} \mathbf{G} \\ \mathbf{0} \end{bmatrix} r, \\
y &= \begin{bmatrix} \mathbf{H} & \mathbf{0} \end{bmatrix} \mathbf{z}.
\end{aligned}$$

Observe that the system is now decomposed into controllable and uncontrollable parts. Hence, we have shown that it is an uncontrollable system.

(c) The transfer function is,

$$T(s) = \mathbf{H}[s\mathbf{I} - (\mathbf{F}-\mathbf{GK})]^{-1}\mathbf{G}.$$

38. This problem is intended to give you more insight into controllability and observability. Consider the circuit in Fig. 7.92, with an input voltage source $u(t)$ and an output current $y(t)$.

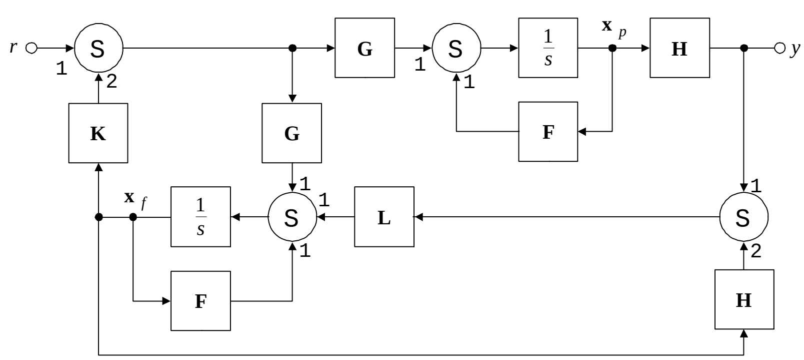


Figure 7.91: Block diagram for Problem 7.37.

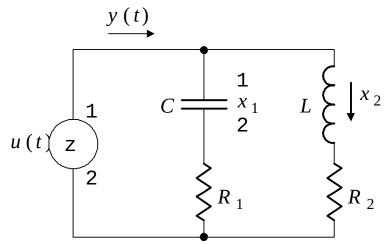


Figure 7.92: Electric circuit for Problem 7.38.

a) Using the capacitor voltage and inductor current as state variables, write state and output equations for the system.
b) Find the conditions relating R_1, R_2, C, and L that render the system uncontrollable. Find a similar set of conditions that result in an unobservable system.
c) Interpret the conditions found in part (b) physically in terms of the time constants of the system.
d) Find the transfer function of the system. Show that there is a pole-zero cancellation for the conditions derived in part (b) (that is, when the system is uncontrollable or unobservable).

Solution:

(a) From Figure 7.92,

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -\frac{1}{R_1 C} & 0 \\ 0 & \frac{R_2}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1 C} \\ \frac{1}{L} \end{bmatrix} u, \\ y &= \begin{bmatrix} -\frac{1}{R_1} & 1 \end{bmatrix} \mathbf{x} + \frac{1}{R_1} u. \end{aligned}$$

(b) First, form the controllability matrix,

$$\begin{aligned} \mathcal{C} &= \begin{bmatrix} \mathbf{G} & \mathbf{FG} \end{bmatrix} = \begin{bmatrix} \frac{1}{R_1 C} & -\frac{1}{(R_1 C)^2} \\ \frac{1}{L} & -\frac{R_2}{L^2} \end{bmatrix}, \\ \det(\mathcal{C}) &= -R_2/R_1 C L^2 + 1/L(R_1 C)^2. \end{aligned}$$

For uncontrollability, $\det(\mathcal{C}) = 0$ implies $R_1 R_2 C = L$.

Next, form the observability matrix,

$$\begin{aligned} \mathcal{O} &= \begin{bmatrix} \mathbf{H} \\ \mathbf{HF} \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_1} & 1 \\ \frac{1}{R_1^2 C} & -\frac{R_2}{L} \end{bmatrix}, \\ \det(\mathcal{O}) &= R_2/R_1 L - 1/R_1^2 C. \end{aligned}$$

For unobservability, $\det(\mathcal{O}) = 0$ implies, again that, $R_1 R_2 C = L$.

(c) When the system is unobservable/uncontrollable, we have $1/R_1 C = R_2/L$ so that:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -\frac{1}{R_1 C} & 0 \\ 0 & -\frac{1}{R_1 C} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{R_1 C} \\ \frac{1}{L} \end{bmatrix} u, \\ y &= \begin{bmatrix} -\frac{1}{R_1} & 1 \end{bmatrix} \mathbf{x} + [1/R_1] u. \end{aligned}$$

The two modes of the system have the same time constant and hence cannot be changed independently using only one control u, i.e., it is not controllable. The system output is a linear combination of modes having the same frequency. It is thus impossible to determine the composition of the combination, i.e., it is not observable.

(d)

$$\begin{aligned} G(s) &= \frac{s^2 + \frac{R_1+R_2}{L}s + \frac{1}{LC}}{R_1(s+1/R_1C)(s+R_2/L)} \\ &= \frac{(s+a)(s+b)}{R_1(s+1/R_1C)(s+R_2/L)}, \end{aligned}$$

where,

$$a, b = \frac{1}{2}\left[\frac{R_1+R_2}{L} \pm \sqrt{\frac{(R_1+R_2)^2}{L^2} - \frac{4}{LC}}\right].$$

Substituting $1/C = R_1R_2/L$, we find $a = \frac{R_2}{L}$, and $b = \frac{R_1}{L}$. Thus,

$$\begin{aligned} G(s) &= \frac{(s+R_2/L)(s+R_1/L)}{R_1(s+1/R_1C)(s+R_2/L)} \\ &= \frac{s+R_1/L}{R_1(s+1/R_1C)}. \end{aligned}$$

Observe the pole-zero *cancellation* in the transfer function.

39. The linearized equations of motion for a satellite are,

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{F}\mathbf{x} + \mathbf{G}\mathbf{u}, \\ \mathbf{y} &= \mathbf{H}\mathbf{x}, \end{aligned}$$

where

$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix}, \; \mathbf{G} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \; \mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$
$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \; \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

The inputs u_1 and u_2 are the radial and tangential thrusts, the state variables x_1 and x_3 are the radial and angular deviations from the reference (circular) orbit, and the outputs y_1 and y_2 are the radial and angular measurements, respectively.
a) Show that the system is controllable using both control inputs.
b) Show that the system is controllable using only a single input. Which one is it?
c) Show that the system is observable using both measurements.
d) Show that the system is observable using only one measurement. Which one is it?

Solution:

(a) Checking the controllability matrix:

$$\mathcal{C} = \begin{bmatrix}\mathbf{G} \ \mathbf{FG} \ \mathbf{F}^2\mathbf{G} \ \mathbf{F}^3\mathbf{G}\end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & \\ 1 & 0 & 0 & 2\omega & \cdots \\ 0 & 0 & 0 & 1 & \\ 0 & 1 & -2\omega & 0 & \end{bmatrix}.$$

Considering only the first four columns of the controllability matrix, the rank is already 4 and hence it is controllable. Incidentally, you could also show this part of the problem by first doing part (b) and then recognizing that if the system is controllable from a single actuator, it will surely be controllable from the same actuator plus any other additional actuators you care to add. This could be useful in multivariable system design. For example, when actuators are expensive, one design criterion could be to minimize the number of actuators while maintaining controllability of the system.

(b) Consider only the first (radial) thruster, u_1, (i.e., $u_2 = 0$). Then $\mathbf{G}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T$. So we have,

$$\mathcal{C}_1 = \begin{bmatrix}\mathbf{G}_1 \ \mathbf{FG}_1 \ \mathbf{F}^2\mathbf{G}_1 \ \mathbf{F}^3\mathbf{G}_1\end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -\omega^2 \\ 1 & 0 & -\omega^2 & 0 \\ 0 & 0 & -2\omega & 0 \\ 0 & -2\omega & 0 & 2\omega^3 \end{bmatrix}.$$

The rank is 3, hence the satellite is uncontrollable using only the radial thruster, u_1. Now consider the second (tangential) thruster, u_2, (i.e., $u_1 = 0$). Then, $\mathbf{G}_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T$. So we have,

$$\mathcal{C}_2 = \begin{bmatrix}\mathbf{G}_2 \ \mathbf{FG}_2 \ \mathbf{F}^2\mathbf{G}_2 \ \mathbf{F}^3\mathbf{G}_2\end{bmatrix} = \begin{bmatrix} 0 & 0 & 2\omega & 0 \\ 0 & 2\omega & 0 & -2\omega^3 \\ 0 & 1 & 0 & -4\omega^2 \\ 1 & 0 & -4\omega^2 & 0 \end{bmatrix}.$$

The rank is 4, hence the satellite is controllable using only the tangential thruster, u_2. The tangential thruster can be used to control the angular velocity of the satellite, and hence radial deviations can be controlled.

(c) Checking the observability matrix:

$$\mathcal{O} = \begin{bmatrix} \mathbf{H} \\ \mathbf{HF} \\ \mathbf{HF}^2 \\ \mathbf{HF}^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ & \vdots & & \end{bmatrix}.$$

Since the rank of the first four rows is already 4, the system is observable.

(d) Using only the first measurement, y_1, we have,

$$\mathcal{O}_1 = \begin{bmatrix} \mathbf{H}_1 \\ \mathbf{H}_1\mathbf{F} \\ \mathbf{H}_1\mathbf{F}^2 \\ \mathbf{H}_1\mathbf{F}^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & -\omega^2 & 0 & 0 \end{bmatrix}.$$

This has a rank of 3, hence the system's state is unobservable using only a radial measurement. Now considering the tangential measurement,

$$\mathcal{O}_2 = \begin{bmatrix} \mathbf{H}_2 \\ \mathbf{H}_2\mathbf{F} \\ \mathbf{H}_2\mathbf{F}^2 \\ \mathbf{H}_2\mathbf{F}^3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \\ -6\omega^3 & 0 & 0 & -4\omega^2 \end{bmatrix}.$$

Since this matrix has a rank of 4, the system is observable using only the tangential measurement.

40. Consider the system in Fig. 7.93.

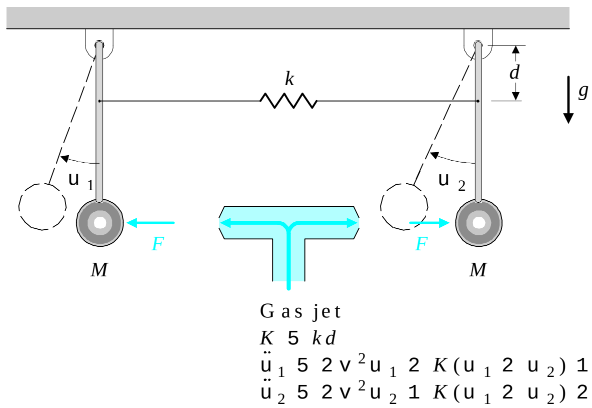


Figure 7.93: Coupled pendulums for Problem 40.

a) Write the state-variable equations for the system, using $[\theta_1\ \theta_2\ \dot{\theta}_1\ \dot{\theta}_2]^T$ as the state vector and F as the single input.
b) Show that all the state variables are observable using measurements of θ_1 alone.
c) Show that the characteristic polynomial for the system is the product of the polynomials for two oscillators. Do so by first writing a new set of system equations involving the state variables

$$\begin{bmatrix} y_1 \\ y_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \theta_1 + \theta_2 \\ \theta_1 - \theta_2 \\ \dot{\theta}_1 + \dot{\theta}_2 \\ \dot{\theta}_1 - \dot{\theta}_2 \end{bmatrix}.$$

Hint: If $\mathbf{A}$ and $\mathbf{D}$ are invertible matrices, then,

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^{-1} \end{bmatrix}.$$

d) Deduce the fact that the spring mode is controllable with F but the pendulum mode is not.

Solution:

The equations of motion for the system given in Fig. 7.93

$$\begin{aligned} ml^2\ddot{\theta}_1 &= -kd^2(\theta_1-\theta_2)-mgl\theta_1+lu, \\ ml^2\ddot{\theta}_2 &= -kd^2(\theta_2-\theta_1)-mgl\theta_2-lu, \end{aligned}$$

where u is the force from the gas jet, l is the pendulum length, m is the pendulum mass, and d is as given in the figure. Letting $\omega^2 = g/l$, $K = kd^2/ml^2$, and $F = (1/ml)u$, we obtain:

$$\begin{aligned} \ddot{\theta}_1 &= -\omega^2\theta_1 - K(\theta_1-\theta_2)+F, \\ \ddot{\theta}_2 &= -\omega^2\theta_2 - K(\theta_2-\theta_1)-F. \end{aligned}$$

(a) Using the state vector $\mathbf{x} = [\theta_1\ \theta_2\ \dot{\theta}_1\ \dot{\theta}_2]^T$,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(\omega^2+K) & K & 0 & 0 \\ K & -(\omega^2+K) & 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} F.$$

(b) Considering only the measurement of θ_1, then:

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}\mathbf{x}.$$

Observability:

$$\mathcal{O} = \begin{bmatrix} \mathbf{H} \\ \mathbf{HF} \\ \mathbf{HF}^2 \\ \mathbf{HF}^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -(\omega^2+K) & K & 0 & 0 \\ 0 & 0 & -(\omega^2+K) & K \end{bmatrix}.$$

Since $K \neq 0$, $\det(\mathcal{O}\} \neq 0$. Hence, the state is observable with θ_1.

(c) Define a state vector, $\mathbf{x}$, such that: $\mathbf{x} = [y_1\ \dot{y}_1\ y_2\ \dot{y}_2]^T = [\theta_1+\theta_2\ \ \dot{\theta}_1+\dot{\theta}_2\ \theta_1-\theta_2\ \dot{\theta}_1-\dot{\theta}_2]^T$. Note that the order of the state variables is chosen such that the resulting plant matrix is block diagonal. With this state vector,

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -(\omega^2+2K) & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} F.$$

The characteristic equation of the system is,

$$\det(s\mathbf{I}-\mathbf{F}) = (s^2+\omega^2)(s^2+\omega^2+2K).$$

Thus, it is the product of two oscillators with frequencies ω and $\sqrt{\omega^2+2K}$.

(d) From the state equations in part (c), note that they are block diagonal. Thus there is no coupling between the *spring mode* $(\theta_1-\theta_2)$ and the *pendulum mode* $(\theta_1+\theta_2)$. Because the gas jets, via F, are only connected to the spring mode, we conclude that spring mode is controllable while the pendulum mode is not.

41. A certain fifth-order system is found to have a characteristic equation with roots at 0, -1, -2, and $-1 \pm 1j$. A decomposition into controllable and uncontrollable parts discloses that the controllable part has a characteristic equation with roots 0, and $-1 \pm 1j$. A decomposition into observable and nonobservable parts discloses that the observable modes are at 0, -1, and -2.
a) Where are the zeros of $b(s) = \mathbf{H}\text{adj}(s\mathbf{I} - \mathbf{F})\mathbf{G}$ for this system?
b) What are the poles of the reduced-order transfer function that includes only controllable and observable modes?

Solution:

(a) $b(s) = \mathbf{H}\text{adj}(s\mathbf{I} - \mathbf{F})\mathbf{G}$

controllable modes: $0, -1 \pm j$

observable modes: $0\ , -1, -2$.

Hence, mode 0 is the only mode that is both controllable and observable. Therefore, $b(s)$ has zeros at $s = -1$, $s = -2$, and $s = -1 \pm j$.

(b) Reduced transfer function has only one pole which is at the origin, i.e., $s = 0$.

42. Consider the systems shown in Fig. 7.94, employing series, parallel, and feedback configurations.

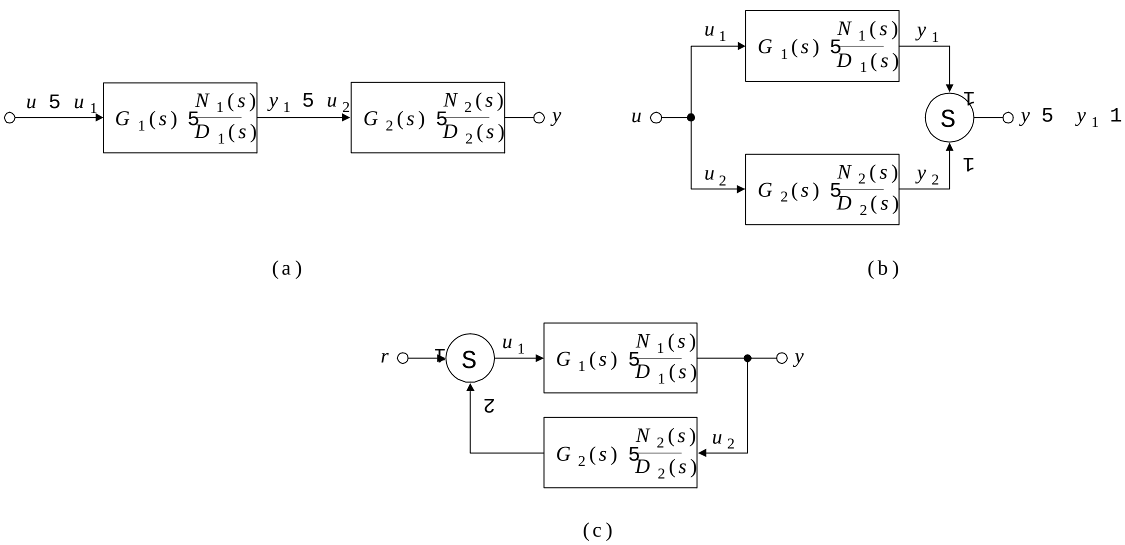


Figure 7.94: Block diagrams for Problem 7.42: (a) series; (b) parallel; (c) feedback

a) Suppose we have controllable-observable realizations for each subsystem:

$$\dot{\mathbf{x}}_i = \mathbf{F}_i\mathbf{x}_i + \mathbf{G}_i u_i,$$
$$y_i = \mathbf{H}_i\mathbf{x}_i, \quad \text{where} \quad i = 1, 2.$$

Give a set of state equations for the combined systems in Fig. 7.94.
b) For each case, determine what condition(s) on the roots of the polynomials N_i and D_i is necessary for each system to be controllable and observable. Give a brief reason for your answer in terms of pole-zero cancellations.

Solution:

(a) Series connection,

$$\begin{aligned}
\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{F}_1 & \mathbf{0} \\ \mathbf{G}_2\mathbf{H}_1 & \mathbf{F}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{0} \end{bmatrix} u, \\
y &= \begin{bmatrix} \mathbf{0} & \mathbf{H}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}.
\end{aligned}$$

See the MATLAB series command.

(b) Parallel connection,

$$\begin{aligned}
\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{F}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{G}_2 \end{bmatrix} u, \\
y &= \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}.
\end{aligned}$$

See the MATLAB parallel command.

(c) Feedback connection,

$$\begin{aligned}
\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} &= \begin{bmatrix} \mathbf{F}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{G}_1(r - y_2) \\ \mathbf{G}_2 y_1 \end{bmatrix}, \\
&= \begin{bmatrix} \mathbf{F}_1 & -\mathbf{G}_1\mathbf{H}_2 \\ \mathbf{G}_2\mathbf{H}_1 & \mathbf{F}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{G}_1 \\ \mathbf{0} \end{bmatrix} r, \\
y &= \begin{bmatrix} \mathbf{H}_1 & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}.
\end{aligned}$$

See the MATLAB feedback command.

(d) Since each sub-system is in controllable-observable realization, each transfer function $G_1(s)$ and $G_2(s)$ is minimal (i.e., no cancellations).

(e) Series connection:

$$\frac{Y(s)}{U(s)} = G_2(s)G_1(s) = \frac{N_2(s)}{D_2(s)} . \frac{N_1(s)}{D_1(s)}$$

For observability, $N_2(s)$ and $D_1(s)$ must be coprime (i.e., no common factors). Otherwise, a mode of $D_1(s)$ is masked from the output. For controllability, $N_1(s)$ and $D_2(s)$ must be coprime. Otherwise, a mode of $D_2(s)$ is masked from the output.

(f) Parallel connection:

$$\frac{Y(s)}{U(s)} = G_1(s) + G_2(s) = \frac{N_1(s)}{D_1(s)} + \frac{N_2(s)}{D_2(s)} = \frac{N_1(s)D_2(s) + D_1(s)N_2(s)}{D_1(s)D_2(s)}.$$

For observability (controllability), $D_1(s)$ and $D_2(s)$ must be coprime. Otherwise, the two modes appear as a single mode from the output (input).

(g) Feedback connection:

$$\frac{Y(s)}{R(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)} = \frac{N_1(s)D_2(s)}{D_1(s)D_2(s) + N_1(s)N_2(s)}.$$

For observability, $N_1(s)$ and $D_2(s)$ must be coprime (i.e., no common factors). For controllability, $N_2(s)$ and $D_1(s)$ must be coprime.

43. Consider the system $\ddot{y} + 3\dot{y} + 2y = \dot{u} + u$.
a) Find the state matrices $\mathbf{F}_c$, $\mathbf{G}_c$, and $\mathbf{H}_c$ in control canonical form that correspond to the given differential equation.
b) Sketch the eigenvectors of $\mathbf{F}_c$ in the (x_1, x_2) plane, and draw vectors that correspond to the completely observable ($\mathbf{x}_0$) and the completely unobservable ($\mathbf{x}_{\bar{0}}$) state variables.
c) Express $\mathbf{x}_0$ and $\mathbf{x}_{\bar{0}}$ in terms of the observability matrix $\mathcal{O}$.
d) Give the state matrices in observer canonical form and repeat parts (b) and (c) in terms of controllability instead of observability.

Solution:

(a) The Laplace transform of the differential equation gives the transfer function,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s+1}{s^2+3s+2}.$$

Hence in controller canonical form,

$$\mathbf{F}_c = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \ \mathbf{G}_c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{H}_c = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

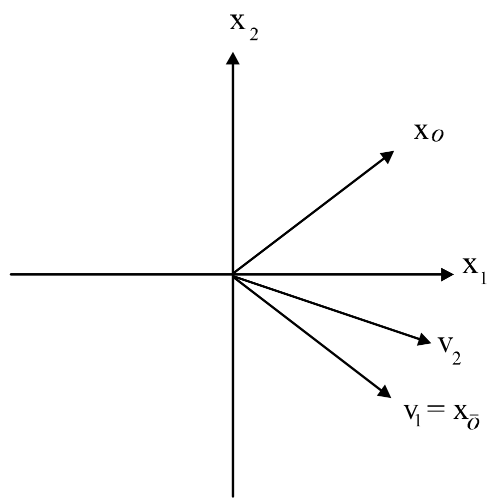
(b) First, we find the eigenvectors of $\mathbf{F}_c$ or the modal directions of the system,

$$\det(s\mathbf{I} - \mathbf{F}_c) = 0 \Longrightarrow s = -1, -2.$$

$$\begin{aligned} (\mathbf{F}_c + \mathbf{I})\mathbf{v}_1 &= 0 \Longrightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \\ (\mathbf{F}_c + 2\mathbf{I})\mathbf{v}_2 &= 0 \Longrightarrow \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}. \end{aligned}$$

Using partial-fraction expansion of $G(s)$, we can determine which modes are unobservable and which are observable. The mode $s = -1$, this mode has a pole-zero cancellation in $G(s)$, so $\mathbf{v}_1$ is the unobservable mode shape. The mode $s = -2$, does appear in the minimal transfer function, so $\mathbf{v}_2$ is the observable mode shape. Therefore, the completely unobservable direction is equal to $\mathbf{v}_1$. The completely observable direction is $\mathbf{v}_2^{\perp}$, where $\mathbf{v}_2^{\perp}$is the projection of $\mathbf{v}_2$ on the orthogonal direction to $\mathbf{v}_1$. These vectors are drawn in the figure below. Note from the figure that $\mathbf{x}_o$ is, in fact, the same as $\mathbf{H}$. Also, the observable mode, $\mathbf{v}_2$ *is observable* since the projection of $\mathbf{v}_2$ onto $\mathbf{x}_o$ is not zero, i.e.,

$$\mathbf{x}_o^T \mathbf{v}_2 = \mathbf{H}_c \mathbf{v}_2 \neq 0.$$



Observable and unobservable state directions for Problem 7.43(b).

(c) Observability measures the ability to reconstruct the realization state variables given an output and its derivatives. Consider the determination of the state initial condition, $\mathbf{x}(\mathbf{0})$, given the initial output measurement and its derivatives, $\mathbf{Y}(\mathbf{0})$, where,

$$\mathbf{Y}(\mathbf{0}) = \begin{bmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{H}_c \\ \mathbf{H}_c\mathbf{F}_c \\ \vdots \\ \mathbf{H}_c\mathbf{F}_c^{(n-1)} \end{bmatrix} \mathbf{x}(\mathbf{0}) = \mathcal{O}\mathbf{x}(\mathbf{0}).$$

So the determination of $\mathbf{x}(\mathbf{0})$ is equivalent to the solution of $Y(\mathbf{0}) = \mathcal{O}\mathbf{x}(\mathbf{0})$. From linear algebra, the unobservable state variables are in the null space of the observability matrix, and the observable state variables are in the left-range space of $\mathcal{O}$:

$$\mathbf{x}_{\bar{o}} \in \mathcal{N}(\mathcal{O}), \quad \mathbf{x}_o \in \mathcal{R}(\mathcal{O}^T).$$

So that,

$$\mathbf{x}_{\bar{o}} = \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \ \mathbf{x}_o = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

(d) In observer canonical form,

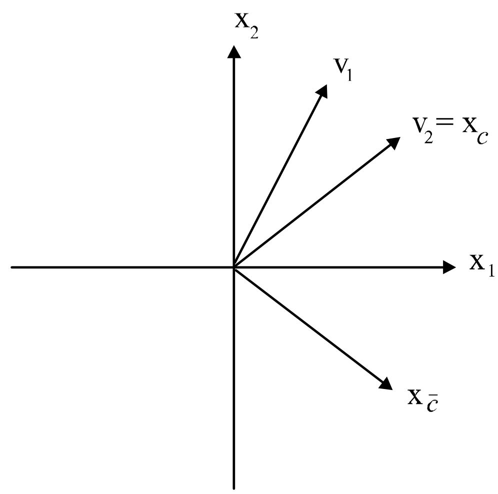
$$\mathbf{F}_o = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}, \ \mathbf{G}_o = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{H}_o = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The eigenvectors of $\mathbf{F}_o$ are,

$$\det(s\mathbf{I} - \mathbf{F}_o) = 0 \Longrightarrow s = -1, -2.$$

$$\begin{aligned} (\mathbf{F}_o + \mathbf{I})\mathbf{v}_1 &= 0 \Longrightarrow \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \\ (\mathbf{F}_o + 2\mathbf{I})\mathbf{v}_2 &= 0 \Longrightarrow \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

The mode $s = -1$, has a pole-zero cancellation from $G(s)$, so $\mathbf{v}_1$ is the uncontrollable mode shape. The mode $s = -2$, appears in $G(s)$, so $\mathbf{v}_2$ is the controllable mode shape. These vectors are drawn in the figure below.



Controllable and uncontrollable state directions for Problem 7.43(d).

Controllability measures the ability to drive the states to arbitrary values. Consider the use of $u(t)$ to move the state vector, $\mathbf{x}(0-)$, to an arbitrary value, say $\mathbf{x}(0+)$, where,

$$u(t) = g_1\delta(t) + g_2\dot{\delta}(t) + \cdots + g_n\delta^{(n-1)}(t).$$

So that,

$$\mathbf{x}(0+) - \mathbf{x}(0-) = \mathcal{C}u^*,$$

where $u^* = [g_1 \;\; g_2 \cdots + g_n]^T$. Hence, a controllable state is one in which some vector u^*exists such that $\mathbf{x}_c = \mathcal{C}u^*$. From linear algebra,

$$\mathbf{x}_{\overline{c}} \in \mathcal{N}(\mathcal{C}^T), \quad \mathbf{x}_c \in \mathcal{R}\{\mathcal{C}\}.$$

So that,

$$\mathbf{x}_{\overline{c}} = \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \; \mathbf{x}_c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

44. The equations of motion for a station-keeping satellite (such as a weather satellite) are

$$\ddot{x} - 2\omega\dot{y} - 3\omega^2 x = 0, \quad \ddot{y} + 2\omega\dot{x} = u,$$

where,

$$\begin{aligned} x &= \text{radial perturbation}, \\ y &= \text{longitudinal position perturbation}, \\ u &= \text{ engine thrust in the y} - \text{direction}, \end{aligned}$$

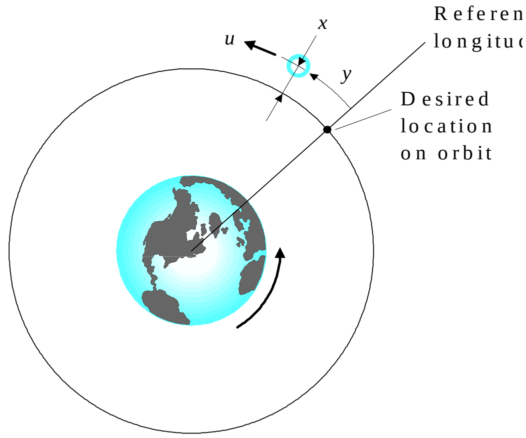


Figure 7.95: Diagram of a station-keeping satellite in orbit for Problem 44.

as depicted in Fig. 7.94. If the orbit is synchronous with the earth's rotation, then $\omega = 2\pi/(3600 \times 24)$ rad/sec.
a) Is the state $[x \quad \dot{x} \quad y \quad \dot{y}]^T$ observable?
b) Choose $\mathbf{x} = [\ x \quad \dot{x} \quad y \quad \dot{y}\ ]^T$ as the state vector and y as the measurement, and design a full-order observer with poles placed at $s = -2\omega$, -3ω, and $-3\omega \pm 3\omega j$.

Solution:

(a) There is not enough information to answer this question. Recall, as mentioned in the chapter, that both observability and controllability are properties of realizations. Thus if you are only given differential equations or transfer functions you will *not* be able to conclude anything about the observability or controllability of the system. This problem was designed to heighten the readers awareness of this issue.

(b) Choosing $x_1 = x$, $x_2 = \dot{x}$, $x_3 = y$, $x_4 = \dot{y}$, and z as the output of the system (so that it doesn't conflict with the variable y, we have the following in state space equations.

$$\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2\omega \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u, \\
z &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}.
\end{aligned}$$

Now that we have a realization for the system, we can check the observability to verify that we can arbitrarily place the estimator poles. The observability matrix is,

$$\mathcal{O} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2\omega & 0 & 0 \\ -6\omega^3 & 0 & 0 & -4\omega^2 \end{bmatrix}.$$

Since $\mathcal{O}$ is full rank, the realization can now be declared observable. The desired and actual estimator characteristic equations are,

$$\begin{aligned}
\alpha_{e,desired}(s) &= (s+2\omega)(s+3\omega)(s+3\omega-j3\omega)(s+3\omega+j3\omega) \\
&= s^4+11\omega s^3+54\omega^2 s^2+126\omega^3 s+108\omega^4 \\
\alpha_e(s) &= \det(s\mathbf{I}-\mathbf{F}+\mathbf{LH}) = s^4+l_3 s^3+(l_4+\omega^2)s^2+(-2\omega l_2+\omega^2 l_3)s+(-3\omega^2 l_4-6\omega^3 l_1).
\end{aligned}$$

Equating coefficients gives,

$$l_1=-44.5\omega,\ l_2=-57.5\omega^2,\ l_3=11\omega,\ l_4=53\omega^2.$$

45. The linearized equations of motion of the simple pendulum in Fig. 7.96 are

$$\ddot{\theta}+\omega^2\theta=u.$$

a) Write the equations of motion in state-space form.
b) Design an estimator (observer) that reconstructs the state of the pendulum given measurements of $\dot{\theta}$. Assume $\omega=5$ rad/sec, and pick the estimator roots to be at $s=-10\pm 10j$.
c) Write the transfer function of the estimator between the measured value of $\dot{\theta}$ and the estimated value of θ.
d) Design a controller (that is, determine the state feedback gain $\mathbf{K}$) so that the roots of the closed-loop characteristic equation are at $s=-4\pm 4j$.

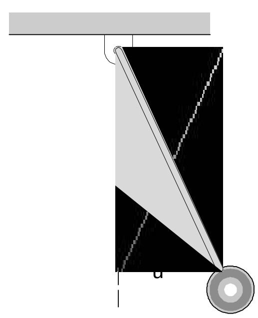


Figure 7.96: Pendulum diagram for Problem 7.45.

Solution:

(a) Defining $x_1=\theta$ and $x_2=\dot{\theta}$, and anticipating that the measured variable in part (b) is $\dot{\theta}$, we have,

$$\begin{aligned}
\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} &= \begin{bmatrix}0 & 1\\ -\omega^2 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u, \\
y &= \begin{bmatrix}0 & 1\end{bmatrix}\mathbf{x}.
\end{aligned}$$

(b) From,

$$\begin{aligned}
\det(s\mathbf{I}-\mathbf{F}+\mathbf{LH}) &= 0, \\
\det\left\{\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix}-\begin{bmatrix}0 & 1\\ -\omega^2 & 0\end{bmatrix}+\begin{bmatrix}l_1\\ l_2\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}\right\} &= s^2+l_2 s+\omega^2(-l_1+1)=0.
\end{aligned}$$

Using $\omega = 5$ and the specified roots for the estimator, we calculate $l_1 = -7$, and $l_2 = 20$. This result can be verified using MATLAB's place command.

(c) To find the transfer function from the measured value of $\dot{\theta}$, y, to the estimated value of θ, $\hat{\theta}$, we use the estimator equations,

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \mathbf{F}\hat{\mathbf{x}} + \mathbf{G}u + \mathbf{L}(y - \mathbf{H}\hat{x}) \\ &= (\mathbf{F} - \mathbf{LH})\hat{\mathbf{x}} + \mathbf{G}u + \mathbf{L}y.\end{aligned}$$

Since this is in state space form, we can now directly compute the transfer function from y to $\hat{\theta}$. It is simply,

$$\begin{aligned}\frac{\hat{\Theta}(s)}{Y(s)} &= \begin{bmatrix} 1 & 0 \end{bmatrix}(s\mathbf{I} - \mathbf{F} + \mathbf{LH})^{-1}\mathbf{L} \\ &= \frac{-7(s - 20/7)}{s^2 + 20s + 200}.\end{aligned}$$

(d) For controller gain $\mathbf{K} = [k_1\ k_2]$, we require,

$$\det(s\mathbf{I} - \mathbf{F} + \mathbf{GK}) = 0 \Longrightarrow s^2 + k_2 s + \omega^2 + k_1 = 0.$$

Comparing this with the specified roots equation:

$$(s + 4 + j4)(s + 4 - j4) = s^2 + 8s + 32 = 0,$$

we obtain $k_1 = 7$, and $k_2 = 8$. This result can be verified using MATLAB's place command.

46. An error analysis of an inertial navigator leads to the set of normalized state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u,$$

where

$$\begin{aligned} x_1 &= \text{east} - \text{velocity error}, \\ x_2 &= \text{platform tilt about the north axis}, \\ x_3 &= \text{north} - \text{gyro drift}, \\ u &= \text{gyro drift rate of change}. \end{aligned}$$

Design a reduced-order estimator with $y = x_1$ as the measurement, and place the observer error poles at -0.1 and -0.1. Be sure to provide all the relevant estimator equations.

Solution:

Partitioning the system matrices yields,

$$\begin{aligned}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} \mathbf{F}_{aa} & \mathbf{F}_{ab} \\ \mathbf{F}_{ba} & \mathbf{F}_{bb} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u, \\ &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u, \\ y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}.\end{aligned}$$

The characteristic equation of the reduced order estimator is then given by,

$$\det(s\mathbf{I} - \mathbf{F}_{bb} + \mathbf{L}\mathbf{F}_{ab}) = s^2 - l_1 s - l_2 = 0.$$

The desired characteristic equation for the reduced order estimator poles is

$$\alpha_e(s) = (s + 0.1)^2 = s^2 + 0.2s + 0.01.$$

Thus, $l_1 = -0.2$, and $l_2 = -0.01$. This result can be verified using MATLAB's acker command.

Problems and Solutions for Section 7.8: Compensator Design: Combined Control Law and Estimator

47. A certain process has the transfer function $G(s) = 4/(s^2 - 4)$.
a) Find **F, G**, and **H** for this system in observer canonical form.
b) If $u = -\mathbf{K}\mathbf{x}$, compute **K** so that the closed-loop control poles are located at $s = -2 \pm 2j$.
c) Compute **L** so that the estimator-error poles are located at $s = -10 \pm 10j$.
d) Give the transfer function of the resulting controller (for example, using Eq. (7.177)).
e) What are the gain and phase margins of the controller and the given open-loop system?

Solution:

(a) From the transfer function, we can read off the elements that will give observer canonical form,

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{F}_o\mathbf{x} + \mathbf{G}_o u, \\
y &= \mathbf{H}_o\mathbf{x}, \\
\mathbf{F}_o &= \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}, \ \mathbf{G}_o = \begin{bmatrix} 0 \\ 4 \end{bmatrix}, \ \mathbf{H}_o = \begin{bmatrix} 1 & 0 \end{bmatrix}.
\end{aligned}$$

(b) With $u = -[k_1\ k_2][x_1\ x_2]^T$, we want to achieve the following closed-loop characteristic equation:

$$\alpha_c(s) = (s + 2 + 2j)(s + 2 - 2j) = s^2 + 4s + 8 = 0.$$

From $\det(s\mathbf{I} - \mathbf{F} + \mathbf{G}\mathbf{K}) = 0$, we obtain,

$$s^2 + 4k_2 s + 4k_1 - 4 = 0.$$

Comparing the coefficients yields $k_1 = 3$, and $k_2 = 1$. This result can be verified using MATLAB's place command.

(c) The estimator roots are determined by the equation $\alpha_e(s) = 0$. We want to find l_1 and l_2 such that,

$$\alpha_e(s) = (s + 10 + 10j)(s + 10 - 10j) = s^2 + 20s + 200.$$

$$\begin{aligned}
\alpha_e(s) &= \det(s\mathbf{I} - \mathbf{F} + \mathbf{L}\mathbf{H}) \\
&= \det\left(\begin{bmatrix} s & -1 \\ -4 & s \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}\right) \\
&= \det\begin{bmatrix} s + l_1 & -1 \\ -4 + l_2 & s \end{bmatrix} = s^2 + l_1 s + l_2 - 4.
\end{aligned}$$

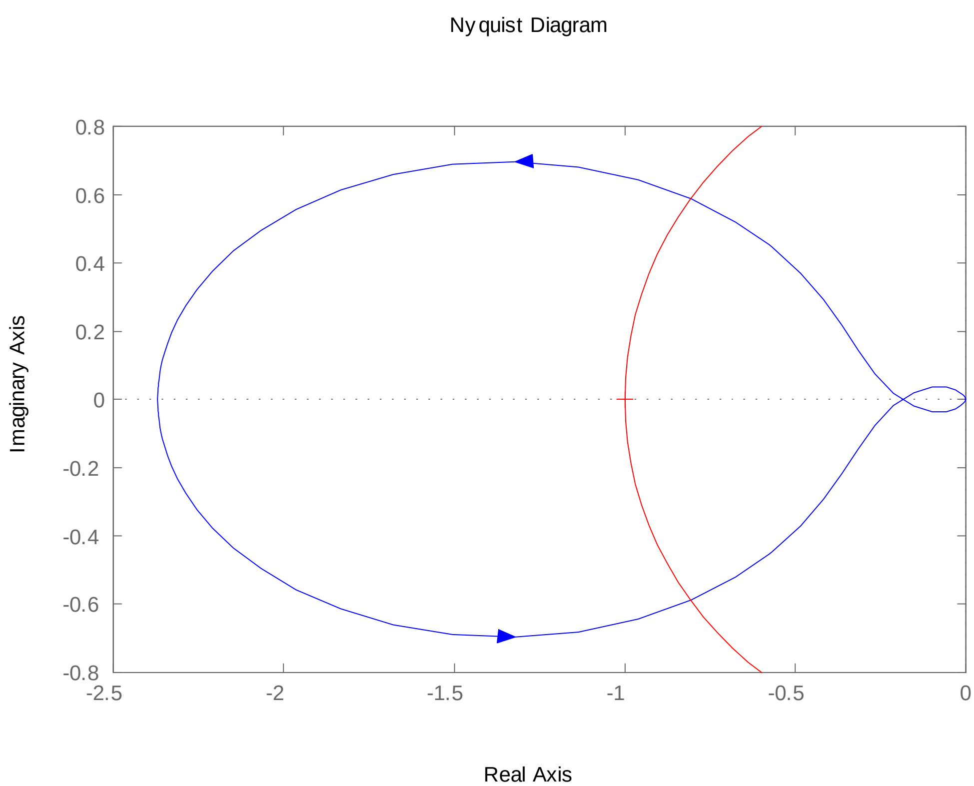
Comparing the coefficients yields $l_1 = 20$, $l_2 = 204$. This result can be verified using MATLAB's place command.

(d) The transfer function of the resulting compensator is,

$$
\begin{aligned}
D(s) &= \frac{U(s)}{Y(s)} = -\mathbf{K}(s\mathbf{I} - \mathbf{F} + \mathbf{GK} + \mathbf{LH})^{-1}\mathbf{L}, \\
&= -\begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} s+20 & -1 \\ 212 & s+4 \end{bmatrix}^{-1} \begin{bmatrix} 20 \\ 204 \end{bmatrix} = \frac{-264s - 692}{s^2 + 24s + 292}.
\end{aligned}
$$

This result can be verified using MATLAB's ss2tf command.

(e) The next figure shows the Nyquist plot generated by MATLAB (using the nyquist command), note that there is both a positive and negative gain margin. The Nyquist plot has a positive gain margin of 0.4220 (i.e., the gain can be increased by $1/0.422 = 2.37$) and a negative margin of 5.46 (i.e., the gain can be decreased by $1/5.46 = 0.183$) before the number of encirclements of the -1 point changes.



Nyquist plot for Problem 7.47.

48. The linearized longitudinal motion of a helicopter near hover (Fig. 7.97) can be modeled by the

normalized third-order system,

$$\begin{bmatrix} \dot{q} \\ \dot{\theta} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} -0.4 & 0 & -0.01 \\ 1 & 0 & 0 \\ -1.4 & 9.8 & -0.02 \end{bmatrix} \begin{bmatrix} q \\ \theta \\ u \end{bmatrix} + \begin{bmatrix} 6.3 \\ 0 \\ 9.8 \end{bmatrix} \delta,$$

where,

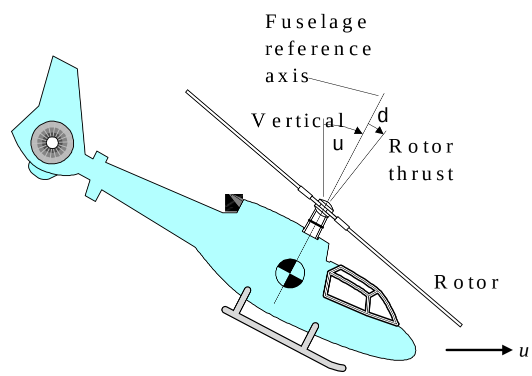


Figure 7.97: Helicopter for Problem 7.48.

$q =$ pitch rate,

$\theta =$ pitch angle of fuselage,

$u =$ horizontal velocity (standard aircraft notation),

$\delta =$ rotor tilt angle (control variable).

Suppose our sensor measures the horizontal velocity u as the output; that is, $y = u$.
a) Find the open-loop pole locations.
b) Is the system controllable?
c) Find the feedback gain that places the poles of the system at $s = -1 \pm 1j$ and $s = -2$.
d) Design a full-order estimator for the system, and place the estimator poles at -8 and $-4 \pm 4\sqrt{3}j$.
e) Design a reduced-order estimator with both poles at -4. What are the advantages and disadvantages of the reduced-order estimator compared with the full-order case?
f) Compute the compensator transfer function using the control gain and the full-order estimator designed in part (d), and plot its frequency response using MATLAB. Draw a Bode plot for the closed-loop design, and indicate the corresponding gain and phase margins.
g) Repeat part (f) with the reduced-order estimator.
h) Draw the symmetrical root locus (SRL) and select roots for a control law that will give a control bandwidth matching the design of part (c), and select roots for a full-order estimator that will result in an estimator error bandwidth comparable to the design of part (d). Draw the corresponding Bode plot and compare the pole placement and SRL designs with respect to bandwidth, stability margins, step response, and control effort for a unit-step rotor-angle input. Use MATLAB for the computations.

Solution:

Again, the equations of motion for the helicopter are,

$$\begin{bmatrix} \dot{q} \\ \dot{\theta} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} -0.4 & 0 & -0.01 \\ 1 & 0 & 0 \\ -1.4 & 9.8 & -0.02 \end{bmatrix} \begin{bmatrix} q \\ \theta \\ u \end{bmatrix} + \begin{bmatrix} 6.3 \\ 0 \\ 9.8 \end{bmatrix} \delta.$$

(a) The open-loop poles are the eigenvalues of $\mathbf{F}$. Solving $\det(s\mathbf{I} - \mathbf{F}) = 0$ gives the open-loop poles as $s = -0.6565$ and $s = 0.1183 \pm j0.3678$. In MATLAB, use eig(F). We also note that the zeros of the plant are in the RHP at $0.25 \pm j2.5$ and can be computed using the MATLAB tzero command.

(b) To determine controllability, we want to look at the rank of the controllability matrix. For the helicopter,

$$\text{rank}\{\mathcal{C}\} = \text{rank} \begin{bmatrix} \mathbf{G} & \mathbf{FG} & \mathbf{F}^2\mathbf{G} \end{bmatrix} = 3.$$

Thus, the system is controllable. Alternatively, you can find the singular values of the matrix $\mathcal{C}$ using the MATLAB svd command. This will give an indication of how large the actuator signals will need to be.

(c) When the order of the system gets larger than two, it is often convenient to let the computer do the necessary calculations. Using MATLAB's place command and the specified pole locations, we find the control gains,

$$\mathbf{K} = \begin{bmatrix} 0.4706 & 1.0 & 0.0627 \end{bmatrix}.$$

If we have to do this computation by hand, the approach would be the following. Form the desired characteristic equation and compare it with the equation,

$$\det(s\mathbf{I} - \mathbf{F} + \mathbf{GK}) = 0.$$

to obtain the values for $\mathbf{K}$.

(d) Using the duality principle, we find the estimator gains using MATLAB's place command as well. We find,

$$\mathbf{L} = \begin{bmatrix} 44.7097 & 18.8130 & 15.5800 \end{bmatrix}^T.$$

(e) The notation of this solution follows Equation 7.139 in the text. Reordering the system matrix, we have,

$$\begin{aligned} \begin{bmatrix} \dot{u} \\ \dot{q} \\ \dot{\theta} \end{bmatrix} &= \begin{bmatrix} -0.02 & -1.4 & 9.8 \\ -0.01 & -0.4 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ q \\ \theta \end{bmatrix} + \begin{bmatrix} 9.8 \\ 0 \\ 6.3 \end{bmatrix} \delta \\ &= \begin{bmatrix} \mathbf{F}_{aa} & \mathbf{F}_{ab} \\ \mathbf{F}_{ba} & \mathbf{F}_{bb} \end{bmatrix} \begin{bmatrix} x_a \\ x_b \end{bmatrix} + \begin{bmatrix} G_a \\ G_b \end{bmatrix} \delta. \end{aligned}$$

To design the reduced order estimator, we need to solve the characteristic equation,

$$\det(s\mathbf{I} - \mathbf{F}_{bb} + \mathbf{L}\mathbf{F}_{ab}) = 0.$$

So that the estimator gains, $\mathbf{L}$, place the poles at the desired locations. Using MATLAB's place command, we find,

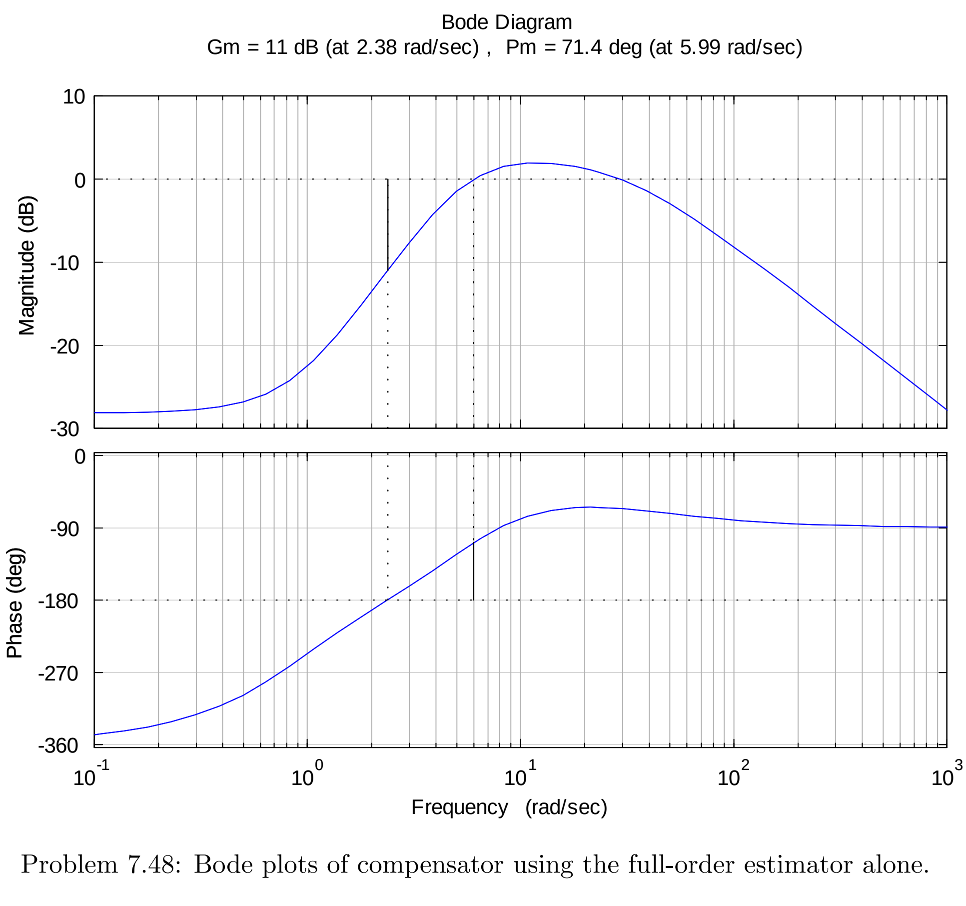
$$\mathbf{L} = \begin{bmatrix} 1.2510 & 0.9542 \end{bmatrix}^T.$$

The advantages of the reduced order estimator are that the resulting estimator is simpler (in terms of the number of flops, Floating point operations, count necessary to implement the estimator on a real system). Another issue is that you are using the measurement of the state directly. This would be advantageous if the measured signal was relatively noise free. However, if the signal was noisy, then it would be better to use the full order estimator because it provides filtering of noisy measurement.

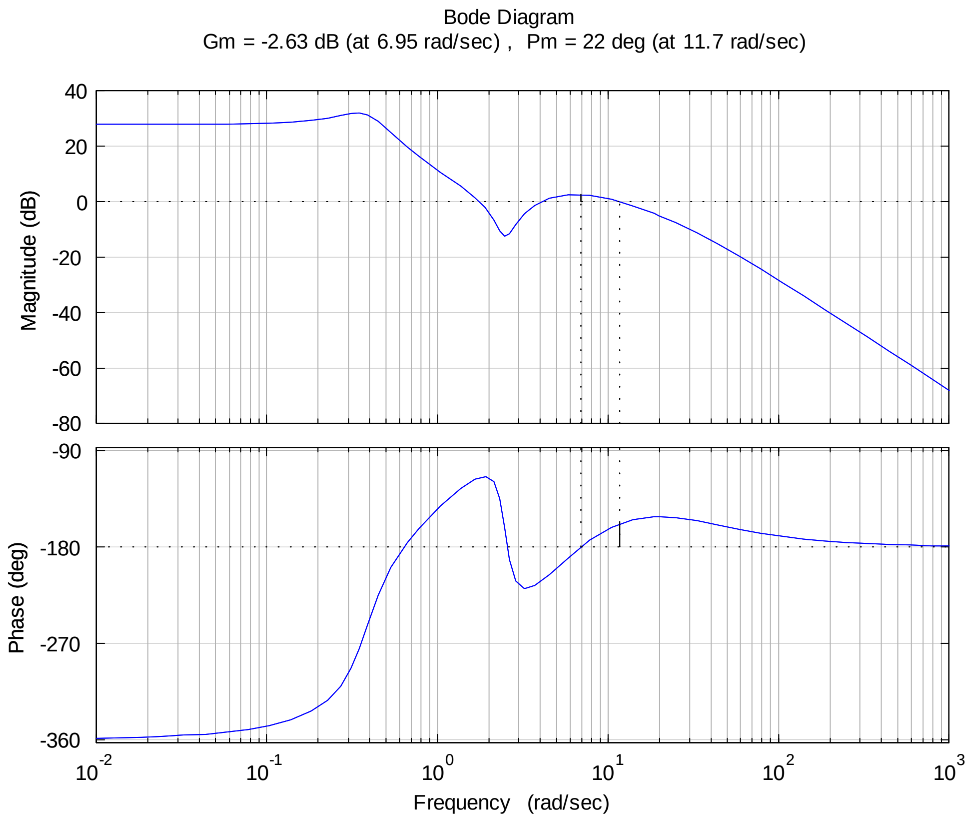
(f) The compensator for the controller in part (c) and estimator in part (d) is,

$$\begin{aligned} D_c(s) &= -\mathbf{K}(s\mathbf{I}-\mathbf{F}+\mathbf{GK}+\mathbf{LH})^{-1}\mathbf{L} = \frac{40.8s^2+61.0s+31.9}{s^3+19.58s^2-210.4s+814.7} \\ &= \frac{40.8(s+0.75\pm j0.47)}{(s+28.1)(s-4.26\pm j3.29)}. \end{aligned}$$

The figures on the next page show the Bode plot of the compensator transfer function using the full-order estimator, and the Bode plot of the plant and compensator. The Phase and Gain margins for the system are -2.6 db and 22.0 degrees respectively.



Problem 7.48: Bode plots of compensator using the full-order estimator alone.



Problem 7.48: Bode plot of plant and compensator combined.

(g) Compensator for the controller in part (c) and estimator in part (e) (i.e., the reduced-order estimator) is,

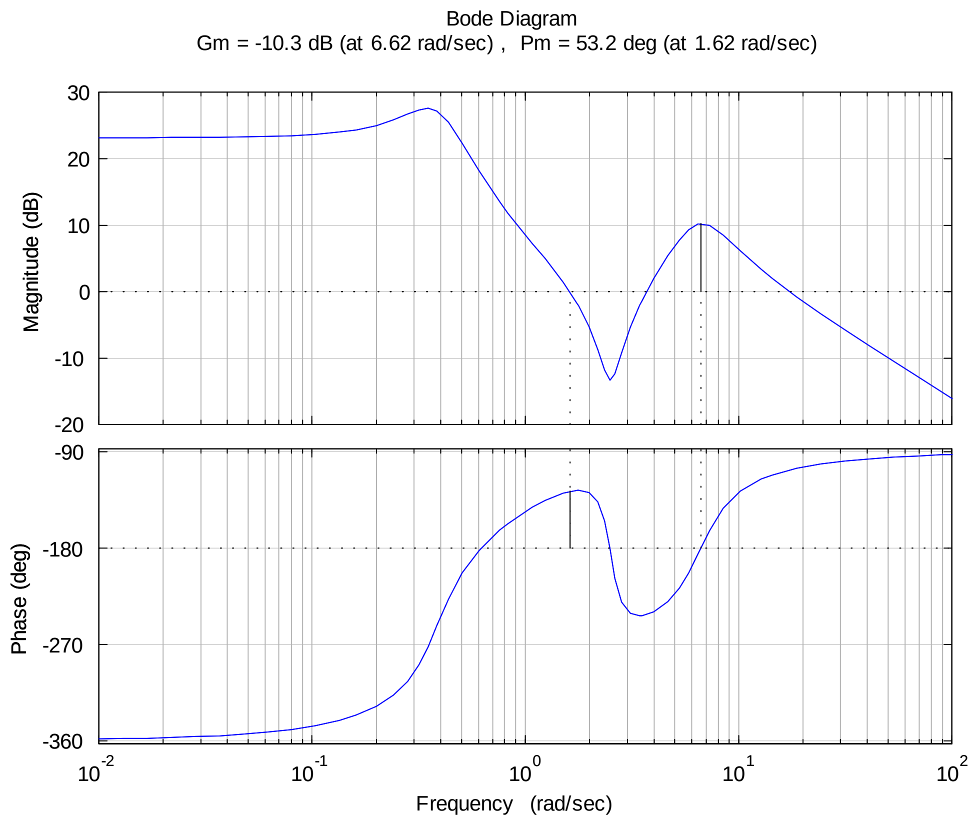
$$D_{cr}(s) = \mathbf{C}_r(s\mathbf{I} - \mathbf{A}_r)^{-1}\mathbf{B}_r + D_r.$$

$$\begin{aligned}
K_a &= 0.0627,\ \mathbf{K}_b = [0.4706, 1], \\
\mathbf{A}_r &= \mathbf{F}_{bb} - \mathbf{L}\mathbf{F}_{ab} - (\mathbf{G}_b - \mathbf{L}\mathbf{G}_a)\mathbf{K}_b = \begin{bmatrix} 4.16 & -6.30 \\ 6.74 & 0.00 \end{bmatrix}, \\
\mathbf{B}_r &= \mathbf{A}_r\mathbf{L} + \mathbf{F}_{ba} - \mathbf{L}\mathbf{F}_{aa} - (\mathbf{G}_b - \mathbf{L}\mathbf{G}_a)K_a = \begin{bmatrix} -0.423 \\ 9.034 \end{bmatrix}, \\
\mathbf{C}_r &= -\mathbf{K}_b = \begin{bmatrix} -0.4706 & -1 \end{bmatrix}, \\
D_r &= -K_a - \mathbf{K}_b\mathbf{L} = -1.61.
\end{aligned}$$

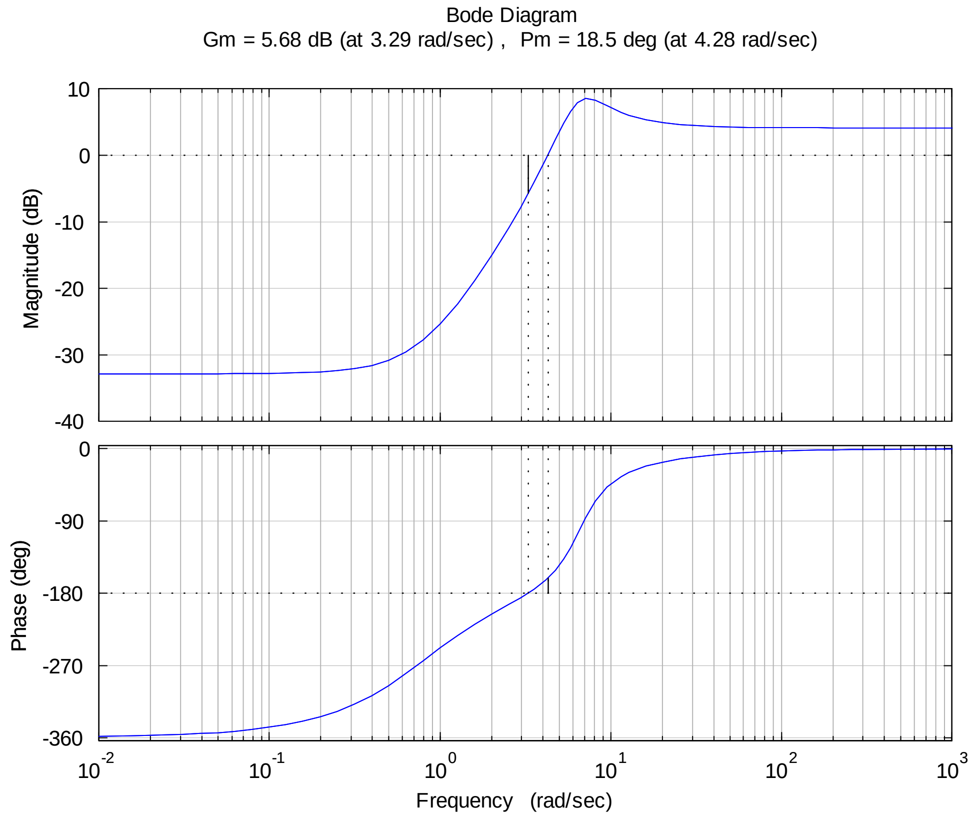
(h) Thus, the transfer function for the compensator using the reduced-order estimator is,

$$D_{cr}(s) = \mathbf{C}_r(s\mathbf{I} - \mathbf{A}_r)^{-1}\mathbf{B}_r + D_r = \frac{-1.61s^2 - 2.16s + 0.97}{s^2 - 4.16s + 42.44}.$$

The figure below shows the open-loop transfer function for the compensator designed using the reduced-order estimator. The Bode plot for the plant and the compensator is also shown on the next page.



Problem 7.48: Bode plot of the compensator transfer function using the reduced-order estimator.



Problem 7.48: Bode plot of the compensated system.

49. Suppose a DC drive motor with motor current u is connected to the wheels of a cart in order to control the movement of an inverted pendulum mounted on the cart. The linearized and normalized equations of motion corresponding to this system can be put in the form

$$\ddot{\theta} = \theta + v + u,$$
$$\dot{v} = \theta - v - u,$$

where,

$$\theta = \text{angle of the pendulum},$$
$$v = \text{velocity of the cart}.$$

a) We wish to control θ by feedback to u of the form,

$$u = -K_1\theta - K_2\dot{\theta} - K_3 v.$$

Find the feedback gains so that the resulting closed-loop poles are located at -1, $-1 \pm j\sqrt{3}$.
b) Assume that θ and v are measured. Construct an estimator for θ and $\dot{\theta}$ of the form,

$$\dot{\hat{\mathbf{x}}} = \mathbf{F}\hat{\mathbf{x}} + \mathbf{L}(y - \hat{y}),$$

where $\mathbf{x} = [\ \theta \quad \dot{\theta}\ ]^T$ and $y = \theta$. Treat both v and u as known. Select $\mathbf{L}$ so that the estimator poles are at -2 and -2.
c) Give the transfer function of the controller, and draw the Bode plot of the closed-loop system, indicating the corresponding gain and phase margins.
d) Using MATLAB, plot the response of the system to an initial condition on θ, and give a physical explanation for the initial motion of the cart.

Solution:

(a) Defining the state $\mathbf{x} = [\theta\ v]^T$, the system is written as,

$$\begin{aligned} \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{v} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} u, \\ \dot{\mathbf{x}} &= \mathbf{F}\mathbf{x} + \mathbf{G}u. \end{aligned}$$

Using $\det(s\mathbf{I} - \mathbf{F} + \mathbf{G}\mathbf{K}) = 0$ with $\mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$, we find the characteristic equation,

$$s^3 + s^2(1 - k_3 + k_2) + s(k_1 - 1) + 2(k_3 - 1) = 0.$$

The desired characteristic equation is,

$$(s+1)((s+1)^2 + 3) = s^3 + 3s^2 + 6s + 4 = 0.$$

Comparing coefficients, $\mathbf{K} = \begin{bmatrix} 7 & 5 & 3 \end{bmatrix}$. This result can be verified using the MATLAB `place` command.

(b) The estimator equations (both explicitly and symbolically) for estimating $\hat{\mathbf{x}} = \begin{bmatrix} \theta & \dot{\theta} \end{bmatrix}^T$ are,

$$\begin{aligned} \begin{bmatrix} \hat{\dot{\theta}} \\ \hat{\ddot{\theta}} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\theta} \\ \hat{\dot{\theta}} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \mathbf{L}(y - \hat{y}), \\ &= \mathbf{F}_e \widehat{\mathbf{x}} + \mathbf{G}_v v + \mathbf{G}_u u + \mathbf{L}(y - \hat{y}). \end{aligned}$$

where u and v are assumed to be known. The output equations for the plant and the estimator are,

$$\begin{aligned} y &= \mathbf{H}\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}, \\ \hat{y} &= \mathbf{H}_e \hat{\mathbf{x}} = \begin{bmatrix} 1 & 0 \end{bmatrix} \hat{\mathbf{x}}. \end{aligned}$$

With $\mathbf{L} = \begin{bmatrix} l_1 & l_2 \end{bmatrix}^T$, the characteristic equation becomes,

$$\det(s\mathbf{I} - \mathbf{F}_e + \mathbf{L}\mathbf{H}_e) = s^2 + sl_1 + l_2 - 1 = 0.$$

Equating with the desired characteristic equation,

$$(s+2)(s+2) = s^2 + 4s + 4,$$

we have $\mathbf{L} = \begin{bmatrix} 4 & 5 \end{bmatrix}^T$. This result can be verified using the MATLAB `place` command.

(c) Construct the feedback u in terms of both the measured signal v and the estimated state $\hat{\mathbf{x}}$. Using the feedback gains from (a), we have,

$$\begin{aligned} u &= -K_1\hat{\theta} - K_2\dot{\hat{\theta}} - K_3 v, \\ &= -\mathbf{K}_e\hat{\mathbf{x}} - K_3 v. \end{aligned}$$

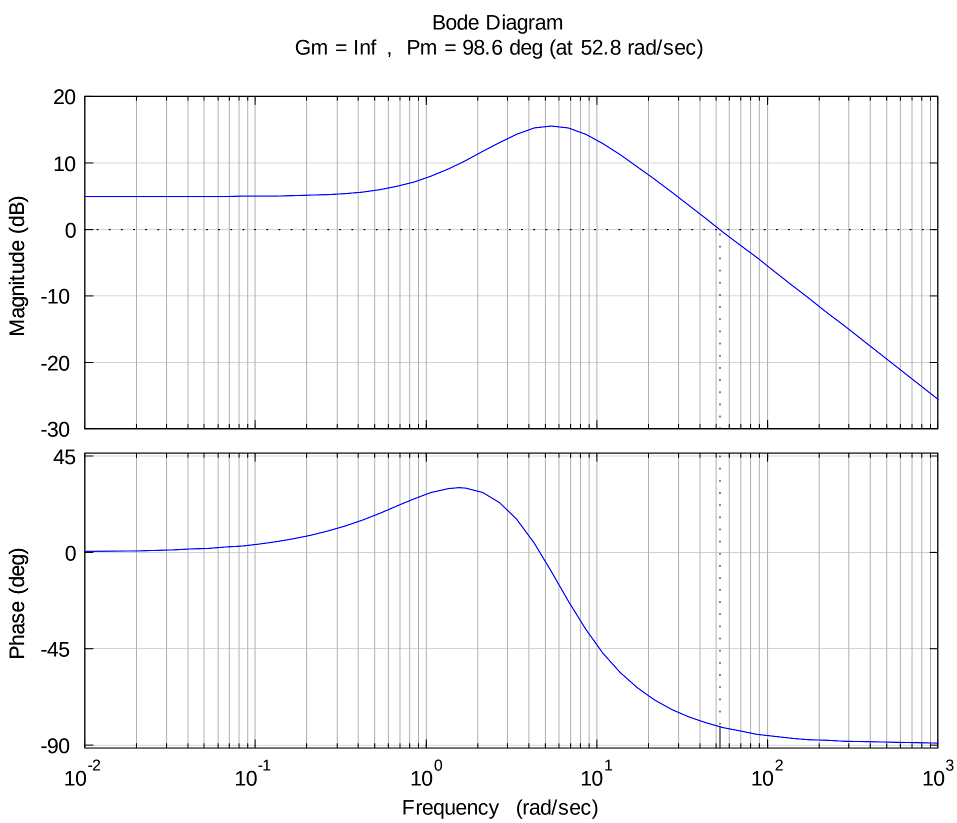
Plugging this expression for u into the estimator equation we have,

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= (\mathbf{F}_e - \mathbf{G}_u\mathbf{K}_e - \mathbf{L}\mathbf{H}_e)\hat{\mathbf{x}} + (\mathbf{G}_v - \mathbf{G}_u K_3)v + \mathbf{L}y, \\ u &= -\mathbf{K}_e\hat{\mathbf{x}} - K_3 v. \end{aligned}$$

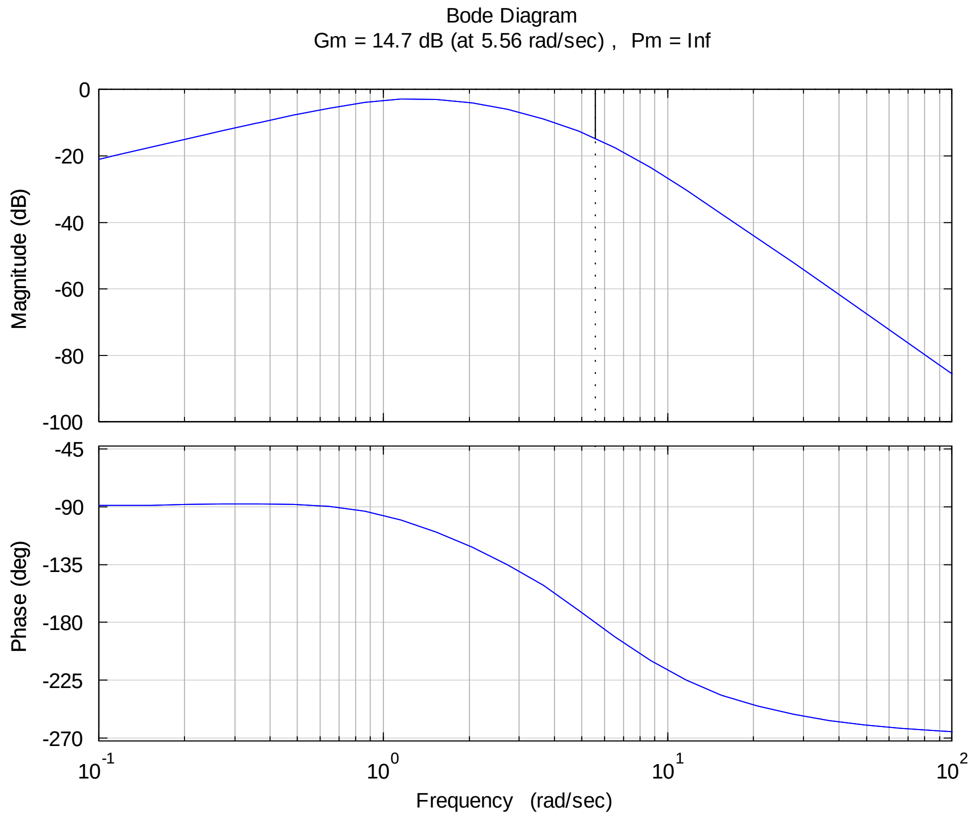
The transfer function from y to u can now be read directly from these two equations by setting all of the auxiliary inputs to zero, i.e., $v = 0$. Thus,

$$D_c(s) = -\mathbf{K}_e(s\mathbf{I} - \mathbf{F}_e + \mathbf{G}_u\mathbf{K}_e + \mathbf{L}\mathbf{H}_e)^{-1}\mathbf{L} = \frac{-(53s+55)}{s^2+9s+31}.$$

The Bode plots are shown next.



Open-loop Bode plot of compensator transfer function for Problem 7.49.

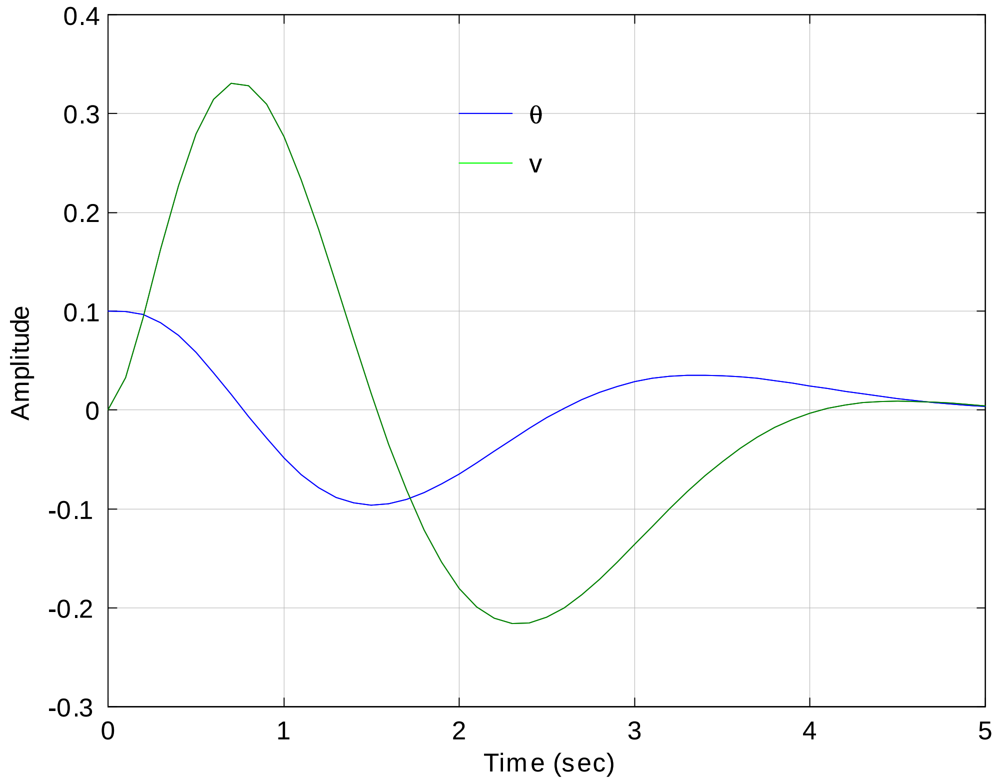


Bode plot of the compensator and plant together for Problem 7.49.

(d) One approach to simulating the system is to augment the plant and estimator equations into one matrix. Recognizing that $v = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \mathbf{H}_v \mathbf{x}$, we can eliminate u and v.

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{F}\mathbf{x} + \mathbf{G}u = (\mathbf{F} - \mathbf{G}K_3\mathbf{H}_v)\mathbf{x} - \mathbf{G}\mathbf{K}_e\hat{\mathbf{x}} \\
\dot{\hat{\mathbf{x}}} &= \mathbf{F}_e\hat{\mathbf{x}} + \mathbf{G}_v v + \mathbf{G}_u u + \mathbf{L}(y - \hat{y}) \\
&= (\mathbf{G}_v\mathbf{H}_v - \mathbf{G}_u K_3 \mathbf{H}_v + \mathbf{L}\mathbf{H})\mathbf{x} + (\mathbf{F}_e - \mathbf{L}\mathbf{H}_e - \mathbf{G}_u\mathbf{K}_e)\hat{\mathbf{x}}.
\end{aligned}
$$

This is now easily implemented using the MATLAB command `lsim`. The figure on the next page shows the closed-loop system response due to an initial angle of $\theta = 0.1$ rad with respect to a vertical line. The initial motion of the cart is in the direction that the pendulum is leaning (due to the initial condition). Physically, if the cart moved away from the direction that the pendulum was leaning, then it would cause the angle to increase eventually toppling the pendulum.



Angle of pendulum and the velocity of the cart given an initial angle for Problem 7.49.

50. Consider the control of

$$G(s) = \frac{Y(s)}{U(s)} = \frac{10}{s(s+1)}.$$

a) Let $y = x_1$ and $\dot{x}_1 = x_2$, and write state equations for the system.
b) Find K_1 and K_2 so that $u = -K_1 x_1 - K_2 x_2$ yields closed-loop poles with a natural frequency $\omega_n = 3$ and a damping ratio $\zeta = 0.5$.
c) Design a state estimator for the system that yields estimator error poles with $\omega_{n1} = 15$ and $\zeta_1 = 0.5$.
d) What is the transfer function of the controller obtained by combining parts (a) through (c)?
e) Sketch the root locus of the resulting closed-loop system as plant gain (nominally 10) is varied.

Solution:

The state equations are,

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}. \end{aligned}$$

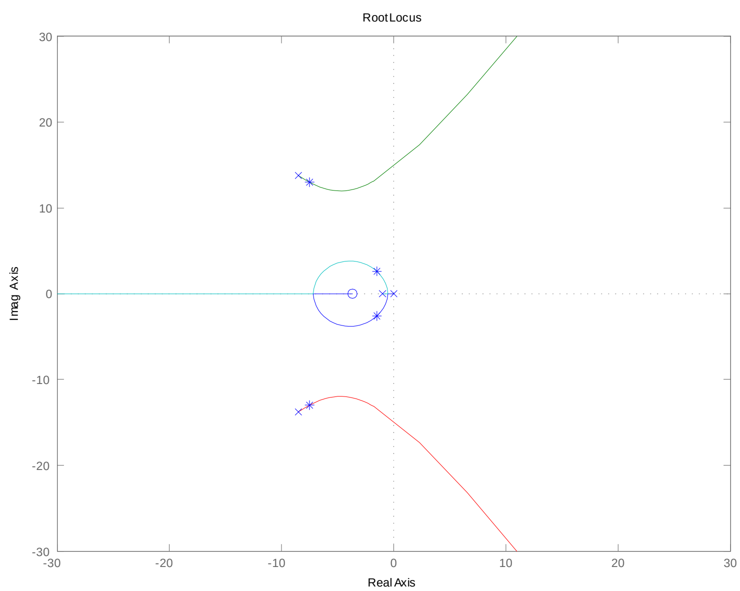
(b) $K =$ place(F, G, roots([1 2 * zeta * wn wn^2]))= [0.9 0.2].

(c) $L = \mathsf{place}(\mathsf{F}', \mathsf{H}', \mathsf{roots}([1\ 2*\mathsf{zeta}*\mathsf{wn}\ \mathsf{wn}\hat{\ }2]))' = [14\ 211]^T$.

(d) The transfer function for the controller is,

$$\begin{aligned} D_c(s) &= -\mathbf{K}(s\mathbf{I} - \mathbf{F} + \mathbf{G}\mathbf{K} + \mathbf{L}\mathbf{H})^{-1}\mathbf{L} \\ &= \frac{-(54.8s + 202.5)}{s^2 + 17s + 262}. \end{aligned}$$

(e) The figure below shows the root locus around a nominal gain of 10, which is indicated by asterisk.



Problem 7.50: Root locus of the closed-loop system as plant gain is varied.

51. Unstable equations of motion of the form,

$$\ddot{x} = x + u,$$

arise in situations where the motion of an upside-down pendulum (such as a rocket) must be controlled.
a) Let $u = -Kx$ (position feedback alone), and sketch the root locus with respect to the scalar gain K.
b) Consider a lead compensator of the form,

$$U(s) = K\frac{s+a}{s+10}X(s).$$

Select a and K so that the system will display a rise time of about 2 sec and no more than 25% overshoot. Sketch the root locus with respect to K.
c) Sketch the Bode plot (both magnitude and phase) of the uncompensated plant.
d) Sketch the Bode plot of the compensated design, and estimate the phase margin.
e) Design state feedback so that the closed-loop poles are at the same locations as those of the

design in part (b).
f) Design an estimator for x and $\dot{x}$ using the measurement of $x = y$, and select the observer gain $\mathbf{L}$ so that the equation for $\tilde{x}$ has characteristic roots with a damping ratio $\zeta = 0.5$ and a natural frequency $\omega_n = 8$ rad/sec.
g) Draw a block diagram of your combined estimator and control law, and indicate where $\hat{x}$ and $\dot{x}$ appear. Draw a Bode plot for the closed-loop system, and compare the resulting bandwidth and stability margins with those obtained using the design of part (b).

Solution:

(a) The root locus using position feedback alone is shown below. Notice that no matter how large the gain is made, the closed-loop roots are never strictly in the LHP.

(b) First of all, we need to translate the specifications into values for ω_n and ζ. Although the closed system with a lead compensator is third-order, we assume the rules of thumb for a second-order system are valid and then validate our design after settling on values for a and K.

$$M_p < 25\% \Longrightarrow \zeta > 0.4, \omega_n = \frac{1.8}{t_r} = \frac{1.8}{2} = 0.9.$$

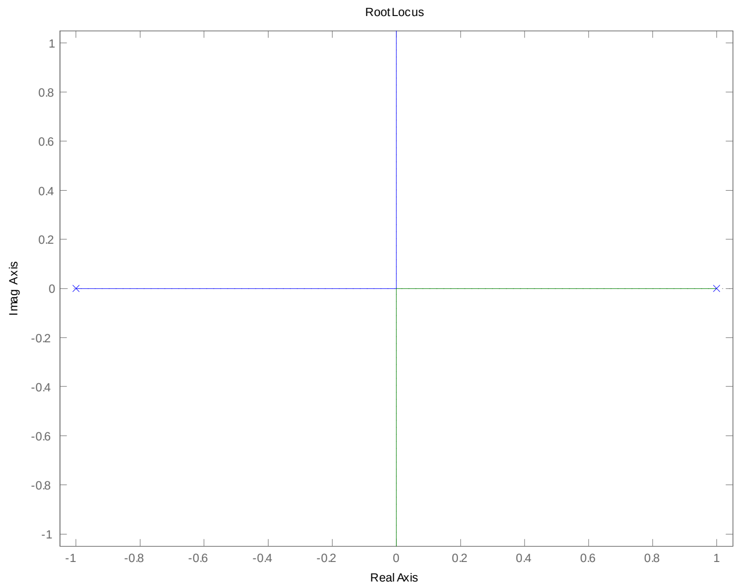
Try $\zeta = 0.4$ and $\omega_n = 1$ for the design. Because the form of the compensator is specified, we can calculate the closed-loop transfer function to be,

$$\frac{Y(s)}{R(s)} = T(s) = \frac{s+10}{s^3 + 10s^2 + (K-1)s + (Ka - 10)}.$$

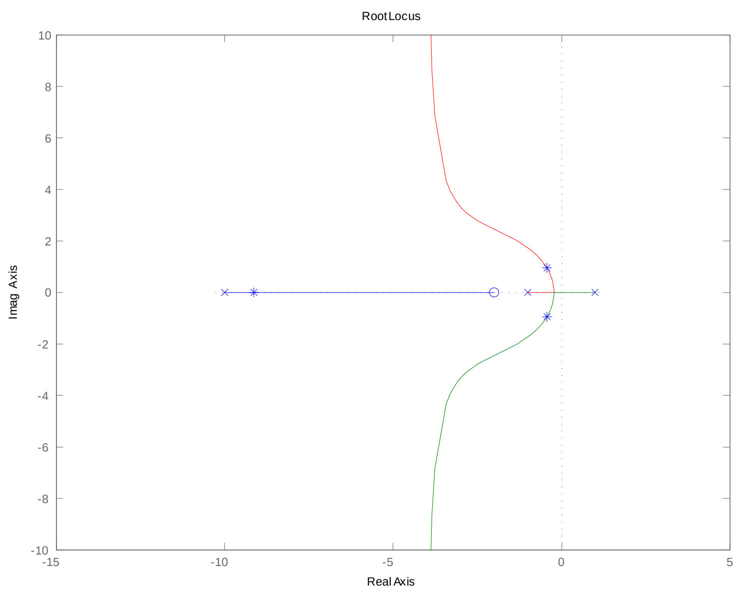
Note that we have subtly introduced r as a reference input to the plant. The desired closed loop poles should be placed at (taking $\alpha = 10$),

$$(s+\alpha)(s^2 + 2\zeta\omega_n s + \omega_n^2) = (s+10)(s^2 + 0.8s + 1) = s^3 + 10.8s^2 + 9s + 10.$$

Although the coefficient for the s^2 term doesn't match exactly, we just want to get a ballpark estimate for K and a. So comparing the other coefficients, we find $K = 10$ and $a = 2$. Using these values, the root locus for using the lead compensator is shown. To verify that our design is acceptable, we also check the step response of the system. This is shown on the last figure in this section.



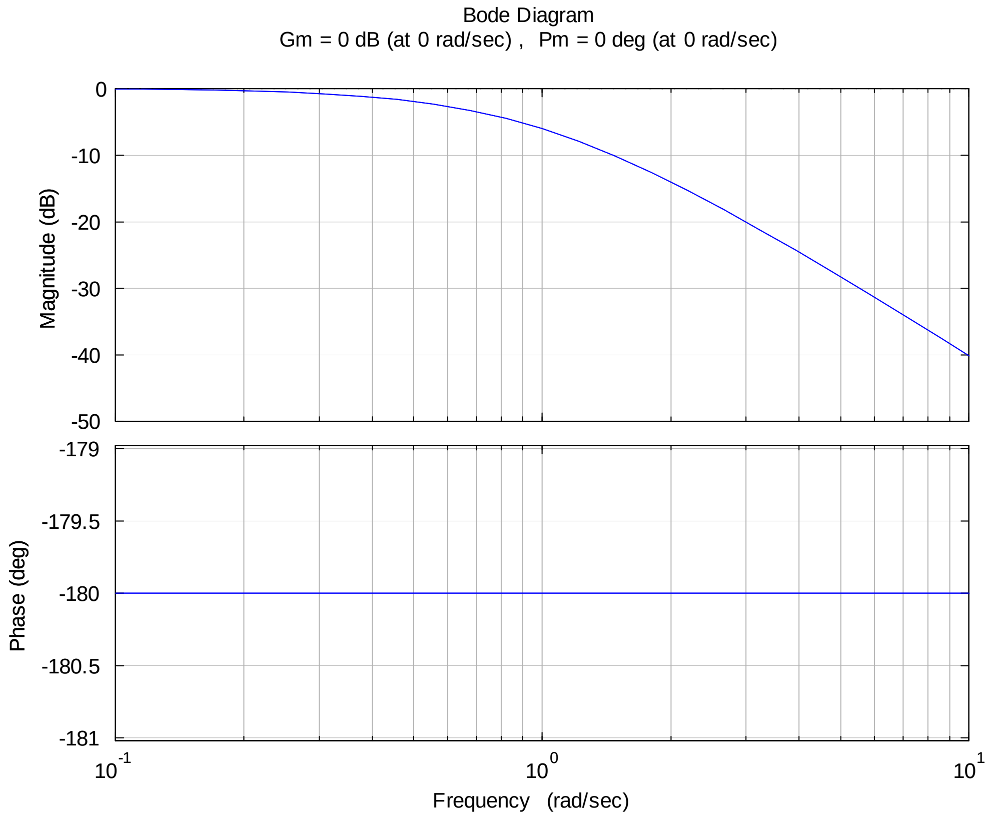
Problem 7.51: Root locus with position feedback alone.



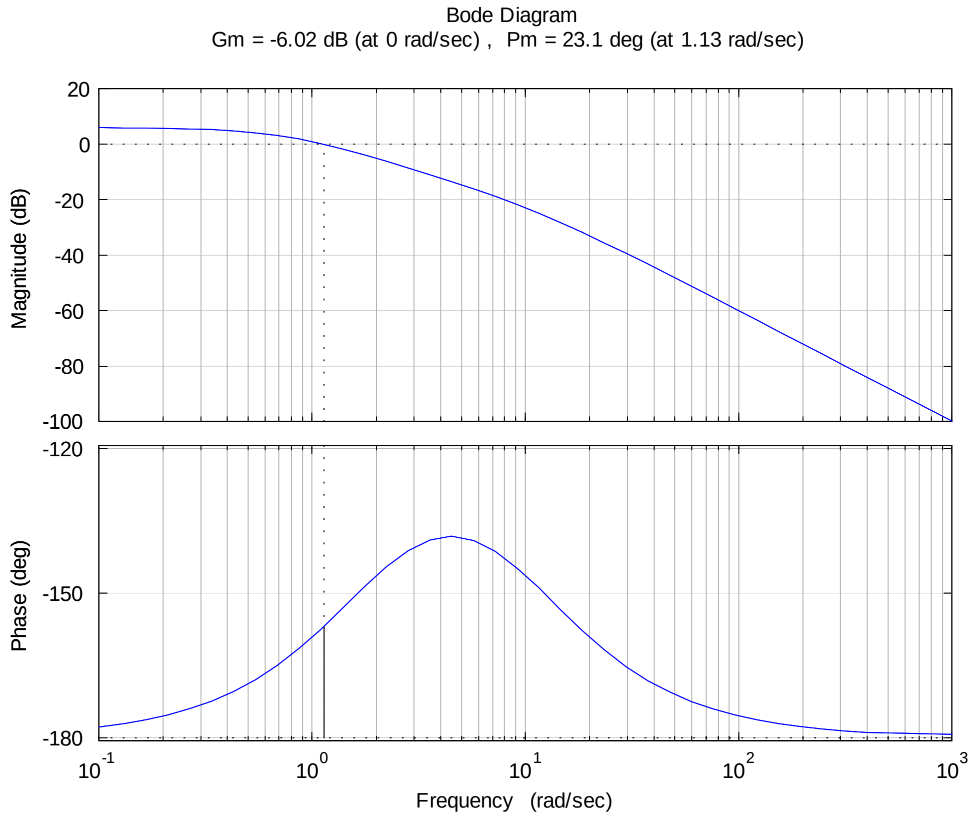
Root locus for Problem 7.51.

(c) The Bode plot of $\frac{1}{s^2-1}$ is shown below.

(d) The Bode plot of the compensated design is also shown on the next page. The phase margin is approximately 23°. The gain margin is 0.5.



Problem 7.51: Bode plots for the open-loop system.



Problem 7.51: Compensator and plant combined.

(e) Although the design in part (b) has three closed-loop poles (due to the lead compensator), full state feedback on a second-order system does not introduce an extra pole. Recognizing this, we keep the poles closest to the plant's open loop poles, $-0.433 \pm 0.953j$. The feedback gains $\mathbf{K}$ can now be determined using MATLAB's place command,

$$\mathsf{K} = \mathsf{place}(\mathsf{F}, \mathsf{G}, [-0.433 + 0.953 * \mathsf{j}; -0.433 - 0.953 * \mathsf{j}]) = [2.09\ 0.87].$$

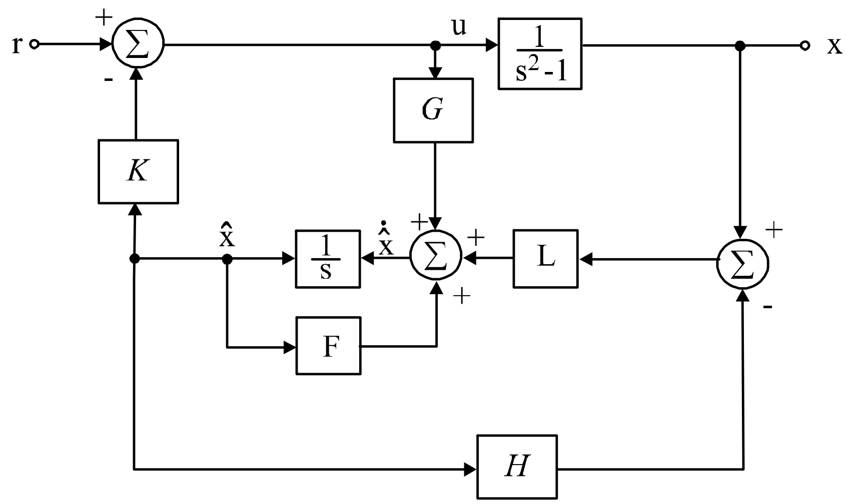
(f) The estimator gains are just as easy to produce. With $\zeta = 0.5$ and $\omega_n = 8$, we have,

$$\begin{aligned} &[\mathsf{F}, \mathsf{G}, \mathsf{H}, \mathsf{J}] = \mathsf{tf2ss}([0\ 0\ 1], [1\ 0\ -1]) \\ &\mathsf{pe} = [1\ 2 * \mathsf{zeta} * \mathsf{omegan}\ \mathsf{omegan\hat{\ }2}] \\ &\mathsf{L} = \mathsf{place}(\mathsf{F}\prime, \mathsf{H}\prime, \mathsf{pe})\prime = [8\ 65]^T. \end{aligned}$$

(g) The estimator equations are,

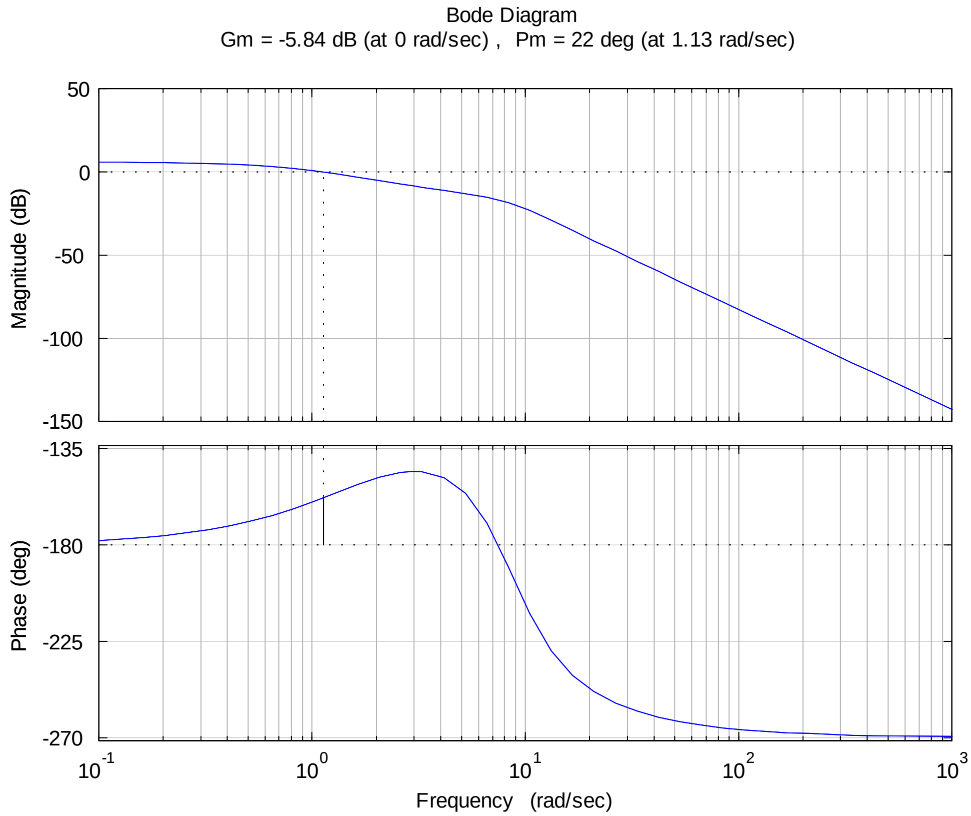
$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \mathbf{F}\hat{\mathbf{x}} + \mathbf{G}u + \mathbf{L}(y - \mathbf{H}\hat{\mathbf{x}}), \\ u &= -\mathbf{K}\hat{\mathbf{x}}. \end{aligned}$$

and are shown in block diagram form on top of the next page.



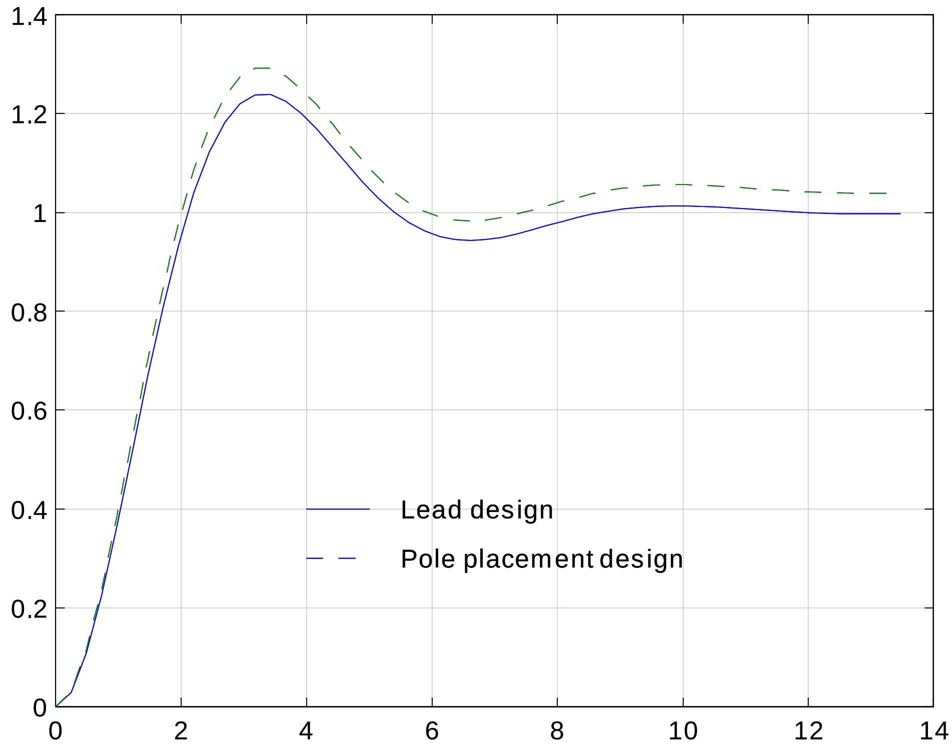
Block diagram of the combined estimator and control law in Problem 7.51.

The Bode plot of the controller and plant designed using pole placement techniques is shown below. The phase margin is approximately 22° and the gain margin now has a limitation both for increasing and decreasing the gain. The gain can be increased by a factor of $1/0.14 = 7.14 = 17$ db and decreased by a factor of $1/1.96 = 0.51 = -5.8$ db. So the lead compensator has roughly equivalent stability margins.



Problem 7.51: Bode plot of plant and compensator design with pole placement.

The step responses for both designs are shown on the next page using the MATLAB `step` command. They differ slightly because the DC gain of the compensator designed using pole placement hasn't been adjusted for unity gain. Also the specification for less than 25% overshoot has not been met with the pole placement design. This can be attributed to an estimator roots which are too slow. Increasing the ω_n of the estimator to 10 rad/sec will meet the specification.



Problem 7.51: Closed-loop step responses.

52. A simplified model for the control of a flexible robotic arm is shown in Fig. 7.98, where

$$k/M = 900 \text{ rad/sec}^2,$$
$$y = \text{output, the mass position,}$$
$$u = \text{input, the position of the end of the spring.}$$

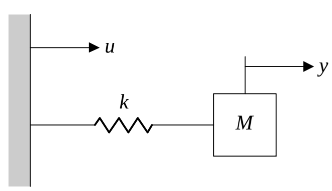


Figure 7.98: Simple robotic arm for Problem 7.52.

a) Write the equations of motion in state-space form.
b) Design an estimator with roots as $s = -100 \pm 100j$.

c) Could both state variables of the system be estimated if only a measurement of $\dot{y}$ was available?
d) Design a full-state feedback controller with roots at $s = -20 \pm 20j$.
e) Would it be reasonable to design a control law for the system with roots at $s = -200 \pm 200j$? State your reasons.
f) Write equations for the compensator, including a command input for y. Draw a Bode plot for the closed-loop system, and give the gain and phase margins for the design.

Solution:

(a) Defining $x_1 = y$ and $x_2 = \dot{y}$, we have,

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -k/M & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ k/M \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}. \end{aligned}$$

(b) Comparing coefficients of like powers of s,

$$\begin{aligned} (s + 100 + j100)(s + 100 - j100) &= s^2 + 200s + 20000 = 0 \\ &= \det(s\mathbf{I} - \mathbf{F} + \mathbf{LH}) = s^2 + l_1 s + l_2 + 900 = 0, \end{aligned}$$

yields $\mathbf{L} = \begin{bmatrix} 200 & 19100 \end{bmatrix}^T$. This result can be verified using the MATLAB place command.

(c) Let's check if x_1 is observable with $y = x_2$. $\det(\mathcal{O}) = k/M \neq 0$. So y is observable and (not surprisingly) both state variables can be estimated from $\dot{y}$.

(d) Comparing coefficients of like powers of s,

$$\begin{aligned} (s + 20 + j20)(s + 20 - j20) &= s^2 + 40s + 800 = 0 \\ &= \det(s\mathbf{I} - \mathbf{F} + \mathbf{GK}) = s^2 + 900k_2 s + 900(k_1 + 1) = 0, \end{aligned}$$

yields $\mathbf{K} = \begin{bmatrix} -0.111 & 0.044 \end{bmatrix}$. This result can be verified using MATLAB's place command.

(e) No.

(i) The bandwidth of the spring is about 30 rad/sec and system roots at 200 rad/sec means that large control levels will be required.

(ii) When using an estimated state feedback, you would like the estimates of the state to have converged "to some extent" before generating a control signal from the estimate. This is the reason for the rule of thumb about picking the estimator roots 3 to 10 times faster than the control roots.

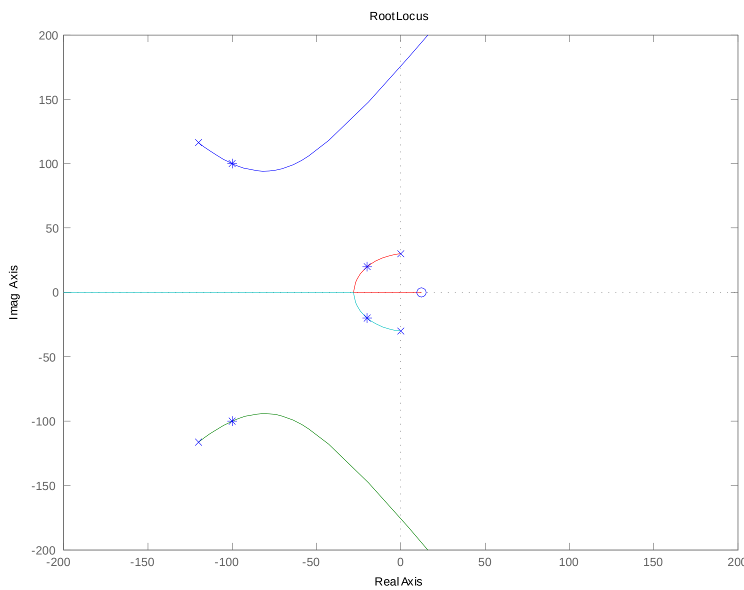
(f) We can express the compensator as,

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= (\mathbf{F} - \mathbf{GK} - \mathbf{LH})\hat{\mathbf{x}} + \mathbf{L}y, \\ u &= -\mathbf{K}\hat{\mathbf{x}}. \end{aligned}$$

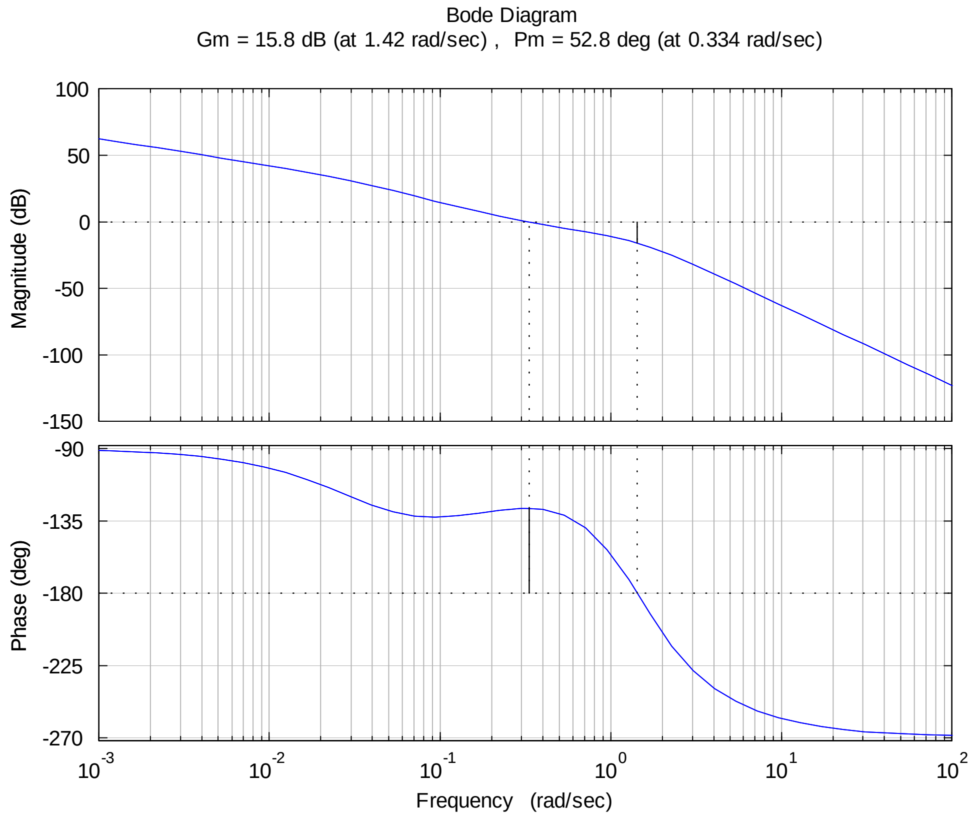
Thus the loop gain is,

$$D(s)G(s) = -\frac{744000(s - 12.2)}{(s + 120 \pm 116j)(s \pm 30j)}.$$

Note that the compensator has a zero in the RHP (*non-minimum phase*). This result can be verified using MATLAB's ss2tf command. The root locus of the compensated system is shown below. From the Bode plot of the loop gain, we find that the gain margin is approximately 8.8 db (can be verified from the root locus as well) and the phase margin is approximately $-64°$. Note that the closed-loop system is stable despite the fact that the phase margin is negative. This is true because the closed-loop system is non-minimum phase.



Root locus for Problem 7.52.



Problem 7.52: Bode plot of plant and compensator for robot arm.

53. The linearized differential equations governing the fluid-flow dynamics for the two cascaded tanks in Fig. 7.99 are

$$\delta \dot{h}_1 + \sigma \delta h_1 = \delta u,$$
$$\delta \dot{h}_2 + \sigma \delta h_2 = \sigma \delta h_1,$$

where,

$$\delta h_1 = \text{deviation of depth in tank 1 from the nominal level,}$$
$$\delta h_2 = \text{deviation of depth in tank 2 from the nominal level,}$$
$$\delta u = \text{deviation in fluid inflow rate to tank 1 (control).}$$

a) *Level Controller for Two Cascaded Tanks:* Using state feedback of the form,

$$\delta u = -K_1 \delta h_1 - K_2 \delta h_2,$$

choose values of K_1 and K_2 that will place the closed-loop eigenvalues at,

$$s = -2\sigma(1 \pm j).$$

b) *Level Estimator for two Cascaded Tanks:* Suppose that only the deviation in the level of tank 2 is measured (that is, $y = \delta h_2$). Using this measurement, design an estimator that will give continuous, smooth estimates of the deviation in levels of tank 1 and tank 2, with estimator error poles at $-8\sigma(1 \pm j)$.
c) *Estimator/Controller for Two Cascaded Tanks:* Sketch a block diagram (showing individual integrators) of the closed-loop system obtained by combining the estimator of part (b) with the controller of part (a).
d) Using MATLAB, compute and plot the response at y to an initial offset in δh_1. Assume $\sigma = 1$ for the plot.

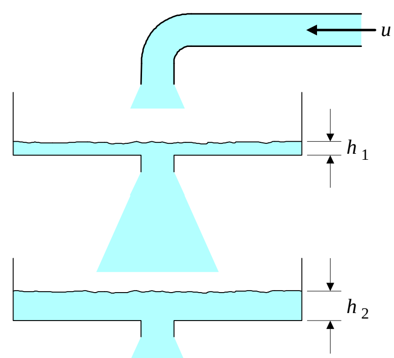


Figure 7.99: Coupled tanks for Problem 7.53.

Solution:

(a) Comparing coefficients of like powers of s,

$$\det \begin{bmatrix} s+\sigma+K_1 & K_2 \\ -\sigma & s+\sigma \end{bmatrix} = s^2 + (2\sigma + K_1)s + \sigma^2 + \sigma(K_1 + K_2) = 0.$$
$$= (s + 2\sigma + 2\sigma j)(s + 2\sigma - 2\sigma j) = s^2 + 4\sigma s + 8\sigma^2 = 0,$$

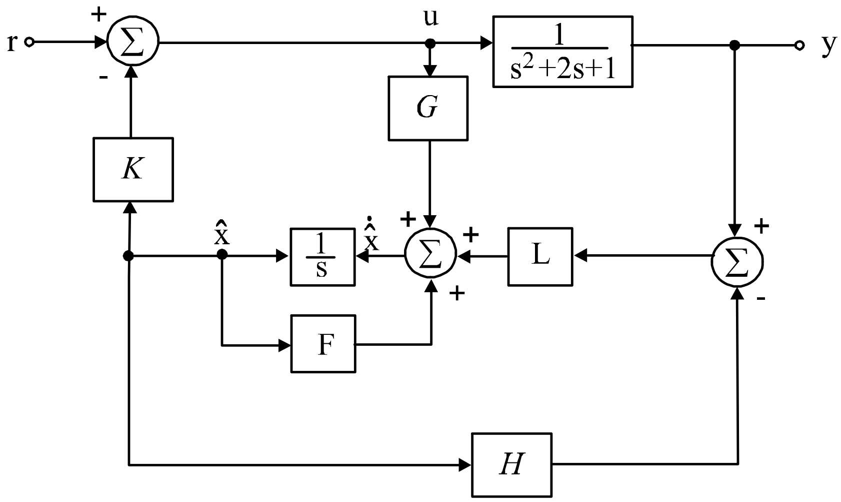
gives $K_1 = 2\sigma$ and $K_2 = 5\sigma$.

(b) Comparing coefficients of like powers of s,

$$\det \begin{bmatrix} s+\sigma & l_1 \\ -\sigma & s+\sigma+l_2 \end{bmatrix} = s^2 + (2\sigma + l_1)s + \sigma(l_1 + l_2) + \sigma^2 = 0$$
$$= (s + 8\sigma + 8\sigma j)(s + 8\sigma - 8\sigma j) = s^2 + 16\sigma s + 128\sigma^2 = 0.$$

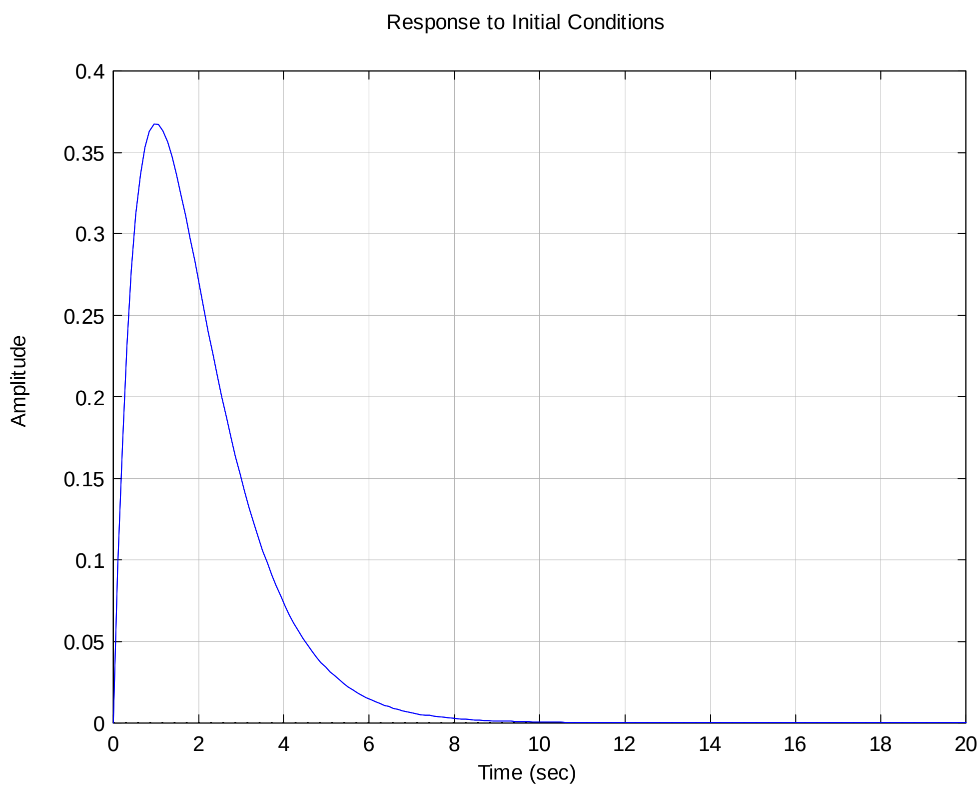
gives $l_1 = 113\sigma$ and $l_2 = 14\sigma$.

(c) The figure below shows a block diagram of the system.



Block diagram for of closed-loop system for Problem 7.53.

(d) The response to an initial condition on $\delta h_1(0)$ is shown on the next page using the MATLAB `initial` command.



Problem 7.53: Initial condition response for offset in $\delta h_1(0)$.

54. The lateral motions of a ship that is 100 m long, moving at a constant velocity of 10 m/sec, are described by

$$\begin{bmatrix} \dot{\beta} \\ \dot{r} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -0.0895 & -0.286 & 0 \\ -0.0439 & -0.272 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ r \\ \psi \end{bmatrix} + \begin{bmatrix} 0.0145 \\ -0.0122 \\ 0 \end{bmatrix} \delta,$$

where

$$\begin{aligned} \beta &= \text{sideslip angle, deg,} \\ \psi &= \text{heading angle,} \\ \delta &= \text{rudder angle, deg,} \\ r &= \text{yaw rate. See Fig. 7.100.} \end{aligned}$$

a) Determine the transfer function from δ to ψ and the characteristic roots of the uncontrolled ship.
b) Using complete state feedback of the form,

$$\delta = -K_1\beta - K_2 r - K_3(\psi - \psi_d),$$

where ψ_d is the desired heading, determine values of K_1, K_2, and K_3 that will place the closed-loop roots at $s = -0.2, -0.2 \pm 0.2j$.
c) Design a state estimator based on the measurement of ψ (obtained from a gyrocompass, for example). Place the roots of the estimator error equation at $s = -0.8$ and $-0.8 \pm 0.8j$.
d) Give the state equations and transfer function for the compensator $D_c(s)$ in Fig. 7.101, and plot its frequency response.
e) Draw the Bode plot for the closed-loop system, and compute the corresponding gain and phase margins.
f) Compute the feed-forward gains for a reference input, and plot the step response of the system to a change in heading of 5°.

Solution:

(a) With ψ as the measurement,

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta \\ r \\ \psi \end{bmatrix} = \mathbf{H}\mathbf{x}.$$

The transfer function from δ to ψ is (using MATLAB's ss2tf command),

$$\frac{\psi(s)}{\delta(s)} = \mathbf{H}(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G} = \frac{-0.0122(s + 0.142)}{s(s + 0.326)(s + 0.036)}.$$

The roots of the uncontrolled ship are the poles of the above transfer function:$s = 0, -0.326, -0.036$.

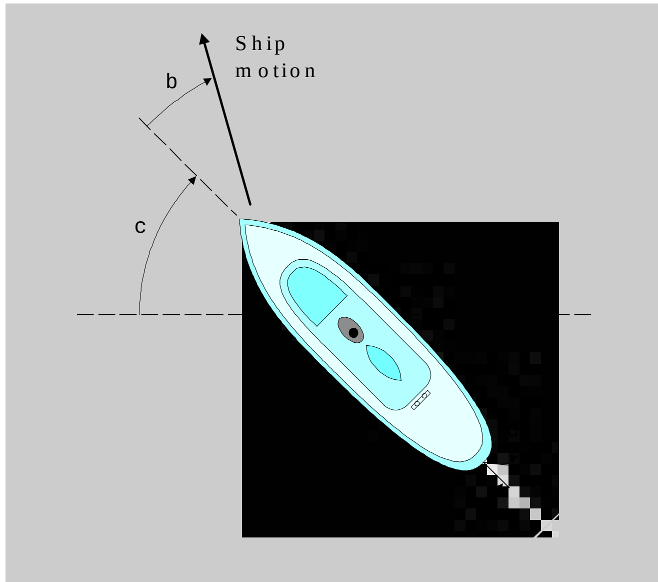


Figure 7.100: View of ship from above for Problem 7.54.

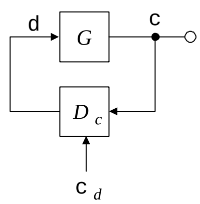


Figure 7.101: Ship control block diagram for Problem 7.54.

(b) Define $\mathbf{K} = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}$ and let $\delta = -\mathbf{K}\mathbf{x}$. Then,

$$\det[s\mathbf{I} - \mathbf{F} + \mathbf{G}\mathbf{K}] = \det \begin{bmatrix} s + 0.0895 + 0.0145K_1 & 0.286 + 0.0145K_2 & 0.0145K_3 \\ 0.0439 - 0.0122K_1 & s + 0.272 - 0.0122K_2 & -0.0122K_3 \\ 0 & -1 & s \end{bmatrix},$$

gives roots at $s = -0.2, -0.2 \pm j0.2$, when,

$$K_1 = 0.276, \; K_2 = -19.22, \; K_3 = -9.26.$$

This result can be verified using MATLAB's place command.

(c) With $\mathbf{L} = \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}^T$,

$$\det[s\mathbf{I} - \mathbf{F} + \mathbf{L}\mathbf{H}] = \det \begin{bmatrix} s + 0.0895 & 0.286 & l_1 \\ 0.0439 & s + 0.272 & l_2 \\ 0 & -1 & s + l_3 \end{bmatrix},$$

gives roots at $s = -0.8, : -0.8 \pm j0.8$, when

$$l_1 = -19.09,\ l_2 = 1.81,\ l_3 = 2.04.$$

Again, this result can be verified using MATLAB's place command.

(d) The compensator state equations are,

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \mathbf{F}\hat{\mathbf{x}} + \mathbf{G}u + \mathbf{L}(y - \mathbf{H}\hat{\mathbf{x}}) = (\mathbf{F} - \mathbf{GK} - \mathbf{LH})\hat{\mathbf{x}} + \mathbf{L}y, \\ u &= -\mathbf{K}\hat{\mathbf{x}}.\end{aligned}$$

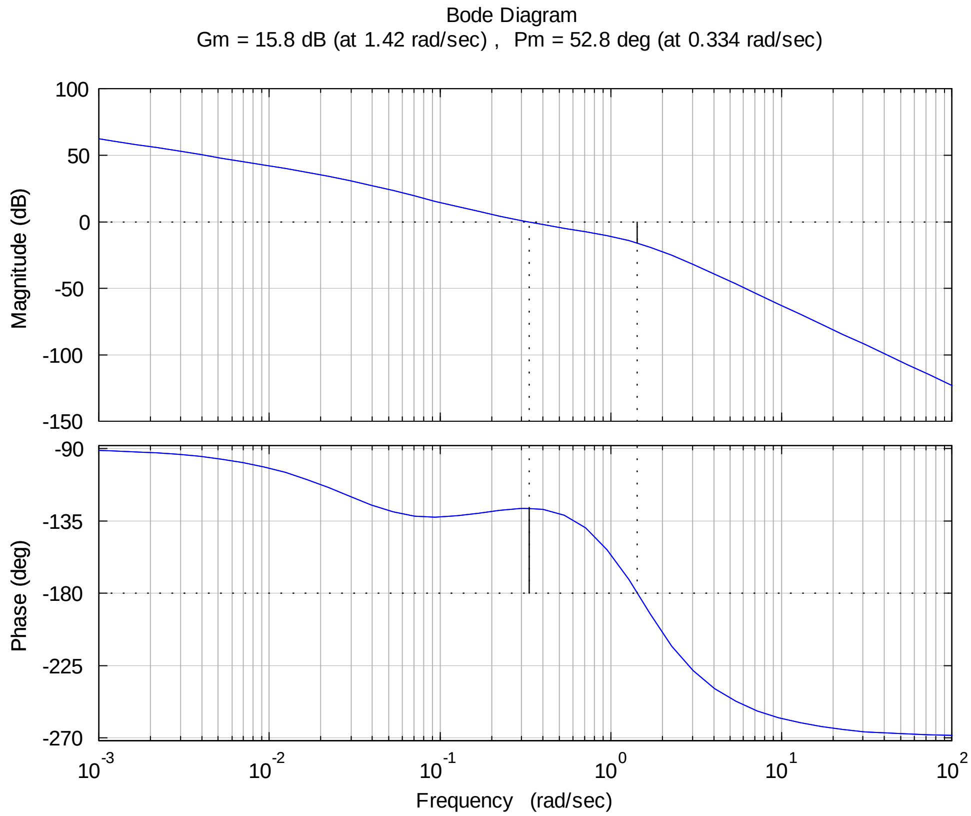
The compensator transfer function is given by (using the MATLAB ss2tf command),

$$D_c(s) = -\mathbf{K}(s\mathbf{I} - \mathbf{F} + \mathbf{GK} + \mathbf{LH})^{-1}\mathbf{L} = 58.96\frac{s^2 + 0.753s + 0.161}{s^3 + 2.64s^2 + 3.2s + 1.05}.$$

(e) The Bode plot of the closed-loop system is shown on the next page. The MATLAB command Bode or Margin can be used to create this figure. Note that when you find the Bode plot, the gain and phase margins only make sense if you consider the transfer function:

$$G(s) = -D_c(s)\frac{\psi(s)}{\delta(s)}.$$

Since the margins on a Bode plot assume negative feedback, the negative sign incorporated in $D_c(s)$ must be removed. The gain and phase margins are Gain Margin = 6.15 db, Phase Margin = 52.8°.



Problem 7.54: Bode plot of closed-loop system for ship control.

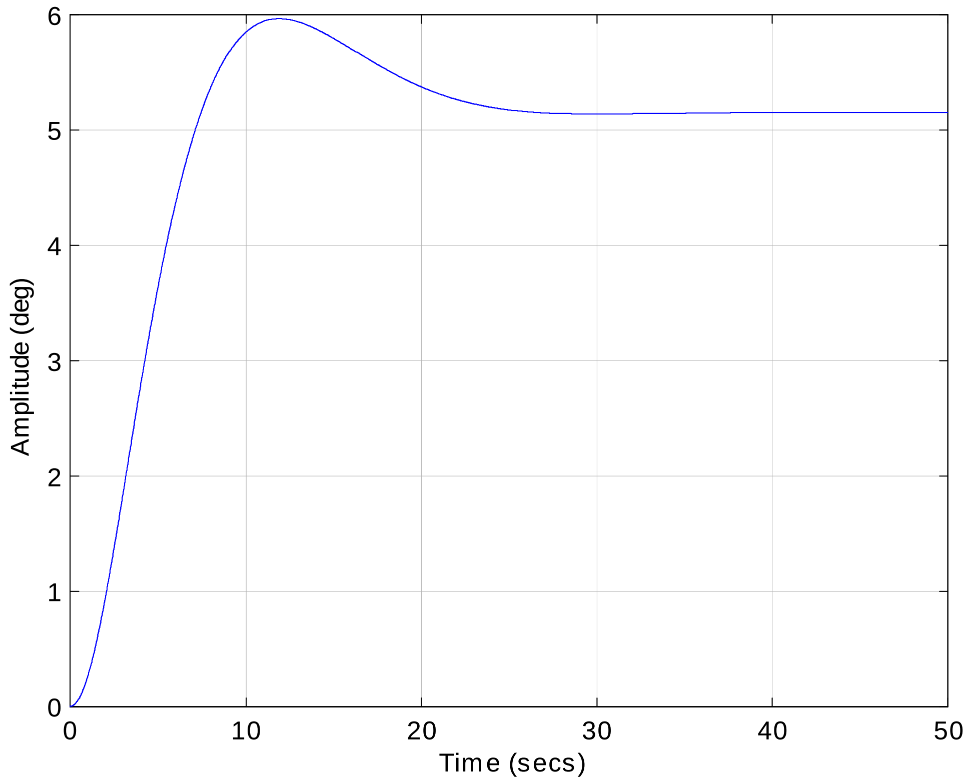
(f) Consider the determination of the feedforward gains $N_{\mathbf{x}}$ and N_u by,

$$\begin{bmatrix} N_{\mathbf{x}} \\ N_u \end{bmatrix} = \begin{bmatrix} \mathbf{F} & \mathbf{G} \\ \mathbf{H} & J \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This gives $N_u = 0$, $N_{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. Hence the control becomes,

$$\delta = \mathbf{K}(N_{\mathbf{x}}\psi_d - \hat{\mathbf{x}}) = \mathbf{K}N_{\mathbf{x}}\psi_d - \mathbf{K}\hat{\mathbf{x}}.$$

The complete closed-loop system step response is shown on the next page. The MATLAB command `step` can be used to create this figure.



Problem 7.54: 5° step response of closed-loop system for ship control.

Problems and Solutions for Section 7.9: Introduction of the Reference Input with the Estimator

55. ▲As mentioned in footnote 11 in Section 7.9.2, a reasonable approach for selecting the feedforward gain in Eq. (7.205) is to choose $\bar{N}$ such that when r and y are both unchanging, the DC gain from r to u is the negative of the DC gain from y to u. Derive a formula for $\bar{N}$ based on this selection rule. Show that if the plant is type 1, this choice is the same as that given by Eq. (7.205).

Solution:

The system equations with the feedforward gains included are,

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= (\mathbf{F} - \mathbf{GK} - \mathbf{LH})\hat{\mathbf{x}} + \mathbf{L}y + \mathbf{M}r,\\ u &= -\mathbf{K}\hat{\mathbf{x}} + \bar{N}r.\end{aligned}$$

To find the DC gain from y to u, we let,

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}_0,\ r = 0,\ y = y_0,\ u = u_0.$$

Then,

$$\begin{aligned}\mathbf{0} &= (\mathbf{F} - \mathbf{GK} - \mathbf{LH})\hat{\mathbf{x}}_0 + \mathbf{L}y_0 \\ u_0 &= -\mathbf{K}\hat{\mathbf{x}}_0.\end{aligned}$$

So that the DC gain from y to u is given by,

$$u_0 = \mathbf{K}(\mathbf{F} - \mathbf{GK} - \mathbf{LH})^{-1}\mathbf{L}y_0.$$

Similarly, to find the DC gain from r to u, we let,

$$\hat{\mathbf{x}} = \hat{\mathbf{x}}_0,\ y = 0,\ r = r_0,\ u = u_{0.}$$

Then,

$$\begin{aligned}\mathbf{0} &= (\mathbf{F} - \mathbf{LK} - \mathbf{LH})\hat{\mathbf{x}}_0 + Mr_0, \\ u_0 &= -\mathbf{K}\hat{\mathbf{x}}_0 + \bar{N}r_0.\end{aligned}$$

So that the DC gain from r to u is given by,

$$u_0 = (\mathbf{K}(\mathbf{F} - \mathbf{GK} - \mathbf{LH})^{-1}M + \bar{N})r_0.$$

From the footnote in the Servodesign section, we set the DC gain from r to u equal to the negative of the DC gain from y to u,

$$-\mathbf{K}(\mathbf{F} - \mathbf{GK} - \mathbf{LH})^{-1}\mathbf{L} = \mathbf{K}(\mathbf{F} - \mathbf{GK} - \mathbf{LH})^{-1}M + \bar{N}.$$

Therefore,

$$\bar{N} = -\mathbf{K}(\mathbf{F} - \mathbf{GK} - \mathbf{LH})^{-1}(\mathbf{L} + M).$$

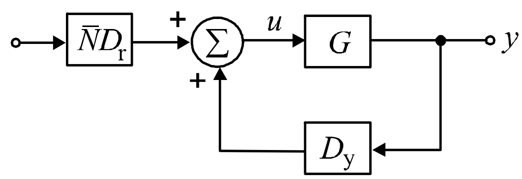
We can show, in general, if $\bar{N}$ is chosen as the footnote implies, then the system DC gain is unity for a type I plant. Consider the general closed-loop system block diagram shown below. Assuming a Type I plant,

$$G(s) = \frac{1}{s}\bar{G}(s),$$

then the closed-loop DC gain is simply,

$$\begin{aligned}\frac{Y(0)}{R(0)} &= \lim_{s\to 0} \bar{N}D_r \frac{G(s)}{1 - \bar{G}(s)D_y} \\ &= \lim_{s\to 0} \bar{N}D_r \frac{G(0)}{1 - \bar{G}(0)D_y} \\ &= -\bar{N}\frac{D_r(0)}{D_y(0)}.\end{aligned}$$

So, if $\bar{N}D_r = -D_y$, then the DC gain of the system is unity. The selection approach of $\bar{N}$ mentioned in the servodesign section is exactly the condition $\bar{N}D_r = -D_y$. Hence, since Eq. 7.205 is a direct result of setting the DC gain to unity, then the above expression for $\bar{N}$, that was derived from the footnote hint, is equivalent to Eq. 7.205.



General closed-loop system block diagram for Problem 7.55.

Problems and Solutions for Section 7.10: Integral Control and Robust Tracking

56. Assume that the linearized and time-scaled equation of motion for the ball-bearing levitation device is $\ddot{x} - x = u + w$. Here w is a constant bias due to the power amplifier. Introduce integral error control, and select three control gains $\mathbf{K} = [\ K_1 \quad K_2 \quad K_3\ ]$ so that the closed-loop poles are at -1 and $-1 \pm j$ and the steady-state error to w and to a (step) position command will be zero. Let $y = x$ and the reference input $r \triangleq y_{\text{ref}}$ be a constant. Draw a block diagram of your design showing the locations of the feedback gains K_i. Assume that both $\dot{x}$ and x can be measured. Plot the response of the closed-loop system to a step command input and the response to a step change in the bias input. Verify that the system is type 1. Use MATLAB (Simulink) software to simulate the system responses.

Solution:

The equations of motion are given by,

$$\begin{aligned} \ddot{x} - x &= u + w, \\ \dot{w} &= 0. \end{aligned}$$

A realization of these equations is,

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w, \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix}. \end{aligned}$$

In order to incorporate integral control, we augment the state vector with an integral state, x_I, such that,

$$\dot{x}_I = y - r.$$

With the augmented state vector, $\mathbf{z} = [x_I\ x\ \dot{x}]^T$, the augmented state matrices become,

$$\mathbf{F}_a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \ \mathbf{G}_a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{H}_a = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}.$$

The design of the state feedback vector, $\mathbf{K}$, is now done using the above augmented state matrices. For closed-loop poles of $s = -1, -1 \pm j$,

$$\det(s\mathbf{I} - \mathbf{F}_a + \mathbf{G}_a\mathbf{K}) = 0,$$

when,

$$\mathbf{K} = [\ K_1 \quad K_2 \quad K_3\ ] = [\ 2 \quad 5 \quad 3\ ].$$

This result can be verified using the MATLAB place command.

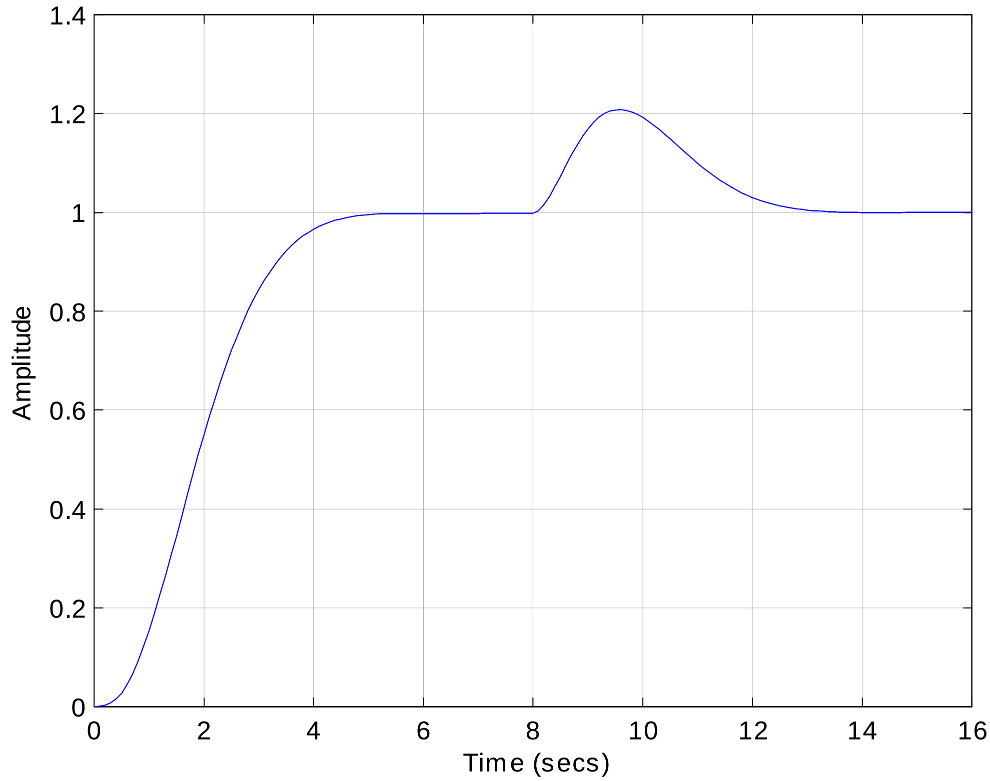
The closed-loop system is given by,

$$\begin{aligned} \dot{\mathbf{z}} &= (\mathbf{F}_a - \mathbf{G}_a\mathbf{K})\mathbf{z} + \mathbf{G}_a w + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} r, \\ y &= \mathbf{H}_a\mathbf{z}. \end{aligned}$$

To show that the system is Type I, show that $y = 0$ for any constant w in the steady-state, i.e., $\dot{\mathbf{z}} = 0$. For the closed-loop system we have,

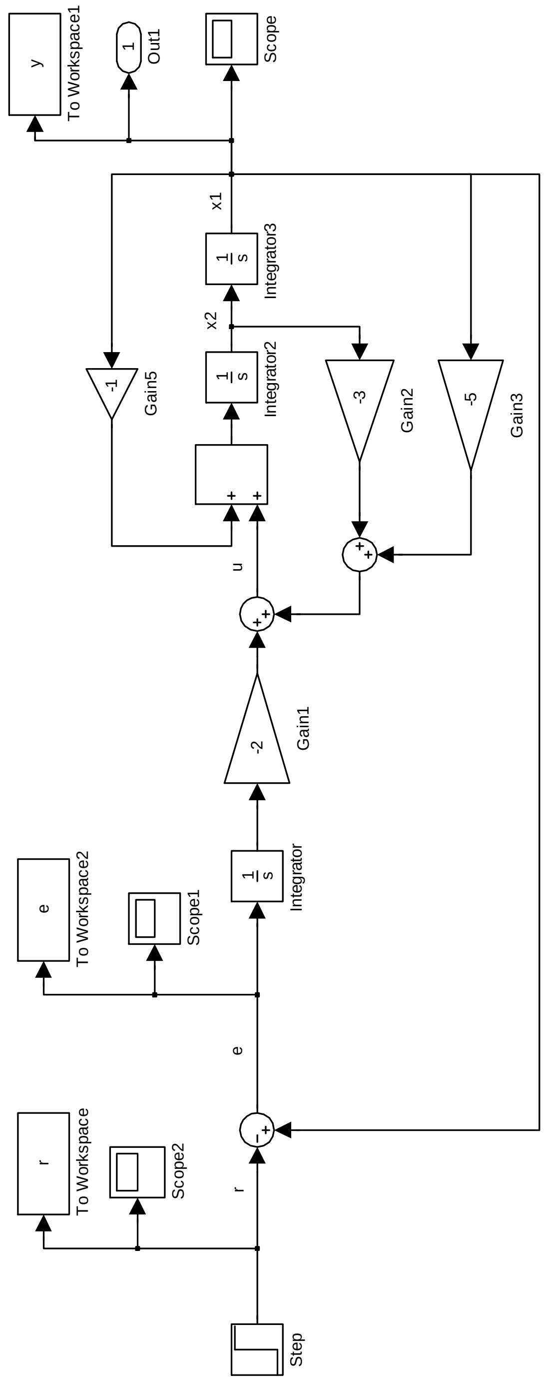
$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -K_1 & 1-K_2 & -K_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w.$$

This immediately gives $z_2 = 0$ and $y = z_2 = 0$. Thus, in steady-state $y = 0$ for any constant w in the steady-state. The figure below shows a simulation of the closed-loop system to a commanded step r, at $t = 0$. At $t = 8$, a step in the constant bias w is applied. This figure was generated using the MATLAB lsim command.



Problem 7.56: Response of closed-loop system to a unit step input at $t = 0$ and step disturbance at $t = 8$.

The simulation of the closed-loop system in Simulink is shown on the next page.



Simulink simulation for Problem 7.56.

57. Consider a system with state matrices,

$$\mathbf{F} = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \mathbf{H} = [\ 1 \quad 3\ ].$$

a) Use feedback of the form $u(t) = -\mathbf{K}\mathbf{x}(t) + \bar{N}r(t)$, where $\bar{N}$ is a nonzero scalar, to move the poles to $-3 \pm 3j$.
b) Choose $\bar{N}$ so that if r is a constant, the system has zero steady-state error; that is $y(\infty) = r$.
c) Show that if $\mathbf{F}$ changes to $\mathbf{F} + \delta\mathbf{F}$, where $\delta\mathbf{F}$ is an arbitrary 2×2 matrix, then your choice of $\bar{N}$ in part (b) will no longer make $y(\infty) = r$. Therefore, the system is not robust under changes to the system parameters in $\mathbf{F}$.
d) The system steady-state error performance can be made robust by augmenting the system with an integrator and using unity feedback; that is, by setting $\dot{x}_I = r - y$, where x_I is the state of the integrator. To see this, first use state feedback of the form $u = -\mathbf{K}\mathbf{x} - K_1 x_I$ so that the poles of the augmented system are at $-3, -2 \pm j\sqrt{3}$.
e) Show that the resulting system will yield $y(\infty) = r$ no matter how the matrices $\mathbf{F}$ and $\mathbf{G}$ are changed, as long as the closed-loop system remains stable.
f) For part (d), use MATLAB (Simulink) software to plot the time response of the system to a constant input. Draw Bode plots of the controller as well as the sensitivity function ($\mathcal{S}$) and the complementary sensitivity function ($\mathcal{T}$).

Solution:

(a) Using feedback of the form, $u = -\mathbf{K}\mathbf{x} + Nr$, we have,

$$\det(s\mathbf{I} - \mathbf{F} + \mathbf{G}\mathbf{K}) = (s + 2 + k_1)(s + 3 + k_2) + k_1(1 - k_1) = s^2 + 6s + 18,$$

when $\mathbf{K} = [5\ \ -4]$. This result can be verified using the MATLAB `place` command.

(b) We can find the desired value for N by setting the DC gain from r to y equal to unity. The closed-loop system equations are,

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{F}\mathbf{x} + \mathbf{G}(-\mathbf{K}\mathbf{x} + Nr) = (\mathbf{F} - \mathbf{G}\mathbf{K})\mathbf{x} + \mathbf{G}Nr, \\ y &= \mathbf{H}\mathbf{x}. \end{aligned}$$

Therefore, the transfer function is,

$$D(s) = \mathbf{H}(s\mathbf{I} - \mathbf{F} + \mathbf{G}\mathbf{K})^{-1}\mathbf{G}N,$$

and the DC gain is simply,

$$D(0) = \mathbf{H}(-\mathbf{F} + \mathbf{G}\mathbf{K})^{-1}\mathbf{G}N = \frac{5}{9}N = 1.$$

Hence, we choose $N = \frac{9}{5}$.

(c) Change $\mathbf{F}$ to $(\mathbf{F} + \delta\mathbf{F})$, and let the value of N that keeps the tracking error at zero be N'. Then letting $T'(s)$ be the transfer function associated with the perturbed system,

$$\begin{aligned} N'^{-1} &= T'(0) = -\mathbf{H}(\mathbf{F} + \delta\mathbf{F} - \mathbf{G}\mathbf{K})^{-1}\mathbf{G}, \\ &= -\mathbf{H}[(\mathbf{F} - \mathbf{G}\mathbf{K})(\mathbf{I} - (\mathbf{F} - \mathbf{G}\mathbf{K})^{-1}\delta\mathbf{F})]^{-1}\mathbf{G}, \\ &= -\mathbf{H}(\mathbf{I} - (\mathbf{F} - \mathbf{G}\mathbf{K})^{-1}\delta\mathbf{F})^{-1}(\mathbf{F} - \mathbf{G}\mathbf{K})^{-1}\mathbf{G}. \end{aligned}$$

For $\delta\mathbf{F}$ small,

$$(\mathbf{I} - (\mathbf{F} - \mathbf{GK})^{-1}\delta\mathbf{F})^{-1} = \mathbf{I} + (\mathbf{F} - \mathbf{GK})^{-1}\delta\mathbf{F}.$$

Hence,

$$N^{'-1} = \underbrace{-\mathbf{H}(\mathbf{F} - \mathbf{GK})^{-1}\mathbf{G}}_{N^{-1}} - \mathbf{H}(\mathbf{F} - \mathbf{GK})^{-1}\delta\mathbf{F}(\mathbf{F} - \mathbf{GK})^{-1}\mathbf{G}.$$

And for arbitrary $\delta\mathbf{F}$ we arrive at,

$$N^{'-1} \neq N^{-1}.$$

Therefore, small changes in the plant matrix $\mathbf{F}$ prevent the steady-state error from reaching zero. The control system is not robust with respect to changes in $\mathbf{F}$.

(d) Augmenting the system equations with an integrator state, x_I, the state equation become,

$$\begin{aligned}
\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{x}_I \end{bmatrix} &= \begin{bmatrix} \mathbf{F} & \mathbf{0} \\ -\mathbf{H} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x_I \end{bmatrix} + \begin{bmatrix} \mathbf{G} \\ 0 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r, \\
y &= [\mathbf{H} \;\; \mathbf{0}] \begin{bmatrix} \mathbf{x} \\ x_I \end{bmatrix}.
\end{aligned}$$

or with $\mathbf{z} = [\mathbf{x} \; x_I]^T$,

$$\begin{aligned}
\dot{\mathbf{z}} &= \mathbf{F}_a\mathbf{z} + \mathbf{G}_a u + \mathbf{G}_r r, \\
y &= \mathbf{H}_a\mathbf{z}.
\end{aligned}$$

Using feedback of the form $u = -\mathbf{Kx} - k_I x_I = -\mathbf{K}_a\mathbf{z}$, we have,

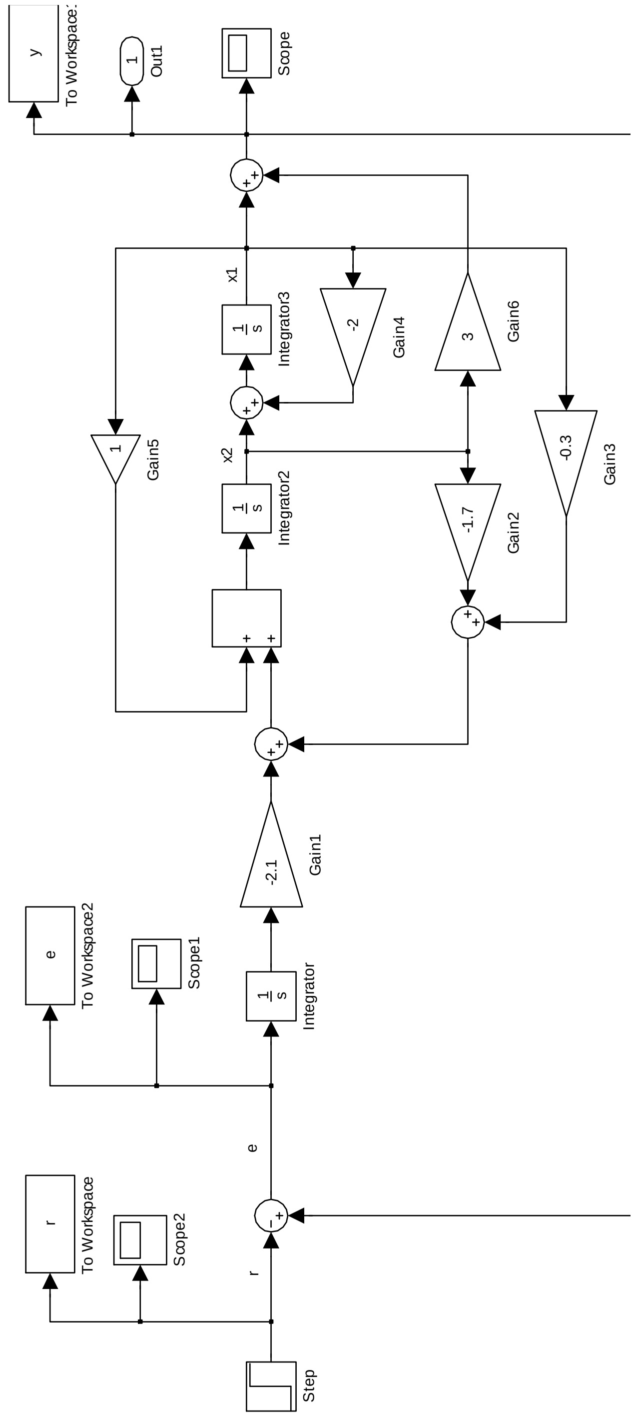
$$\det(s\mathbf{I} - \mathbf{F}_a + \mathbf{G}_a\mathbf{K}_a) = 0 \;\; \text{for } s = -3, -2 \pm j\sqrt{3},$$

when $\mathbf{K}_a = \begin{bmatrix} 0.3 & 1.7 & -2.1 \end{bmatrix}$. This result can be verified using the MATLAB `place` command.

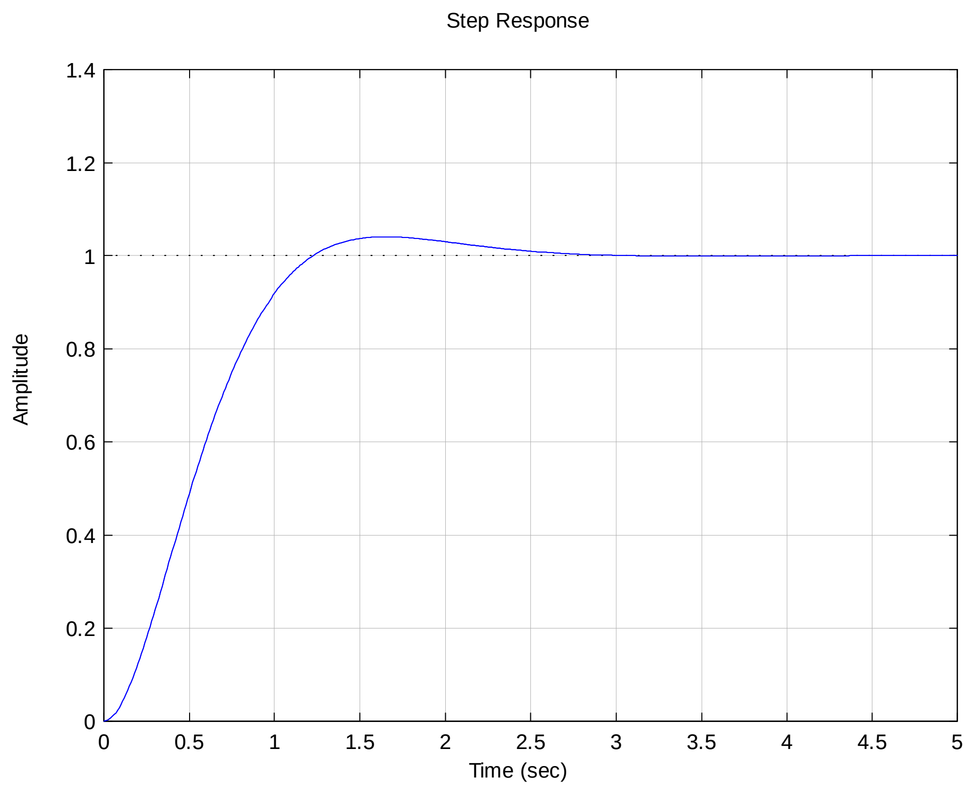
(e) We can show that the closed-loop DC gain from r to y is independent of $\mathbf{F}$,

$$\begin{aligned}
y_\infty &= T(0)r_\infty = [\mathbf{H} \;\; \mathbf{0}] \begin{bmatrix} -\mathbf{F} + \mathbf{GK} & \mathbf{G}k_I \\ \mathbf{H} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r_\infty \\
&= [\mathbf{H} \;\; \mathbf{0}] \begin{bmatrix} * & (\mathbf{F} - \mathbf{GK})^{-1}\mathbf{G}k_I[\mathbf{H}(\mathbf{F} - \mathbf{GK})^{-1}\mathbf{G}k_I]^{-1} \\ * & * \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r_\infty \\
&= [\mathbf{H}(\mathbf{F} - \mathbf{GK})^{-1}\mathbf{G}k_I][\mathbf{H}(\mathbf{F} - \mathbf{GK})^{-1}\mathbf{G}k_1]^{-1} r_\infty = r_\infty \text{ independent of } \mathbf{F}, \mathbf{G}, \mathbf{H}.
\end{aligned}$$

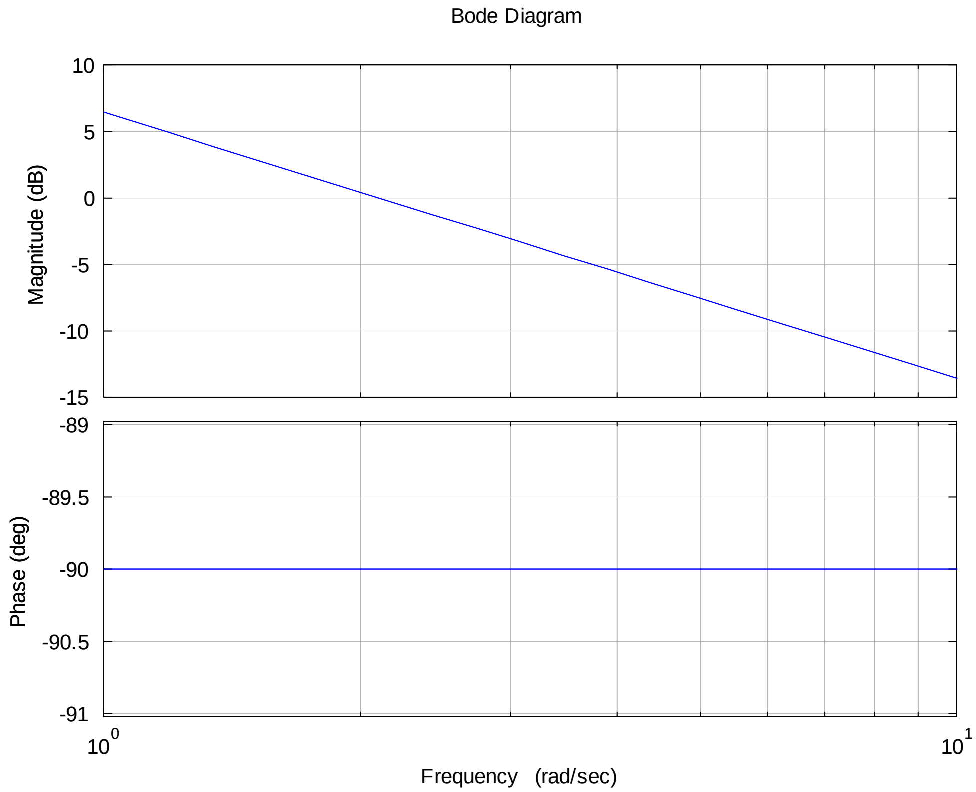
(f) The simulation of the closed-loop system in Simulink is shown on the next page. The closed-loop step response is shown next. The Bode plot of the controller, the sensitivity function ($\mathcal{S}$), as well as the complementary sensitivity function ($\mathcal{T}$), are also shown.



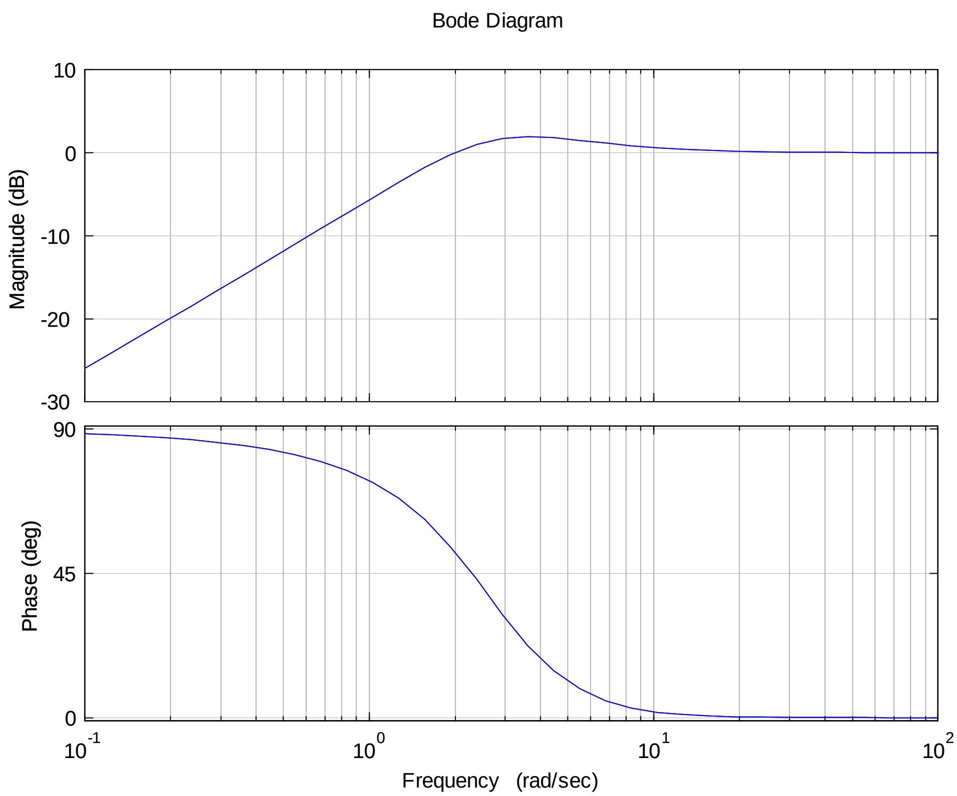
Simulink simulation for Problem 7.57.



Closed-loop step response for Problem 7.57.



Bode plot of the controller for Problem 7.57.



Bode plot of the sensitivity function for roblem 7.57.

58. ▲Consider a servomechanism for following the data track on a computer-disk memory system. Because of various unavoidable mechanical imperfections, the data track is not exactly a centered circle, and thus the radial servo must follow a sinusoidal input of radian frequency ω_0 (the spin rate of the disk). The state matrices for a linearized model of such a system are

$$\mathbf{F} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{H} = [\ 1 \quad 0\ ].$$

The sinusoidal reference input satisfies $\ddot{r} = -\omega_0^2 r$.
a) Let $\omega_0 = 1$, and place the poles of the error system for an internal model design at,

$$\alpha_c(s) = (s + 2 \pm j2)(s + 1 \pm 1j),$$

and the pole of the reduced-order estimator at

$$\alpha_e(s) = (s + 6).$$

b) Draw a block diagram of the system, and clearly show the presence of the oscillator with frequency ω_0 (the internal model) in the controller. Also verify the presence of the blocking

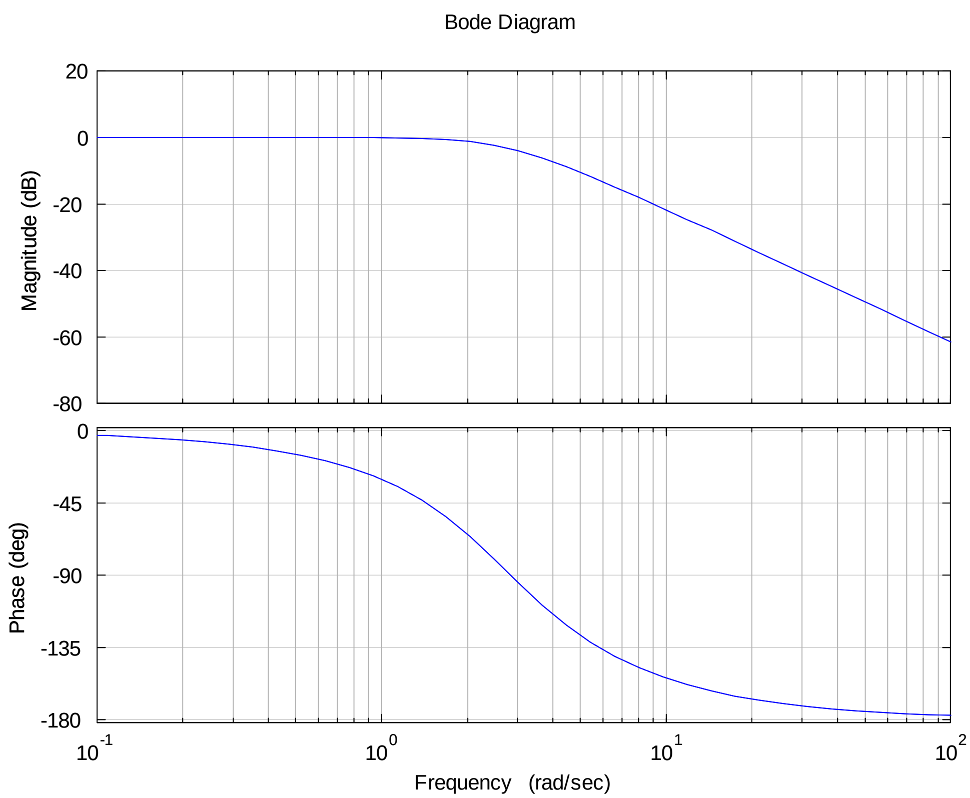


Figure 7.102: Bode plot of the complementary sensitivity function for Problem 7.57.

zeros at $\pm j\omega_0$.
c) Use MATLAB (Simulink) software to plot the time response of the system to a sinusoidal input at frequency $\omega_0 = 1$.
d) Draw a Bode plot to show how this system will respond to sinusoidal inputs at frequencies different from but near ω_0.

Solution:

(a) The compensator design consists of two parts: a feedback design using an internal model approach, and a reduced-order estimator. Let $\mathbf{x}$ be the plant state vector where,

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u.$$

Since the reference input satisfies $\ddot{r} = -\omega_0^2 r$, we can write out the error-state equations (using $e = y - r$) as,

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\mu,$$

with,

$$\mathbf{z} = \begin{bmatrix} e \\ \dot{e} \\ \xi \\ \dot{\xi} \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

With $\mu = -\mathbf{K}\mathbf{z}$, we can find $\mathbf{K}$ from pole placement. In this case,

$$\det(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}) = 0.$$

when $s = -2 \pm j2, -1 \pm j1$, for,

$$\begin{aligned} \mathbf{K} &= [K_2 \quad K_1 \quad K_{01} \quad K_{02}] \\ &= \begin{bmatrix} -1. & 18 & 17 & 5 \end{bmatrix}. \end{aligned}$$

This result can be verified using the MATLAB `place` command.

To design the estimator, consider the plant matrices given in the problem,

$$\mathbf{F} = \begin{bmatrix} \mathbf{F}_{aa} & \mathbf{F}_{ab} \\ \mathbf{F}_{ba} & \mathbf{F}_{bb} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} G_a \\ G_b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

So that the equation for the estimate of only x_2 is,

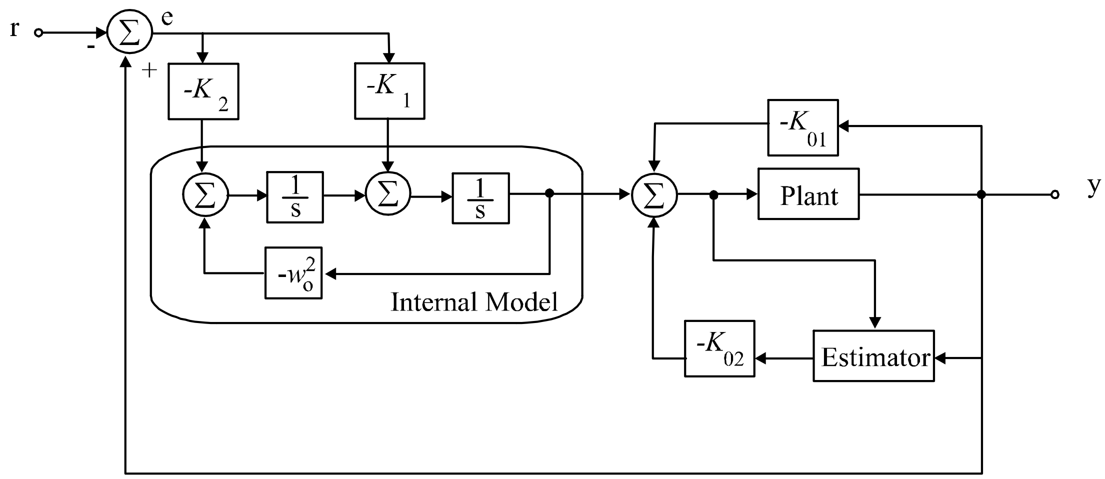
$$\begin{aligned} \hat{x}_b &= \hat{x}_c + Ly, \\ \dot{\hat{x}}_c &= -(L+1)\hat{x}_b + u, \end{aligned}$$

where the value for L is chosen from the estimate error characteristic equation

$$\det(s - (\mathbf{F}_{bb} - L\mathbf{F}_{ab})) = s + 1 + L.$$

For an estimator pole at $s = -6$, we have $L = 5$.

(b) The block diagram for this compensator is given below. Note that the internal model of the oscillator is plainly seen.



Problem 7.58: Compensator structure for robust following of sinusoid using an internal model controller and reduced-order estimator.

To see the "blocking zeros", we compute the transfer function from r to e. The system equations from r to e are,

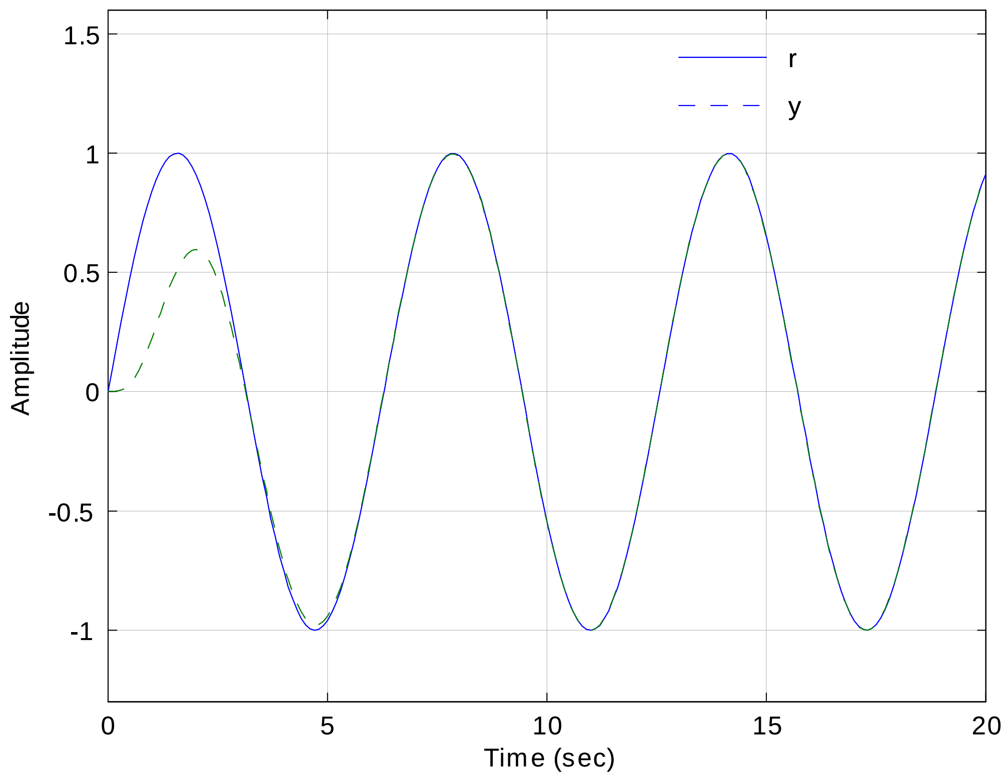
$$\begin{aligned}
\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_I \\ \dot{\tilde{x}}_b \end{bmatrix} &= \begin{bmatrix} \mathbf{F}-\mathbf{G}\mathbf{K}_{01} & \mathbf{G}\mathbf{C}_c & \mathbf{G}\mathbf{K}_{02} \\ \mathbf{B}_c\mathbf{H} & \mathbf{A}_c & 0 \\ 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_I \\ \tilde{x}_b \end{bmatrix} + \begin{bmatrix} \mathbf{G} \\ -\mathbf{B}_c \\ 0 \end{bmatrix} r, \\
e &= \begin{bmatrix} \mathbf{H} & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_I \\ \tilde{x}_b \end{bmatrix} - r.
\end{aligned}$$

where,

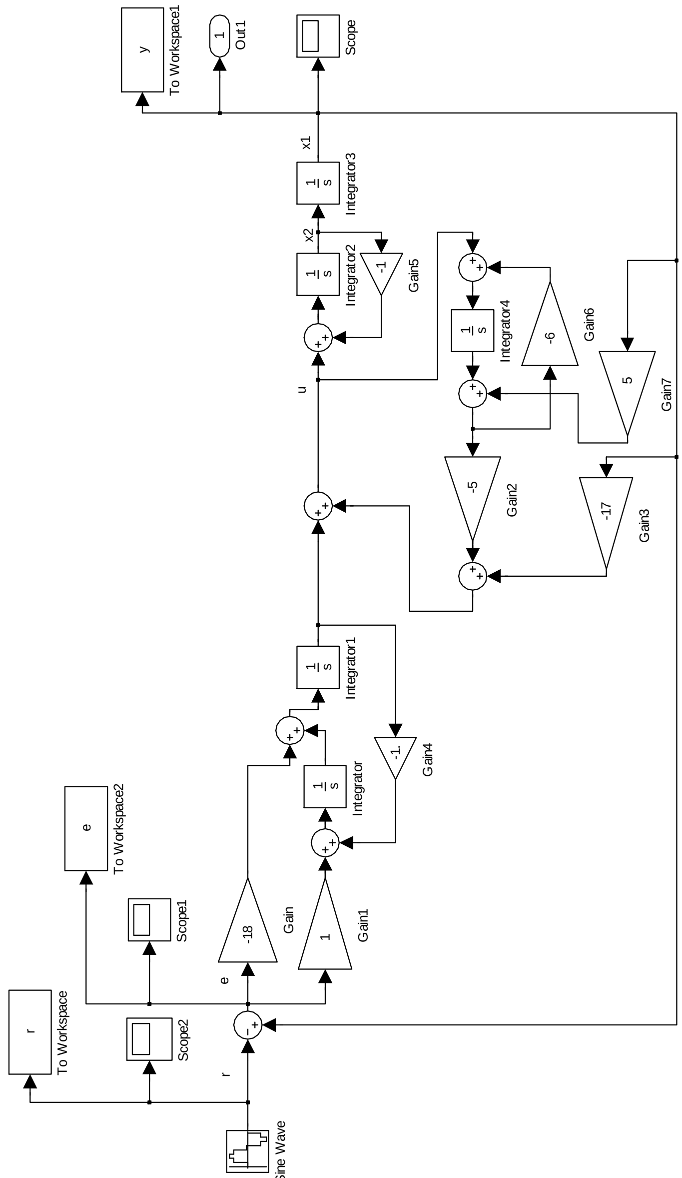
$$\begin{aligned}
\mathbf{A}_c &= \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}, \ \mathbf{B}_c = \begin{bmatrix} -K_1 \\ -K_2 \end{bmatrix}, \\
\mathbf{C}_c &= \begin{bmatrix} 1 & 0 \end{bmatrix}.
\end{aligned}$$

The transmission zeros of this system realization are at $s = -6, -3 \pm j2.645, \pm j$ with the last two as the blocking zeros. The zero locations are computed using the MATLAB tzero command.

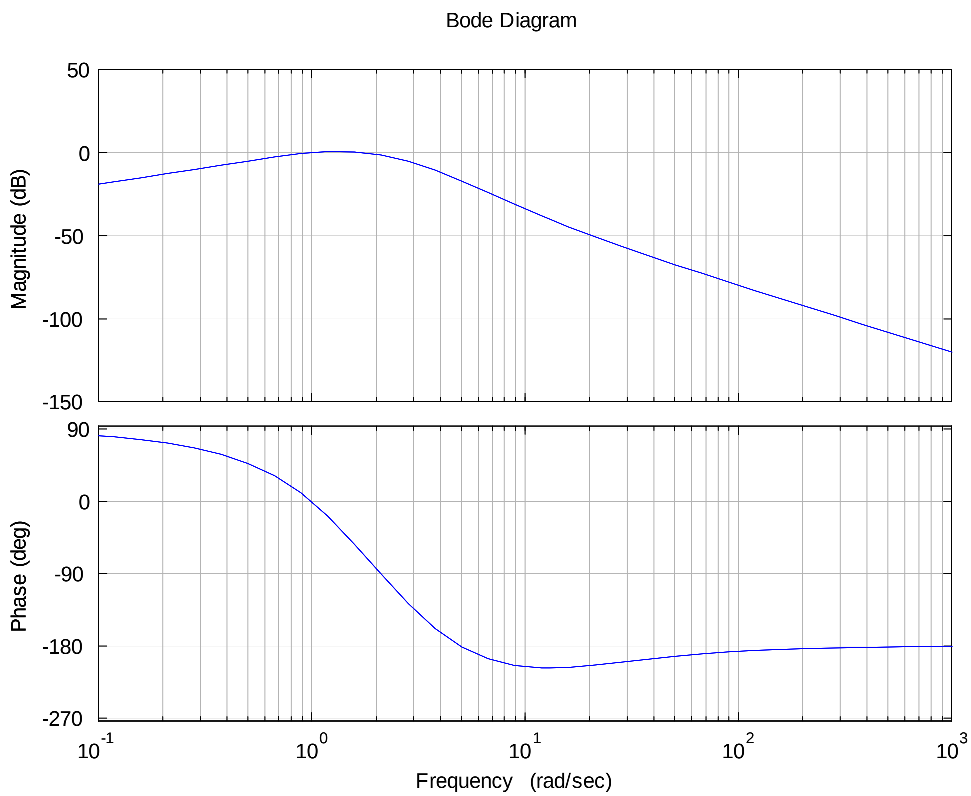
(c) The time response of the closed-loop system subjected to a sinusoid at a frequency of ω_0 is shown below using the MATLAB lsim command. The simulation of the closed-loop system in Simulink is shown on the next page.



Problem 7.58: Time history of closed-loop system with a sinusoidial input.



Simulink simulation for Problem 7.58.



Problem 7.58: Bode plot of closed-loop system for sinusoidal following.

(d) A Bode plot of the compensated system is given above.

59. ▲Compute the controller transfer function (from $Y(s)$ to $U(s)$) in Example 7.38. What is the prominent feature of the controller that allows tracking and disturbance rejection?

Solution:

The related equations from the Text are,

$$\begin{aligned}\dot{\hat{\rho}} &= l_1(e-\hat{x}),\\ \dot{\hat{x}} &= -3\hat{x}+\hat{\rho}+u+l_2(e-\hat{x}),\\ u &= -K\hat{x}-\hat{\rho}.\end{aligned}$$

To find the transfer function from $Y(s)$ to $U(s)$, we re-write the equations as,

$$\begin{aligned}\begin{bmatrix}\dot{\hat{\rho}}\\ \dot{\hat{x}}\end{bmatrix} &= \begin{bmatrix}0 & -l_1\\ 0 & -3-K-l_2\end{bmatrix}\begin{bmatrix}\hat{\rho}\\ \hat{x}\end{bmatrix}+\begin{bmatrix}l_1\\ l_2\end{bmatrix}y,\\ u &= \begin{bmatrix}-1 & -K\end{bmatrix}\begin{bmatrix}\hat{\rho}\\ \hat{x}\end{bmatrix}.\end{aligned}$$

The controller transfer function is,

$$\frac{U(s)}{Y(s)} = \frac{-(l_1 + Kl_2)s - (3l_1 + Kl_1)}{s(s + 3 + K + l_2)} = \frac{-279(s + 4.0323)}{s(s + 32)},$$

and shows the presence of an *integrator*!

60. ▲Consider the pendulum problem with control torque T_c and disturbance torque T_d:

$$\ddot{\theta} + 4\theta = T_c + T_d,$$

(here $g/l = 4$.) Assume that there is a potentiometer at the pin that measures the output angle θ, but with a constant unknown bias b. Thus the measurement equation is $y = \theta + b$.
a) Take the "augmented" state vector to be

$$\begin{bmatrix} \theta \\ \dot{\theta} \\ w \end{bmatrix},$$

where w is the input-equivalent bias. Write the system equations in state-space form. Give values for the matrices **F, G**, and **H**.
b) Using state-variable methods, show that the characteristic equation of the model is $s(s^2+4) = 0$.
c) Show that w is observable if we assume that $y = \theta$, and write the estimator equations for

$$\begin{bmatrix} \hat{\theta} \\ \dot{\hat{\theta}} \\ \hat{w} \end{bmatrix}.$$

Pick estimator gains $[\ l_1 \quad l_2 \quad l_3\ ]^T$ to place all the roots of the estimator-error characteristic equation at -10.
d) Using full-state feedback of the estimated (controllable) state-variables, derive a control law to place the closed-loop poles at $-2 \pm 2j$.
e) Draw a block diagram of the complete closed-loop system (estimator, plant, and controller) using integrator blocks.
f) Introduce the estimated bias into the control so as to yield zero steady-state error to the output bias b. Demonstrate the performance of your design by plotting the response of the system to a step change in b; that is, b changes from 0 to some constant value.

Solution:

(a) The system with equivalent input disturbance which replaces the actual disturbance, b, with the equivalent disturbance w at the control input is,

$$\begin{aligned} \ddot{\theta} &= -4\theta + T_c + w + T_d, \\ \dot{w} &= 0, \\ y &= \theta. \end{aligned}$$

In state-space form,

$$\begin{aligned}
\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \\ \dot{w} \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ -4 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ w \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} T_c + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} T_d, \\
y &= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \\ w \end{bmatrix},
\end{aligned}$$

(b)

$$\det(s\mathbf{I} - \mathbf{F}) = \det \begin{bmatrix} s & -1 & 0 \\ 4 & s & -1 \\ 0 & 0 & s \end{bmatrix} = s(s^2+4) = 0.$$

(c) Forming the observability matrix, we have,

$$\mathcal{O} = \begin{bmatrix} \mathbf{H} \\ \mathbf{HF} \\ \mathbf{HF}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}.$$

Clearly, det($\mathcal{O}$)= $1 \neq 0$ so that w is observable. The estimator gains can be determined by solving,

$$\det(s\mathbf{I} - \mathbf{F} + \mathbf{LH}) = (s+10)^3 = s^3 + 30s^2 + 300s + 1000,$$

We find that $\mathbf{L} = \begin{bmatrix} 30 & 296 & 1000 \end{bmatrix}^T$. This result can be verified using MATLAB's `place` command.

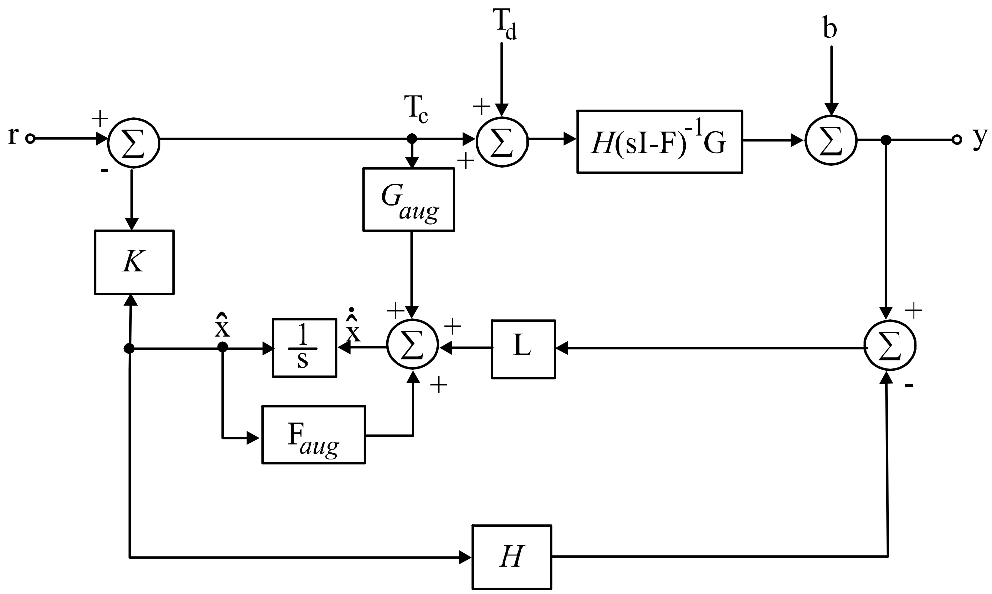
(d) The bias state variable w is not controllable, we cannot move the pole at 0. So state feedback should place the poles at $-2 \pm 2j$ and 0. Equating,

$$\det(s\mathbf{I} - \mathbf{F} + \mathbf{GK}) = s(s^2 + 4s + 8),$$

we find that $\mathbf{K} = \begin{bmatrix} 4 & 4 & 0 \end{bmatrix}$. (Note: the zero was chosen arbitrarily).

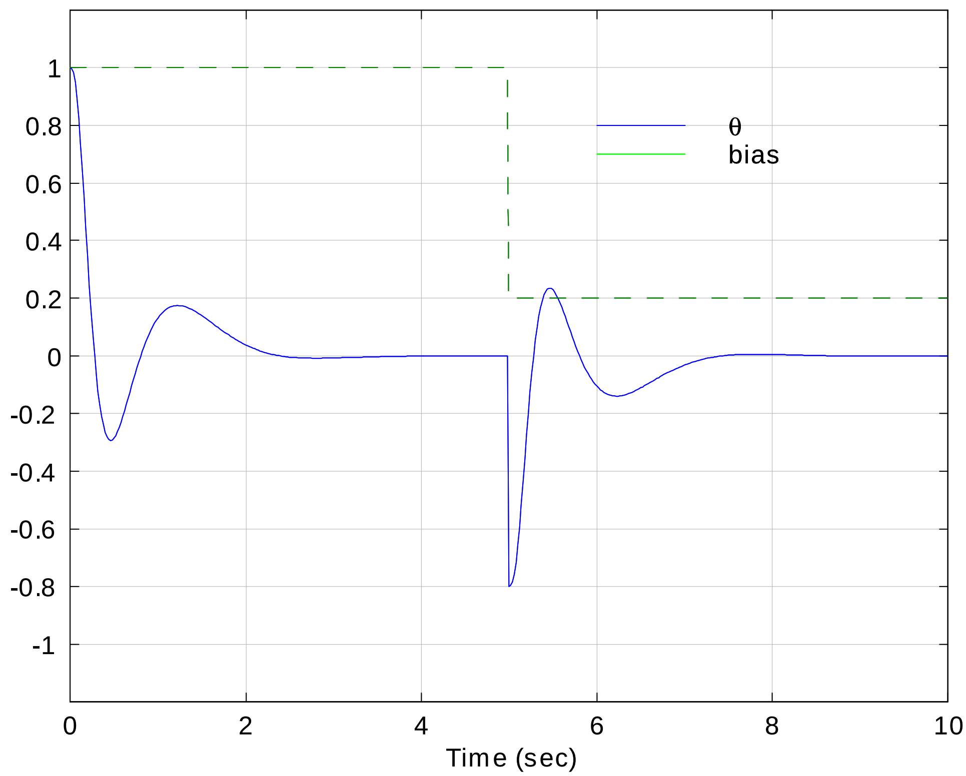
(e) The equations shown in the figure are,

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{Fx} + \mathbf{G}T_c + \mathbf{G}T_d, \\
\dot{\hat{\mathbf{x}}} &= \mathbf{F}_{aug}\hat{\mathbf{x}} + \mathbf{G}_{aug}T_c + \mathbf{L}(y - \mathbf{H}\hat{\mathbf{x}}), \\
y &= \mathbf{Hx}, \\
T_c &= -\mathbf{K}\hat{\mathbf{x}} + r.
\end{aligned}$$



Block diagram for Problem 7.60(e).

(f) The performance of the system is shown to a step change in the bias b from 0 to 1 at $t = 0$ sec and then another step change from 1 to 0.2 at $t = 5$ sec.



Problem 7.60: Step change in bias.

Problems and Solutions for Section 7.13: Design for Systems with Pure Time Delay

61. ▲Consider the system with the transfer function $e^{-Ts}G(s)$, where,

$$G(s) = \frac{1}{s(s+1)(s+2)}.$$

The Smith compensator for this system is given by

$$D_c'(s) = \frac{D_c}{1 + (1 - e^{-sT})G(s)D_c}.$$

Plot the frequency response of the compensator for $T = 5$ and $D_c = 1$, and draw a Bode plot that shows the gain and phase margins of the system.[2]

Solution:

This problem can be solved using a few different approaches. A computer tool such as MATLAB (see the MATLAB command `pade`) can be used to calculate the exact magnitude and phase of

[2] This problem was given by Åström (1977).

$D_c'(s)$, or a pade approximation can be made for the e^{-sT} terms,

$$e^{-sT} \approx \frac{2 - sT + (-sT)^2/2! + (-sT)^3/3! + \ldots}{2 + sT + (sT)^2/2! + (sT)^3/3! + \ldots}.$$

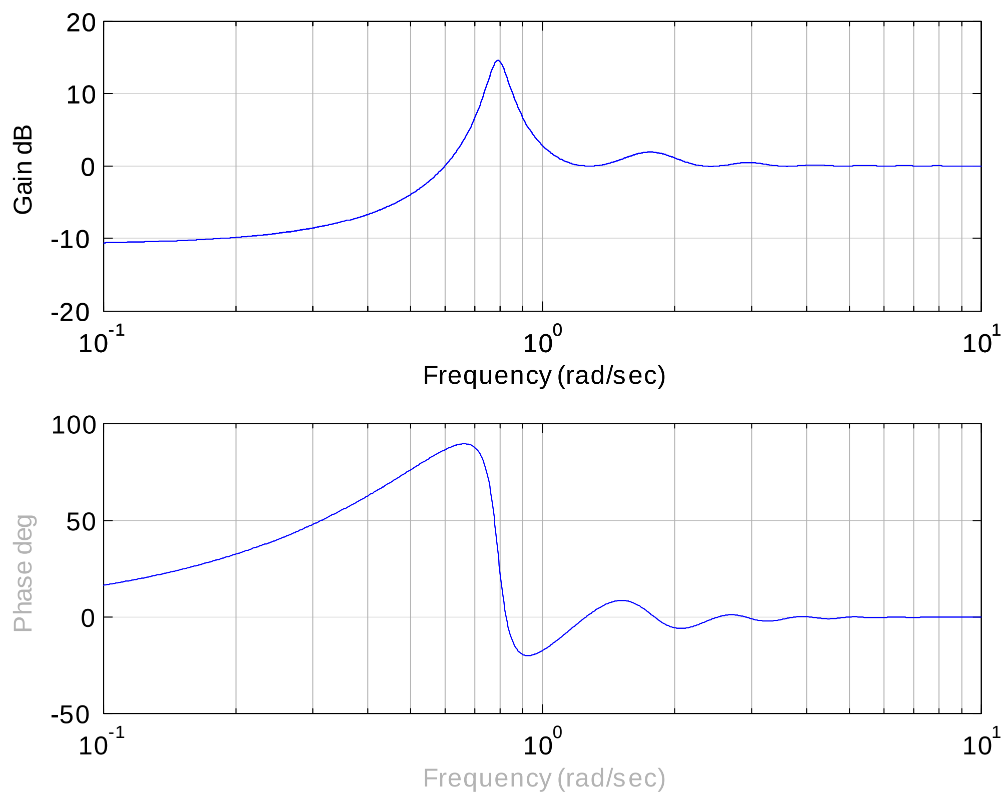
We will show both the exact calculation and Bode plots using a fourth-order pade approximation. The Bode plot of the compensator

$$D_c(s) = \frac{1}{1 + [1 - e^{-sT}]G(s)},$$

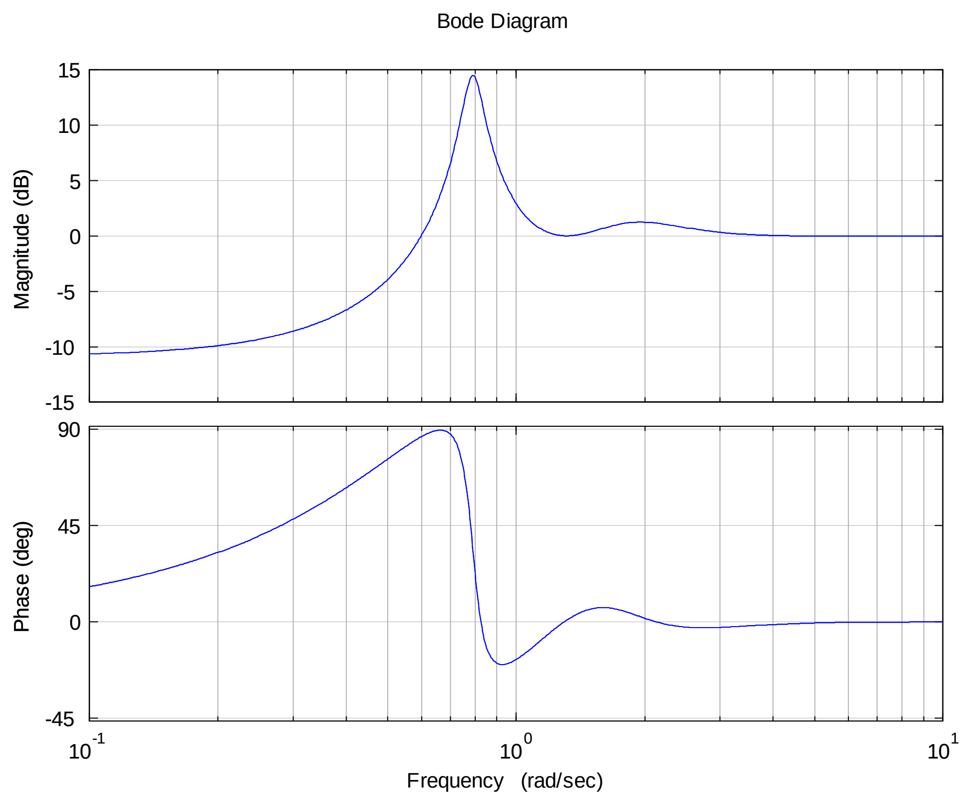
appears in the figures, with,

$$G(s) = \frac{1}{s(s+1)(s+2)}, \text{ and } T = 5.$$

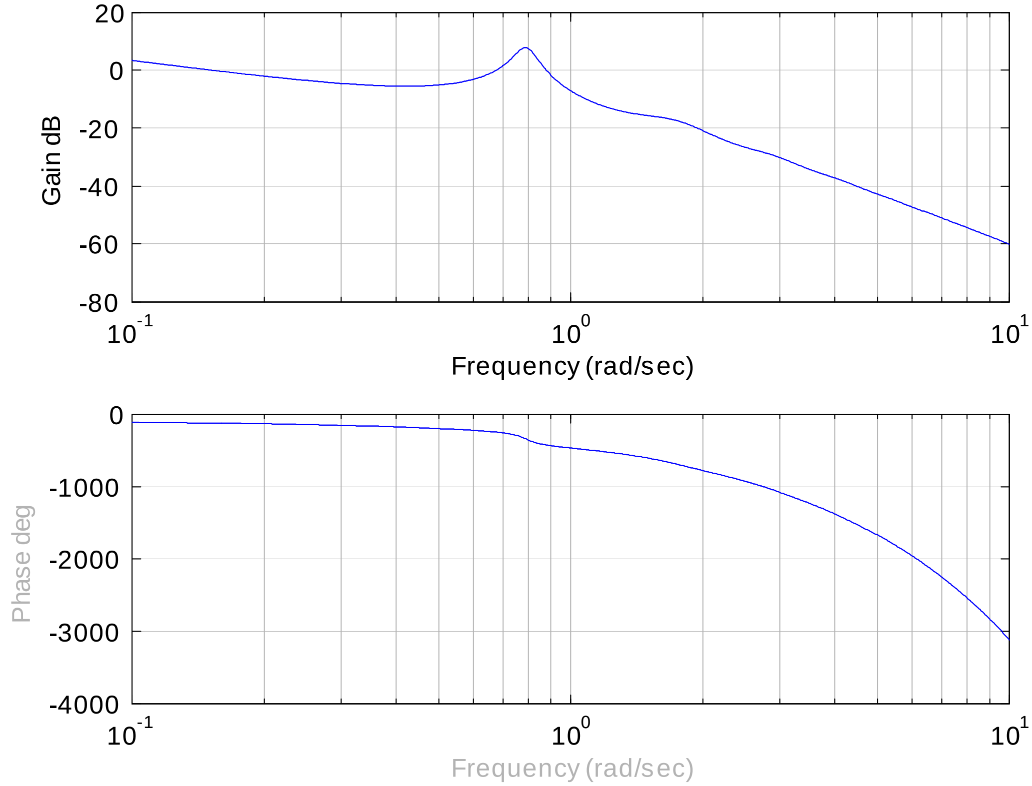
The Bode plot of the closed-loop system is also shown.



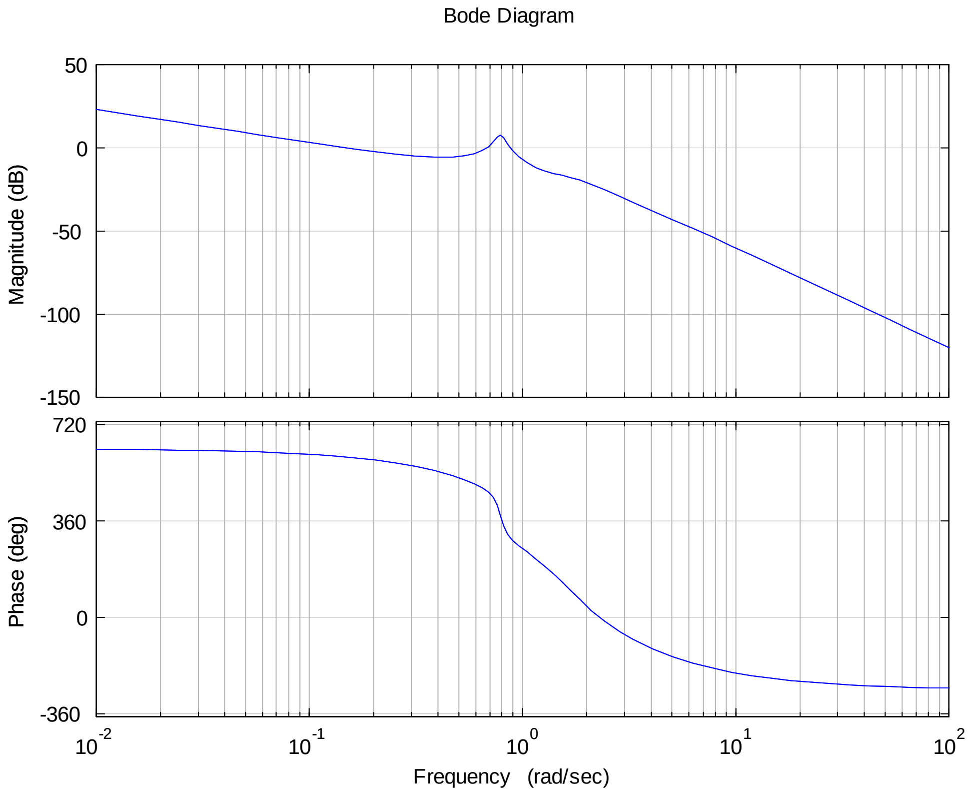
Problem 7.61 Bode plot of compensator $D_c'(s)$: exact.



Problem 7.61: Pade approximation.



Problem 7.61 Bode plot of closed-loop system: exact.



Problem 7.61 Bode plot of closed-loop system: Pade approximation.

Remark: Note that the Smith compensator, $D_c^{'}(s)$, is structured such that the closed-loop transfer function is,

$$\mathcal{T}(s) = \frac{G(s)}{1 + G(s)D_c(s)} e^{-Ts}.$$

Chapter 8

Digital Control

Problems and Solutions for Section 8.2

1. The z-transform of a discrete-time filter $h(k)$ at a $1Hz$ sample rate is

$$H(z) = \frac{1 + (1/2)z^{-1}}{[1 - (1/2)z^{-1}][1 + (1/3)z^{-1}]}.$$

   (a) Let $u(k)$ and $y(k)$ be the discrete input and output of this filter. Find a difference equation relating $u(k)$ and $y(k)$.

   (b) Find the natural frequency and damping coefficient of the filter's poles

   (c) Is the filter stable?

   Solution:

   (a) Find a difference equation :

$$H(z) = \frac{Y(z)}{U(z)} = \frac{1 + (1/2)\,z^{-1}}{[1 - (1/2)z^{-1}]\,[1 + (1/3)z^{-1}]}$$

$$\Longrightarrow \quad Y(z) - \frac{1}{6}z^{-1}Y(z) - \frac{1}{6}z^{-2}y(z) = U(z) + \frac{1}{2}z^{-1}U(z)$$

$$\Longrightarrow \quad y(k) - \frac{1}{6}y(k-1) - \frac{1}{6}y(k-2) = u(k) + \frac{1}{2}u(k-1)$$

   (b) Two poles at $z = 1/2$ and $z = -1/3$ in z-plane.

$$z = e^{sT} \Longrightarrow s = \frac{-0.693}{T} \text{ and } s = \frac{-1.10 + 3.14j}{T} \text{ in s-plane,}$$

   where T is the sampling period. Since the sample rate is 1 Hz, $T = 1$ sec.

$$\begin{aligned} \text{For } z &= \frac{1}{2},\ \omega_n = \frac{0.693}{T} = 0.693 \text{ rad/sec, } \zeta = 1.0 \\ \text{For } z &= \frac{-1}{3},\ \omega_n = \frac{3.33}{T} = 3.33 \text{ rad/sec, } \zeta = 0.330 \end{aligned}$$

(c) Yes, both poles are inside the unit circle.

2. Use the z-transform to solve the difference equation

$$y(k) - 3y(k-1) + 2y(k-2) = 2u(k-1) - 2u(k-2),$$

where

$$u(k) = \begin{cases} k, & k \geq 0, \\ 0, & k < 0, \end{cases}$$
$$y(k) = 0, \qquad k < 0.$$

Solution:

$$\begin{aligned} \frac{Y(z)}{U(z)} &= \frac{2(z^{-1} - z^{-2})}{1 - 3z^{-1} - 2z^{-2}} = \frac{2}{z-2} \\ u(k) &= \begin{Bmatrix} k & k \geq 0 \\ 0 & k < 0 \end{Bmatrix} \end{aligned}$$

$$\Longrightarrow U(z) = \frac{z}{(z-1)^2}$$

$$Y(z) = \frac{2}{z-2} \times \frac{z}{(z-1)^2} = \frac{2z}{z-2} - \frac{2z}{z-1} - \frac{2z}{(z-1)^2}$$

Taking the inverse z-transform from Table 8.1,

$$y(k) = 2(2^k - 1 - k) \quad (k \geq 0)$$

3. The one-sided z-transform is defined as

$$F(z) = \sum_0^\infty f(k) z^{-k}.$$

(a) Show that the one-sided transform of $f(k+1)$ is $\mathcal{Z}\{f(k+1)\} = zF(z) - zf(0)$.

(b) Use the one-sided transform to solve for the transforms of the Fibonacci numbers generated by the difference equation $u(k+2) = u(k+1) + u(k)$. Let $u(0) = u(1) = 1$. [*Hint:* You will need to find a general expression for the transform of $f(k+2)$ in terms of the transform of $f(k)$].

(c) Compute the pole locations of the transform of the Fibonacci numbers.

(d) Compute the inverse transform of the Fibonacci numbers.

(e) Show that, if $u(k)$ represents the kth Fibonacci number, then the ratio $u(k+1)/u(k)$ will approach $(1+\sqrt{5})/2$. This is the golden ratio valued so highly by the Greeks.

Solution:

(a)

$$\begin{aligned}
\mathcal{Z}\{f()k+1)\} &= \sum_{k=0}^{\infty} f(k+1)z^{-1} = \sum_{j=1}^{\infty} f(j)z^{1(j-1)},\ k+1=j \\
&= z\sum_{0}^{\infty} f(j)z^{-1} - zf(0) \\
&= zF(z) - zf(0)
\end{aligned}$$

(b)

$$u(k+2) - u(k+1) - u(k) = 0$$

We have :

$$\mathcal{Z}\{f(k+2)\} = z^2F(z) - z^2f(0) - zf(1)$$

Taking the z-transform,

$$z^2U(z) - z^2u(0) - zu(1) - [zU(z) - zu(0)] - U(z) = 0$$

$$\Longrightarrow (z^2 - z - 1)U(z) = (z^2 - z)u(0) + zu(1)$$

Since $u(0) = u(1) = 1$, we have :

$$U(z) = \frac{z^2}{z^2 - z - 1}$$

(c) The poles are at :

$$z = \frac{1 \pm \sqrt{5}}{2} = 1.618,\ -0.618 \triangleq \alpha_1, \alpha_2$$

(d) (i) By long division :

$$\begin{array}{r}
1 + z^{-1} + 2z^{-2} + 3z^{-3} + \cdots \\
1 - z^{-1} - z^{-2}\,\overline{)\,1 \qquad\qquad\qquad\qquad} \\
\underline{1 - z^{-1} - z^{-2}} \qquad\qquad \\
z^{-1} + z^{-2} \qquad\qquad \\
\underline{z^{-1} - z^{-2} - z^{-3}} \qquad \\
2z^{-2} + z^{-3} \qquad \\
\underline{2z^{-2} - 2z^{-3} - 2z^{-4}} \\
3z^{-3} + 2z^{-4} \\
\cdots
\end{array}$$

$$u(k) = 1,\ 1,\ 2,\ 3,\ 5, \cdots$$

(ii) By partial fraction expansion :

$$\begin{aligned}
U(z) &= \frac{1}{1-z^{-1}-z^{-2}} = \frac{1}{(1-\alpha_1 z^{-1})(1-\alpha_2 z^{-1})} \\
&= \frac{\left(\dfrac{\alpha_1}{\alpha_1-\alpha_2}\right)}{1-\alpha_1 z^{-1}} + \frac{\left(\dfrac{\alpha_2}{\alpha_2-\alpha_1}\right)}{1-\alpha_2 z^{-1}} \\
u(k) &= \frac{\alpha_1}{\alpha_1-\alpha_2}\alpha_1^k + \frac{\alpha_2}{\alpha_2-\alpha_1}\alpha_2^k \\
&= \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^k
\end{aligned}$$

(e) Since $|\alpha_2| < 1$, for large k the second term is $\cong 0$, and the ratio of $u(k+1)$ to $u(k)$ is $\alpha_1 = (1+\sqrt{5})/2$.

4. Prove the seven properties of the s-plane-to-z-plane mapping listed in Section 8.2.3.

Solution

(a) The stability boundary in s-plane is :

$$s = j\omega, \text{ for all } \omega \text{ between } [-\infty, \infty]$$

By $z = e^{sT}$, this boundary is mapped to :

$$\begin{aligned}
z &= e^{j\omega T} = \cos\omega T + j\sin\omega T \\
\Longrightarrow & \quad |z| = |\cos\omega T + j\sin\omega T|
\end{aligned}$$

Thus, the unit circle in z-plane represents the stability boundary.

(b) In the small vicinity around $s = 0$ in the s-plane,

$$s = -\sigma \pm j\omega_d$$

where $\sigma \ll \omega_s = \frac{2\pi}{T}$ and $\omega_d \ll \omega_s = \frac{2\pi}{T}$.
By $z = e^{sT}$, corresponding locations relative to 1 in the z-plane are :

$$\begin{aligned}
z-1 &= e^{(-\sigma \pm j\omega_d)T} - 1 \\
&= e^{-\sigma T}(\cos\omega_d T \pm j\sin\omega_d T) - 1 \\
&\cong \left\{1 + \frac{(-\sigma T)}{1!} + \frac{(-\sigma T)^2}{2!} + ...\right\}(1 \pm j\omega_d T) - 1 \\
&= 1 - \sigma T \pm j\omega_d T \mp j\sigma T\omega_d T - 1 \\
&\cong -\sigma T \pm j\omega_d T
\end{aligned}$$

Thus, is approximately mapped to $z - 1 = -\sigma T \pm j\omega_d T$. The small vicinity around $z = +1$ in the z-plane is identical to the vicinity around $s = 0$ in the s-plane by a factor of $T = \frac{2\pi}{\omega_s}$.

(c) An arbitrary location in the s-plane is represented by :

$$\begin{aligned} s &= -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} \\ &= x_s(\omega_n) + jy_s(\omega_n) \end{aligned}$$

where ω_n is in rad/sec and is nondimensional. Thus, the s-plane locations give response information in terms of frequency.
By $z = e^{sT} = e^{s\frac{2\pi}{\omega_s}}$, the corresponding location in the z-plane is :

$$\begin{aligned} z &= e^{-2\pi\frac{\omega_n}{\omega_s}} \cos\left(2\pi\sqrt{1-\zeta^2}\frac{\omega_n}{\omega_s}\right) + je^{-2\pi\frac{\omega_n}{\omega_s}} \sin\left(2\pi\sqrt{1-\zeta^2}\frac{\omega_n}{\omega_s}\right) \\ &= x_z\left(\frac{\omega_n}{\omega_s}\right) + jy_z\left(\frac{\omega_n}{\omega_s}\right) \end{aligned}$$

Since $\frac{\omega_n}{\omega_s}$ is nondimensional, the z-plane locations give response information normalized to the sample rate.

(d) Locations in the s-plane, $s = -\sigma \pm j\omega_d$, are mapped to z-plane locations :

$$z = e^{-\sigma T}(\cos\omega_d T \pm j\sin\omega_d T)$$

If z is on the negative real axis, we need :

$$\begin{aligned} \cos\omega_d T &< 0,\ \sin\omega_d T = 0 \\ &\Longrightarrow \omega_d T = 2\pi n + \pi,\ n = 0, 1, 2, ... \\ &\Longrightarrow \omega_d = (2n+1)\frac{\omega_s}{2},\ n = 0, 1, 2, ... \end{aligned}$$

Indeed, if $\omega_d = (2m+1)\frac{\omega_s}{2},\ n = 0, 1, 2, ...,$

$$z = -e^{-\sigma T} \Longrightarrow \text{negative real axis}$$

Thus, the negative real z-axis represents a horizontal line with a damped frequency :

$$\omega_d = (2n+1)\frac{\omega_s}{2},\ n = 0, 1, 2, ...$$

(e) An arbitrary vertical line in the left half of the s-plane is represented by :

$$s = -\sigma \pm j\omega_d,\ \sigma > 0,\ \text{for all } \omega_d \text{ between } [-\infty, \infty]$$

By $z = e^{sT}$, the vertical line is mapped to :

$$z = -e^{-\sigma T}e^{\pm j\omega_d T} = -e^{-\sigma T}(\cos\omega_d T \pm j\sin\omega_d T)$$

$$\Longrightarrow |z| = r = |-e^{-\sigma T}|\ = \text{constant} < 1$$

$$\angle z = 0 \longrightarrow 2\pi \text{ as } \omega_d = 2n\pi \longrightarrow (2n+1)\pi$$

Thus, vertical lines in the left half of the s-plane map into circles within the unit circle of the z-plane.

(f) An arbitrary horizontal line in the s-plane is represented by :

$$s = -\sigma \pm j\omega_d, \text{ for } \sigma \text{ between } [-\infty, \infty], \text{ at a given } \omega_d$$

By $z = e^{sT}$, the horizontal line is mapped to :

$$z = -e^{-\sigma T} e^{\pm j\omega_d T} = -e^{-\sigma T}(\cos\omega_d T \pm j \sin\omega_d T)$$

$$\Longrightarrow \angle z = \tan^{-1}\left(\frac{e^{-\sigma T}\sin\omega_d T}{e^{-\sigma T}\cos\omega_d T}\right) = \omega_d T = \text{ constant}$$

$$|z| = r = 0 \longrightarrow \infty \text{ as } \sigma = \infty \longrightarrow -\infty$$

Thus, a horizontal line in the s-plane maps into radial lines in the z-plane.

(g) Let s-plane locations s_1 and s_2 be :

$$\begin{aligned} s_1 &= -\sigma \pm j\omega_d \\ s_2 &= -\sigma \pm j\left(\omega_d + m\omega_s\right), \quad m = 1, 2, 3, ... \end{aligned}$$

where ω_d is between $\left[-\frac{2\pi}{\omega_s},\ \frac{\omega_s}{2}\right]$, which is called the "primary strip".

By $z = e^{sT} = e^{s\frac{2\pi}{\omega_s}}$, these s-plane locations are mapped to z-plane locations :

$$\begin{aligned} z_1 &= e^{-\sigma\frac{2\pi}{\omega_s}}\left(\cos\omega_d\frac{2\pi}{\omega_s} + j\sin\omega_d\frac{2\pi}{\omega_s}\right) \\ z_2 &= e^{-\sigma\frac{2\pi}{\omega_s}}\left\{\cos\left(\omega_d + 2m\frac{\omega_s}{2}\right)\frac{2\pi}{\omega_s} + j\sin\left(\omega_d + 2m\frac{\omega_s}{2}\right)\frac{2\pi}{\omega_s}\right\} \\ &= e^{-\sigma\frac{2\pi}{\omega_s}}\left\{\cos\left(\omega_d\frac{2\pi}{\omega_s}\right)\cos 2m\pi - \sin\left(\omega_d\frac{2\pi}{\omega_s}\right)\sin 2m\pi + j\sin\left(\omega_d\frac{2\pi}{\omega_s}\right)\cos 2m\pi + ...\right\} \\ &= e^{-\sigma\frac{2\pi}{\omega_s}}\left\{\cos\left(\omega_d\frac{2\pi}{\omega_s}\right) + j\sin\left(\omega_d\frac{2\pi}{\omega_s}\right)\right\} \\ &= z_1 \end{aligned}$$

Thus, frequencies greater than $\frac{\omega_s}{2}$ appear in the z-plane on top of corresponding lower frequencies. Physically, this means that frequencies sampled faster than $\frac{\omega_s}{2}$ will appear in the samples to be at a much lower frequency. This is called "aliasing".

Problems and Solutions for Section 8.3

5. A unity feedback system has an open-loop transfer function given by

$$G(s) = \frac{250}{s[(s/10)+1]}.$$

The following lag compensator added in series with the plant yields a phase margin of 50°:

$$D(s) = \frac{s/1.25+1}{50s+1}.$$

Using the matched pole-zero approximation, determine an equivalent digital realization of this compensator.

Solution:

(a) For the compensated closed-loop system, $\dfrac{D_c(s)G(s)}{1+D_c(s)G(s)}$, the bandwidth is approximately 3 rad/sec. A very safe sample rate would be faster than ω_n by a factor of 20. So choose a sample rate, ω_s, to be :

$$\omega_s = 20 \times 3 = 60 \text{ rad/sec} \cong 10 \text{ Hz}$$

So the sample rate is $T = 0.1$ sec.
Since

$$D_c(s) = \frac{1+s/1.25}{1+s/0.02} = 0.016\frac{s+1.25}{s+0.02},$$

an equivalent $D_c(z)$ is found for the matched pole-zero method by using method summarized in Section 8.3.1. Step 1 maps the pole and zero, while Eq. (8.22) shows how to compute the gain. The result is,

$$D_c(s) = 0.0170\frac{z-0.8825}{z-0.9980}.$$

6. The following transfer function is a lead network designed to add about 60° of phase at $\omega_1 = 3$ rad/sec:

$$H(s) = \frac{s+1}{0.1s+1}.$$

(a) Assume a sampling period of $T = 0.25$ sec, and compute and plot in the z-plane the pole and zero locations of the digital implementations of $H(s)$ obtained using (1) Tustin's method and (2) pole-zero mapping. For each case, compute the amount of phase lead provided by the network at $z_1 = e^{j\omega_1 T}$

(b) Using a log-log scale for the frequency range $\omega = 0.1$ to $\omega = 100$ rad/sec, plot the magnitude Bode plots for each of the equivalent digital systems you found in part (a), and compare with $H(s)$. (*Hint:* Magnitude Bode plots are given by $|H(z)| = |H(e^{j\omega T})|$.)

Solution:

(a)

$$H(s) = \frac{s+1}{0.1s+1}, \ \angle H(j\omega)|_{\omega=3} = 54.87^\circ$$

From MATLAB, [mag,phasew1] = bode([1 1],[.1 1],3) yields phasew1 = 54.87.

(1) Tustin's method, analytically :

$$\begin{aligned} H(z) &= H(s)|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{(2+T)+(T-2)z^{-1}}{(0.2+T)+(T-0.2)z^{-1}} \\ &= 5\frac{z-0.7778}{z+0.1111} \end{aligned}$$

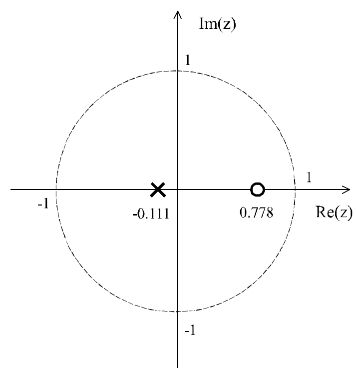
or, via MATLAB:

```
sysC = tf([1 1],[.1 1]);
sysDTust = c2d(sysC,T,'tustin')
```

Phase lead at $\omega_1 = 3$: $\angle H(e^{j\omega_1 T}) = 54.90^\circ$, which is most easily obtained by MATLAB

```
[mag,phasew1] = bode(sysDTust,3)
```

The pole-zero plot is:



(2) Matched pole-zero method, analytically :

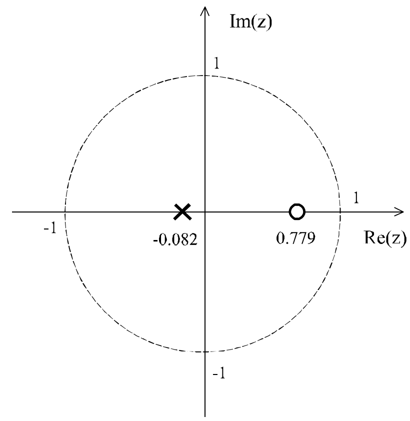
$$\begin{aligned} H(z) &= K\frac{z-e^{-1T}}{z-e^{-10T}} = 4.150\frac{z-0.7788}{z-0.0821} \\ K &= 4.150 \Longrightarrow |H(z)|_{z=1} = |H(s)|_{s=0} \end{aligned}$$

or, alternatively via MATLAB

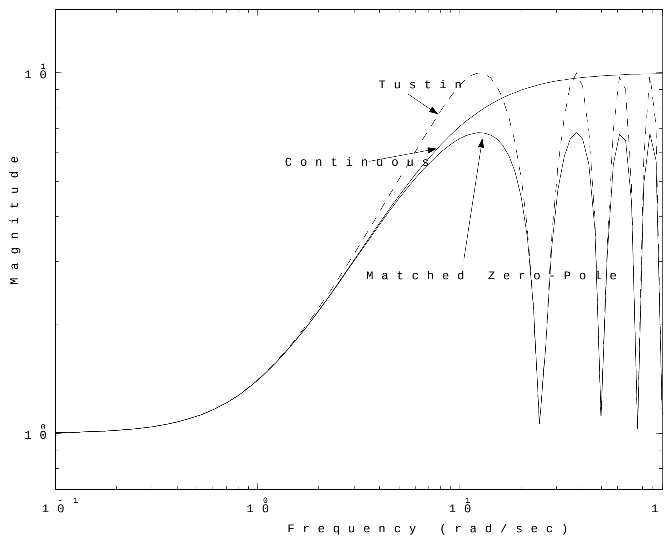
```
sysDmpz = c2d(sysC,T,'matched')
```

will produce the same result.

Phase lead at $\omega_1 = 3$: $\angle H(e^{j\omega_1 T}) = 47.58^\circ$ is obtained from [mag,phasew1] = bode(sysDmpz,3). The pole-zero plot is below. Note how similar the two pole-zero plots are.



(b) The Bode plots match fairly well until the frequency approaches the half sample frequency ($\cong$ 12 rad/sec), at which time the curves diverge.



7. The following transfer function is a lag network designed to introduce a gain attenuation of 10(-20dB) at $\omega_1 = 3$ rad/sec:

$$H(s) = \frac{10s+1}{100s+1}.$$

(a) Assume a sampling period of $T = 0.25\,\text{sec}$, and compute and plot in the z-plane the pole and zero locations of the digital implementations of $H(s)$ obtained using (1) Tustin's method and (2) pole-zero mapping. For each case, compute the amount of gain attenuation provided by the network at $z_1 = e^{j\omega_1 T}$.

(b) For each of the equivalent digital systems in part (a), plot the Bode magnitude curves over the frequency range $\omega = 0.01$ to 10 rad/sec.

Solution:

(a) First, we'll compute the attenuation of the continuous system,

$$H(s) = \frac{10s+1}{100s+1}, \; |H(j\omega)|_{\omega=3} = 0.1001 \quad (-20 \text{ db})$$

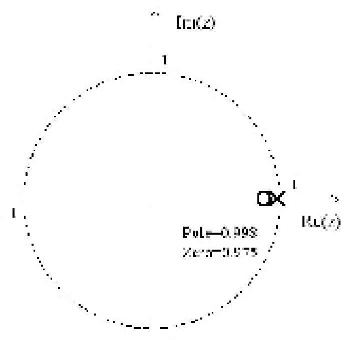
(1) Tustin's method :

$$\begin{aligned} H(z) &= H(s)|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{(20+T)+(T-20)z^{-1}}{(200+T)+(T-200)z^{-1}} \\ &= 0.10112\frac{z-0.97531}{z+0.99750} \end{aligned}$$

or, use c2d as shown for problem 5.
Gain attenuation at $\omega_1 = 3$: $|H(e^{j\omega_1 T})| = 0.1000$ $(-20$ db), most easily computed from: [mag,phase]=bode(sysDTust,T,3).
(2) Matched pole-zero method :

$$\begin{aligned} H(z) &= K\frac{z-e^{-0.1T}}{z-e^{-0.01T}} = 0.10113\frac{z-0.97531}{z-0.99750} \\ K &= 0.10113 \longleftarrow |H(z)|_{z=1} = |H(s)|_{s=0} \end{aligned}$$

Gain attenuation at $\omega_1 = 3$: $|H(e^{j\omega_1 T})| = 0.1001$ $(-20$ db), most easily computed from: [mag,phase]=bode(sysDmpz,T,3).



In this case, the sampling rate is so fast compared to the break frequencies that both methods give essentially the same equivalent, and both have a gain attenuation of a factor of 10 at $\omega_1 = 3$ rad/sec.

(b) All three are essentially the same and indistinguishable on the plot because the range of interest is below the half sample frequency (= 12 rad/sec).

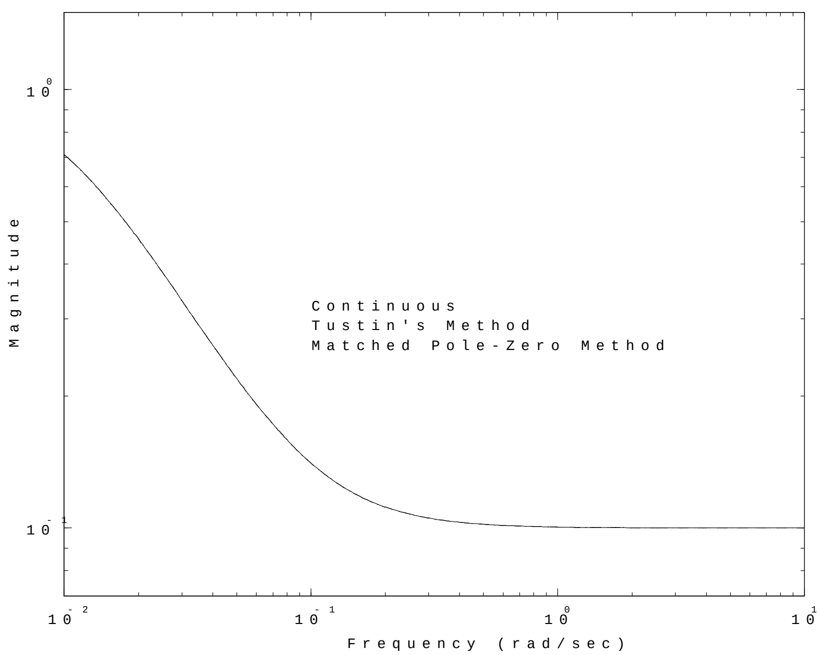
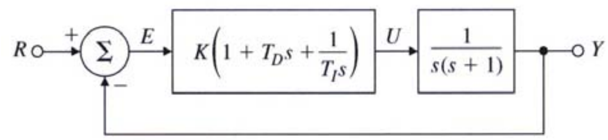


Figure 8.20: Control system for Problem 8



Problems and Solutions for Section 8.5

8. For the system shown in Fig. 8.20, find values for K, T_D, and T_I so that the closed-loop poles satisfy $\zeta > 0.5$ and $\omega_n > 1$ rad/sec. Discretize the PID controller using:

(a) Tustin's method

(b) matched pole-zero method

Use MATLAB to simulate the step response of each of these digital implementations for sample times of $T = 1$, 0.1, and 0.01 sec.

Solution

(a) Continuous PID-controller design

Plant transfer function :

$$G(s) = \frac{1}{s(s+1)}$$

Continuous PID controller :

$$D(s) = K\left(1 + T_D s + \frac{1}{T_I s}\right)$$

There is no requirement that there be an integral term, so first let's look at a design without the integral term. To understand the difficulty, a sketch of the root locus with only proportional control (T_D =0) shows that $K = 1$ yields roots at $s = -0.5 \pm 0.86j$ which means that $\omega = 1$ rad/sec and $\zeta = 0.5$. If we lower or raise the gain, one of the specs will not be met. So the design specifications are marginally met with proportional control only. It would certainly be useful to add a little derivative control in order to pull the locus to the left and provide some margin above the specs. One approach is to try some values of T_D and iterate with rlocus in MATLAB until a comfortable margin is reached on the two specs. Generally, it is also a good design feature to have some integral control in order to reduce steady state errors, so it would make sense to include the integral term. This term can also be designed iteratively by introducing a small amount (large T_I) and adjusting the other gains as needed to meet the specs. Clearly, this problem is underdetermined and there are many ways to meet the specs, a typical situation in control system design.

Another approach for those more mathematically inclined is to evaluate the characteristic equation :

$$\begin{aligned} 1 + D(s)G(s) &= 1 + K\frac{s + T_D s^2 + \frac{1}{T_I}}{s}\frac{1}{s(s+1)} = 0 \\ &\Longrightarrow s^3 + (1 + KT_D)s^2 + K_P s + \frac{K}{T_I} = 0 \end{aligned}$$

Specification :

$$\omega_n > 1 \text{ rad/sec}, \zeta > 0.5$$

Choose the desired dominant closed-loop poles to exceed the specs, a reasonable choice is :

$$s = -0.8 \pm j \Longrightarrow \omega = 1.28 > 1,\ \zeta = 0.625 > 0.5$$

Evaluate the characteristic equation at $s = -0.8 + j$:

$$\left\{1.888 - 0.36(1 + KT_D) - 0.8K_P + \frac{K}{T_D}\right\} + \{0.92 - 1.6(1 + KT_D) + K\}j = 0$$

so the real and complex terms must each equal zero. We somewhat arbitrarily select $T_I = 10.0$, which will provide a fairly low gain on the integral term. Evaluating the expression above yields the K and T_D. So we have:

$$K = 1.817,\ T_D = 0.3912,\ T_I = 10.0$$

Re-arranging some, we have the continuous PID controller transfer function:

$$D(s) = \frac{KT_D(s+\alpha)(s+\beta)}{s}$$

where

$$\begin{aligned} \alpha &= \frac{1}{2T_D} + \frac{1}{2T_D}\sqrt{1 - 4\frac{T_D}{T_I}} \\ \beta &= \frac{1}{2T_D} - \frac{1}{2T_D}\sqrt{1 - 4\frac{T_D}{T_I}} \end{aligned}$$

(b) Discrete PID controller by Tustin's method can be obtained analytically as below or by using `c2d` in MATLAB :

$$
\begin{aligned}
D(z) &= D(s)|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} \\
&= \frac{K\left\{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + T_D\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + \frac{1}{T_I}\right\}}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} \\
&= \frac{\left(K + KT_D\frac{2}{T} + \frac{KT}{2T_I}\right) + \left(-2KT_D\frac{2}{T} + 2\frac{KT}{2T_I}\right)z^{-1} + \left(-K + KT_D\frac{2}{T} + \frac{KT}{2T_I}\right)z^{-1}}{1 - z^{-2}} \\
&= \left\{\begin{array}{cc} \frac{3.3300-2.662z^{-1}-0.305z^{-2}}{1-z^{-2}}, & T = 1 \\ \frac{16.042-28.414z^{-1}+12.408z^{-2}}{1-z^{-2}} & T = 0.1 \\ \frac{143.980-284.322z^{-1}+140.346z^{-2}}{1-z^{-2}} & T = 0.01 \end{array}\right\}
\end{aligned}
$$

(c) For the Matched Pole-zero approximation, note there is one more zero than pole, hence we need to add a pole at $z = -1$,

$$D(z) = K_d\frac{(z - e^{-\alpha T})(z - e^{-\beta T})}{(z + 1)(z - 1)}$$

There is no DC gain for this transfer function, so we can either match the K_v of $D(z)$ with that of $D(s)$ or match the gain at some other frequency. A good choice would be to match the gains at $s = j\omega_n$ for example. (ω_n is the closed-loop natural frequency.) Carrying this out,

$$
\begin{aligned}
D(s)|_{s=j\omega_n} &= KT_D\left\{\frac{1}{T_D} + \left(\omega_n - \frac{1}{T_I T_D\omega_n}\right)j\right\} \\
|D(s)|_{s=j\omega_n}| &= KT_D\sqrt{\left(\frac{1}{T_D}\right)^2 + \left(\omega_n - \frac{1}{T_I T_D\omega_n}\right)^2}
\end{aligned}
$$

$$
\begin{aligned}
D(z)|_{z=e^{j\omega_n T}} &= K_d\frac{A + Bj}{\{\cos(2\omega_n T) - 1\}^2 + \{\sin(2\omega_n T)\}^2} \\
|D(z)|_{z=e^{j\omega_n T}}| &= K_d\frac{\sqrt{A^2 + B^2}}{2 + 2\cos(2\omega_n T)}
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \left\{\cos(2\omega_n T) - \left(e^{-\alpha T} + e^{-\beta T}\right)\cos(\omega_n T) + e^{-(\alpha+\beta)T}\right\}\{\cos(2\omega_n T) - 1\} \\
&\quad + \left\{\sin(2\omega_n T) - (e^{-\alpha T} + e^{-\beta T})\sin(\omega_n T)\right\}\sin(2\omega_n T) \\
B &= \left\{\cos(2\omega_n T) - \left(e^{-\alpha T} + e^{-\beta T}\right)\cos(\omega_n T) + e^{-(\alpha+\beta)T}\right\}\sin(2\omega_n T) \\
&\quad + \left\{\sin(2\omega_n T) - (e^{-\alpha T} + e^{-\beta T})\sin(\omega_n T)\right\}\{\cos(2\omega_n T) - 1\}
\end{aligned}
$$

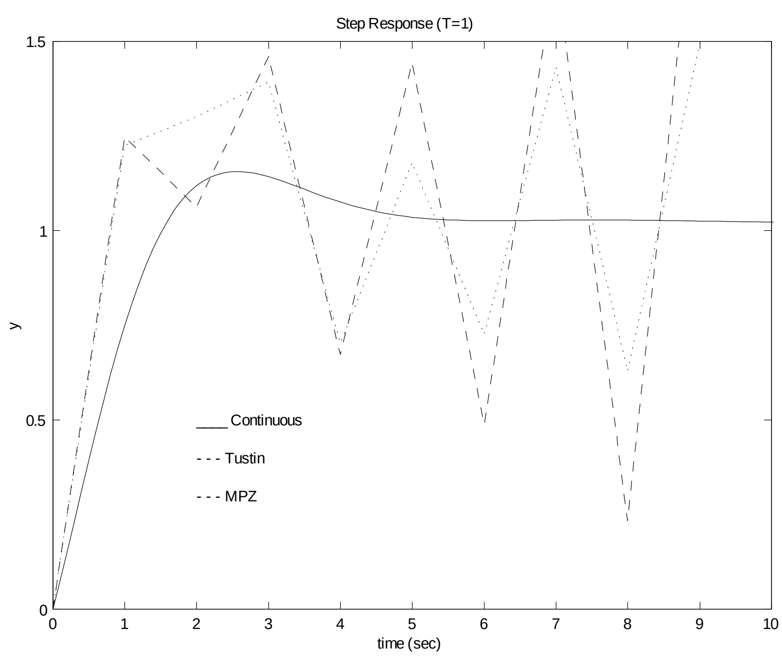
$$|D(s)|_{s=j\omega_n}| = |D(z)|_{z=e^{j\omega_n T}}|$$

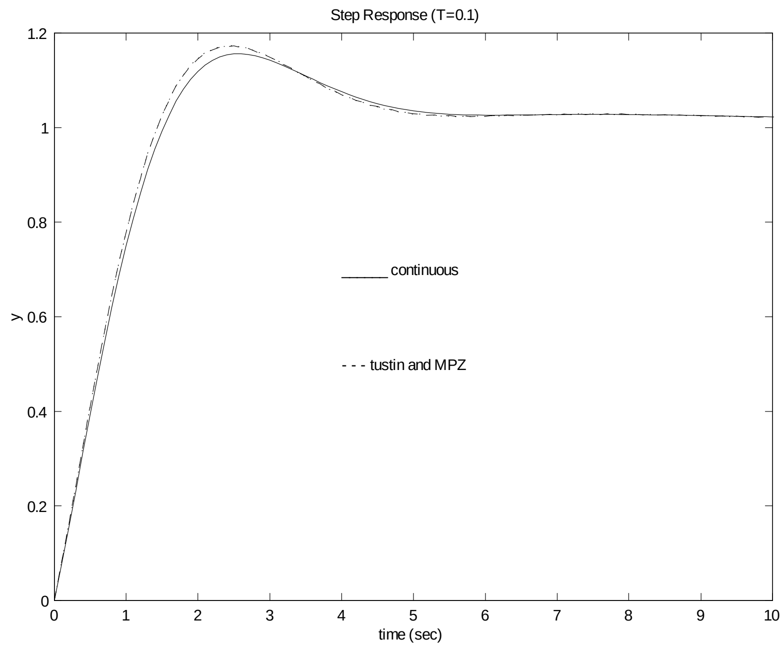
$$\Longrightarrow K_d = KT_D\sqrt{\left(\frac{1}{T_D}\right)^2 + \left(\omega_n - \frac{1}{T_I T_D \omega_n}\right)^2}\frac{2+2\cos(2\omega_n T)}{\sqrt{A^2+B^2}}$$

Thus,

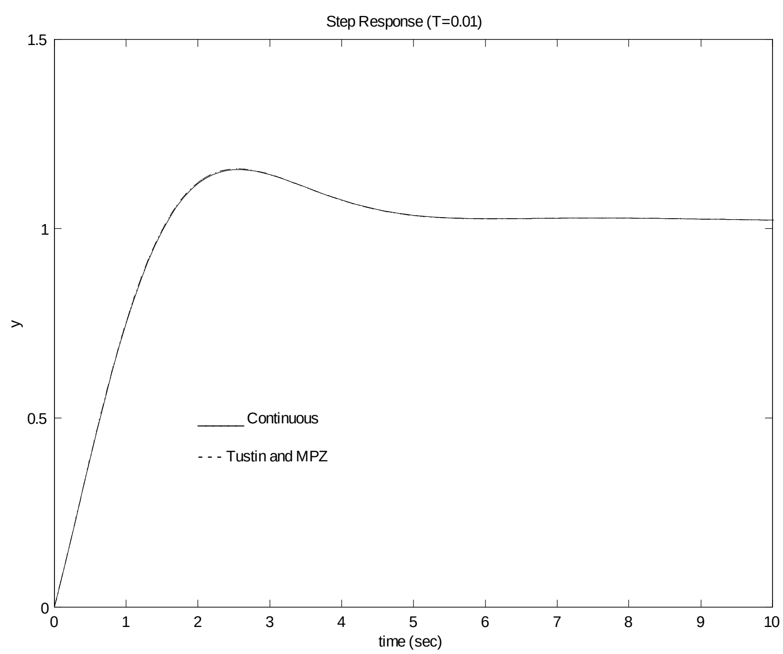
$$\begin{aligned} D(z) &= K_d \frac{1-\left(e^{-\alpha T}+e^{-\beta T}\right)z^{-1}+e^{-(\alpha+\beta)T}z^{-2}}{1-z^{-2}} \\ &= \left\{ \begin{array}{cc} \frac{3.339-3.349z^{-1}-0.263z^{-2}}{1-z^{-2}}, & T=1 \\ \frac{16.092-28.518z^{-1}+12.462z^{-2}}{1-z^{-2}} & T=0.1 \\ \frac{143.985-284.333z^{-1}+140.351z^{-2}}{1-z^{-2}} & T=0.01 \end{array} \right\} \end{aligned}$$

Step responses $(T=1,\ T=0.1,\ T=0.01)$:



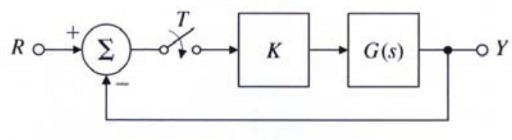


Step Response (T=0.1)
1.2
1
0.8
0.6
0.4
0.2
0
y
0
1
2
3
4
5
6
7
8
9
10
time (sec)
______ continuous
- - - tustin and MPZ



Step Response (T=0.01)
1.5
1
0.5
0
y
0
1
2
3
4
5
6
7
8
9
10
time (sec)
______ Continuous
- - - Tustin and MPZ

Figure 8.21: Control system for Problem 8.8



Problems and Solutions for Section 8.6

9. Consider the system configuration shown in Fig. 8.21, where

$$G(s) = \frac{40(s+2)}{(s+10)(s^2-1.4)}.$$

   (a) Find the transfer function $G(z)$ for $T = 1$ assuming the system is preceded by a ZOH.

   (b) Use MATLAB to draw the root locus of the system with respect to K.

   (c) What is the range of K for which the closed-loop system is stable?

   (d) Compare your results of part (c) to the case where an analog controller is used (that is, where the sampling switch is always closed). Which system has a larger allowable value of K?

   (e) Use MATLAB to compute the step response of both the continuous and discrete systems with K chosen to yield a damping factor of $\zeta = 0.5$ for the continuous case.

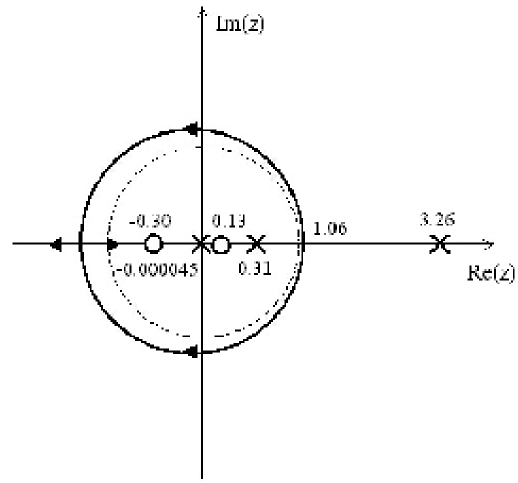
   Solution

(a) Using partial fraction expansion along with Table 8.1,

$$\begin{aligned}
G(z) &= \frac{z-1}{z}\mathcal{Z}\left\{\frac{G(s)}{s}\right\} = \frac{z-1}{z}\mathcal{Z}\left\{\frac{40(s+2)}{s(s+10)(s^2-1.4)}\right\} \\
&= \frac{z-1}{z}\mathcal{Z}\left\{40\left(-\frac{0.1429}{s} + \frac{0.0081}{s+10} + \frac{0.0331}{s+\sqrt{1.4}} + \frac{0.1017}{s-\sqrt{1.4}}\right)\right\} \\
&= \frac{z-1}{z}\mathcal{Z}\left\{40\left(-0.1429\frac{z}{z-1} + 0.0081\frac{z}{z-e^{-10}}\right.\right. \\
&\quad \left.\left. +0.0331\frac{z}{z-e^{-\sqrt{1.4}}} + 0.1017\frac{z}{z-e^{\sqrt{1.4}}}\right)\right\} \\
&= \frac{7.967z^{-1} + 1.335z^{-2} - 0.3245z^{-3}}{1 - 3.571z^{-1} + 1.000z^{-2} - 0.00004540z^{-3}}
\end{aligned}$$

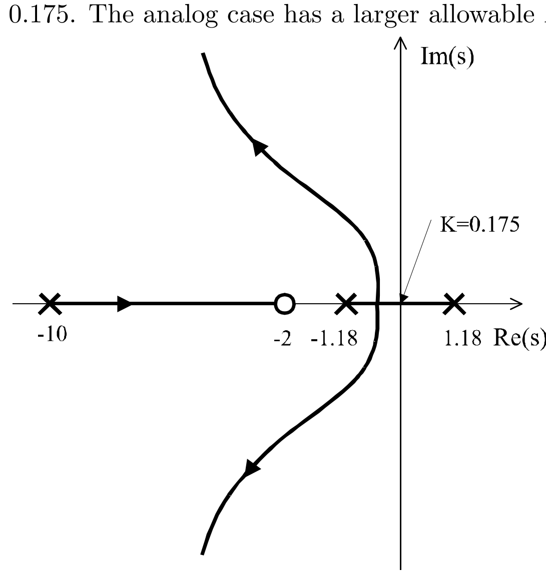
Alternately, we could compute the same result using `c2d` in MATLAB with $G(s)$.

(b) The z-plane root locus is shown.



(c) A portion of the locus is outside the unit circle for any value of K; therefore, the closed-loop system for the discrete case is unstable for all K.

(d) The s-plane root locus is shown. The closed-loop system is stable for $K > 0.175$. The analog case has a larger allowable K.



(e) Since $\zeta = 0.5$ must be achieved, an analytical approach would be to let a desired closed-loop pole be :

$$s_d = \sigma + \sqrt{3}\sigma j$$

Evaluate the continuous characteristic equation at s_d :

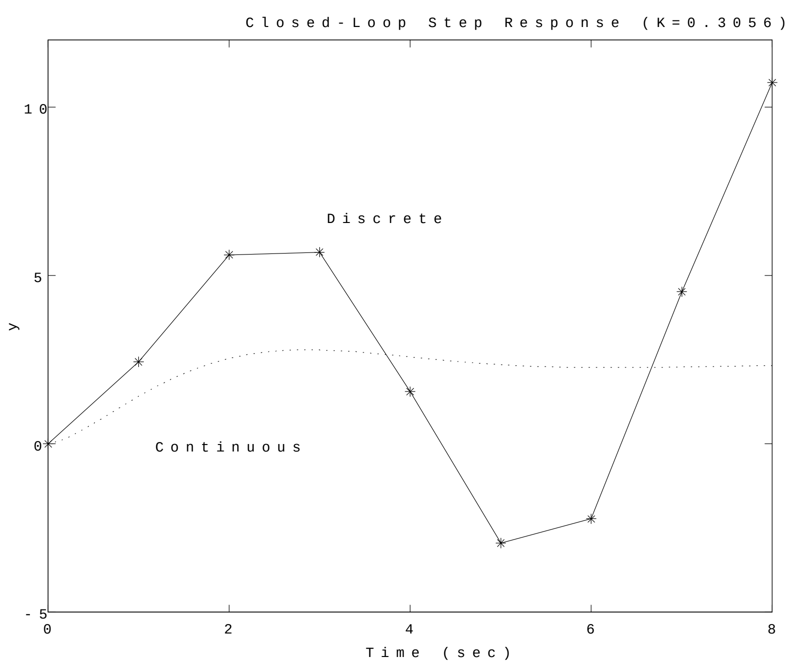
$$\left\{ 1 + \frac{40K(s+2)}{s(s+10)(s^2-1.4)} \right\} |_{s=\sigma+\sqrt{3}\sigma j} = 0$$

and find that a cubic results, i.e., there are three places on the locus where $\zeta = 0.5$.

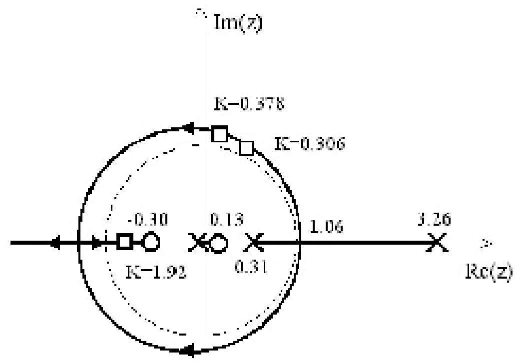
$$\Longrightarrow (-8\sigma^3 - 20\sigma^2 + 40K\sigma - 1.4\sigma + 80K - 14)$$
$$+(34.641\sigma^2 + 40\sqrt{3}K - 1.4\sqrt{3})j = 0$$

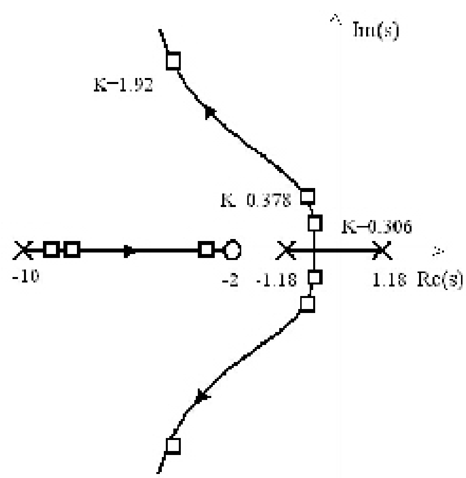
$$\Longrightarrow \quad \sigma = \begin{bmatrix} -3.7732 \\ -0.6857 \\ -0.5411 \end{bmatrix}, \quad K = \begin{bmatrix} 1.9216 \\ 0.3778 \\ 0.3056 \end{bmatrix}$$
$$\Longrightarrow \quad s = \begin{bmatrix} -3.7732 - 6.5354j \\ -0.6857 - 1.1876j \\ -0.5411 - 0.9373j \end{bmatrix}$$
$$\Longrightarrow \quad \omega_n = \begin{bmatrix} 7.5456 \\ 1.3713 \\ 1.0823 \end{bmatrix}, \quad \zeta = 0.5$$

Any of these gains yield a damping factor of $\zeta = 0.5$ for the continuous case; however, we will use the lowest value of K. Alternatively, we could use rlocfind from MATLAB to determine K at the desired $\zeta = 0.5$.
Step responses for $K = 0.3056$:



As expected from the root loci, the discrete case is unstable for this case of quite slow sampling. The z-plane / s-plane root loci with closed-loop poles for 1.9216, 0.3778, 0.3056 marked are shown below:



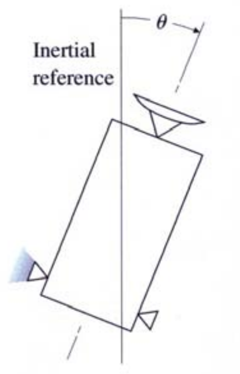


10. *Single-axis Satellite Attitude Control:* Satellites often require attitude control for proper orientation of antennas and sensors with respect to Earth. Figure 2.6 shows a communication satellite with a three-axis attitude-control system. To gain insight into the three-axis problem we often consider one axis at a time. Figure 8.23 depicts this case where motion is only allowed about an axis perpendicular to the page. The equations of motion of the system are given by

$$I\ddot{\theta} = M_C + M_D,$$

where

Figure 8.22: Satellite control schematic for Problem 10



I = moment of inertia of the satellite about its mass center,

M_C =control torque applied by the thrusters,

M_D =disturbance torques,

θ =angle of the satellite axis with respect to an inertial reference with no angular acceleration.

We normalize the equations of motion by defining

$$u = \frac{M_C}{I}, \quad w_d = \frac{M_D}{I},$$

and obtain

$$\ddot{\theta} = u + w_d.$$

Taking the Laplace transform yields

$$\theta(s) = \frac{1}{s^2}[u(s) + w_d(s)],$$

which with no disturbance becomes

$$\frac{\theta(s)}{u(s)} = \frac{1}{s^2} = G_1(s).$$

In the discrete case where u is applied through a ZOH, we can use the methods described in this chapter to obtain the discrete transfer function

$$G_1(z) = \frac{\theta(z)}{u(z)} = \frac{T^2}{2}\left[\frac{z+1}{(z-1)^2}\right].$$

(a) Sketch the root locus of this system by hand assuming proportional control.

(b) Draw the root locus using MATLAB to verify the hand sketch.

(c) Add a discrete velocity feedback to your controller so that the dominant poles correspond to $\zeta = 0.5$ and $\omega_n = 3\pi/(10T)$.

(d) What is the feedback gain if $T = 1\,\text{sec}$? If $T = 2\,\text{sec}$.

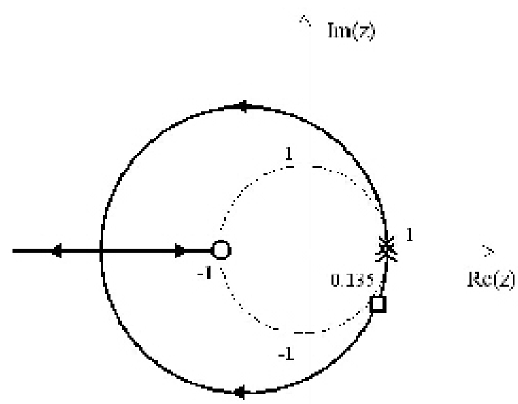
(e) Plot the closed-loop step response and the associated control time history for $T = 1\,\text{sec}$.

Solution

(a) The hand sketch will show that the loci branches depart vertically from $z = 1$; therefore, the system is marginally stable or unstable for any value of gain.

(b) The MATLAB version below confirms the situation.

$$G_1(z) = \frac{T^2}{2}\frac{(z+1)}{(z-1)^2} = K_0\frac{(z+1)}{(z-1)^2}$$



(c) To obtain the desired damping and frequency, Fig. 8.4 shows that a root locus branch should go through the desired poles at $z = 0.44 \pm 0.44j$. After some trial and error, you can find that this can be accomplished with the lead compensation :

$$D(z) = K\frac{(z-0.63)}{(z+0.44)}$$

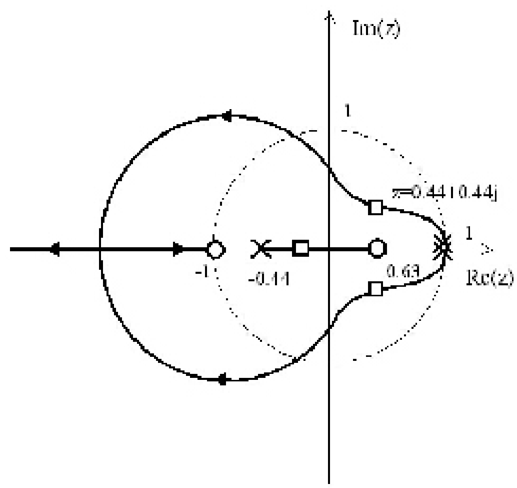
The specific value of K that yields the closed-loop poles are at :

$$z = 0.44 \pm 0.44j,\ -0.113$$

is $K = \frac{0.692}{K_0}$. The second-order pair give :

$$\begin{aligned}\omega_n &= \frac{0.917}{T}\ \text{rad/sec}\\ \zeta &= 0.519\end{aligned}$$

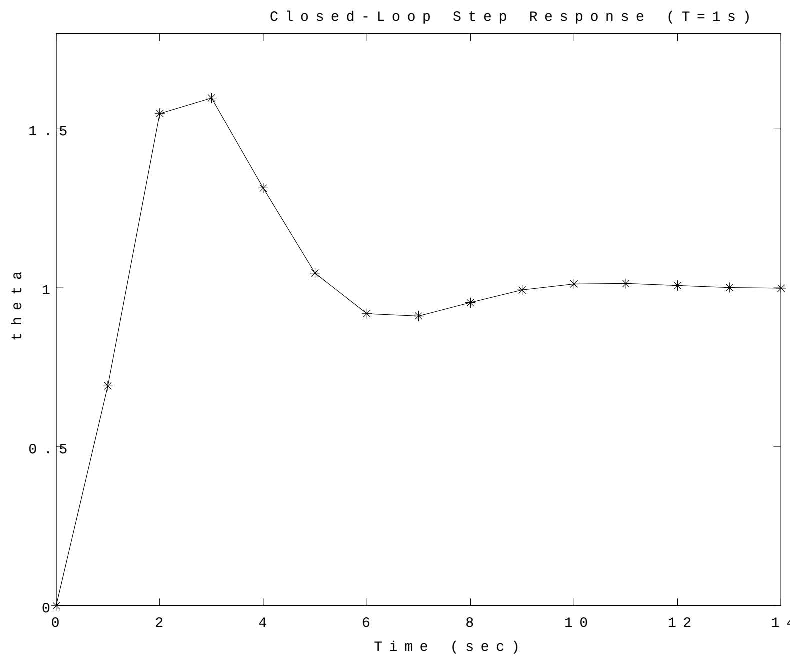
which is close enough to the desired specifications. The root locus for the compensated design is:

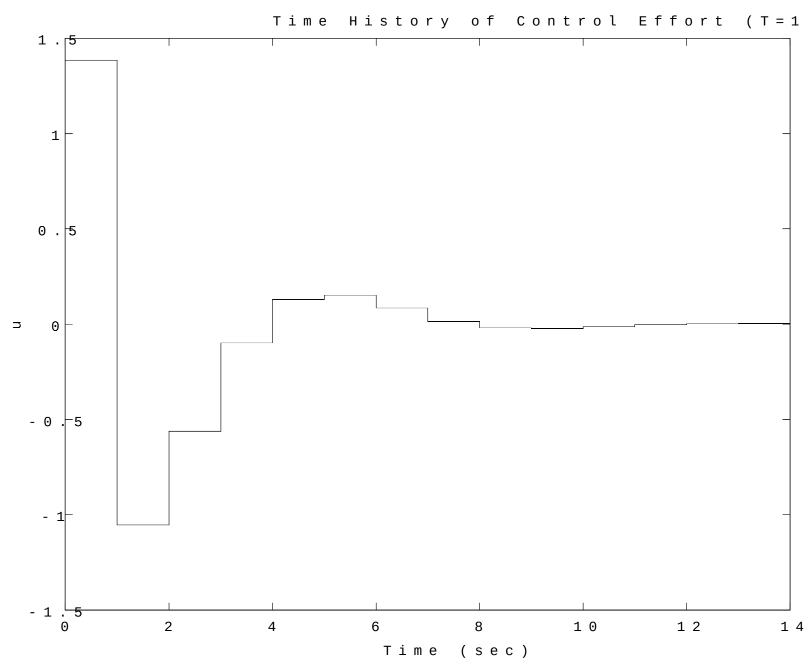


(d)

$$
\begin{aligned}
K &= \frac{0.692}{\frac{T^2}{2}} \\
&= \left\{ \begin{array}{cc} 1.383 & \text{for } T = 1 \text{ sec} \\ 0.3458 & \text{for } T = 2 \text{ sec} \end{array} \right\}
\end{aligned}
$$

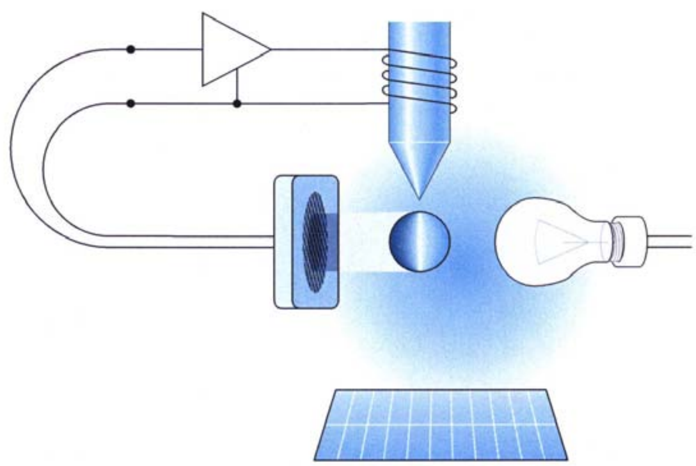
(e) Closed-loop step response :





11. It is possible to suspend a mass of magnetic material by means of an

Figure 8.23: Schematic of magnetic levitation device for Problems11



electromagnet whose current is controlled by the position of the mass (Woodson and Melcher, 1968). The schematic of a possible setup is shown in Fig. 8.23, and a photo of a working system at Stanford University is shown in Fig. 9.2. The equations of motion are

$$m\ddot{x} = -mg + f(x, I),$$

where the force on the ball due to the electromagnet is given by $f(x, I)$. At equilibrium the magnet force balances the gravity force. Suppose we let I_0 represent the current at equilibrium. If we write $I = I_0 + i$, expand f about $x = 0$ and $I = I_0$, and neglect higher-order terms, we obtain the linearized equation

$$m\ddot{x} = k_1 x + k_2 i. \tag{1}$$

Reasonable values for the constants in Eq. (1) are $m = 0.02$ kg, $k_1 = 20$ N/m, and $k_2 = 0.4$ N/A.

(a) Compute the transfer function from I to x, and draw the (continuous) root locus for the simple feedback $i = -Kx$.

(b) Assume the input is passed through a ZOH, and let the sampling period be 0.02 sec. Compute the transfer function of the equivalent discrete-time plant.

(c) Design a digital control for the magnetic levitation device so that the closed-loop system meets the following specifications: $t_r \leq 0.1$ sec, $t_s \leq 0.4$ sec, and overshoot $\leq 20\%$.

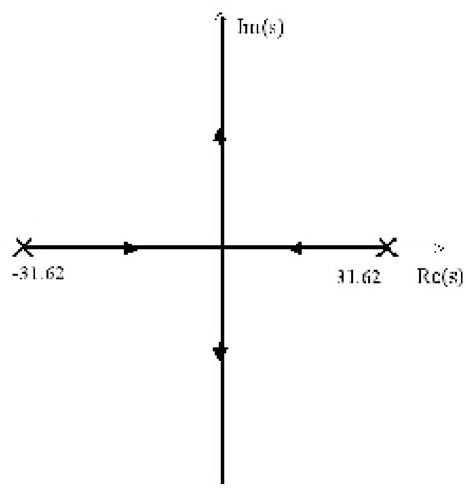
(d) Plot a root locus with respect to k_1 for your design, and discuss the possibility of using your closed-loop system to balance balls of various masses.

(e) Plot the step response of your design to an initial disturbance displacement on the ball, and show both x and the control current i. If the sensor can measure x only over a range of $\pm 1/4$cm and the amplifier can only provide a current of 1 A, what is the *maximum* displacement possible for control, neglecting the nonlinear terms in $f(x, I)$?

Solution:

(a)

$$G(s) = \frac{X(s)}{I(s)} = \frac{k_2/m}{s^2 - k_1/m} \tag{2}$$
$$= \frac{20}{s^2 - 1000}$$



(b) $T = 0.02$ sec,

$$G(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$
$$= 0.004135\frac{z+1}{(z-0.5313)(z-1.8822)}$$

(c) The specifications imply that :

$$\begin{aligned}
t_r &\leq 0.1 \text{ sec} \Longrightarrow \omega_n \geq \frac{1.8}{0.1} = 18 \text{ rad/sec} \\
t_s &\leq 0.4 \text{ sec} \Longrightarrow \sigma \geq \frac{4.6}{0.4} = 11.5 \text{ rad/sec} \\
&\Longrightarrow r = |z| \leq e^{-11.5 \times 0.02} = 0.7945 \ (\longleftarrow z = e^{sT}) \\
M_p &\leq 20\% \ \Longrightarrow \zeta \geq 0.48
\end{aligned}$$

Thus, the closed-loop poles must be pulled into the unit circle near $r = 0.8$ and $\zeta = 0.5$. Using the template of Fig. 8.4, we experiment with lead compensation and select,

$$D(z) = 116 \frac{z - 0.5313}{z - 0.093}$$

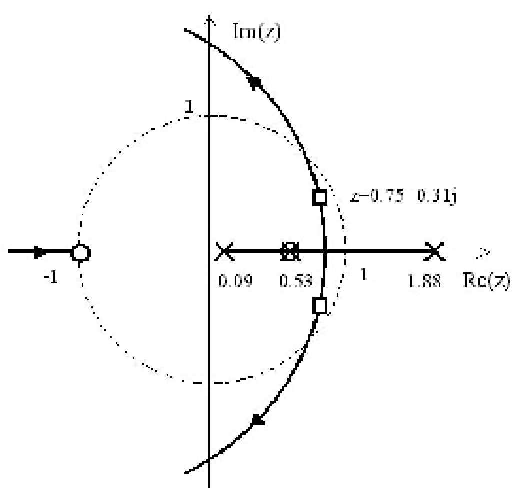
The closed-loop poles are :

$$z = 0.75 \pm 0.39j, \ 0.53$$

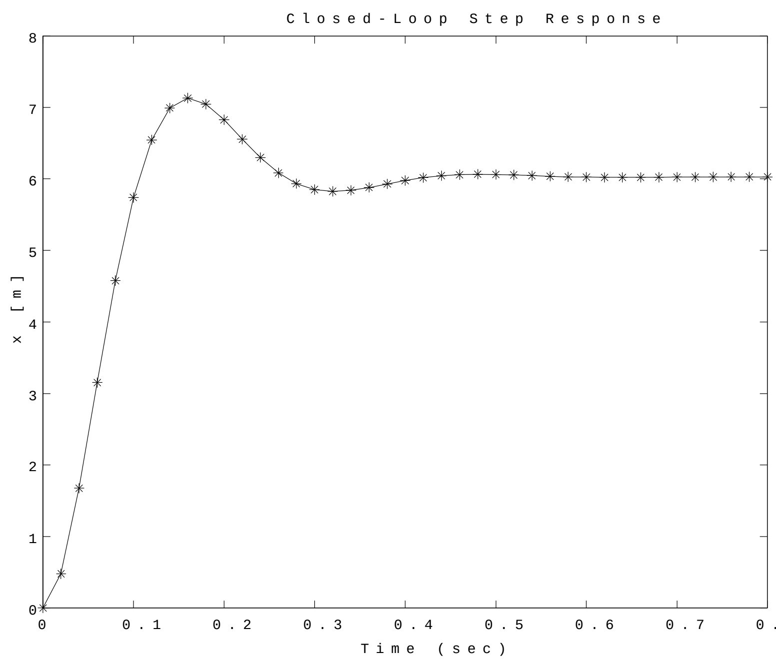
Performance :

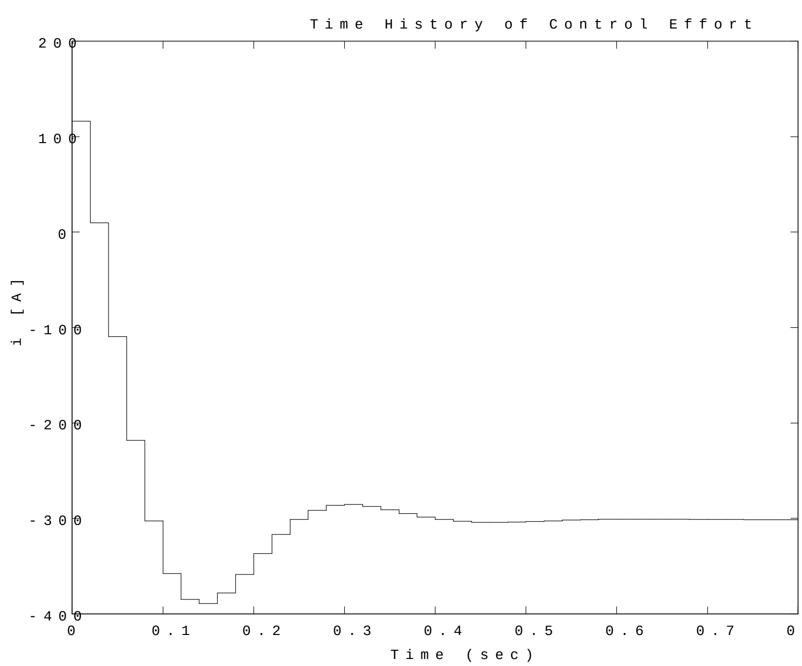
$$\begin{aligned}
t_r &= 0.072 \\
t_s &= 0.397 \\
M_p &= 18.3\%
\end{aligned}$$

which meet all the specifications. The root locus is below.

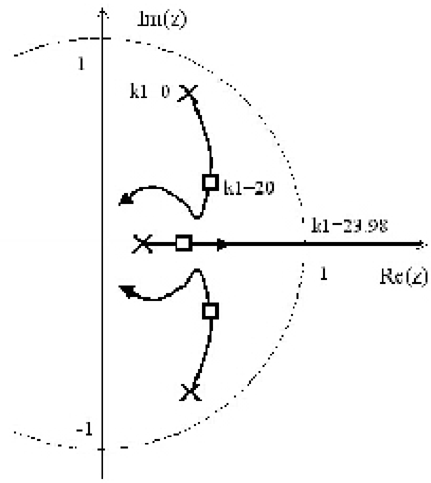


The step response shows M_p



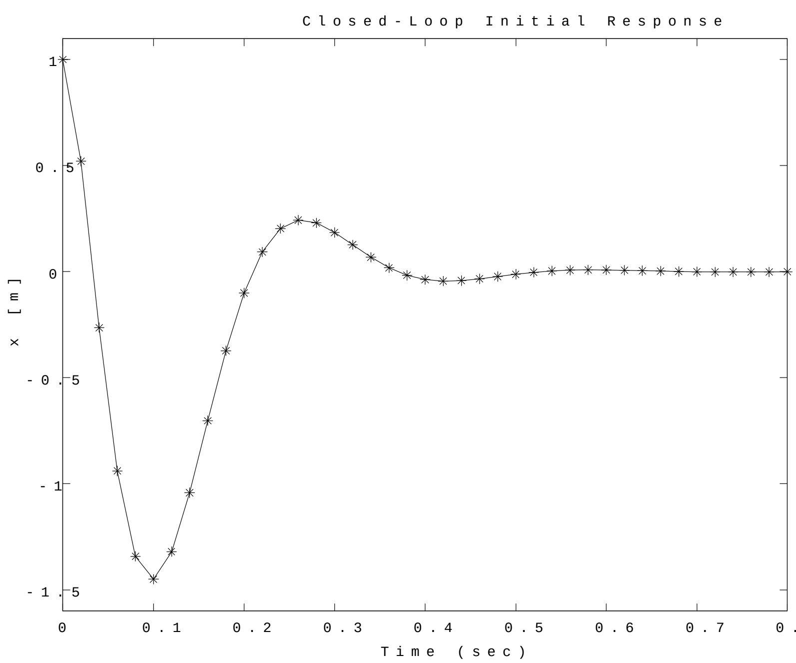


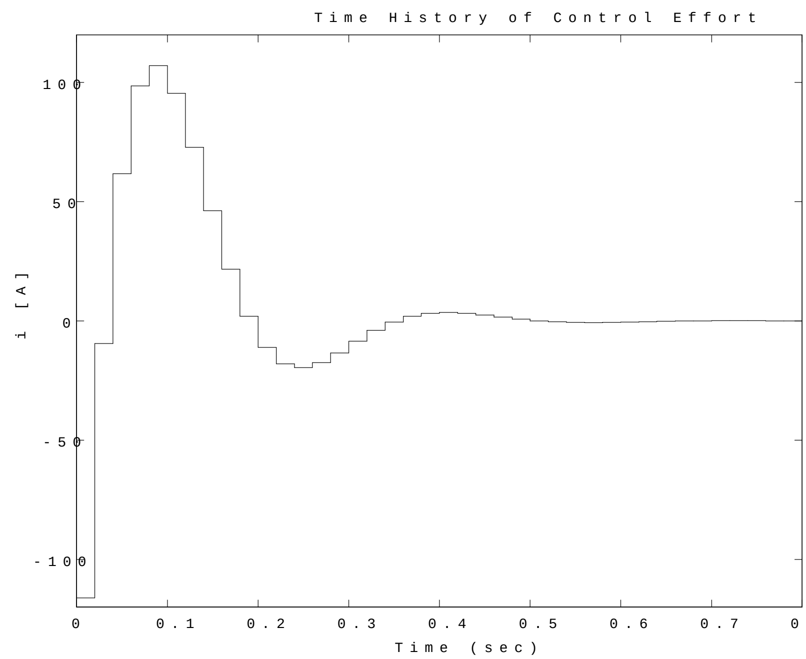
(d) As can be seen from Eq. (2), the loop gain and the open loop pole locations depend on the mass of the ball. Changing the mass will therefore affect the dynamic characteristics of the system and may render it unstable. A root locus of the closed-loop poles versus k_1 shows how the locus changes as a function of the mass:



The closed-loop system becomes unstable for $k_1 \geq 24$. Since a small increase in k_1 makes the system unstable and a decrease in m has the same effect on the system, it is difficult to balance balls of smaller masses.

(e) The response to an initial x displacement is shown :





The assumption here is that an allowable transient must stay in the range of the sensor and not require more than the limit of the current. From $I(z) = D(z)(0 - X(z))$, we have a difference equation :

$$i(k) - 0.093i(k-1) = -116\{x(k) - 0.5313x(k-1)\}$$

For $k = 0$, $i(0) = -116x(0)$. We see that if $x(0) = 1$ then $i(0) = -116$. Note that $i(0) = -D(\infty)x(0)$.
Thus, if i is to be kept below $1A$ then $x(0)$ must be kept below $1/116 = 0.00862$ m $= 0.862$ cm $= 0.339$ inch, which is greater than the sensor range. The current control can handle any displacement in the range of ± 0.25 inch.

12. Repeat Problem 5.27 in Chapter 5 by constructing discrete root loci and performing the designs directly in the z-plane. Assume that the output y is sampled, the input u is passed through a ZOH as it enters the plant, and the sample rate is 15 Hz.

Solution

(a) The most effective discrete design method is to start with some idea what the continuous design looks like, then adjust that as necessary with the discrete model of the plant and compensation. We refer to the solution for Probelm 5.27 for the starting point. It shows that the specs can be met with a lead compensation,

$$D_1(s) = K\frac{(s+1)}{(s+60)}$$

and a lag compensation,

$$D_2(s) = \frac{(s+0.4)}{(s+0.032)}.$$

Although it is stated in the solution to Problem 5.27 that a gain, $K = 240$ will satisfy the constraints, in fact, a gain of about $K = 270$ is actually required to meet the rise time constraint of $t_r \leq 0.4$ sec. So we will assume here that our reference continuous design is

$$D_1(s) = 270\frac{(s+1)}{(s+60)}\frac{(s+0.4)}{(s+0.032)}$$

It yields a rise time, $t_r \cong 0.38$, $M_p \cong 15\%$, and $K_v = (270)(\frac{1}{60})(\frac{0.4}{0.32}) = 56$. So all specs are met. For interest, use of the damp function shows that $\omega_n = 6.4$ rad/sec for the dominant roots, and those roots have a damping ratio, $\zeta \cong 0.7$. For the discrete case with $T = 15$ Hz, we should expect some degradation in performance, especially the damping, because the sample rate is approximately $15{\times}\omega_n$.

The discrete transfer function for the plant described by $G(s)$ and preceeded by a ZOH is:

$$\begin{aligned} G(z) &= (1-z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\} \\ &= \frac{z-1}{z}\mathcal{Z}\left\{\frac{10}{s(s+1)(s+10)}\right\} \end{aligned}$$

This is most easily determined via MATLAB,

```
sysC = tf([10],[1 11 10 0]);
T=1/15;
sysD = c2d(sysC,T,'zoh');
```

which produces:

$$G(z) = 0.00041424\frac{(z+3.136)(z+0.2211)}{(z-1)(z-0.9355)(z-0.5134}$$

The essential elements of the compensation are that the lead provides velocity feedback with a $T_D = 1$ and the lag provides some high frequency gain. The discrete equivalent of the proportional plus lead would be (see Eq. 8.42):

$$D_1(z) = K(1+\frac{T_D}{T}(1-z^{-1})) = K\frac{(1+T_D/T)z - T_D/T}{z}$$

which for $T = 1/15 = 0.0667$ and $T_D = 1$ reduces to

$$D_1(z) = 270\frac{16z - 15z^{-1}}{z} = 4320\frac{z - 0.9375}{z}.$$

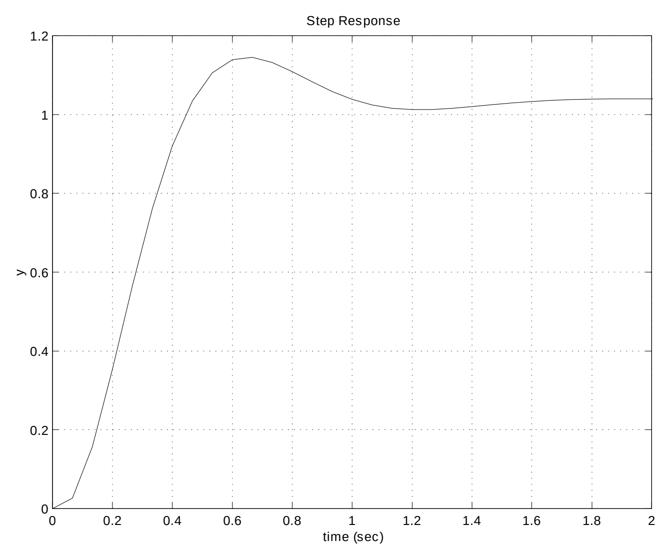
The lag equivalent is best introduced by use of one of the approximation techniques, such as the matched pole-zero:

$$D_2(z) = \frac{z - e^{-0.4T}}{z - e^{-0.032T}} = \frac{z - .9737}{z - .9979}$$

as its whole function is to raise the gain at very low frequencies for error reduction. Examining the resulting discrete root locus and picking roots with rlocfind to yield the required damping shows that the gain, $K = 60$. While the use of damp indicates that the frequency and damping are acceptable, the time response shows an overshoot of about 20% and the rise time is slightly below spec. We therefore need to increase the lead (move the lead zero closer to $z = +1$) to decrease the overshoot and increase gain to speed up the rise time. Several iterations on these two quantities indicates that moving the lead zero from $z = 0.9375$ to $z = 0.96$ and increasing the gain from $K = 60$ to $K = 65$ meets both specs. The velocity coefficient is found from and is also satisfied.

$$K_v = \frac{\lim}{z \to 1}\frac{(z-1)D(z)G(z)}{Tz}$$

The time response of the final design below shows that all specs are met.



13. Design a digital controller for the antenna servo system shown in Figs. 3.61 and 3.62 and described in Problem 3.31. The design should provide a step

response with an overshoot of less than 10% and a rise time of less than 80 sec.

(a) What should the sample rate be?

(b) Use the matched pole-zero discrete equivalent method.

(c) Use discrete design and the z-plane root locus.

Solution

(a) The equation of motion is :

$$J\ddot{\theta} + B\theta = T_c$$

where

$$J = 600000 \text{ kg.m}^2, \ B = 20000 \text{ N.m.sec}$$

If we define :

$$a = \frac{B}{J} = \frac{1}{30}, \ U = \frac{T_c}{B}$$

after Laplace transform, we obtain :

$$G(s) = \frac{\theta(s)}{u(s)} = \frac{1}{s(30s+1)}$$

From the specifications,

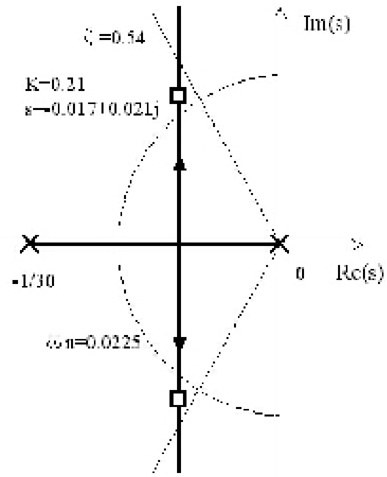
$$\begin{aligned} M_p &< 10\% \Longrightarrow M_P \cong \left(1 - \frac{\zeta}{0.6}\right)100 \Longrightarrow \zeta > 0.54 \\ t_r &< 80 \text{ sec} \Longrightarrow t_r \cong \frac{1.8}{\omega_n} < 80 \Longrightarrow \omega_n \cong \omega_{BW} > 0.0225 \end{aligned}$$

Note that $\omega_{pn} \cong 1/30 < \omega_n$.
If designing by emulation, a sample rate of 20 times the bandwidth is recommended. If using discrete design, the sample rate can be lowered somewhat to perhaps as slow as 10 times the bandwidth. However, to reject random disturbances, best results are obtained by sampling at 20 times the closed-loop bandwidth or faster. Thus, for both design methods, we choose :

$$\begin{aligned} T &= 10 \text{ sec} \\ \omega_s &= 0.628 \text{ rad/sec, which is} > 20\omega_n = 0.45 \text{ rad/sec} \end{aligned}$$

(b) Continuous design :
Use a proportional controller :

$$u(s) = D(s)(\theta_r(s) - \theta(s)) = K(\theta_r(s) - \theta(s))$$



Root locus :
Choose $K = 0.210$.
The closed-loop pole location in s-plane :

$$s = -0.0167 \pm 0.0205j$$

The corresponding natural frequency and damping :

$$\omega_n = 0.0265, \ \zeta = 0.6299$$

Digitized the continuous controller with matched pole-zero method :

$$\begin{aligned} D(z) &= 0.0210 \\ T_c(z) &= Bu(z) = 420(\theta_r(z) - \theta(z)) \end{aligned}$$

Performance :

$$\begin{aligned} M_p &= 0.119 \\ t_r &= 67.3 \text{ sec} \end{aligned}$$

(c) With $u(k)$ applied through a ZOH, the transfer function for an equivalent discrete-time system is :

$$G(z) = K\frac{z+b}{(z-1)(z-e^{aT})}$$

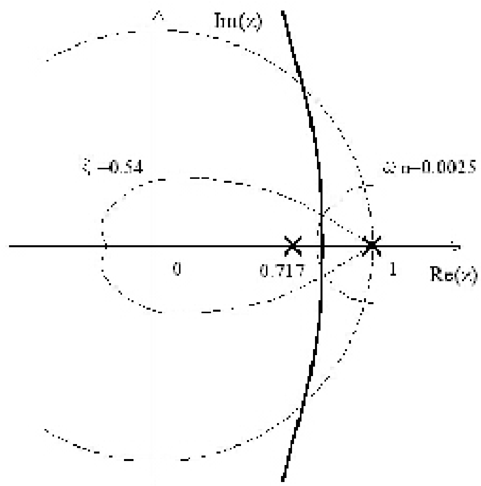
where

$$\begin{aligned} K &= \frac{aT - 1 + e^{-aT}}{a}, \ b = \frac{1 - e^{-aT} - aTe^{-aT}}{aT - 1 + e^{-aT}} \\ \Longrightarrow & \ G(z) = 1.4959\frac{z + 0.8949}{(z-1)(z-0.7165)} \end{aligned}$$

Use a proportional control of the form :

$$D(z) = K$$

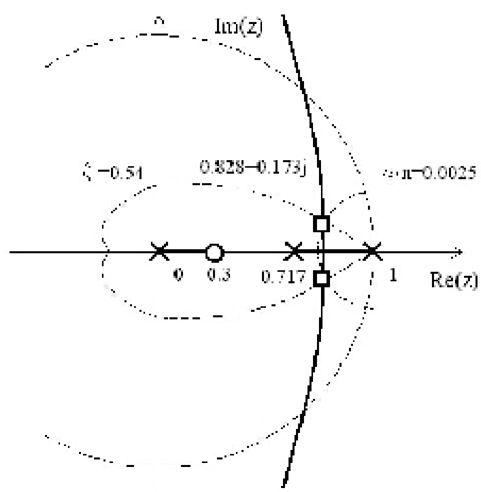
Root locus :



The specification can be achieved with the proportional control. However, we try to achieve the same closed-loop poles as the emulation design (part (b)) for comparison. These closed-loop pole locations are denoted by " + " in the root locus.
Use a PD control of the form :

$$D(z) = K\frac{z-\alpha}{z}$$

Root locus :



Choose $K = 0.0294,\ \alpha = 0.3$.
The resulting z-plane roots :

$$z = 0.8280 \pm 0.1725j,\ 0.0165$$

This corresponds to the s-plane roots :

$s = -0.0167 \pm 0.0205j$ (the design point of emulation design), -0.4104

which satisfy the specification :

$$\begin{aligned} \omega_n &= 0.0265,\ 0.4104 \\ \zeta &= 0.6321,\ 1.000 \end{aligned}$$

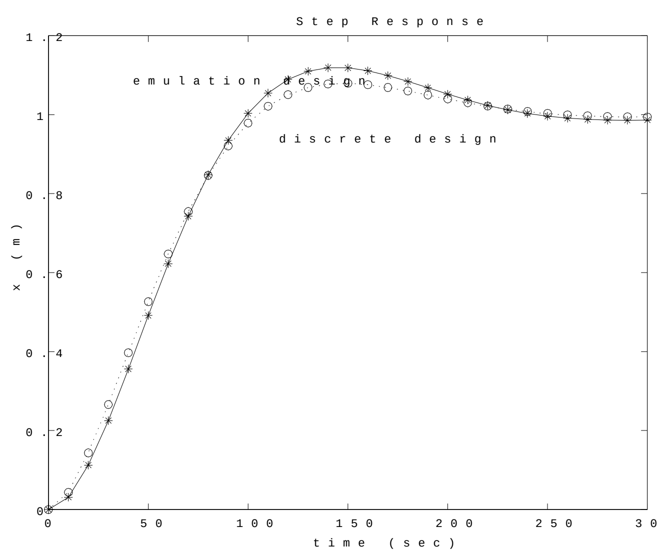
The control law :

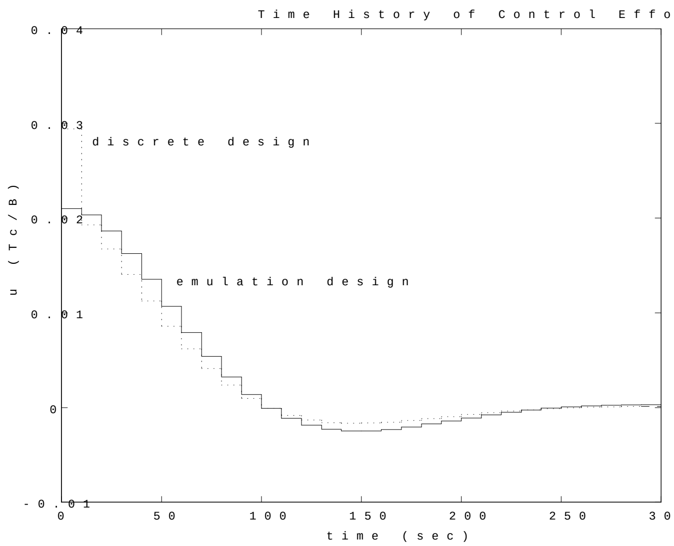
$$\begin{aligned} D(z) &= 0.0294\frac{z-0.3}{z} \\ T_c(z) &= Bu(z) = 588\frac{z-0.3}{z} \end{aligned}$$

Performance :

$$\begin{aligned} M_p &= 0.079 \\ t_r &= 71.3 \text{ sec} \end{aligned}$$

Step response :





14. The system

$$G(s) = \frac{1}{(s+0.1)(s+3)}$$

is to be controlled with a digital controller having a sampling period of $T = 0.1$ sec. Using a z-plane root locus, design compensation that will respond to a step with a rise time $t_r \leq 1$ sec and an overshoot $M_p \leq 5\%$. What can be done to reduce the steady-state error?

Solution

(a) Continuous plant :

$$G(s) = \frac{1}{(s+0.1)(s+3)}, \quad \text{Type 0 system}$$

Discrete model of $G(s)$ preceeded by a ZOH ($T = 0.1$ sec) :

$$G(z) = 0.0045 \frac{z+0.9019}{(z-0.7408)(z-0.99)}$$

Specifications :

$$\begin{aligned} t_r &\leq 1 \text{ sec} \longrightarrow \omega_n \geq 1.8 \text{ rad/sec} \\ M_p &\leq 5\% \longrightarrow \zeta \geq 0.7 \end{aligned}$$

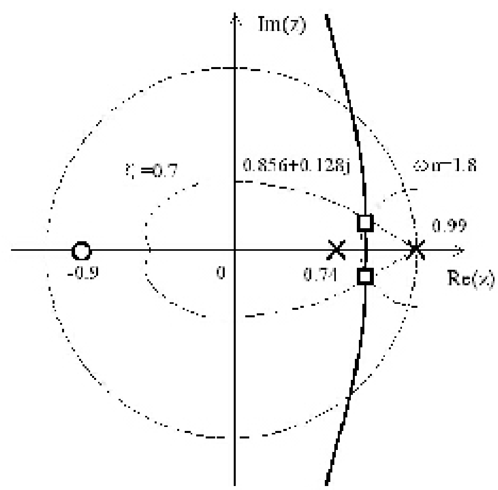
Discrete design : A simple proportional feedback, $D(z) = K = 4.0$, will bring the closed-loop poles to :

$$z = 0.8564 \pm 0.1278j$$

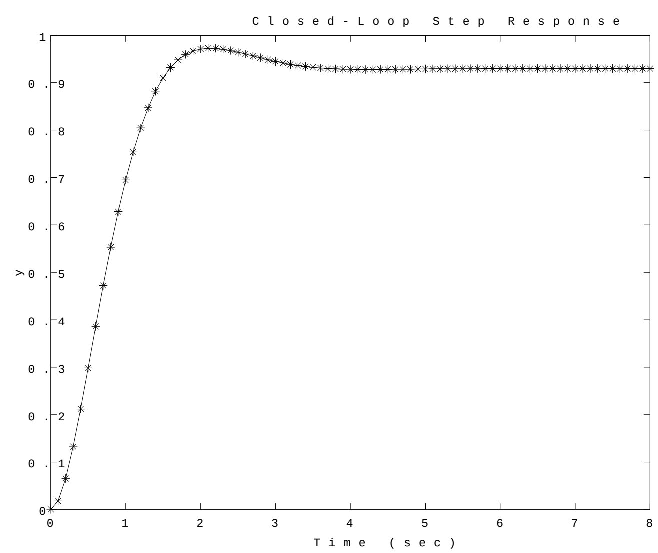
which are inside the specs region.

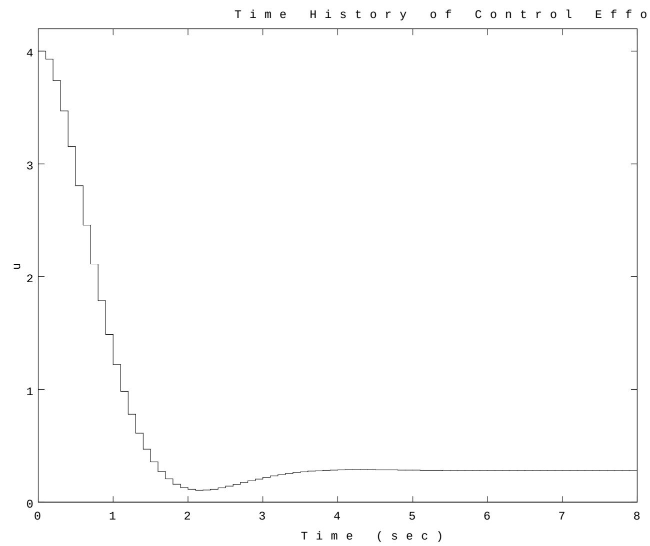
$$\omega_n = 2.07 \text{ rad/sec}, \ \zeta = 0.70$$

Root locus :



Step response :





The step response shows that :

$$\begin{aligned} t_r &\cong 1.02 \text{ sec} \\ M_p &\cong 4.7\% \end{aligned}$$

However, since the system is type 0, steady-state error exists and is 7% in this case. An integral control of the form,

$$D(z) = \frac{K}{T_I} \frac{Tz}{z-1}$$

can be added to the proportional control to reduce the steady-state error, but this typically occurs at the cost of reduced stability.

15. The transfer function for pure derivative control is

$$D(z) = KT_D \frac{z-1}{Tz},$$

where the pole at $z = 0$ adds some destabilizing phase lag. Can this phase lag be removed by using derivative control of the form

$$D(z) = KT_D \frac{(z-1)}{T}?$$

Support your answer with the difference equation that would be required, and discuss the requirements to implement it.

Solution:

(a) No, we cannot use derivative control of the form :

$$D(z) = KT_D \frac{z-1}{T}$$

to remove the phase lag. The difference equation corresponding to

$$D(z) = K_p T_D \frac{z-1}{T} = \frac{U(z)}{E(z)}$$

is

$$u(k) = K_p T_D \frac{e(k+1) - e(k)}{T}$$

This is not a *causal* system since it needs the future error signal to compute the current control. In real time applications, it is not possible to implement a non-causal system.

Chapter 9

Nonlinear Systems

Problems and Solutions for Section 9.2: Analysis By Linearization

1. Figure 9.56 shows a simple pendulum system in which a cord is wrapped around a fixed cylinder. The motion of the system that results is described by the differential equation

$$(l + R\theta)\ddot{\theta} + g\sin\theta + R\dot{\theta}^2 = 0,$$

where

$$l = \text{length of the cord in the vertical (down) position,}$$
$$R = \text{radius of the cylinder.}$$

(a) Write the state-variable equations for this system.

(b) Linearize the equation around the point $\theta = 0$, and show that for small values of θ the system equation reduces to an equation for a simple pendulum, that is,

$$\ddot{\theta} + (g/l)\theta = 0.$$

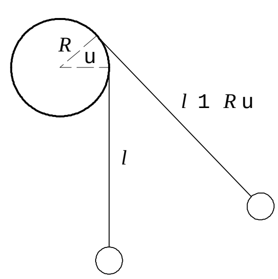


Figure 9.56: Motion of cord wrapped around a fixed cylinder for Problem 1.

Solution:

(a) This is a second order non-linear differential equation in θ. Let $\mathbf{x} = \left[\begin{array}{cc} \dot\theta & \theta \end{array} \right]^T$.

$$\begin{aligned}
\dot x_1 &= \ddot\theta = -\frac{R\dot\theta^2 + g\sin\theta}{(l + R\theta)} = -\frac{Rx_1^2 + g\sin x_2}{(l + Rx_2)}, \\
\dot x_2 &= \dot\theta = x_1.
\end{aligned}$$

(b) For small values of θ.

$$\begin{aligned}
(l + R\theta) &\cong l, \\
\sin\theta &\cong \theta, \\
\dot\theta^2 &\cong 0.
\end{aligned}$$

(a)

$$\begin{aligned}
l\ddot\theta + g\theta &= 0 \\
\ddot\theta + \frac{g}{l}\theta &= 0
\end{aligned}$$

2. The circuit shown in Fig. 9.57 has a nonlinear conductance G such that $i_G = g(v_G) = v_G(v_G - 1)(v_G - 4)$. The state differential equations are

$$\begin{aligned}
\frac{di}{dt} &= -i + v, \\
\frac{dv}{dt} &= -i + g(u - v),
\end{aligned}$$

where i and v are the state variables and u is the input.

(a) One equilibrium state occurs when $u = 1$ yielding $i_1 = v_1 = 0$. Find the other two pairs of v and i that will produce equilibrium.

(b) Find the linearized model of the system about the equilibrium point $u = 1$, $i = v_1 = 0$.

(c) Find the linearized models about the other two equilibrium points.

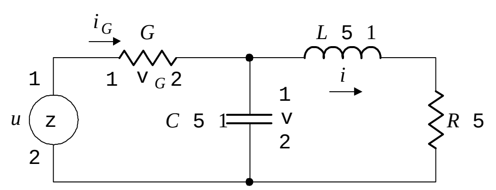


Figure 9.57: Nonlinear circuit for Problem 2.

Solution:

(a) Equilibrium:

$$\begin{aligned}
\frac{di}{dt} &= -i + v = 0, \\
\frac{dv}{dt} &= -i + g(u - v) = 0,
\end{aligned}$$

$$g(u-v)-i=(u-v)\left[(u-v)-1\right]\left[(u-v)-4\right]-v=0,$$

as $u=1$,

$$(1-v)(-v)(-3-v)-v=v\left(v^2+2v-2\right)=0.$$

$$v=0,-1\pm\sqrt{3}.$$

So,

$$i=v=0,-1\pm\sqrt{3}.$$

(b) Let's replace u, v, and i by $1+\delta u$, δv, and δi.

$$\begin{aligned}
\delta\dot{i} &= -\delta i+\delta v,\\
\delta\dot{v} &= -\delta i+g(1+\delta u-\delta v),\\
&= -\delta i+(1+\delta u-\delta v)((1+\delta u-\delta v)-1)((1+\delta u-\delta v)-4),\\
&= -\delta i+(1+\delta u-\delta v)\left(\delta u-\delta v\right)\left(\delta u-\delta v-3\right),\\
&\cong -\delta i-3\delta u+3\delta v.
\end{aligned}$$

$$\frac{d}{dt}\begin{bmatrix}\delta i\\ \delta v\end{bmatrix}=\begin{bmatrix}-1 & 1\\ -1 & 3\end{bmatrix}\begin{bmatrix}\delta i\\ \delta v\end{bmatrix}+\begin{bmatrix}0\\ -3\end{bmatrix}\delta u.$$

(c) In general the linearized form will be,

$$\frac{d}{dt}\begin{bmatrix}\delta i\\ \delta v\end{bmatrix}=\begin{bmatrix}-1 & 1\\ -1 & \frac{\partial g}{\partial v}\end{bmatrix}\begin{bmatrix}\delta i\\ \delta v\end{bmatrix}+\begin{bmatrix}0\\ \frac{\partial g}{\partial u}\end{bmatrix}\delta u,$$

As $u=1$,

$$\begin{aligned}
g(u,v) &= g(1,v)=v\left(v^2+2v-2\right),\\
\frac{\partial g}{\partial v} &= \left(v^2+2v-2\right)+v\left(2v+2\right),\\
&= 5\mp 2\sqrt{3}\text{ when }v=-1\pm\sqrt{3}.
\end{aligned}$$

Also

$$\frac{\partial g\left(u-v\right)}{\partial v}=-g'\left(u-v\right),$$

$$\frac{\partial g\left(u-v\right)}{\partial u}=g'\left(u-v\right)=-\frac{\partial g}{\partial v}=-5\pm 2\sqrt{3}.$$

So,

$$\frac{d}{dt}\begin{bmatrix}\delta i\\ \delta v\end{bmatrix}=\begin{bmatrix}-1 & 1\\ -1 & 5\mp 2\sqrt{3}\end{bmatrix}\begin{bmatrix}\delta i\\ \delta v\end{bmatrix}+\begin{bmatrix}0\\ -5\pm 2\sqrt{3}\end{bmatrix}\delta u.$$

3. Consider the circuit shown in Fig. 9.58; $u_1$and u_2 are voltage and current sources, respectively, and R_1 and R_2 are nonlinear resistors with the following characteristics:

$$\begin{aligned} \text{Resistor 1} : i_1 &= G(v_1) = v_1^3 \\ \text{Resistor 2} : v_2 &= r(i_2), \end{aligned}$$

where the function r is defined in Fig. 9.59.

(a) Show that the circuit equations can be written as

$$\begin{aligned} \dot{x}_1 &= G(u_1 - x_1) + u_2 - x_3 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_1 - x_2 - r(x_3). \end{aligned}$$

Suppose we have a constant voltage source of 1 Volt at u_1 and a constant current source of 27 Amps; i.e., $u_1^o = 1$, $u_2^o = 27$. Find the *equilibrium state* $\mathbf{x}^o = [x_1^o, x_2^o, x_3^o]^T$ for the circuit. For a particular input u^o, an equilibrium state of the system is defined to be any constant state vector whose elements satisfy the relation

$$\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0.$$

Consequently, any system started in one of its equilibrium states will remain there indefinitely until a different input is applied.

(b) Due to disturbances, the initial state (capacitance, voltages, and inductor current) is slightly different from the equilibrium and so are the independent sources; that is,

$$\begin{aligned} u(t) &= u^o + \delta u(t) \\ x(t_0) &= x^o(t_0) + \delta x(t_0). \end{aligned}$$

Do a small-signal analysis of the network about the equilibrium found in (a), displaying the equations in the form

$$\delta\dot{x}_1 = f_{11}\ \delta x_1 + f_{12}\ \delta x_2 + f_{13}\ \delta x_3 + g_1\ \delta u_1 + g_2\ \delta u_2.$$

(c) Draw the circuit diagram that corresponds to the linearized model. Give the values of the elements.

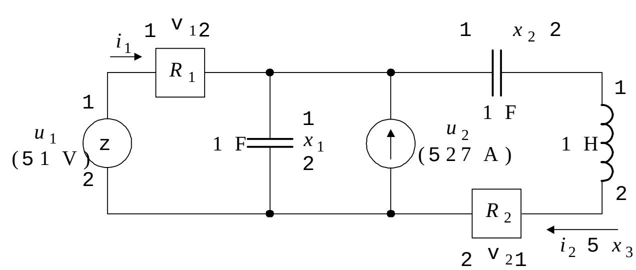


Figure 9.58: A nonlinear circuit for Problem 3.

Solution:

(a)

$$\begin{aligned} i_1 &= G(u_1 - x_1), \\ i_1 - i_2 + u_2 &= C\frac{d}{dt}x_1, \\ i_2 &= C\frac{d}{dt}x_2 = x_3, \\ v_2 &= r(i_2), \\ 0 - x_2 + x_1 - v_2 &= L\frac{d}{dt}i_2. \end{aligned}$$

$$\begin{aligned} C\dot{x}_1 &= G(u_1 - x_1) - x_3 + u_2, \\ C\dot{x}_2 &= x_3, \\ L\dot{x}_3 &= x_1 - x_2 - r(x_3). \end{aligned}$$

As $C = L = 1$,

$$\begin{aligned} \dot{x}_1 &= G(u_1 - x_1) - x_3 + u_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= x_1 - x_2 - r(x_3). \end{aligned}$$

(b) Equilibrium state around $u_1^0 = 1$, $u_2^0 = 27$.

$$\begin{aligned} 0 &= G\left(u_1^0 - x_1^0\right) - x_3^0 + u_2^0, \\ &= G\left(1 - x_1^0\right) - x_3^0 + 27, \\ 0 &= x_3^0, \\ 0 &= x_1^0 - x_2^0 - r(0). \end{aligned}$$

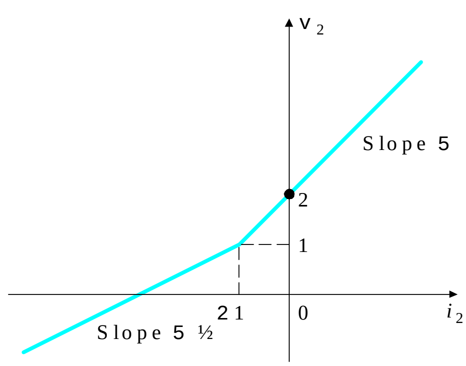


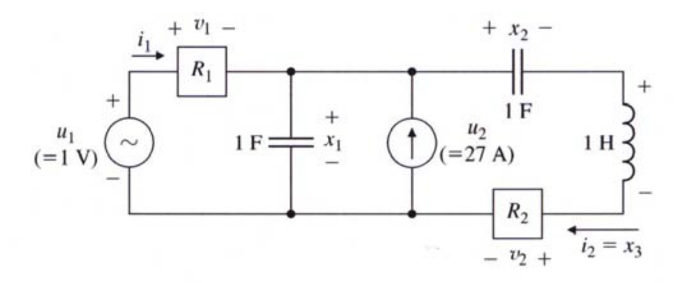
Figure 9.59: Nonlinear resistance for Problem 3.

$$\begin{aligned}
G\left(1-x_1^0\right)-x_3^0+27 &= \left(1-x_1^0\right)^3+27=0,\\
x_1^0 &= 4,\\
x_2^0 &= x_1^0-r(0)=4-2=2,\\
x_3^0 &= 0.
\end{aligned}$$

$$\mathbf{x}^0=\left[\begin{array}{ccc}4 & 2 & 0\end{array}\right]^T.$$

$$\begin{aligned}
\delta\dot{x}_1 &= G\left(\left(u_1^0+\delta u_1\right)-\left(x_1^0+\delta x_1\right)\right)-\left(x_3^0+\delta x_3\right)+\left(u_2^0+\delta u_2\right),\\
&= \left(-3+\delta u_1-\delta x_1\right)^3-\delta x_3+27+\delta u_2,\\
&\cong -27+3\times 9\left(\delta u_1-\delta x_1\right)-\delta x_3+27+\delta u_2,\\
&= -27\delta x_1-\delta x_3+27\delta u_1+\delta u_2.\\
\delta\dot{x}_2 &= x_3^0+\delta x_3,\\
&= \delta x_3.\\
\delta\dot{x}_3 &= \left(x_1^0+\delta x_1\right)-\left(x_2^0+\delta x_2\right)-r\left(x_3^0+\delta x_3\right),\\
&= 4+\delta x_1-2-\delta x_2-\left(\delta x_3+2\right),\\
&= \delta x_1-\delta x_2-\delta x_3.
\end{aligned}$$

(c) Circuit diagram:



Linear circuit model for Problem 9.3.

$L=1\text{H},\ \ R_1=\frac{1}{27}\Omega,\ \ R_2=1\Omega.$

4. Consider the nonlinear system

$$\dot{x}=-x^2e^{-\frac{1}{x}}+\sin u \qquad x(0)=1$$

a) Assume $u^{\mathrm{o}}=0$ and solve for $x^{\mathrm{o}}(t)$.

b) Find the linearized model about the nominal solution in part (a).

Solution:

(a)

$$\begin{aligned}\dot{x}(t) &= -x^2 e^{-\frac{1}{x}},\\ \frac{dx}{dt} &= -x^2 e^{-\frac{1}{x}},\\ -x^{-2} e^{\frac{1}{x}}\, dx &= dt.\end{aligned}$$

Integrate both sides:

$$\begin{aligned}\int_{x(0)}^{x(t)} -\frac{1}{x^2} e^{\frac{1}{x}}\, dx &= \int_0^t dt,\\ e^{\frac{1}{x(t)}} - e^{\frac{1}{1}} &= t,\\ x_o(t) &= \frac{1}{\log(t+e)}.\end{aligned}$$

(b)

$$\begin{aligned}\dot{x} &= -x^2 e^{-\frac{1}{x}} + \sin u,\\ \delta\dot{x} &= -\left[2xe^{-\frac{1}{x}} + e^{-\frac{1}{x}}\right]\delta x + \cos u\ \delta u.\end{aligned}$$

For the equilibrium conditions of part (a),

$$\delta\dot{x} = -\frac{1}{t+e}\left[1 + \frac{2}{\log(t+e)}\right]\delta x + \delta u.$$

5. *Linearizing effect of feedback.* We have seen that feedback can reduce the sensitivity of the input-output transfer function with respect to changes in the plant transfer function , and reduce the effects of a disturbance acting on the plant. In this problem we explore another beneficial property of feedback: it can make the input-output response *more linear* than the open-loop response of the plant alone. For simplicity let us ignore all the dynamics of the plant, and assume that the plant is described by the static nonlinearity

$$y(t) = \begin{cases} u & u \le 1 \\ \frac{u+1}{2} & u > 1 \end{cases}$$

a) Suppose we use proportional feedback

$$u(t) = r(t) + \alpha(r(t) - y(t))$$

where $\alpha \ge 0$ is the feedback gain. Find an expression for $y(t)$ as a function of $r(t)$ for the closed-loop system (This function is called the *nonlinear characteristic* of the system.) Sketch the nonlinear transfer characteristic for $\alpha = 0$ (which is really open-loop), $\alpha = 1$, and $\alpha = 2$.

b) Suppose we use integral control,

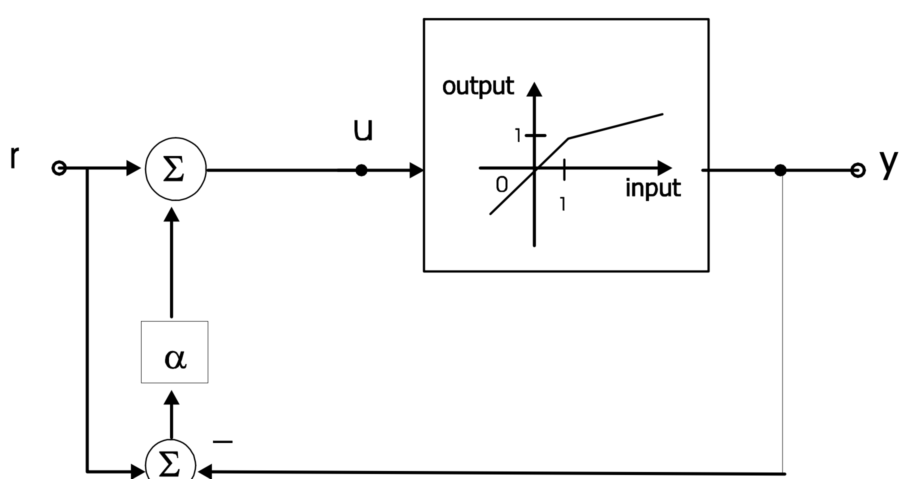
$$u(t) = r(t) + \int_0^t (r(\tau) - y(\tau))d\tau$$

The closed-loop system is therefore nonlinear and dynamic. Show that if $r(t)$ is a constant, say r, then $\lim_{t\to\infty} y(t) = r$. Thus, the integral control makes the steady-state transfer characteristic of the closed-loop system *exactly linear*. Can the closed-loop system be described by a transfer function from r to y?

Solution:

(a)

Fig 10



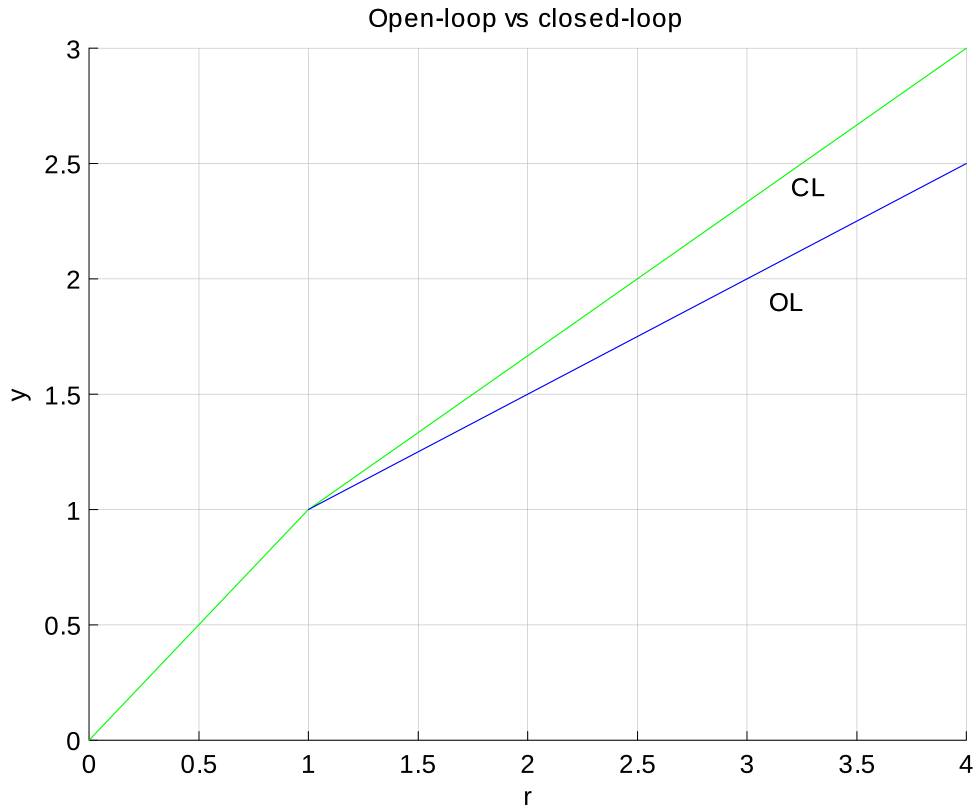
6.pdf
Problem 9.5. Nonlinear system with saturation: proportional control.

For $u \le 1$:

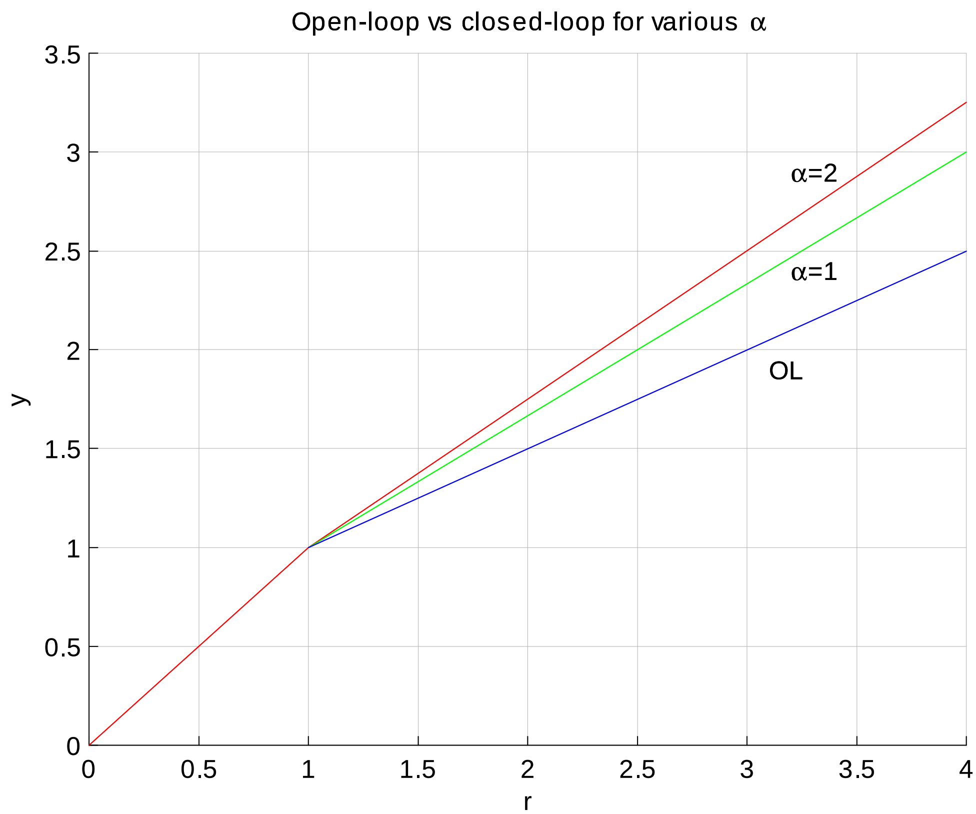
$$y = r + \alpha(r - y) = (1+\alpha)y = (1+\alpha)r;\; y = r = u.$$

For $u > 1$:

$$\begin{aligned}
y &= \frac{u+1}{2} = \frac{1}{2} + \frac{1}{2}[r + \alpha(r-y)],\\
2y &= 1 + r + \alpha r - \alpha y,\\
(2+\alpha)y &= 1 + (1+\alpha)r,\\
y &= \frac{1+(1+\alpha)r}{2+\alpha}. \quad \text{See Figure on top of the next page.}\\
\text{if } \alpha &= 0 \quad \text{then} \quad y = \frac{1+r}{2}.\\
\text{if } \alpha &= 1 \quad \text{then} \quad y = \frac{1}{3} + \frac{2}{3}r.\\
\text{if } \alpha &= 2 \quad \text{then} \quad y = \frac{1}{4} + \frac{3}{4}r. \quad \text{See middle Figure on the next page.}
\end{aligned}$$



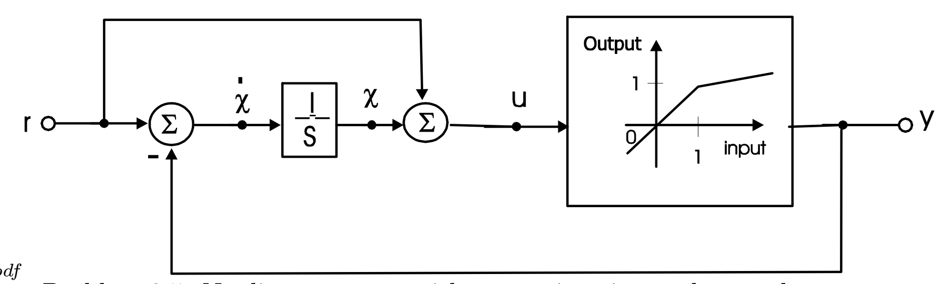
Problem 9.5. Open-loop vs closed-loop response.



Problem 9.5. Open-loop vs closed-loop response for various values of α.

(b)

Fig 11



$_9.pdf$
Problem 9.5. Nonlinear system with saturation: integral control.

$$
\begin{aligned}
u &= r + \int_0^t (r - y)dt, \\
r &= \text{cons tan t},
\end{aligned}
$$

$u < 1$,

$$\begin{aligned} y &= r + \int_0^t (r-y)dt, \\ Y &= R + \frac{R-Y}{s}, \\ (1+s)Y &= (1+s)R; \quad Y = R \end{aligned}$$

Assume stable: y stays bounded,

$$\begin{aligned} y &\rightarrow y_\infty, \\ y &= r + \int_0^t (r - y_\infty)dt \rightarrow \infty \quad \text{if} \quad y_\infty \neq r, \end{aligned}$$

$$\begin{aligned} &\Rightarrow \quad y_\infty = r, \\ \dot{y} &= r - y, \\ \dot{y} + y &= r. \end{aligned}$$

$u > 1$,

$$\begin{aligned} y &= \frac{1}{2} + \frac{1}{2}(r + \int_0^t (r-y)dt, \\ y &= f(u), \\ \dot{x} &= r - y, \\ \dot{y} &= \frac{1}{2}(r-y), \\ 2\dot{y} + y &= r, \\ \frac{Y(s)}{R(s)} &= \frac{1}{2s+1}. \end{aligned}$$

In the steady-state: $\quad y = r$.

6. This problem shows that linearization does not always work. Consider the system

$$\dot{x} = \alpha x^3 \qquad x(0) \neq 0$$

a) Find the equilibrium point and solve for $x(t)$.

b) Assume $\alpha = 1$. Is the linearized model a valid representation of the system?

c) Assume $\alpha = -1$. Is the linearized model a valid representation of the system?

Solution: (a) The equilibrium point is found from:

$$\begin{aligned} \dot{x} &= \alpha x^3 = 0, \\ &\Rightarrow \quad x_e = 0. \end{aligned}$$

To determine $x(t)$ we re-write the system equation as,

$$\frac{dx}{x^3} = \alpha dt,$$

Integrating both sides:

$$\begin{aligned}
\int_{x(0)}^{x(t)} \frac{dx}{x^3} &= \alpha \int_0^t dt, \\
-\frac{1}{2}x^{-2}|_{x(0)}^{x(t)} &= \alpha t, \\
x^2 &= \frac{x(0)^2}{1-2\alpha x(0)^2 t}, \\
x(t) &= \frac{1}{\sqrt{x(0)^{-2}-2\alpha t}}. \qquad (*)
\end{aligned}$$

(b) If $\alpha = 1$ the linearized system is,

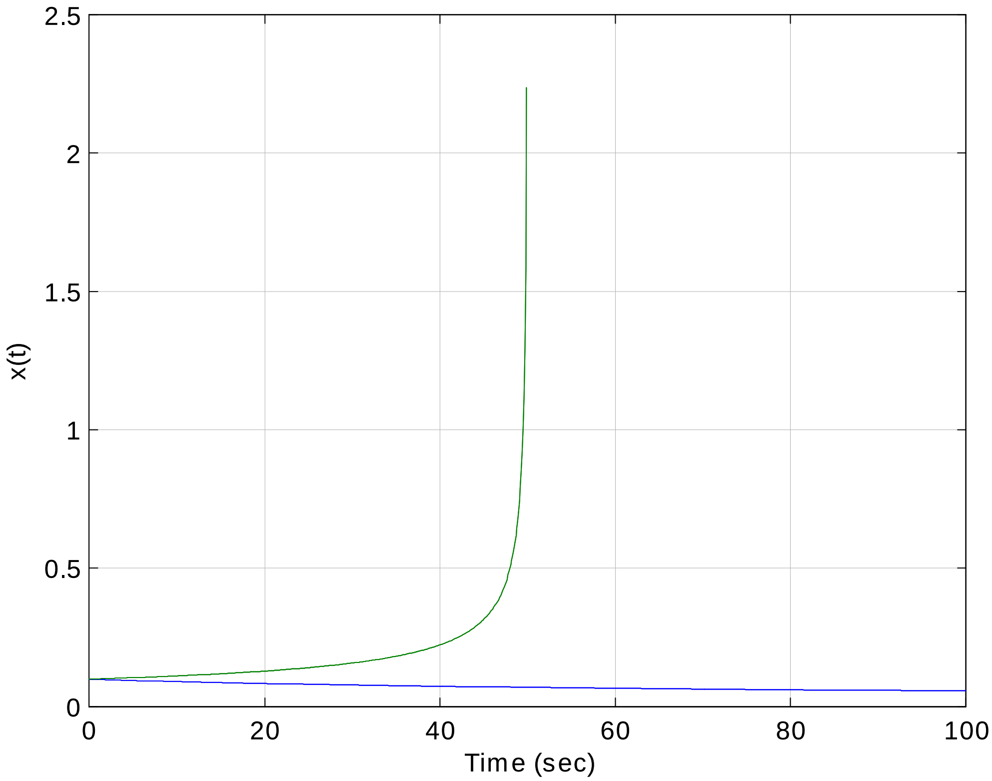
$$\begin{aligned}
\delta \dot{x} &= 3(1)x_e^2 = 0, \\
\delta x &= \text{cons}\tan\text{t},
\end{aligned}$$

that is the linear system is not asymptotically stable (it is neutrally stable). However, we can see from the nonlinear solution given by Equation (*) that the system is unstable and exhibits a finite "escape-time" at $t = \frac{1}{2}x(0)^{-2}$ (i.e., the response of the nonlinear system tends to infinity in finite time;see Figure on the next page). The linear system does not predict qualitative behavior of the nonlinear system. So the linear model is not a valid representation of the system.

(c) If $\alpha = -1$ the linearized system is

$$\begin{aligned}
\delta \dot{x} &= 3(-1)x_e^2 = 0, \\
\delta x &= \text{cons}\tan\text{t},
\end{aligned}$$

that is the linear system is not asymptotically stable (it is neutrally stable). However, we can see from the nonlinear solution given by Equation (*) that the system is asymptotically stable as x^2 starts off at x_o^2 but drops off to zero (see Figure on top of the next page). The linear system does not predict qualitative behavior of the nonlinear system. So the linear model is not a valid representation of the system. The two systems corresponding to $\alpha = +1$ and $\alpha = -1$ have the *same* linearized system but *very different* nonlinear behavior. The conclusion is that the linearized system *usually* gives a good idea of the system behavior around the equilibrium (x_e) *but not always.*



Problem 9.6: Behavior of nonlinear system.

7. Consider the object moving in a straight line with constant velocity shown in Figure 9.60. The only available measurement is the range to the object. The system equations are

$$\begin{bmatrix} \dot{x} \\ \dot{v} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ v \\ z \end{bmatrix}$$

where

$$\begin{aligned} z &= \mathrm{cons\,tan\,t} \\ \dot{x} &= \mathrm{cons\,tan\,t} = v_0 \\ r &= \sqrt{x^2 + z^2} \end{aligned}$$

Derive a linear model for this system.

Solution: This system has only an output nonlinearity,

$$\begin{aligned}
y &= r = h(\mathbf{x}), \\
\delta r &= \frac{\delta h}{\partial \mathbf{x}}\delta \mathbf{x} = \mathbf{H}\delta \mathbf{x}, \\
\mathbf{H} &= \left[\begin{array}{ccc} \frac{\delta h}{\partial x} & \frac{\delta h}{\partial v} & \frac{\delta h}{\partial z} \end{array} \right], \\
&= \left[\begin{array}{ccc} \frac{x}{r} & 0 & \frac{z}{r} \end{array} \right]. \\
\dot{\mathbf{x}} &= \mathbf{F}\mathbf{x}, \\
\delta r &= \left[\begin{array}{ccc} \frac{x}{r} & 0 & \frac{z}{r} \end{array} \right] \mathbf{x}.
\end{aligned}$$

Problems and Solutions for Section 9.3: Equivalent Gain Analysis Using Root Locus

8. Consider the third-order system shown in Fig. 9.61.

   (a) Sketch the root locus for this system with respect to K, showing your calculations for the asymptote angles, departure angles, and so on.

   (b) Using graphical techniques, locate carefully the point at which the locus crosses the imaginary axis. What is the value of K at that point?

   (c) Assume that, due to some unknown mechanism, the amplifier output is given by the following saturation non linearity (instead of by a proportional gain K):

$$u = \begin{cases} e, & |e| \leq 1; \\ 1, & e > 1; \\ -1, & e < -1. \end{cases}$$

   Qualitatively describe how you would expect the system to respond to a unit step input.

   Solution:

(a) The locus branches leave the origin at angles of 180°and ±60°. Two break in at angles of ±90° near $s = -3$. See root locus plot.

(b) The locus crosses the imaginary axis at $\omega = 1$ for $K = 0.5$.

(c) The system is conditionally stable and with saturation would be expected to be stable for small inputs and go unstable for large inputs.

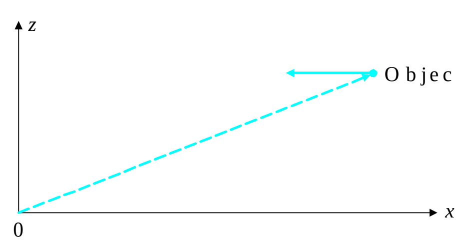
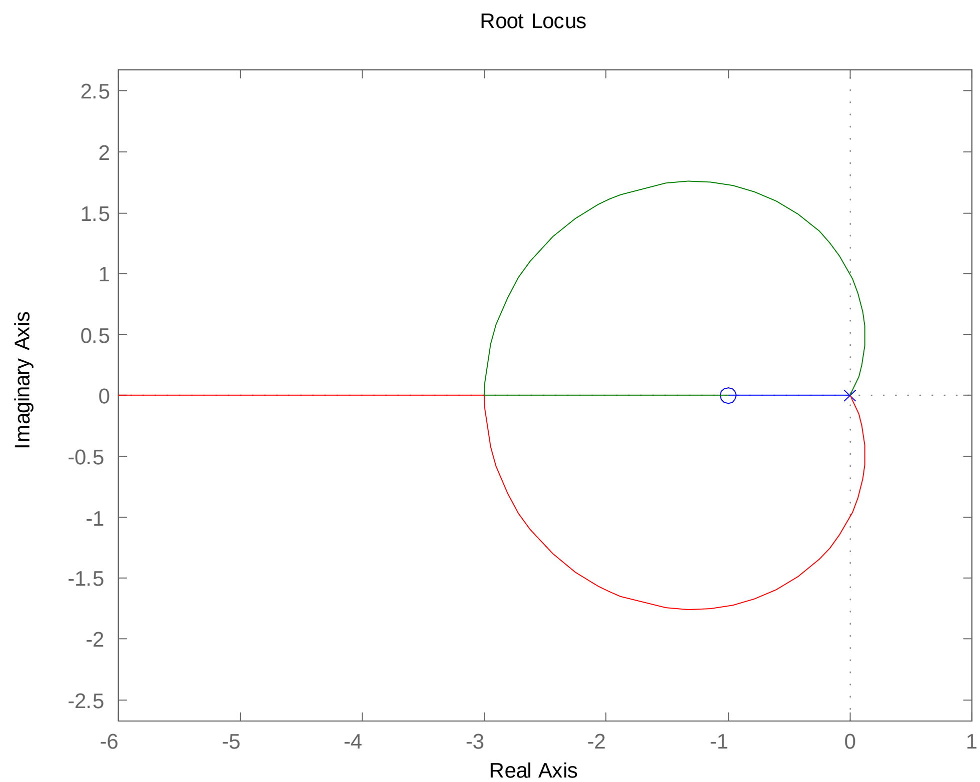


Figure 9.60: Diagram of moving object for Problem 9.7.



Root locus for Problem 9.8.

(a) Problems and Solutions for Section 9.4: Equivalent Gain Analysis Using Frequency Response: Describing Functions

9. Compute the describing function for the relay with deadzone nonlinearity shown in Figure 9.6 (c).

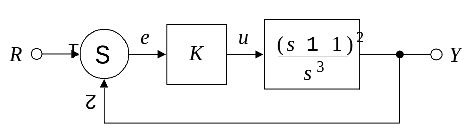


Figure 9.61: Control system for Problem 9.8.

Solution:

$$\begin{aligned} Y_1 &= \frac{1}{\pi}\int_0^{2\pi} y(t)\sin(\omega t)d(\omega t) \\ &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} y(t)\sin(\omega t)d(\omega t) \\ &= \frac{4N}{\pi}\int_{\omega t_1}^{\frac{\pi}{2}} \sin(\omega t)d(\omega t) \\ &= \frac{4N}{\pi}\cos(\omega t_1). \end{aligned}$$

Since $\omega t_1 = \frac{h}{a}$ then $\cos(\omega t_1) = \sqrt{1-\left(\frac{h}{a}\right)^2}$.
The describing function is then given by,

$$DF = \frac{Y_1}{a} = \frac{4N}{\pi a}\sqrt{1-\left(\frac{h}{a}\right)^2}.$$

10. Compute the describing function for gain with dead zone nonlinearity shown in Figure 9.6 (d).

    Solution: This is an odd nonlinearity so that all the cosine terms are zeros and the DF is real:

$$\begin{aligned} Y_1 &= \frac{1}{\pi}\int_0^{2\pi} y(t)\sin(\omega t)d(\omega t), \\ &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} y(t)\sin(\omega t)d(\omega t), \\ &= \frac{4K_o}{\pi}\int_0^{\frac{\pi}{2}} (A\sin(\omega t)-a)\sin(\omega t)d(\omega t). \end{aligned}$$

Since $h = a\sin(\omega t_1)$ then $\omega t_1 = \sin^{-1}\left(\frac{h}{\Delta}\right)$,

$$\begin{aligned} Y_1 &= \frac{4aK_o}{\pi}\left[\int_{\omega t_1}^{\frac{\pi}{2}} (\sin^2(\omega t) - \sin(\omega t_1)\sin(\omega t))d(\omega t)\right], \\ &= \frac{2aK_o}{\pi}\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{a}{A}\right) - \frac{a}{A}\sqrt{1-\left(\frac{a}{A}\right)^2}\right]. \end{aligned}$$

The describing function is then given by,

$$DF = \frac{Y_1}{a} = \frac{2K_o}{\pi}\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{h}{a}\right) - \frac{h}{a}\sqrt{1-\left(\frac{h}{a}\right)^2}\right].$$

11. Compute the describing function for the preloaded spring or Coulomb plus viscous friction nonlinearity shown in Figure 9.6 (e).

    Solution: This is a combination of a gain, K_0, plus a relay nonlinearity (see Example 9.11). Therefore,

$$DF = \frac{K_0 a}{a} + \frac{4N}{\pi a} = K_0 + \frac{4N}{\pi a}.$$

12. Consider the quantizer function shown in Figure 9.62 that resembles a staircase. Find the describing function for this nonlinearity and write a MATLAB .m function to generate it.

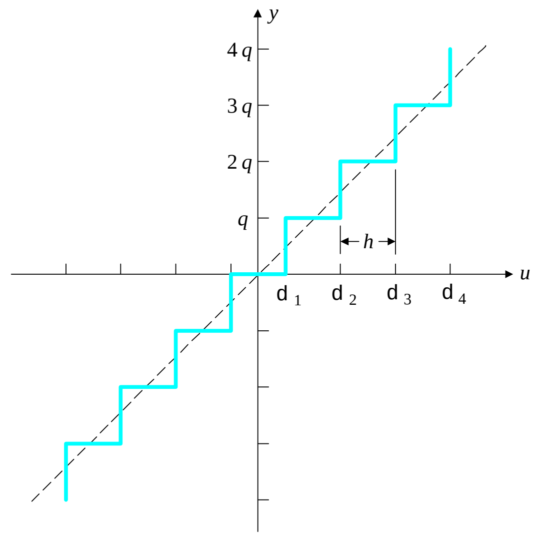


Figure 9.62: Quantizer nonlinearity for Problem 12.

Solution. The abscissa breakpoints are denoted by δ_i. From Eq. 9.23,

$$\begin{aligned} b_1 &= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f\left(a\sin\omega t\right)\sin\omega t\ d(\omega t) \\ &= \frac{4}{\pi}\int_0^{\varphi_1} 0\cdot\sin\omega t\ d(\omega t) + \int_0^{\varphi_2} q\cdot\sin\omega t\ d(\omega t) + ... + \int_0^{\varphi_n} nq\cdot\sin\omega t\ d(\omega t) \\ &= \frac{4}{\pi}(\cos\varphi_1 + \cos\varphi_2 + ... + \cos\varphi_n), \end{aligned}$$

where,

$$\psi_i = \sin\left(\frac{\delta_i}{a}\right) \qquad i = 1, ..., n.$$

The describing function is then given by,

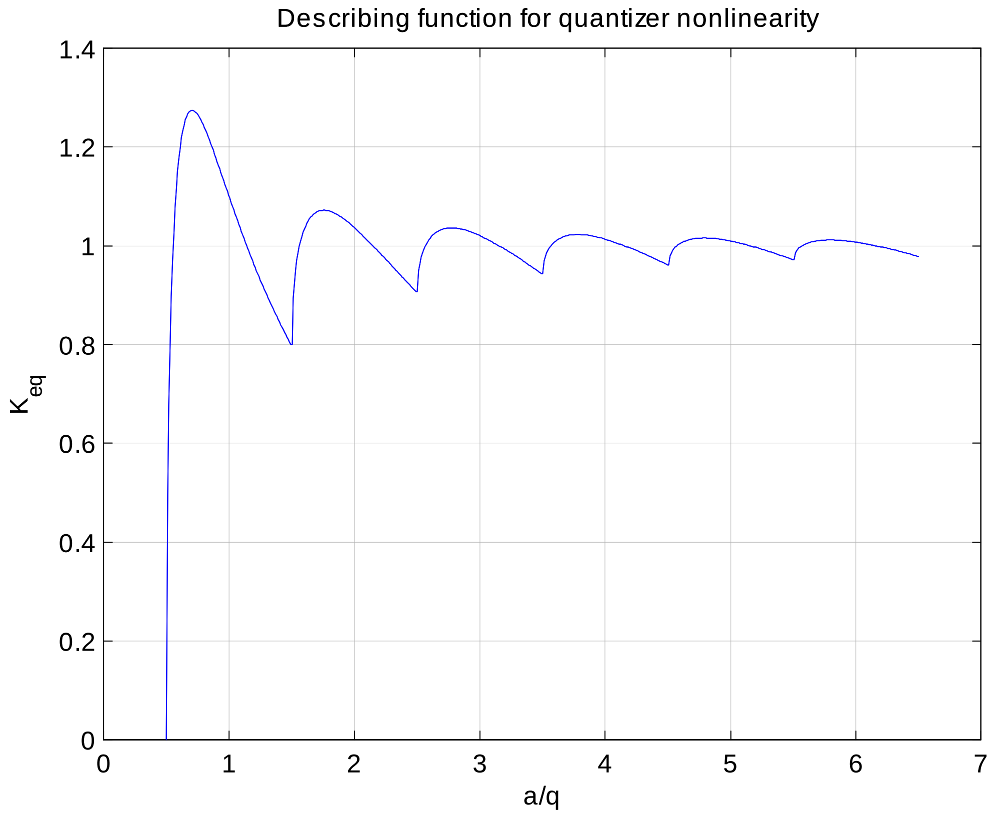
$$K_{eq}(a) = \frac{b_1}{a} = \begin{cases} 0 & 0 < a < \frac{q}{2} \\ \frac{4q}{\pi a}\sum_{i=1}^{n}\sqrt{1-\left(\frac{2i-1}{2a}q\right)^2} & \frac{2n-1}{2}q < a < \frac{2n+1}{2}q \end{cases}$$

The following shows the MATLAB .m function:

```
% Feedback Control of Dynamic Systems, 6e
% Franklin, Powell, Emami
% Problem 9.12
```

```
clear all;

close all;

na = 99;

nn = 6;

Keq=zeros(1,nn*na);

for n=1:nn

ai = linspace((2*n-1)/2, (2*n+1)/2, na);

for ni=1:na

for k=1:n

Keq((n-1)*na+ni) = Keq((n-1)*na+ni) + (4/(pi*ai(ni)))*sqrt(1- ((2*k-1)/(2*ai(ni)))^2);

end;

end;

end;

plot(linspace(1/2,(2*nn+1)/2,na*nn),Keq);

title('Describing function for quantizer nonlinearity')

xlabel('a/q')

ylabel('K_{eq}')

grid;

hold off
```

The describing function is plotted in the figure as a function of $\frac{a}{q}$. The maximum of the DF occurs at $K_{eq} = \frac{4}{\pi} = 1.27$ corresponding to $\frac{a}{q} = 0.7$. Since the staircase can be approximated by a straight line, it is seen that the DF will in the limit approach the slope of the linear approximation, that is one.



13. Derive the describing function for the ideal contactor controller shown in Figure 9.63. Is it frequency dependent? Would it be frequency dependent if it had a time delay or hysteresis? Graphically, sketch the time histories of the output for several amplitudes of the input and determine the describing function values for those inputs.

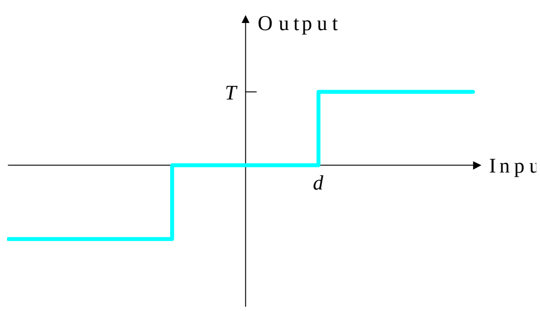


Figure 9.63: Contactor for Problem 9.13.

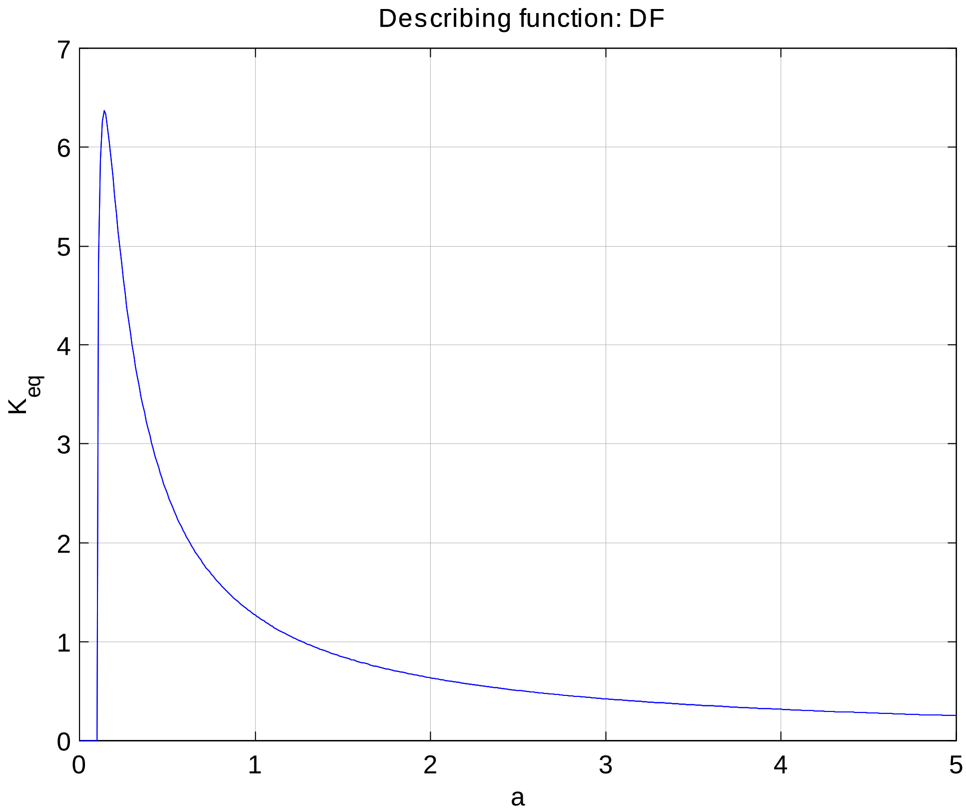
Solution:

$$
\begin{aligned}
Y_1 &= \frac{1}{\pi}\int_0^{2\pi} y(t)\sin(\omega t)d(\omega t) \\
&= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} y(t)\sin(\omega t)d(\omega t) \\
&= \frac{4T}{\pi}\int_{\omega t_1}^{\frac{\pi}{2}} \sin(\omega t)d(\omega t) \\
&= \frac{4T}{\pi}\cos(\omega t_1). \\
\text{Since } \ \omega t_1 &= \frac{d}{a} \ \text{ then } \ \cos(\omega t_1) = \sqrt{1-\left(\frac{d}{a}\right)^2}.
\end{aligned}
$$

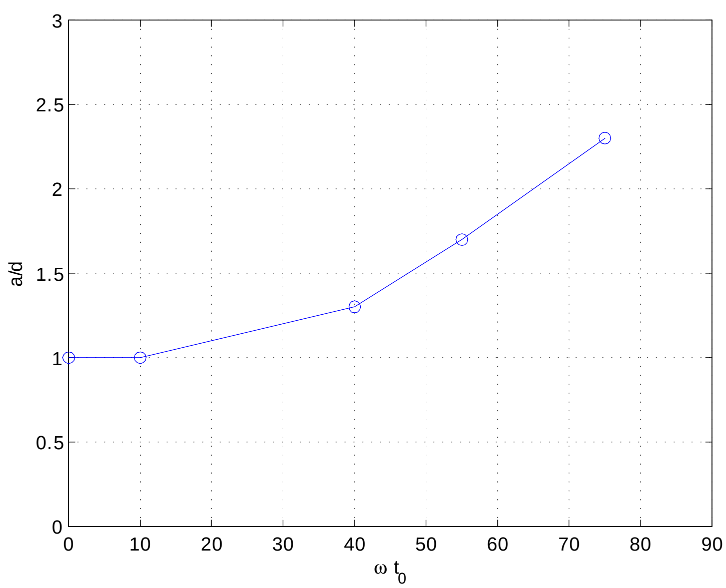
The describing function is then given by,

$$
DF = \begin{cases} \frac{Y_1}{a} = \frac{4T}{\pi a}\sqrt{1-\left(\frac{d}{a}\right)^2} & d < a \\ \qquad 0 & a < d \end{cases}
$$

and is *not* frequency dependent. See Figure on top of the next page. Frequency dependence will be introduced with a delay.



Problem 9.13. DF for $d = 0.1$, $T = 1.0$.



Problem 9.13. DF values for several different input frequencies.

14. A contactor controller of an inertial platform is shown in Figure 9.64 where

$$
\begin{aligned}
I &= 0.1\text{kgm}^2 \\
\frac{I}{B} &= 10\,\text{sec} \\
\frac{h}{c} &= 1 \\
\frac{J}{c} &= 0.01\,\text{sec} \\
\tau_L &= 0.1\,\text{sec} \\
\tau_f &= 0.01\,\text{sec} \\
d &= 10^{-5}\text{rad} \\
T &= 1\text{Nm}
\end{aligned}
$$

The required stabilization resolution is approximately 10^{-6} rad

$$K\varphi_m > d \quad \text{for} \quad \varphi_m > 10^{-6}\text{rad}$$

Discuss the existence, amplitude and frequency of possible limit cycles as a function of the gain K and the DF of the controller. Repeat the problem for a deadband with hysteresis.

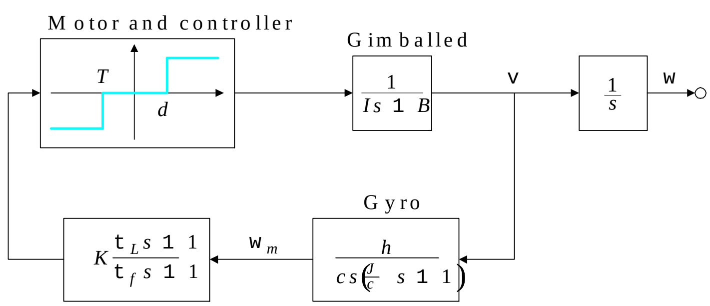


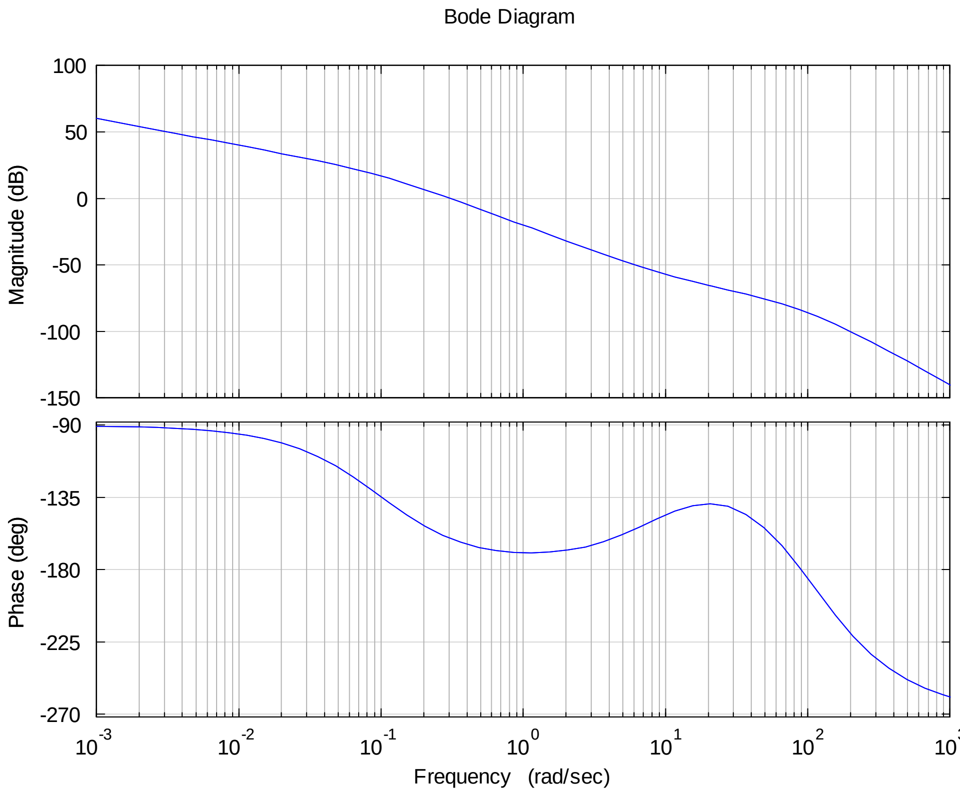
Figure 9.64: Block diagram of the system for Problem 9.14.

Solution: Limit cycles depend on the natural behavior of the closed-loop part. The DF of the switch $= K_{eq}$. Characteristic equation is:

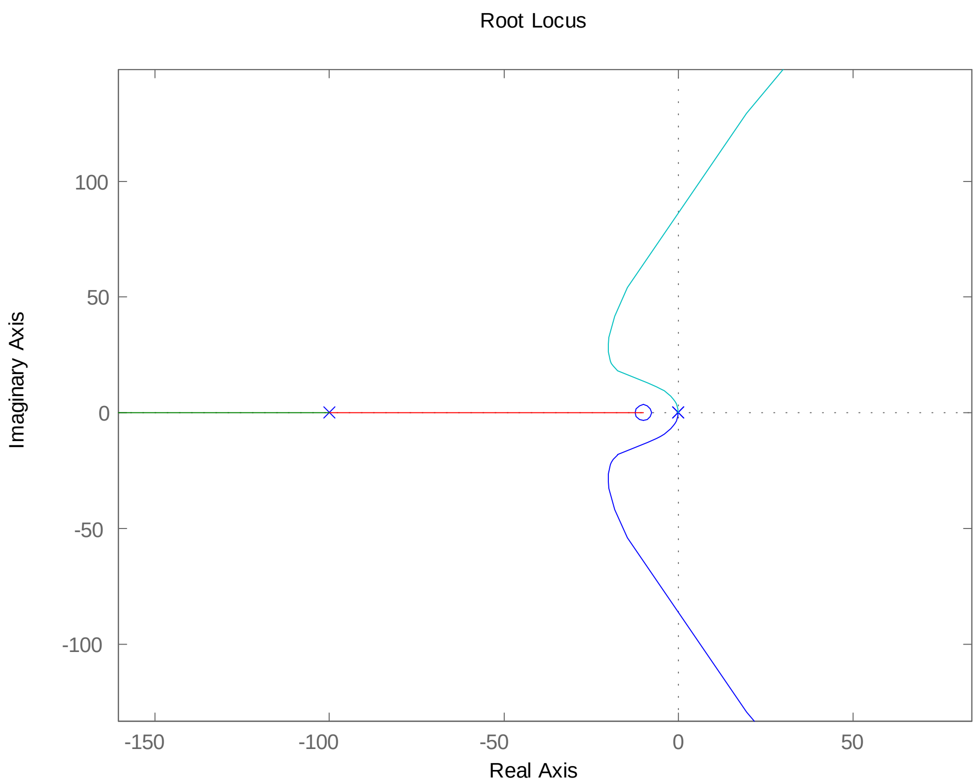
$$KG + 1 = 0$$

$$\frac{K}{B}\frac{h}{c} = \frac{K_{eq}}{s\left[\left(\frac{J}{c}\right)s + 1\right]}\left(\frac{\tau_L s + 1}{\tau_f s + 1}\right)\frac{1}{\left[\left(\frac{I}{B}\right)s + 1\right]} + 1 = 0.$$

$$\left(\frac{K_{eq}K}{0.01}\right)\left[\frac{0.1s + 1}{s(10s + 1)(0.01s + 1)^2}\right] = -1.$$



Frequency response for Problem 9.14.



Root locus for Problem 9.14.

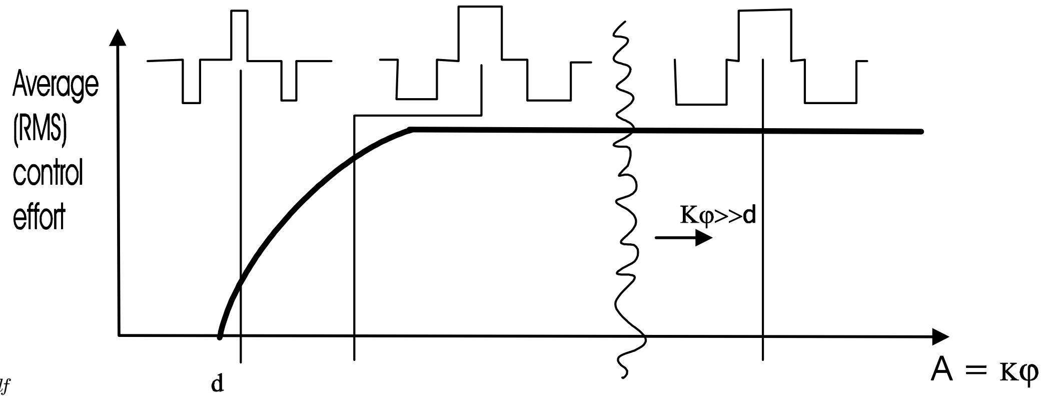
Condition for limit cycle is $\angle = -180^\circ$ which occurs at ~ 95 rad/sec and magnitude $= 1$ (see Figure) but

$$\begin{aligned} \frac{KK_{eq}}{0.01} &\approx \frac{1}{10 \times 10^{-5}} \quad \text{(the value at } \omega \simeq 95 \text{ rad/ sec)}, \\ KK_{eq} &= 100(\text{rad/ sec})(\text{nm sec /rad}) = 100 \text{ nm}, \end{aligned}$$

$$K_{eq} = \frac{4T}{\pi d}\left(\frac{d}{a}\right)\sqrt{1-\left(\frac{d}{a}\right)^2}.$$

with maximum at $\frac{4T}{\pi d}$. But note from the figure below,

Fig



$_2$2.pdf

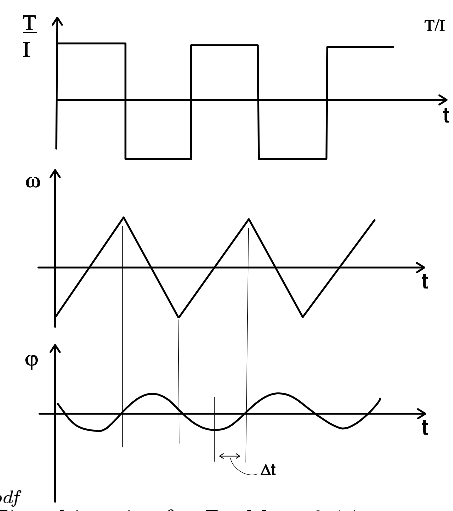
Response for nonlinear system for Problem 9.14.

that for levels of say $a > 3d$, the output is a constant. The frequency of the limit cycle is *fixed* by the *phase*! So φ is constant. The result is *independent of* K, for large enough K to insure that $K\varphi \gg d$. This is consistent with $KK_{eq} = \text{cons}\tan\text{t}$ for $K_{eq} = \frac{\text{cons}\tan\text{t}}{a}$, $a = K\varphi$,

$$K_{eq}K = \left(\frac{\text{cons}\tan\text{t}}{K\varphi}\right)K = \frac{\text{cons}\tan\text{t}}{\varphi} = \text{cons}\tan\text{t} \quad \text{for} \quad K\varphi >> d.$$

The amplitude is:

Fig 9



$_2$3.pdf

Time histories for Problem 9.14.

See above figures:

$$\begin{aligned}
\frac{1}{2}\frac{T}{I}\Delta t^2 &= \varphi = 5\frac{\text{rad}}{\text{sec}^2}(\Delta t)^2, \\
\Delta t &= \frac{P}{4}, \quad \omega = \frac{2\pi}{P} \cong 100\frac{\text{rad}}{\text{sec}}, \\
\Delta t &= \frac{2\pi}{4\omega} = \frac{3.14}{200} = 1.57 \times 10^{-2}\,\text{sec}, \\
\varphi &= 12 \times 10^{-4}\ \text{rad}.
\end{aligned}$$

If the resolution of platform pickoff should be $\sim 10^{-6}$rad and "short" term sensor noise $\ll 10^{-6}$rad, then $K\varphi_m >> d$ is satisfied, say,

$$\begin{aligned}
K \times 10^{-6} &= d = 10^{-5}\ \text{rad}, \\
&\Rightarrow K = 10. \\
KK_{eq} &= 100\ \text{nm}, \\
&\Rightarrow K_{eq} = 10\ \text{nm}. \\
K_{eq} &= \frac{4T}{\pi d}\left(\frac{d}{a}\right)\sqrt{1-\left(\frac{d}{a}\right)^2} \approx \frac{4\times 1}{\pi \times 10^{-5}}\left(\frac{d}{a}\right) = 10. \\
\frac{d}{a} &= 0.8 \times 10^{-5} \times 10 = 8 \times 10^{-5}.
\end{aligned}$$

To check:

$$a = \frac{d}{8 \times 10^{-5}} = 0.125 \times 10^5 \times 10^{-5} = 0.125\ \text{rad}.$$

Here we must consider the mid frequency model because the limit cycle is at $\omega = 100$ rad/sec. ω gets integrated in the gyro below its break frequency but φ_m goes through the lead circuit for a gain of 10 and $K = 10$ so $a = 12.5 \times 10^{-4}\text{rad} \times 100 = 0.125$ rad as before, so it checks.

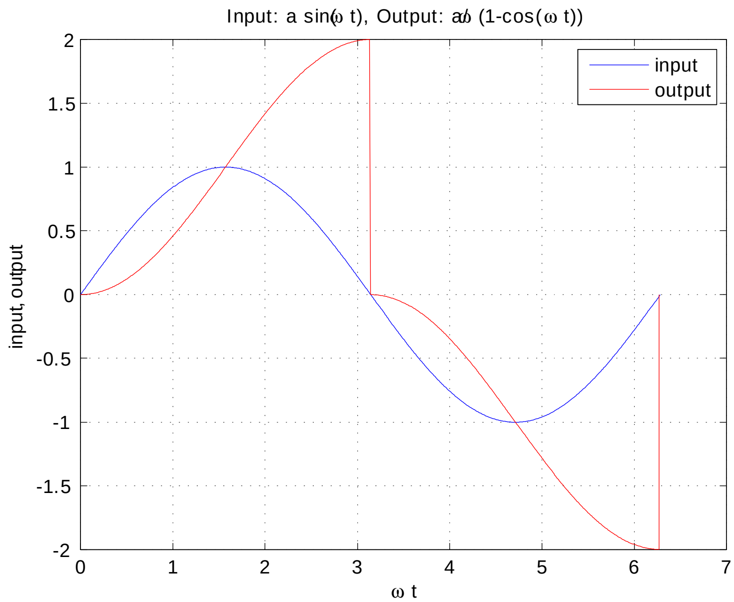
15. *Nonlinear Clegg Integrator* There have been some attempts over the years to improve upon the linear integrator. A linear integrator has the disadvantage of having a phase lag of 90° at all frequencies. In 1958, J. C. Clegg suggested that we modify the linear integrator to reset its state, x, to zero whenever the input to the integrator, e, crosses zero (i.e., changes sign). The Clegg integrator has the property that it acts like a linear integrator whenever its input and output have the same sign. Otherwise, it *resets* it output to zero. The Clegg integrator can be described by

$$\begin{aligned}
x(t) = e(t) &\quad \text{if } e(t) \neq 0, \\
x(t+) = 0 &\quad \text{if } e(t) = 0,
\end{aligned}$$

where the latter equation implies that the state of the integrator, x, is reset to zero immediately after e changes sign. The Clegg integrator can be implemented with op-amps and diodes. A potential disadvantage of the Clegg integrator is that it may induce oscillations. (a) Sketch the output of the Clegg integrator if the input is $e = a\sin(\omega t)$. (b) Prove that the DF for the Clegg integrator is

$$N(a,\omega) = \frac{4}{\pi\omega} - j\frac{1}{\omega}.$$

and this amounts to a phase lag of only 38°. **Solution:** (a) See Figure below.



Plots of the input and output signals.

(b)

$$
\begin{aligned}
a_1 &= \frac{2}{\pi}\int_0^{\pi} x(t)\cos(\omega t)d(\omega t) \\
&= \frac{2}{\pi}\int_0^{\pi} \frac{a}{\omega}\cos(\omega t)d(\omega t) \\
&= \frac{2a}{\pi\omega}\int_0^{\pi}\cos(\omega t)d(\omega t) - \int_0^{\pi}\cos^2(\omega t)d(\omega t) \\
&= \frac{2a}{\pi\omega}\left(0+\frac{\pi}{2}\right) = -\frac{a}{\omega}.
\end{aligned}
$$

$$
\begin{aligned}
b_1 &= \frac{2}{\pi}\int_0^{\pi} x(t)\sin(\omega t)d(\omega t) \\
&= \frac{2}{\pi}\int_0^{\pi} \frac{a}{\omega}(1-\cos(\omega t))\sin(\omega t)d(\omega t) \\
&= \frac{2a}{\pi\omega}\int_0^{\pi}\sin(\omega t)d(\omega t) - \int_0^{\pi}\cos(\omega t)\sin(\omega t)d(\omega t) \\
&= \frac{2a}{\pi\omega}(2+0) = \frac{4a}{\pi\omega}.
\end{aligned}
$$

The describing function is then given by,

$$\begin{aligned} DF &= \frac{Y_1}{a} = \frac{4}{\pi\omega} - j\frac{1}{\omega} \\ &= \frac{1}{\omega}\sqrt{1+\left(\frac{4}{\pi}\right)^2} \arctan\left(\frac{-\pi}{4}\right) \\ &= \frac{1.619}{\pi}\angle - 38.15^\circ. \end{aligned}$$

■Problems and Solutions for Section 9.5: Analysis and Design Based on Stability

16. Compute and sketch the optimal reversal curve and optimal control for the minimal time control of the plant

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 + u \\ |u| &\leq 1 \end{aligned}$$

Use the reverse time method and eliminate the time.

Solution: Use the reverse time method and eliminate the time:

$$\begin{aligned} \dot{x}_1 &= -x_2, \\ \dot{x}_2 &= x_2 - \text{u}. \end{aligned}$$

For $u = +1$,time reversal means that we let $\tau = -t$, and that changes the sign on the system and the input matrices,

$$\frac{d\mathbf{x}}{d\tau} = -\mathbf{F}\mathbf{x} - \mathbf{G}u.$$

In our case,

$$\begin{aligned} \dot{x}_1 &= -x_2, \\ \dot{x}_2 &= x_2 - 1, \\ \frac{dx_2}{d\tau} &= x_2 - 1, \\ \frac{dx_2}{(x_2-1)} &= d\tau. \end{aligned}$$

Integrate both sides:

$$\int_0^{x_2} \frac{dx_2}{(x_2-1)} = \int_0^{\tau} d\tau.$$

$$\ln(x_2 - 1) = \tau + C_1.$$

Since $x_2(0) = 0$ then,

$$\begin{aligned} C_1 &= \ln(-1), \\ \ln(x_2 - 1) - \ln(-1) &= \tau, \\ \ln(1 - x_2) &= \tau, \\ x_2 &= 1 - e^{\tau}, \\ \tau &= \ln(1 - x_2). \end{aligned}$$

Now,

$$\begin{aligned} \dot{x}_1 &= -x_2, \\ \dot{x}_1 &= -1 + e^{\tau}, \\ dx_1 &= (-1 + e^{\tau})d\tau. \end{aligned}$$

Integrate both sides:

$$\begin{aligned} \int_0^{x_1} dx_1 &= \int_0^{\tau} (-1 + e^{\tau})d\tau. \\ x_1 &= e^{\tau} - \tau - 1. \end{aligned}$$

Eliminate τ to get,

$$x_1 = -x_2 - \ln(1 - x_2).$$

This is the reversal curve for $u = 1$, $x_2 < 0$.

For $u = -1$,

$$\begin{aligned} \dot{x}_2 &= x_2 + 1, \\ \frac{dx_2}{x_2 + 1} &= d\tau. \end{aligned}$$

Integrate both sides:

$$\begin{aligned} \int_0^{x_2} \frac{dx_2}{x_2 + 1} &= \int_0^{\tau} d\tau. \\ \ln(x_2 + 1) &= \tau + C_1. \end{aligned}$$

Since $x_2(0) = 0$ then

$$\begin{aligned} \ln(x_2 + 1) &= \tau + C_1, \\ &\Rightarrow C_1 = \ln(1), \\ \ln(x_2 + 1) &= \tau, \\ x_2 + 1 &= e^{\tau}, \\ x_2 &= e^{\tau} - 1, \\ &\Rightarrow \tau = \ln(x_2 + 1). \end{aligned}$$

Now,

$$\begin{aligned}\dot{x}_1 &= -x_2 = 1 - e^{\tau},\\ dx_1 &= (1 - e^{\tau})d\tau.\end{aligned}$$

Integrate both sides:

$$\begin{aligned}\int_0^{x_1} dx_1 &= \int_0^{\tau}(1 - e^{\tau})d\tau,\\ x_1 &= -e^{\tau} + \tau + C_2,\\ x_2(0) &= 0, \quad C_2 = 1,\\ x_1 &= -e^{\tau} + \tau + 1.\end{aligned}$$

Eliminate τ to get,

$$x_1 = \ln(1 + |x_2|) - x_2.$$

This is the reversal curve for $u = -1$, $x_2 > 0$.

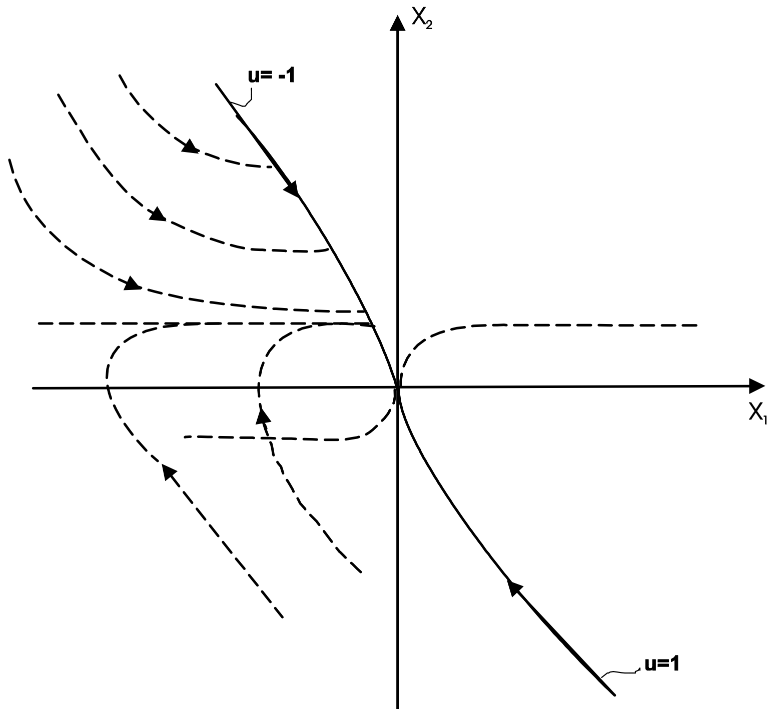
We can then write in general, for all x_2,

$$x_1 = sgn(x_2)\ln(1 + |x_2|) - x_2$$

Therefore, the control law is:

$$u = -sgn[x_1 + x_2 - sgn(x_2)\ln(1 + |x_2|)].$$

Fig 14



$_2 5.pdf$

Optimal reversal curve for Problem 9.15.

17. Sketch the optimal reversal curve for the minimal time control with $|u| \leq 1$ of the linear plant

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -2x_1 - 3x_2 + u.
\end{aligned}
$$

Solution: We reverse time that means $\tau = -t$, and that changes the sign on the system and the input matrices,

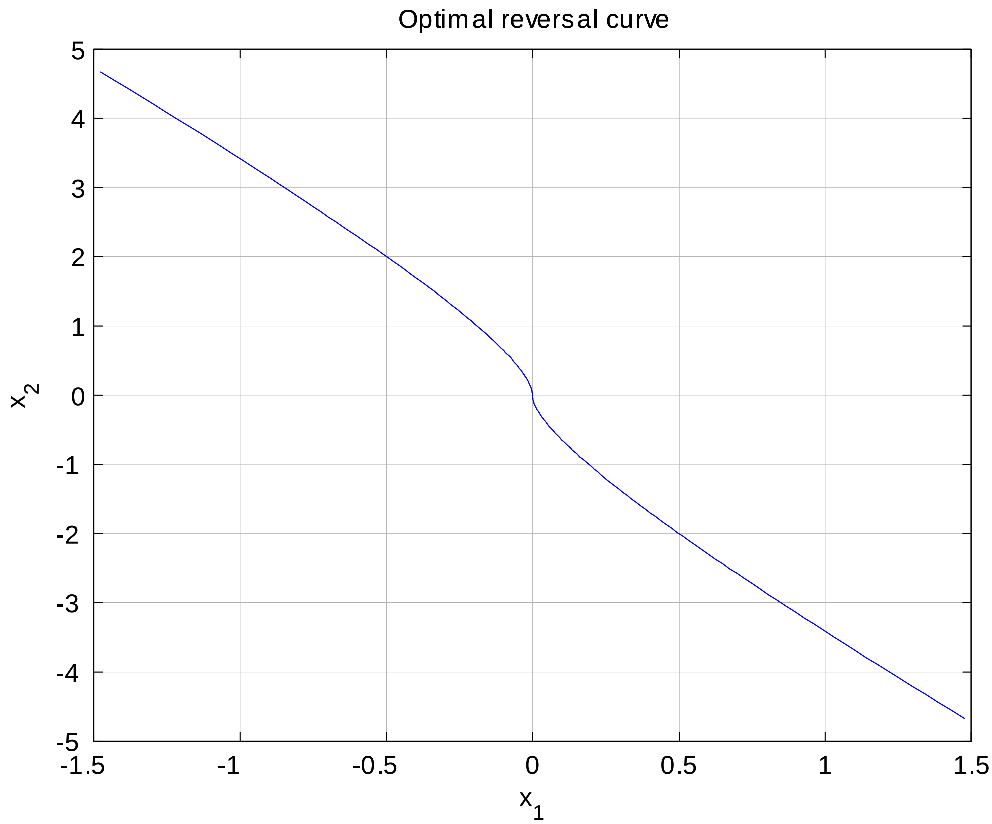
$$\frac{d\mathbf{x}}{d\tau} = -\mathbf{F}\mathbf{x} - \mathbf{G}u.$$

In our case,

$$
\begin{aligned}
\dot{x}_1 &= -x_2 \\
\dot{x}_2 &= +2x_1 + 3x_2 - u.
\end{aligned}
$$

We simulate the system using the MATLAB lsim function with $u = +1$ and store x_1 and x_2, and repeat with $u = -1$ and store x_1 and x_2 and plot the results to obtain the optimal reversal curve shown in the plot on the next page:

```
%Franklin, Powell, Emami 6e
%Problem 9.17
t=0:.01:1;
F=[0 1;-2 -3];
G=[0;1];
H=[1 0];
J=[0];
% Using the reverse time method
sys=ss(-F,-G,H,J);
%u=+1;
[yp,t,xp]=lsim(sys,ones(101,1),t);
plot(xp(:,1),xp(:,2));
hold on;
%u=-1
[ym,t,xm]=lsim(sys,-1*ones(101,1),t);
plot(xm(:,1),xm(:,2));
grid;
xlabel('x_1');
ylabel('x_2');
title('Optimal reversal curve');
```



Optimal reversal curve for Problem 9.16.

18. Sketch the time optimal control law for

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1 + u\\ |u| &\leq 1\end{aligned}$$

and show a trajectory for $x_1(0) = 3$, and $x_2(0) = 0$.

Solution: $u = +1$,

$$\begin{aligned}sX_1(s) &= -X_2(s),\\ sX_2(s) &= X_1(s) - \frac{1}{s},\\ X_2(s) &= -\frac{1}{s^2+1},\\ x_2(t) &= -\sin(t),\\ \dot{x}_2 &= -\cos(t) = x_1 - 1,\\ x_1 &= +1 - \cos(t).\end{aligned}$$

We see that,

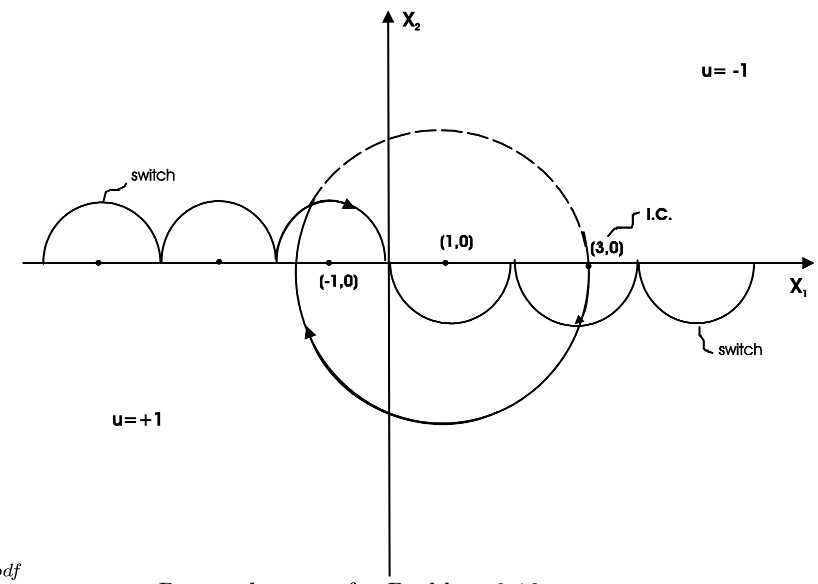
$$(x_1 - 1)^2 + x_2^2 = 1$$

that is a circle with center at $(1, 0)$, $\quad x_2 < 0$.
Similarly for $u = -1$ we get,

$$\begin{aligned} x_2(t) &= \sin(t) \\ x_1(t) &= \cos(t) - 1 \\ (x_1 + 1)^2 + x_2^2 &= 1, \end{aligned}$$

that is a circle with center at $(-1, 0)$, $x_2 > 0$.

Fig 12



$_2$7.pdf

Reversal curves for Problem 9.18.

The trajectories for this system are circles centered at $(\pm 1, 0)$. This is called the Bushaw problem in optimal control literature.

19. Consider the thermal control system shown in Figure 9.65. The physical plant can be a room, an oven, etc.

    (a) What is the limit cycle period?

    (b) If T_r is commanded as a slowly increasing function, sketch the output of the system, T. Show the solution for T_r "large".

    Solution:

    (a) This is a first order system so use $(T,\ t)$ plot. For an oven, it is piecewise linear

    $$\dot{T} + aT = BN\ sgn(e) \quad \text{with hysteresis}$$

    with

    $$T_r = 800°C, \quad \frac{BN}{a} \approx 1000°C \qquad \text{above} \qquad T = 0 \text{ (say room temperature)}$$

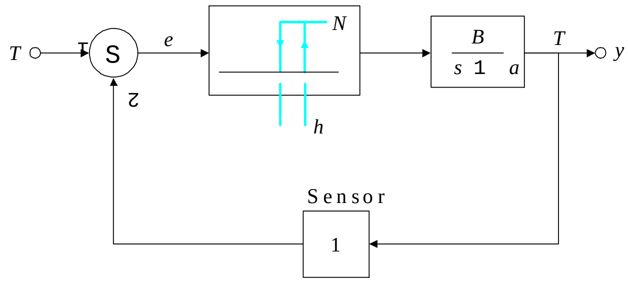


Figure 9.65: Thermal system for Problem 9.19.

$$a \simeq 0.01\,\text{sec}^{-1},\ \text{plot vs } at,\ h \sim 100°C,\ T_o = 0.$$

(b)

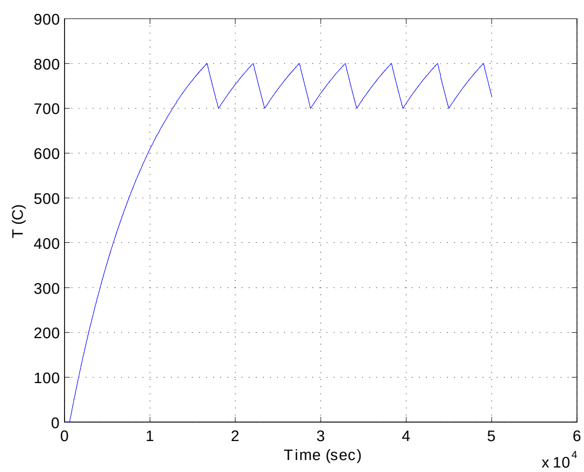
$$\begin{aligned} \text{limit cycle period} &\triangleq P = t_{on} + t_{off}, \\ T_r e^{-at} &= (T_r - h) \end{aligned}$$

gives t_{off},

$$\left[\frac{BN}{a} - (T_r - h)\right] e^{-at} = \left[\frac{BN}{a} - T_r\right]$$

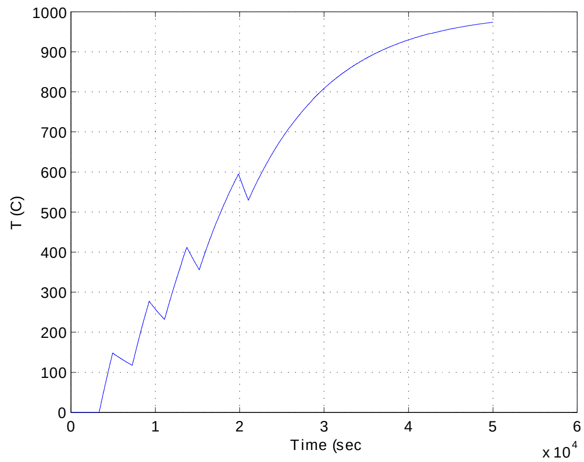
gives t_{on},

$$\begin{aligned} aP &= -\ln\left(\frac{700}{800}\right) - \ln\left(\frac{200}{300}\right), \\ P &= 100(0.058 + 0.176) = 23.4\,\text{sec}\,. \end{aligned}$$



Temperature output for Problem 9.19.

(c) See Figure below.



Temperature output for reference input $T_r = 3t$ for Problem 9.19.

20. Several systems such as spacecraft, spring-mass system with resonant frequency well below the frequency of switching, a large motor driven load with very small friction, etc. can be modeled as just an inertia. For an ideal switching curve, sketch the phase portraits of the system. The switching function is $e = \theta + \tau\omega$. Assume $\tau = 10$ *sec*, and the control signal $= 10^{-3}$ rad/sec^2. Now sketch the results with,

    a) deadband,

    b) deadband plus hysteresis,

    c) deadband plus time delay T,

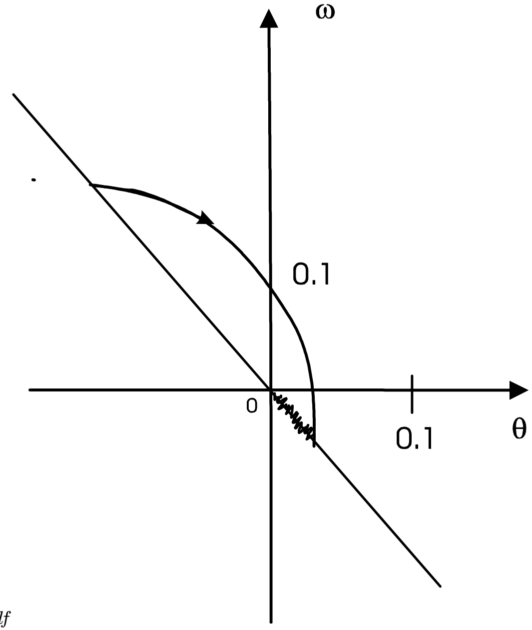
    d) deadband plus a constant disturbance.

    Solution:

    (a)

$$\begin{aligned}
\frac{d\omega}{d\theta} &= \frac{u}{\omega} = \frac{10^{-3}\text{rad}/\sec^2}{\omega}, \\
\frac{\omega^2}{2} &= u\ \theta, \\
\theta &= 500\ \omega^2, \\
\omega &= 10^{-2} \rightarrow \theta = 0.05.
\end{aligned}$$

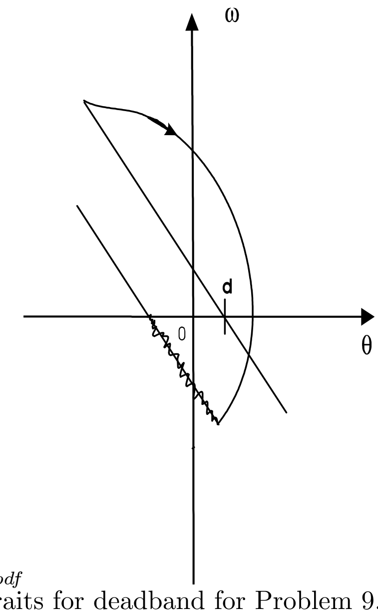
Fig 1



$_3$1.*pdf*
Phase portraits for deadband for Problem 9.20.

(a)

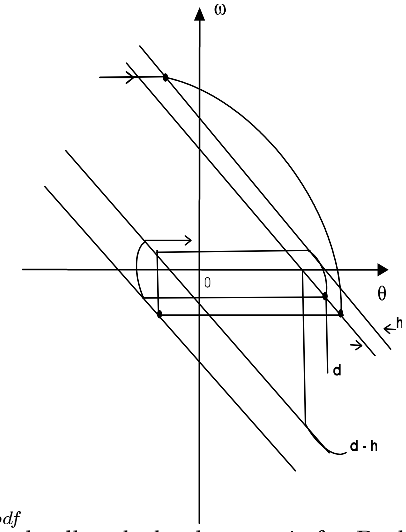
Fig 2



$_3$2.*pdf*
Phase portraits for deadband for Problem 9.20.

(b)

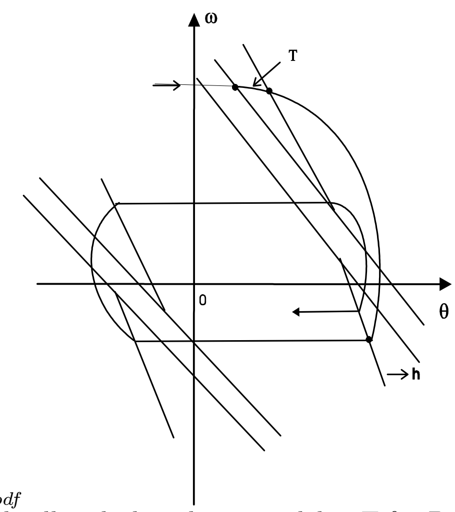
Fig 3



$_3$3.*pdf*
Phase portraits for deadband plus hysteresis for Problem 9.20.

(c)

Fig 4



$_3$4.*pdf*
Phase portraits for deadband plus plus time delay T for Problem 9.20.

(d)

$$
\begin{aligned}
T\omega &= \Delta\theta, \\
\Delta\theta + \tau\omega &= 0, \\
\text{changes slope to} \qquad \Delta\theta &= -\tau\omega + T\omega = -(\tau - T)\omega, \\
\dot{\theta} &= \omega \qquad u \to 0 \quad \text{at} \qquad t \quad \text{for} \qquad (\theta - d + h + \tau\omega = 0) + T, \\
\dot{\omega} &= u + D \qquad t_s \longleftrightarrow \theta_s = -\tau\omega_s + (d - h), \\
\omega &= (u + D)t + \omega_o, \\
&\Rightarrow \theta = (u + D)\frac{t^2}{2} + \omega_s t + \theta_s, \\
\text{for } t &= T, \quad \theta = \omega_s \underbrace{(T - \tau)}_{\text{new slope}} + \underbrace{(d - h) + \left(\frac{u + D}{2}\right) T^2}_{\text{new intercept}}.
\end{aligned}
$$

Reference:

[1] D. Graham and D. McRuer, *Analysis of Nonlinear Control Systems*, John Wiley & Sons, 1961.

21. Compute the amplitude of the limit cycle in the case of satellite attitude control with delay

$$I\,\ddot{\theta} = N\,u(t - \Delta)$$

using

$$u = -sgn(\tau\dot{\theta} + \theta)$$

Sketch the phase plane trajectory of the limit cycle and time history of θ giving the maximum value of θ.

Solution: Since the delay is Δ seconds, $\dot{\theta}$ must travel $\frac{\Delta N}{I}$ units during the delay. We can obtain the following relations:

$$
\begin{aligned}
\dot{\theta} &= \frac{N}{I}(t - t_0) + \dot{\theta}_0, \\
\theta &= \frac{1}{2}\frac{N}{I}(t - t_0)^2 + \dot{\theta}_0(t - t_0) + \theta_0.
\end{aligned}
$$

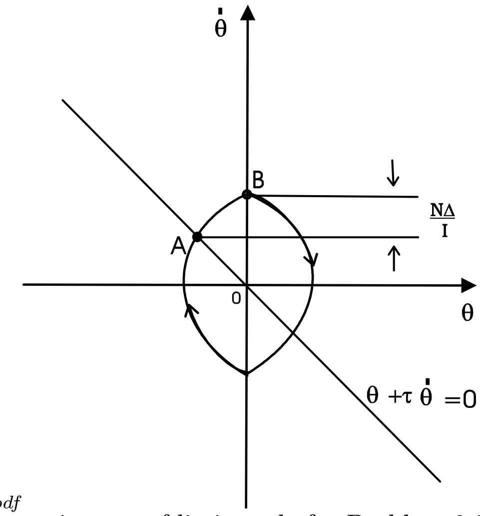
Eliminating $(t - t_0)$, we get

$$
\begin{aligned}
\theta - \theta_0 &= \frac{I}{2N}(\dot{\theta}^2 - \dot{\theta}_0^2) = \frac{I}{2N}(\dot{\theta}^2 - 0), \\
\theta &= \frac{2N}{I}(\theta - \theta_0) + \frac{I}{2N}\dot{\theta}^2),
\end{aligned}
$$

and

$$
\begin{aligned}
\theta + \tau\dot{\theta} &= 0, \\
\tau &= \frac{1}{a}.
\end{aligned}
$$

Fig 6



$_3$5.pdf
Phase-plane trajectory of limit cycle for Problem 9.21.

From the geometry of the limit cycle (see above Figure),

$$\dot{\theta}_A + \frac{\Delta N}{I} = \dot{\theta}_B, \tag{2}$$

At point A,

$$\theta_A - \theta_0 = \frac{I}{2N}\dot{\theta}_A^2, \tag{3}$$

$$\theta_A + \tau\dot{\theta}_A = 0, \tag{4}$$

At point B,

$$\dot{\theta}_B^2 \left(\frac{I}{2N}\right) = -\theta_0, \tag{5}$$

We need to solve the above four equations for θ_0 (for $\theta_{\max} = \theta_0$, $\theta_0 = 0$).
It seems to be easiest to first eliminate θ_0 using Eqs. (3) and (5) to get

$$\theta_A + \frac{I}{2N}\left[\dot{\theta}_B^2 - \dot{\theta}_A^2\right] = 0.$$

If we eliminate $\dot{\theta}_B$ using Eq. (2), then,

$$\theta_A = -\frac{\Delta}{2}\left[2\dot{\theta}_A + \frac{\Delta N}{I}\right] = 0.$$

If we use Eq. (4) to eliminate θ_A,

$$\tau\dot{\theta}_A = \frac{\Delta}{2}\left[2\dot{\theta}_A + \frac{\Delta N}{I}\right] = 0.$$

Solve for $\dot\theta_A$,

$$\dot\theta_A = \frac{\Delta^2 N}{2I}\left(\frac{1}{\tau-\Delta}\right).$$

From Eq. (2),

$$\dot\theta_B + \frac{N\Delta}{I} = \frac{\Delta^2 N}{2I}\left(\frac{1}{\tau-\Delta}\right).$$

Then using Eq. (5),

$$-\theta_0 = \frac{I}{2N}\left[\frac{\Delta^2 N}{2I}\left(\frac{1}{\tau-\Delta}\right) + \frac{\Delta N}{I}\right]^2,$$

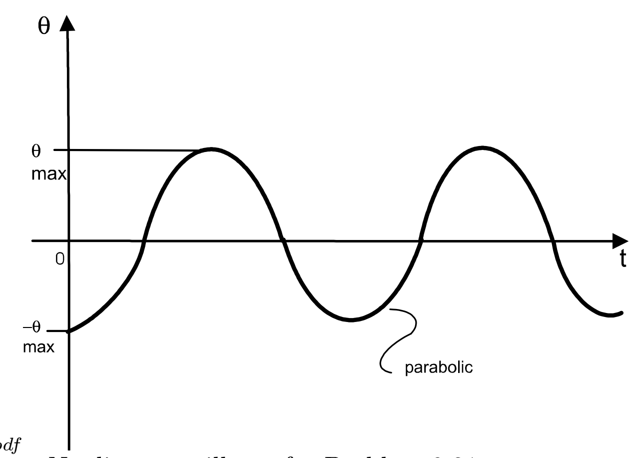
or,

$$|\theta_0| = \frac{I}{2N}\left[\frac{\Delta N(\Delta + 2(\tau-\Delta))}{2I(\tau-\Delta)}\right]^2,$$

$$|\theta_0| = \frac{N\Delta^2}{8I}\left[\frac{2\tau-\Delta}{\tau-\Delta}\right]^2 = |\theta_{\max}|.$$

Time history shown in the Figure below and shows a "nonlinear oscillator."

Fig 7



$_3$6.*pdf*

Nonlinear oscillator for Problem 9.21.

22. Consider the point mass pendulum with zero friction as shown in Figure 9.66. Using the method of isoclines as a guide, sketch the phase-plane portrait of the motion. Pay particular attention to the vicinity of $\theta = \pi$. Indicate a trajectory corresponding to spinning of the bob around and around rather than oscillating back and forth.

    (a) **Solution:** The equations are:

$$\begin{aligned} I\ddot\theta &= mgl\sin\theta, \\ ml^2\ddot\theta &= mgl\sin\theta, \\ \ddot\theta &= \frac{g}{l}\sin\theta, \end{aligned}$$

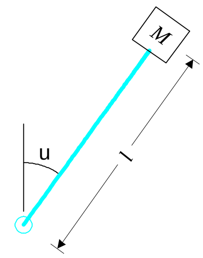


Figure 9.66: Pendulum for Problem 9.21.

For $\theta \sim 0$, $\sin\theta \sim \theta$,

$$
\begin{aligned}
\ddot{\theta} &= \frac{g}{l}\theta, \\
s^2 &= \frac{g}{l}, \\
s &= \pm\sqrt{\frac{g}{l}}.
\end{aligned}
$$

There is a "saddle" point at $\theta = 0, 2\pi, ...$

For $\theta \sim \pi$, $\sin\theta \sim -\theta$,

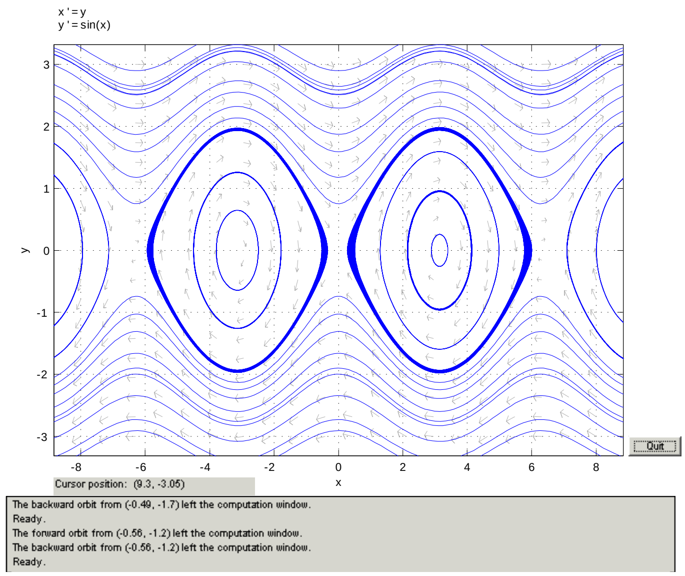
$$
\begin{aligned}
\ddot{\theta} &= -\frac{g}{l}\theta, \\
s^2 &= -\frac{g}{l}, \\
s &= \pm j\sqrt{\frac{g}{l}}.
\end{aligned}
$$

There is a "center" at $\theta = \pm\pi, \pm 3\pi, ...$

Using isoclines:

$$
\begin{aligned}
\ddot{\theta} &= \dot{\theta}\frac{d\dot{\theta}}{d\theta} = \frac{g}{l}\sin\theta \\
\alpha &= \frac{d\dot{\theta}}{d\theta} = \frac{g}{l}\sin\theta.
\end{aligned}
$$

The isoclines are sinusoidal curves. The phase portraits are shown in the Figure below. The upper and lower portraits correspond to the "whirling motion" with pendulum going round and round.



Phase portraits for Problem 9.22.

NOTE: The phase portraits can be generated using the ODE Software in Matlab pplane7.m (for Matlab Version 7) by Professor John C. Polking at Rice University available on the Web at:

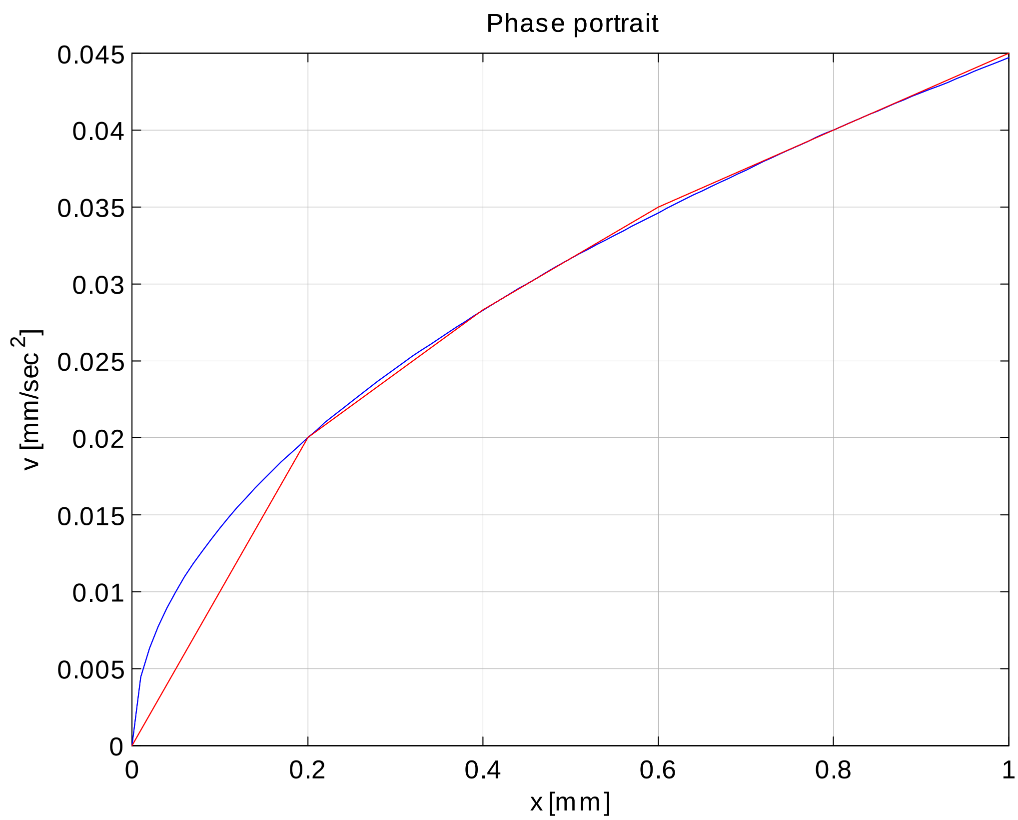
http://math.rice.edu/˜dfield/

23. Draw the phase trajectory for a system

$$\ddot{x} = 10^{-6} \text{ m/ sec}^2$$

between $\dot{x}(0) = 0$, $x(0) = 0$ and $x(t) = 1$mm. Find the transition time, t_f, by graphical means from the parabolic curve by comparing your solution with several different interval sizes and the exact solution.

Solution: The phase portrait is shown in the figure below.



Phase portrait for Problem 9.23.

$$
\begin{aligned}
\ddot{x} &= a, \qquad \frac{\dot{v}}{\dot{x}} = \frac{a}{v}, \\
\frac{v^2}{2} &= ax \ \text{ and } \ a = 10^{-6}, \\
v &= \sqrt{2ax} = at + v_o = at, \\
x(t) &= \frac{a}{2}t^2 + v_o t + x_o = 5 \times 10^{-7} t^2 = 1 \times 10^{-3}, \\
t^2 &= \left(\frac{1}{5}\right) \times 10^4 \Rightarrow t_f = 45\,\text{sec}\,.
\end{aligned}
$$

$$v = \frac{\Delta x}{\Delta t}.$$

To obtain t graphically, i.e., by graphical integration, we write,

$$\Delta t = \frac{\Delta x}{\hat{v}},$$

where $\hat{v}$ is the "average" v in a given interval of Δx. In the figure above, we divide x into five intervals and find (going from left to the right),

$$\begin{aligned} t_f &= \Delta t_1 + \Delta t_2 + \Delta t_3 + \Delta t_4 + \Delta t_5 \\ &= \frac{0.2}{0.01} + \frac{0.2}{0.025} + \frac{0.2}{0.0315} + \frac{0.2}{0.037} + \frac{0.2}{0.0425} \\ &= 20 + 8 + 6 + 5.4 + 4.7 = 44.1\,\text{sec}\,. \end{aligned}$$

which compares well with the exact answer of $t = 45$ sec. Better approximation can be found by finer division of x. Alternatively we can compute the time from,

$$t_f = \int_0^x \frac{1}{v} dx$$

which means the time can also be found by finding the area under the $v(x)^{-1}$ plot.

24. Consider the system with equations of motion,

$$\ddot{\theta} + \dot{\theta} + \sin\theta = 0$$

a) What physical system does this correspond to?

b) Draw the phase portraits for this system.

c) Show a specific trajectory for $\theta_0 = 0.5$ rad $\dot{\theta} = 0$.

Solution:

(a) Physical system is a pendulum with a hinge damping.

$$\begin{aligned} I\,\ddot{\theta} + b\,\dot{\theta} + mgl\sin\theta &= 0, \\ \dot{\theta} &= \omega, \\ \dot{\omega} &= -\frac{b}{I}\omega - \frac{mgl}{I}\sin\theta. \end{aligned}$$

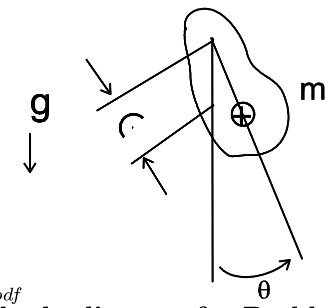
Independent variable scaled for $\tau \triangleq \frac{t}{T}$ where $T^2 = \frac{I}{mgl}$,

$$\frac{d\theta}{T\frac{dt}{T}} = \frac{1}{T}\frac{d\theta}{d\tau} = \omega.$$

Let us define $\Omega \triangleq T\omega$.

$$\begin{aligned} \frac{d\theta}{d\tau} &= \Omega, \\ \frac{d\Omega}{d\tau} &= -\left(\frac{Tb}{I}\right)\Omega - \sin\theta. \end{aligned}$$

Fig 5



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Pendulum free body diagram for Problem 9.24.

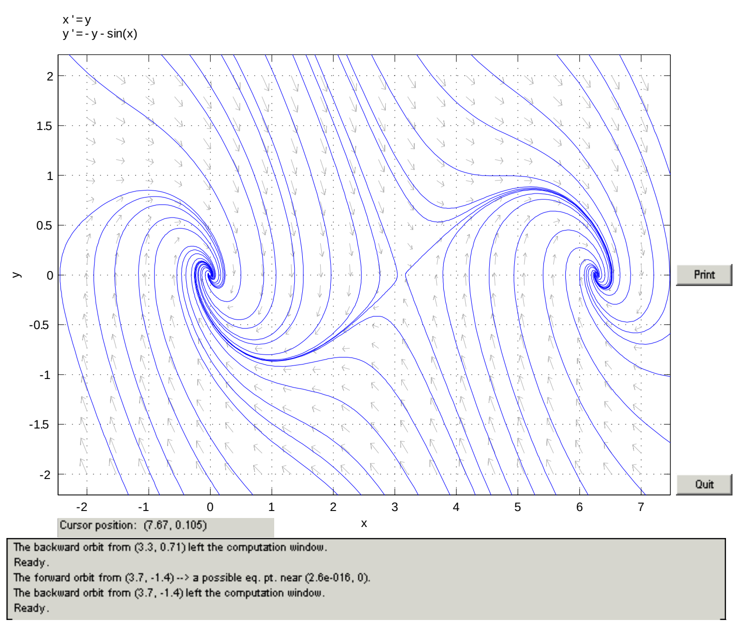
So the problem is a special case with $\frac{Tb}{I} = \sqrt{\frac{b^2}{\text{Im}\, gl}} = 1.$

$$\frac{d\Omega}{d\theta} = -1 - \frac{\sin\theta}{\Omega},$$

isoclines with slope $M = -1 - \frac{\sin\theta}{\Omega}$, where M is the slope.

(b) See phase portrait figure shown in the Figure below. Note the unstable equilibrium corresponding to $\theta = \pi$ and the stable equilibrium corresponding to the origin and $\theta = 2\pi$.

(c) See trajectory corresponding to $(x = \theta_0 = 0.5\text{rad},\ y = \dot{\theta}_0 = 0)$ in the phase portrait in the Figure below using pplane7.m software.



Phase portraits for pendulum with damping for Problem 9.24.

25. Consider the nonlinear upright pendulum with a motor at its base as an actuator. Design a feedback controller to stabilize this system.

Solution:

$$\ddot{\theta} = \sin\theta + u.$$

Using a lead network: $U(s) = -\frac{4(s+1)}{(s+3)}\Theta(s)$

$$\begin{aligned} x_1 &= \theta, \\ x_2 &= \dot{\theta}, \\ x_3 &= x_c. \\ \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} x_2 \\ \sin x_1 - 4x_1 - 4x_3 \\ -3x_3 - 2x_1 \end{bmatrix}. \end{aligned}$$

The linearized system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -3 & 0 & -4 \\ -2 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The system has poles at

$$\det(s\mathbf{I} - \mathbf{F}) = (s+1)^3.$$

The system is asymptotically stable near the origin so if the system starts near the upright position, it will be balanced.

26. Consider the system

$$\dot{x} = -\sin x$$

Prove that the origin is an asymptotically stable equilibrium point.

Solution: We wish to show that

$$\dot{V}(x) \leq -x^T Q x.$$

Select the Lyapunov function for

$$\begin{aligned} P &= 1 : \\ V(x) &= x^2, \end{aligned}$$

then

$$\begin{aligned} \dot{V} &= 2x\dot{x} = -2x\sin x, \\ \text{and} \quad \dot{V}(x) &\leq -x^2 \quad \text{for} \quad |x| \leq 1. \\ \text{Since} \quad \sin x &\geq \frac{1}{2}x \quad \text{for} \quad 0 \leq x \leq 1, \\ \text{we choose } Q &= 1, \end{aligned}$$

and conclude that the origin is an asymptotically stable equilibrium point.

27. A first-order nonlinear system is described by the equation $\dot{x} = -f(x)$, where $f(x)$ is a continuous and differentiable nonlinear function that satisfies the following:

$$\begin{aligned} &f(0) = 0, \\ &f(x) > 0 \quad \text{for} \quad x > 0, \\ &f(x) < 0 \quad \text{for} \quad x < 0. \end{aligned}$$

Use the Lyapunov function $V(x) = x^2/2$ to show that the system is stable near the origin ($x = 0$).

Solution:

$$\begin{aligned}
\dot{x} &= -f(x), \\
V(x) &= \frac{1}{2}x^2, \\
\dot{V}(x) &= x\dot{x} = -xf(x), \\
\text{For } x &> 0 \text{ and } f(x) > 0 \Longrightarrow \dot{V}(x) < 0, \\
\text{For } x &< 0 \text{ and } \ f(x) < 0 \Longrightarrow \dot{V}(x) < 0, \\
\text{For } x &= 0 \text{ and } f(x) = 0 \Longrightarrow \dot{V}(x) = 0.
\end{aligned}$$

Thus, for all $x \neq 0$, $\dot{V} < 0$. So applying Lyapunov's stability criterion, we conclude that the system is stable.

28. Use the Lyapunov equation

$$\mathbf{F}^T\mathbf{P} + \mathbf{P}\mathbf{F} = -\mathbf{Q} = -\mathbf{I}$$

to find the range of K for which the system in Fig. 9.67 will be stable. Compare your answer with the stable values for K obtained using Routh's stability criterion.

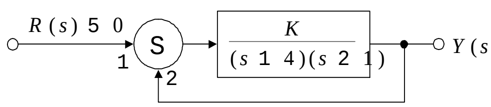


Figure 9.67: Control system for Problem 28

Solution:

Our approach is to set up the continuous Lyapunov equation and check that $\mathbf{P}$ is a positive definite matrix, i.e., $\mathbf{P} > 0$. Let,

$$\mathbf{P} = \begin{bmatrix} p & q \\ q & r \end{bmatrix}.$$

From the figure, the *closed-loop* system matrix $\mathbf{F}$ in controller canonical form is,

$$\mathbf{F} = \begin{bmatrix} -3 & 4-k \\ 1 & 0 \end{bmatrix}.$$

Solving $\mathbf{F}^T\mathbf{P} + \mathbf{P}\mathbf{F} = -\mathbf{I}$ yields,

$$\begin{aligned}
\begin{bmatrix} p & q \\ q & r \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 4-k & 0 \end{bmatrix} + \begin{bmatrix} p & q \\ q & r \end{bmatrix}\begin{bmatrix} -3 & 4-k \\ 1 & 0 \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \\
2q - 6p &= -1, \\
2q(4-k) &= -1, \\
p(4-k) + r - 3q &= 0.
\end{aligned}$$

Hence,

$$\begin{aligned} q &= \frac{-1}{2(4-k)}, \\ p &= \frac{1}{6}\left(\frac{3-k}{4-k}\right), \\ r &= \frac{-k^2+7k-21}{6(4-k)}. \end{aligned}$$

The two conditions for $\mathbf{P} > 0$ are $p > 0$ and $pr - q^2 > 0$, or,

$$p > 0 \Longrightarrow k > 3 \text{ or } k > 4,$$

and,

$$\begin{aligned} pr - q^2 &= \frac{k^3 - 10k^2 + 42k - 72}{36(k-4)^2} > 0, \\ &= \frac{(k-4)(k^2 - 6k + 18)}{36(k-4)^2} > 0. \end{aligned}$$

which is satisfied when $k > 4$, since $(k^2 - 6k + 18)$ is always positive. Thus, $k > 4$ for stability. Forming the Routh array, we have,

$$\begin{array}{cc} 1 & k-4 \\ 3 & 0 \\ 3(k-4) & \end{array}$$

Recall that the condition for stability is that all of the coefficients in the first column must be positive, which agrees with our previous answer above, namely $k > 4$.

Remark: Back of the envelope calculations using the Routh array are handy when the order of the system is low (as in this example). However for higher order systems, use of Lyapunov equation solvers, such as MATLAB's `lyap` command, are recommended.

29. Consider the system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 u \\ x_2(x_2 + u) \end{bmatrix}, \quad y = x_1.$$

Find all values of α and β for which the input $u(t) = \alpha y(t) + \beta$ will achieve the goal of maintaining the output $y(t)$ near 1.

Solution:

(a) It is desired to maintain the output $y(t)$ of the system,

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} x_1 + x_1 + x_2 u \\ x_2(x_2 + u) \end{bmatrix}, \\ y &= x_1. \end{aligned}$$

near 1. Find all values of α and β for which the input $u(t) = \alpha y(t) + \beta$ will achieve this goal. The problem has two parts: First, we investigate the equilibrium points; Next, we investigate the stability of the system by linearizing the nonlinear state equations near these equilibria. The nonlinear, closed-loop system equations are,

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2(\alpha x_1 + \beta),\\ \dot{x}_2 &= x_2(x_2 + \alpha x_1 + \beta).\end{aligned}$$

To find the equilibrium points for the desired output of $y = 1$, we set $x_1 = 1$, $\dot{x}_1 = \dot{x}_2 = 0$, to get,

$$\begin{aligned}0 &= 1 + x_2(\alpha + \beta),\\ 0 &= x_2(x_2 + \alpha + \beta),\end{aligned}$$

which can be solved for the equilibrium values of x_2 and the necessary relationship between α and β. Simultaneous solution yields,

$$x_2 = -\frac{1}{\alpha + \beta}.$$

and,

$$0 = x_2^2 + x_2(\alpha + \beta) = x_2^2 - 1 \Longrightarrow x_2 = \pm 1.$$

Consider the two equilibrium cases:

$$x_1 = 1,\ x_2 = 1 : \text{Let } y_1 = x_1 - 1,\ y_2 = x_2 - 1, \text{and } \alpha + \beta = -1.$$

Substituting these into the nonlinear closed-loop equations, we get,

$$\begin{aligned}\dot{y}_1 &= (1 + \alpha)y_1 - y_2 + \alpha y_1 y_2,\\ \dot{y}_2 &= \alpha y_1 + y_2 + \alpha y_1 y_2 + y_2^2.\end{aligned}$$

The characteristic equation of the linearized system is,

$$s^2 - (\alpha + 2)s + (2\alpha + 1) = 0.$$

There are no values of α which produce stable roots. So we conclude $x_1 = 1$ and $x_2 = 1$ is an unstable equilibrium point.

$$x_1 = 1,\ x_2 = -1 : \text{Let } y_1 = x_1 - 1,\ y_2 = x_2 + 1,\ \text{and } \alpha + \beta = 1.$$

Then,

$$\begin{aligned}\dot{y}_1 &= (1 - \alpha)y_1 + y_2 + \alpha y_1 y_2,\\ \dot{y}_2 &= -\alpha y_1 - y_2 + \alpha y_1 y_2 + y_2^2.\end{aligned}$$

The characteristic equation of the linearized system is,

$$s^2 + \alpha s + (2\alpha - 1) = 0.$$

So the system is stable for small signals near the equilibrium point if,

$$\alpha > 1/2 \text{ and } \alpha + \beta - 1 = 0.$$

30. Consider the nonlinear autonomous system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2(x_3 - x_1) \\ x_1^2 - 1 \\ -x_1 x_3 \end{bmatrix}.$$

a) Find the equilibrium point(s).
b) Find the linearized system about each equilibrium point.
c) For each case in part (b), what does Lyapunov theory tell us about the stability of the nonlinear system near the equilibrium point?

Solution:

(a) Setting $\dot{x}_1 = \dot{x}_2 = \dot{x}_3 = 0$ and solving the nonlinear equations, we obtain $[1, 0, 0]^T$ and $[-1, 0, 0]^T$ as the equilibrium points.

(b) We linearize the nonlinear state equations around the two equilibrium points from the first part.

(i)

$$\mathbf{x} = [1, 0, 0]^T : \text{Let } \ y_1 = x_1 - 1, \ y_2 = x_2, \ \text{and } y_3 = x_3.$$

Then the nonlinear equations become,

$$\begin{aligned} \dot{y}_1 &= -y_2 + y_2 y_3 - y_1 y_2, \\ \dot{y}_2 &= 2y_1 + y_1^2, \\ \dot{y}_3 &= -y_3 - y_1 y_3. \end{aligned}$$

Thus, the linearized system is $\dot{\mathbf{y}} = \mathbf{F}\mathbf{y}$ where,

$$\mathbf{F} = \begin{bmatrix} 0 & -1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(ii)

$$\mathbf{x} = [-1, 0, 0]^T \quad : \text{Let } y_1 = x_1 + 1, \ y_2 = x_2, \text{and } y_3 = x_3.$$

Then the nonlinear equations become,

$$\begin{aligned} \dot{y}_1 &= y_2 + y_2 y_3 - y_1 y_2, \\ \dot{y}_2 &= -2y_1 + y_1^2, \\ \dot{y}_3 &= y_3 - y_1 y_3. \end{aligned}$$

Thus, the linearized system is $\dot{\mathbf{y}} = \mathbf{F}\mathbf{y}$ where,

$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) We can use the linearization from the previous part to determine the stability of the system near the two equilibria.

(i) The characteristic equation is $(s^2+2)(s+1)=0$. The linear system is neutrally (marginally) stable with two poles on the $j\omega$ axis. So Lyapunov theory does not tell us whether this system is stable, and the nonlinear terms will affect the stability at the equilibrium point $[1,0,0]$.

(ii) The characteristic equation is $(s^2+2)(s-1)=0$. Thus the system at the equilibrium point $[-1,0,0]$ is unstable.

31. Van der Pol's equation: Consider the system described by the nonlinear equation

$$\ddot{x}+\varepsilon(1-x^2)\dot{x}+x=0$$

with the constant $\varepsilon>0$.

(a) Show that the equations can be put in the form [Liénard or (x,y) plane]:

$$\begin{aligned}\dot{x} &= y+\varepsilon\left(\frac{x^3}{3}-x\right)\\ \dot{y} &= -x.\end{aligned}$$

(b) Use the Lyapunov function $V=\frac{1}{2}(x^2+\dot{x}^2)$ and sketch the *region* of stability as predicted by this V in the Liénard plane.

(c) Plot the trajectories of part (b) and show the initial conditions that tend to the origin. Simulate the system in Simulink® using various initial conditions on $x(0)$ and $\dot{x}(0)$. Consider two cases with $\varepsilon=0.5$, and $\varepsilon=1.0$.

Solution:

(a) If we differentiate the first Liénard equation, we obtain

$$\begin{aligned}\ddot{x} &= \dot{y}+\varepsilon\left(\frac{3x^2}{3}\dot{x}-\dot{x}\right)=-x+\varepsilon\left(x^2\dot{x}-\dot{x}\right)\\ &= -x-\varepsilon(1-x^2)\dot{x}.\end{aligned}$$

which is the same as van der Pol's equation. Hence the two representations are equivalent. The two coordinate systems are related by the transformation,

$$\begin{bmatrix} x \\ \dot{x} \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ \varepsilon(\frac{x^2}{3}-1) & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

or,

$$\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ -\varepsilon(\frac{x^2}{3}-1) & 1 \end{bmatrix}\begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

(b) If we linearize the system, we obtain

$$\ddot{x}+\varepsilon\dot{x}+x=0$$

which has both roots inside the LHP at $\frac{-\varepsilon\pm\sqrt{\varepsilon^2-4}}{2}$. Therefore, there is a region of stability around the origin. Define $x_1\triangleq x,\ x_2\triangleq\dot{x}$,

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\varepsilon(1-x_1^2)x_2-x_1.\end{aligned}$$

Then,

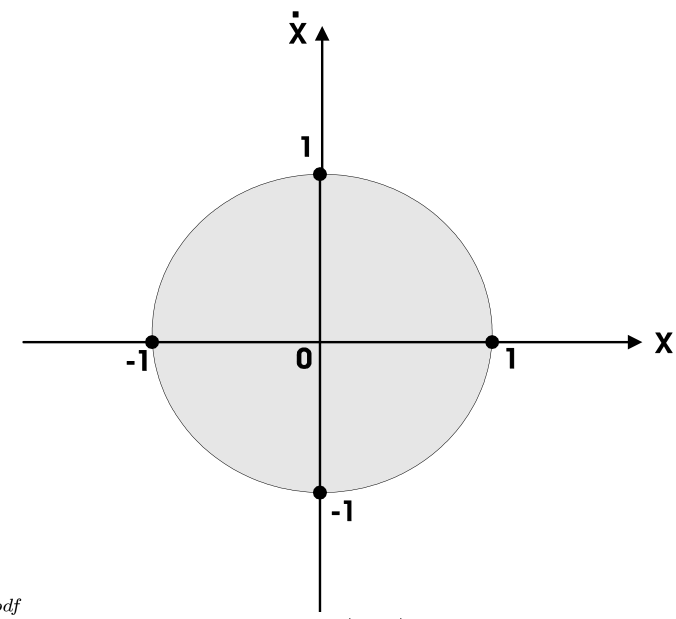
$$\begin{aligned}\dot{V} &= x\,\dot{x} + \dot{x}\,\ddot{x} = \dot{x}\left[x - \varepsilon(1-x^2)\dot{x} - x\right] \\ &= -\varepsilon\dot{x}^2(1-x^2).\end{aligned}$$

Now,

$$\dot{V} \leq 0 \Rightarrow (1-x^2) \geqslant 0 \ \Rightarrow |x| \leq 1.$$

The level curves where $V = c =$ constant are circles centered at the origin of the $(x,\,\dot{x})$ plane. The theory requires that the region of stability be *inside* a level curve where $V < c$. Therefore, all trajectories starting inside a circle of radius one centered at the origin (see Figure below) converge to the origin of the $(x,\,\dot{x})$ plane. This means that the origin is Lyapunov-stable. It also means that the limit cycle must lie outside a circle of radius one centered at the origin.

Fig 18



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Stability region in the $(x,\,\dot{x})$ plane.

The stability region may be mapped into the Liénard plane. The circular boundary in the $(x,\,\dot{x})$ plane can be mapped into the $(x,\,y)$ plane:

$$y = -\varepsilon\left(\frac{x^3}{3} - x\right) + \sqrt{1-x^2}$$

which resembles an ellipsoidal curve as shown in the Figure on top of the next page. An approximate analytical answer is also possible. The stability region in the Liénard $(x,\,y)$

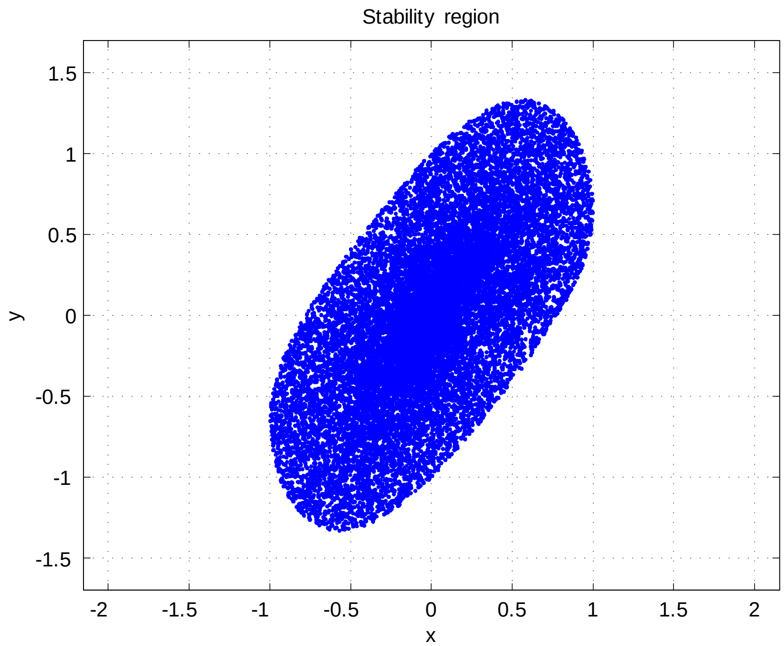


Figure 9.68: Stability region in the Liénard $(x,\, y)$ plane.

plane is roughly a rotated ellipse. This can be seen as follows.

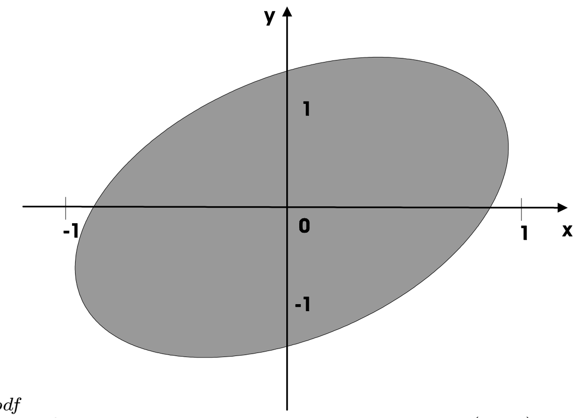
$$\begin{aligned} V &= \frac{1}{2}(x^2+\dot{x}^2)=c \\ &= \frac{1}{2}\left(x^2+\left[y+\varepsilon\left(\frac{x^3}{3}-x\right)\right]^2\right) \\ &= \frac{1}{2}\left(x^2+y^2+\varepsilon^2\left(\frac{x^3}{3}-x\right)^2+2y\varepsilon\left(\frac{x^3}{3}-x\right)\right) \end{aligned}$$

Assuming that ε^2 and x^3 are small and may be neglected,

$$V \approx \frac{1}{2}\left(x^2+y^2-2\varepsilon yx\right) \approx c$$

It is seen that the level curves are roughly ellipses that are rotated by an angle of $+45^\circ$ in the Liénard $(x,\, y)$ plane.

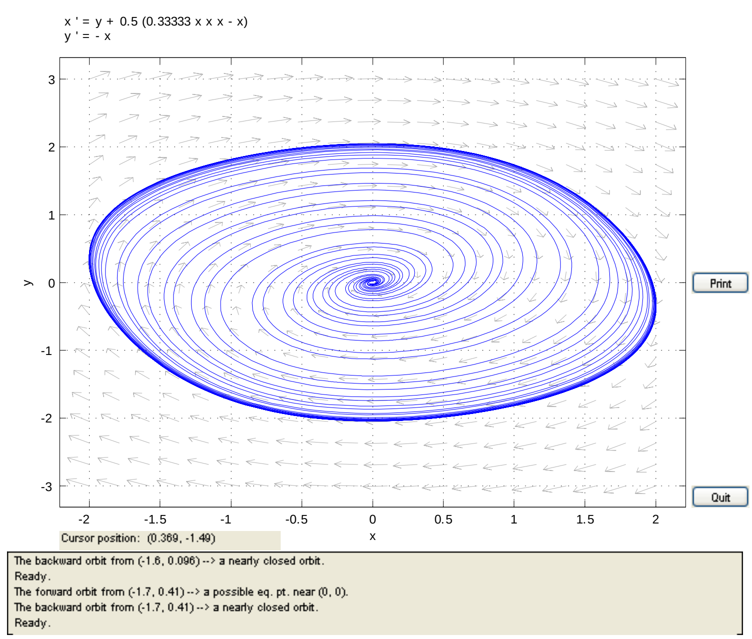
Fig 19



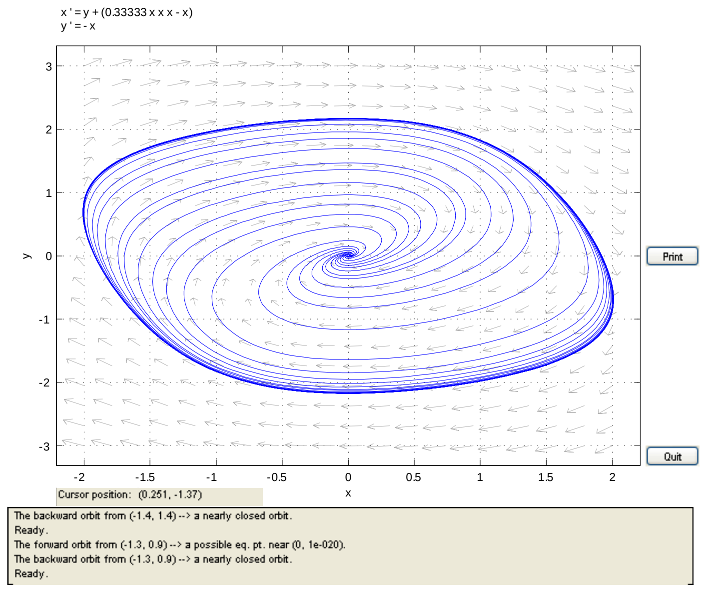
$_4 5.pdf$

Approximation of the stability region in the Liénard $(x,\, y)$ plane.

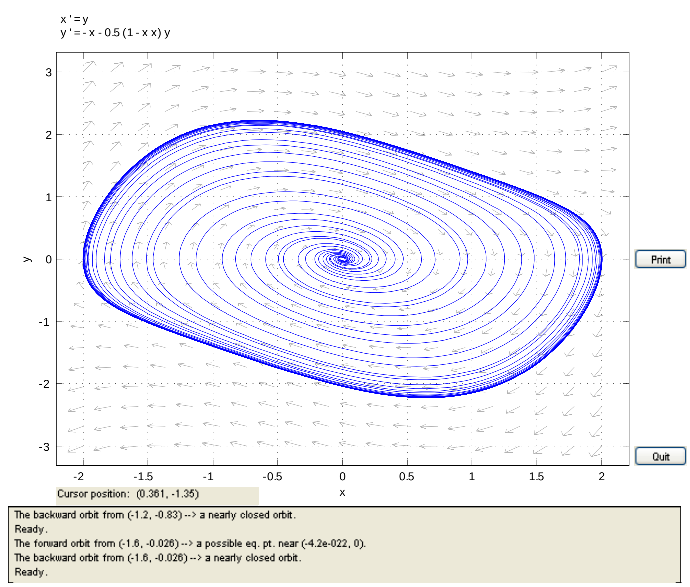
(c) Using pplane7.m software we see that the limit cycle is *nearly circular* with radius 2:



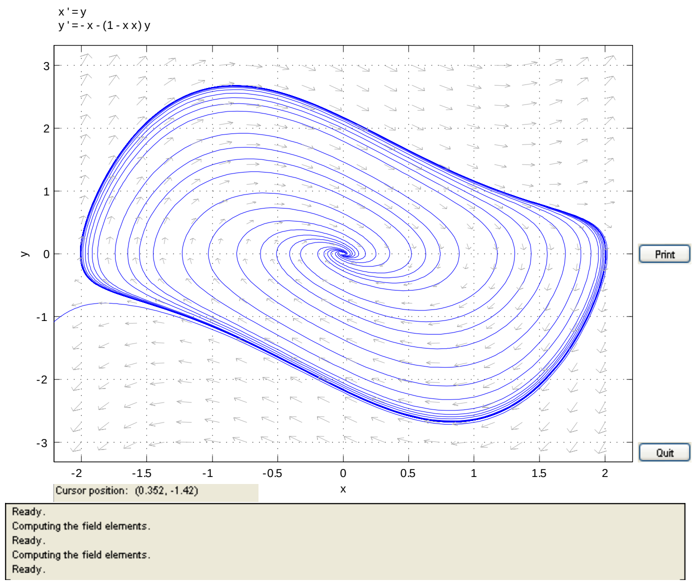
Phase portraits for van der Pol equation in Liénard $(x,\, y)$ form for $\varepsilon = 0.5$.



Phase portraits for van der Pol equation in the Liénard $(x,\, y)$ plane for $\varepsilon = 1.0$.

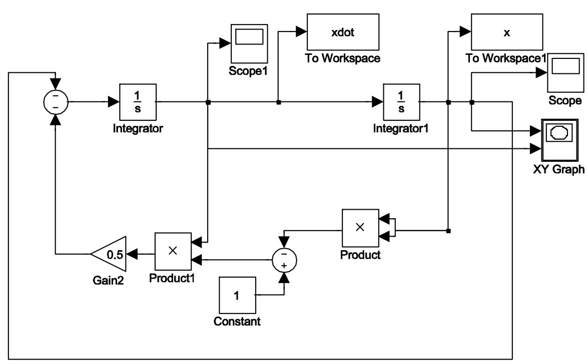


Phase portraits for van der Pol equation in the $(x,\, \dot{x})$ plane for $\varepsilon = 0.5$.

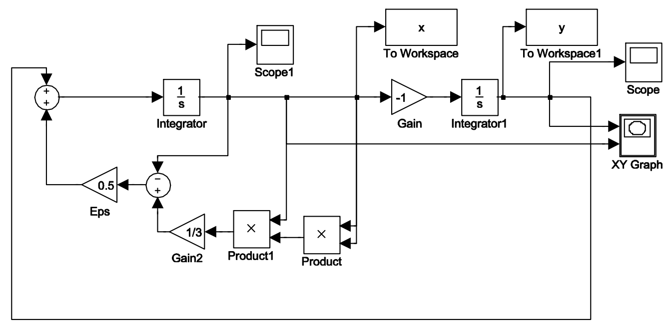


Phase portraits for van der Pol equation in the $(x, \dot{x})$ plane for $\varepsilon = 1.0$.

The Simulink simulations are shown in the attached figures.

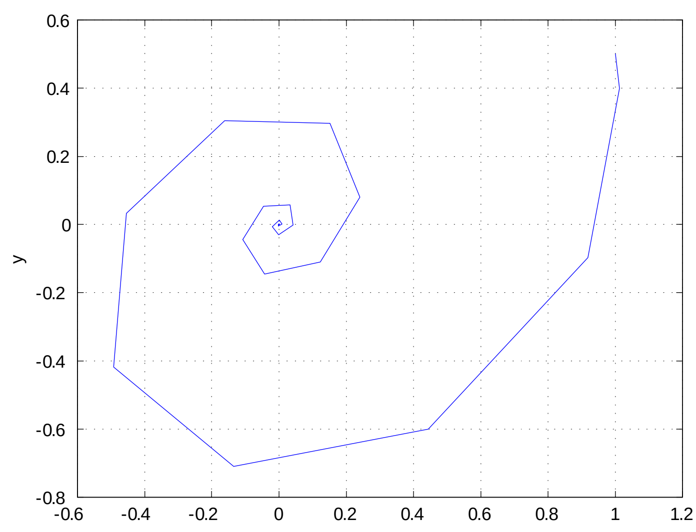


Simulink simulation diagram for van der Pol's equation for $(x, \dot{x})$ plane.

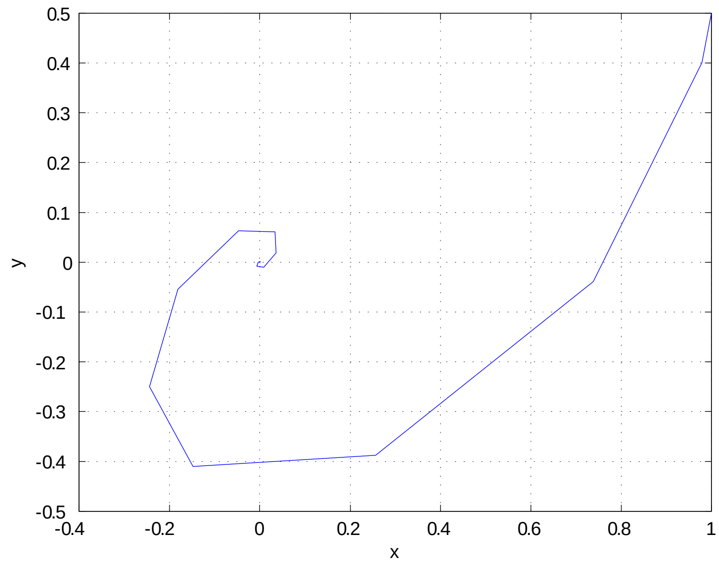


Simulink simulation of van der Pol's equation in the Liénard (x, y) plane.

Sample trajectories are shown in the attached figures.

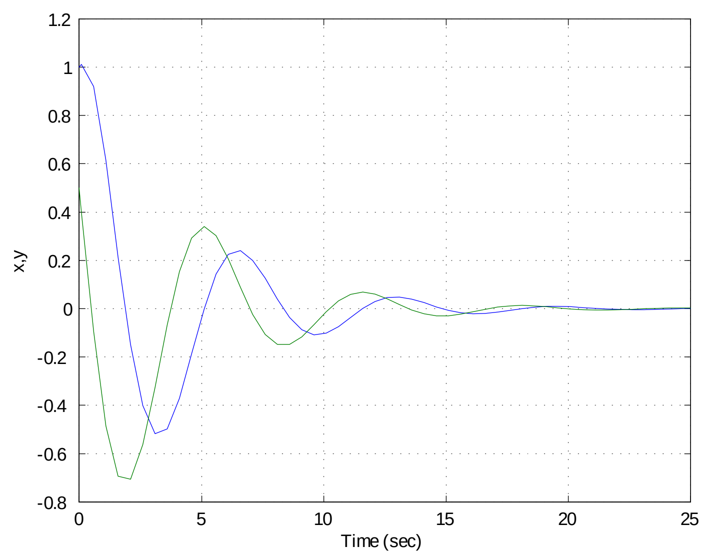


Trajectory for $x(0) = 1$, $y(0) = 0.5$ in the Liénard (x, y) plane for $\varepsilon = 0.5$.

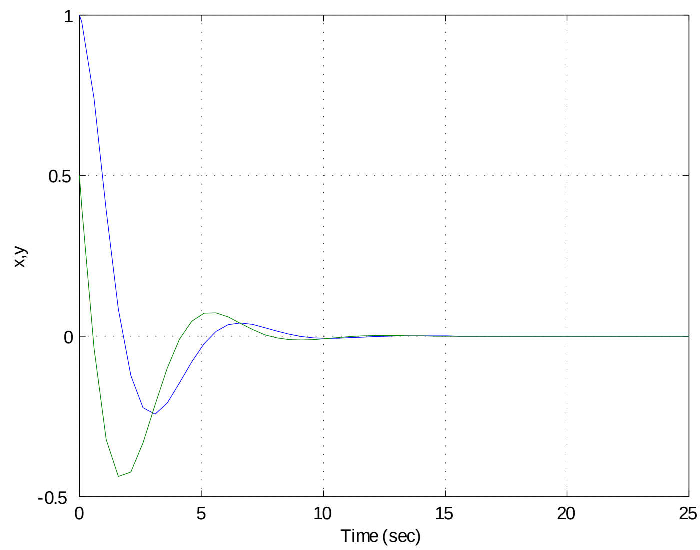


Trajectory for $x(0) = 1,\ \ y(0) = 0.5$ in the Liénard $(x,\, y)$ plane for $\varepsilon = 1.0$.

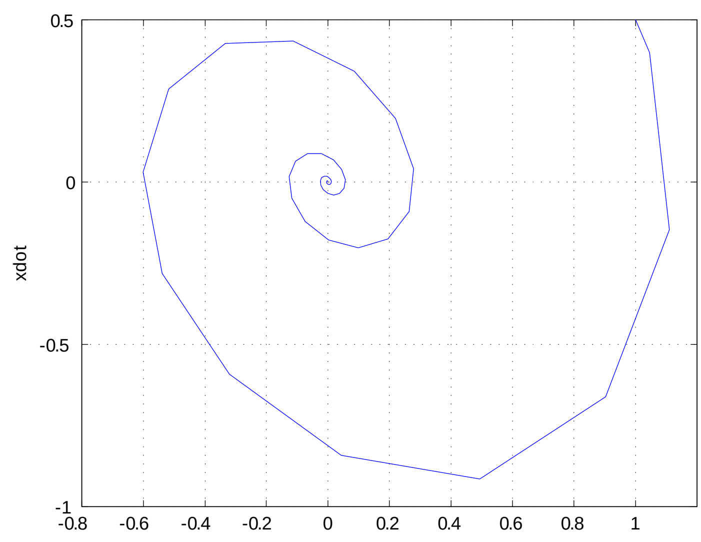
The corresponding time domain responses are shown in the attached figures.



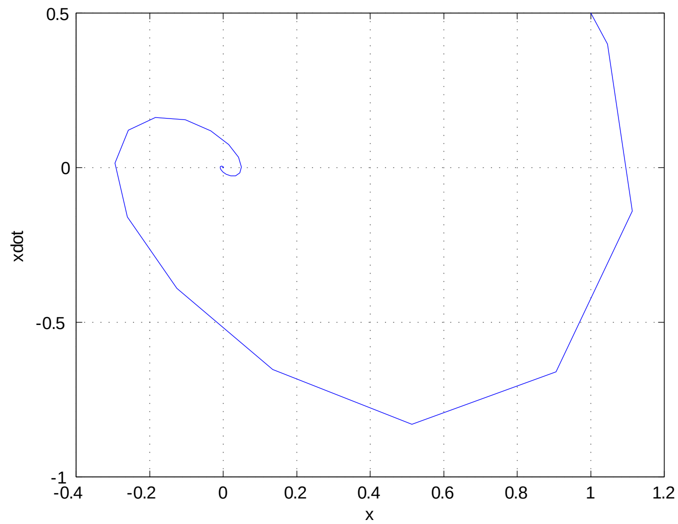
State variables for $x(0) = 1$, $y(0) = 0.5$ in the Liénard $(x,\, y)$ plane for $\varepsilon = 0.5$.



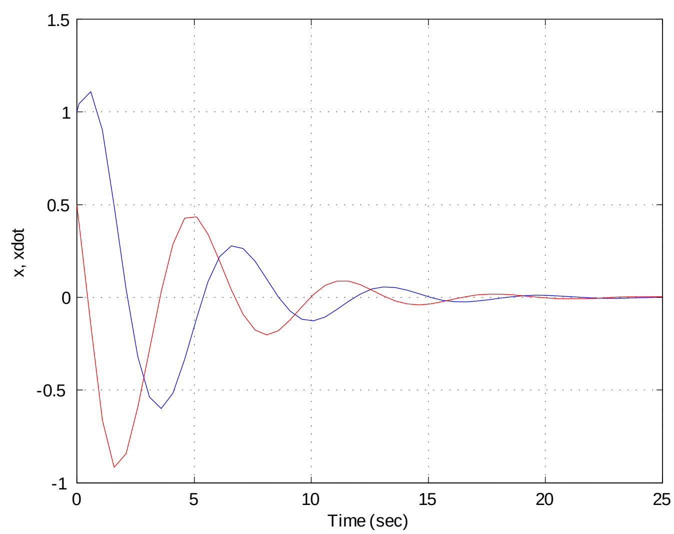
State variables for $x(0) = 1$, $y(0) = 0.5$ in the Liénard plane for $\varepsilon = 1.0$.



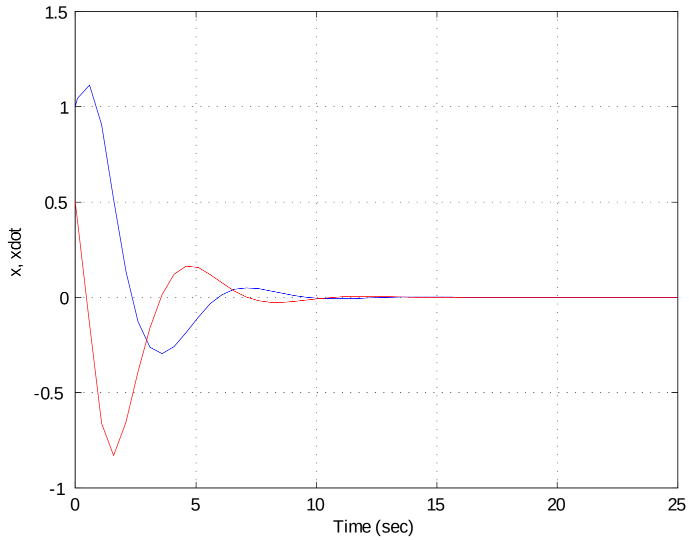
Trajectory for $x(0) = 1$, $\dot{x}(0) = 0.5$ in the $(x,\, \dot{x})$ plane for $\varepsilon = 0.5$.



Trajectory for $x(0) = 1$, $\dot{x}(0) = 0.5$ in the $(x,\, \dot{x})$ plane for $\varepsilon = 1.0$.



State variables for $x(0) = 1$, $\dot{x}(0) = 0.5$ in the $(x,\, \dot{x}\ )$ plane for $\varepsilon = 0.5$.



State variables for $x(0) = 1$, $\dot{x}(0) = 0.5$ in the $(x, \dot{x})$ plane for $\varepsilon = 1.0$.

Chapter 10

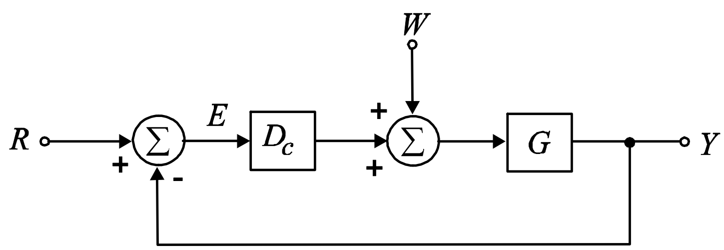
Control-System Design: Principles and Case Studies

Problems and Solutions for Chapter 10

1. Of the three types of PID control (proportional, integral, or derivative), which one is the most effective in reducing the error resulting from a constant disturbance? Explain.

   Solution:

   Integral control is the most effective in reducing the error due to constant disturbances.



Problem 10.1: Block diagram for showing integral control is the most effective means of reducing steady-state errors.

Using the above block diagram,

$$\begin{aligned}
Y &= G(W + ED_c), \\
E &= R - Y = R - G(W + ED_c), \\
E &= \frac{1}{1 + D_c G} R - \frac{G}{1 + D_c G} W, \\
e_\infty &= \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \left(\frac{1}{1 + D_c G} R - \frac{G}{1 + D_c G} W \right).
\end{aligned}$$

Writing $G(s) = \frac{n_G(s)}{d_G(s)}$, and using a step input $R(s) = \frac{k_r}{s}$, and a step disturbance $W(s) = \frac{k_w}{s}$, we can show that integral control leads to zero steady-state error, while proportional and derivative control, in general, do not.

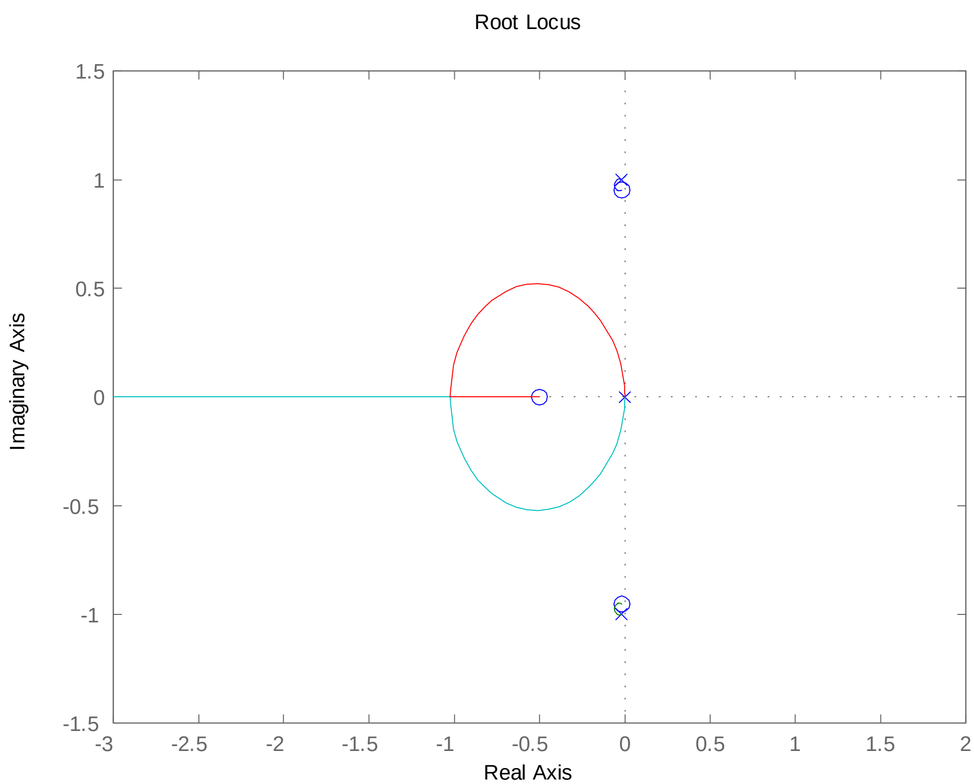
$$\begin{aligned}
\text{Integral control}, D_c(s) &= \frac{1}{s},\ e_\infty = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\left(\frac{d_G s k_r}{d_G s + n_G} + \frac{n_G s k_w}{d_G s + n_G}\right) = 0,\ \text{if } n_G(0) \neq 0,\\
\text{Proportional control}, D_c(s) &= K_p,\ e_\infty \neq 0,\\
\text{Derivative control}, D_c(s) &= s,\ e_\infty = k_r - G(0)k_w \neq 0,\ \text{if } d_G(0) \neq 0.
\end{aligned}$$

This analysis assumes that there are no pole-zero cancellations between the plant, G, and the compensator, D_c. In general, proportional or derivative control will not have zero steady-state error.

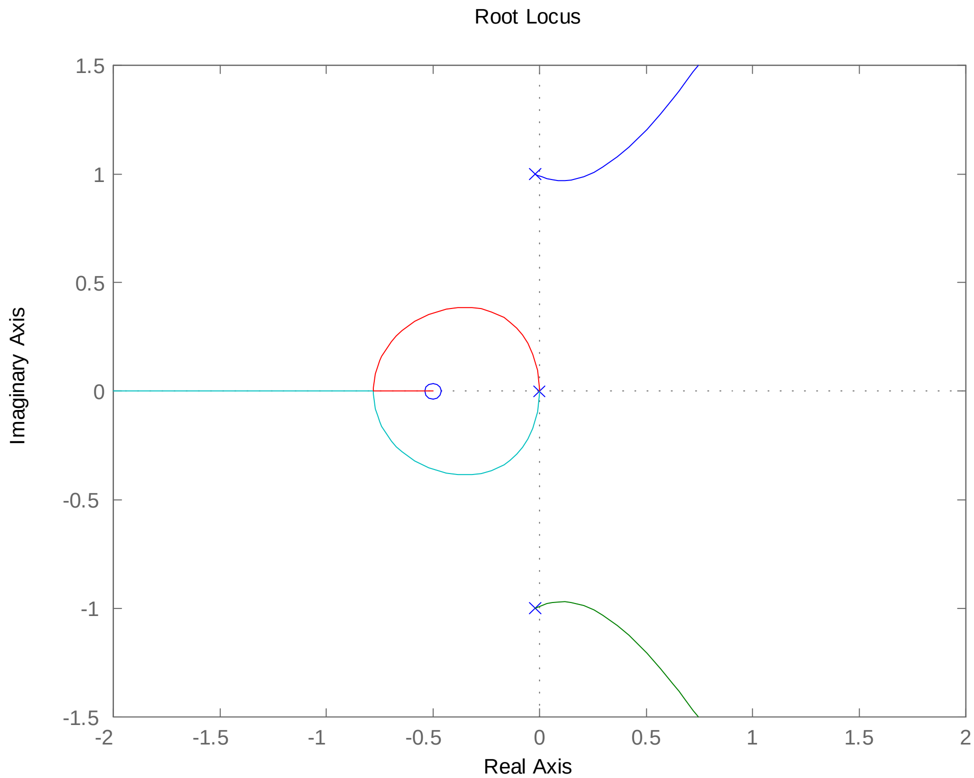
2. Is there a greater chance of instability when the sensor in a feedback control system for a mechanical structure is not collocated with the actuator? Explain.

Solution:

Yes. For comparison, see the following two root loci which were taken from the discussion in the text on satellite attitude control. In Fig. 10.26, the sensor and the actuator are collocated, resulting in a stable closed-loop system with PD control. In Fig. 10.5, the sensor and the actuator are not collocated creating an unstable system with the same PD control.



Problem 10.2: [Text Fig. 10.26] PD control of satellite: collocated.



Problem 10.2: [Text Fig. 10.5] PD control of satellite: non-collocated.

3. Consider the plant $G(s) = 1/s^3$. Determine whether it is possible to stabilize this plant by adding the lead compensator

$$D_c(s) = K\frac{s+a}{s+b}, \quad (a < b).$$

(a) What is the maximum phase margin of the resulting feedback system?

(b) Can a system with this plant, together with any number of lead compensators, be made unconditionally stable? Explain why or why not.

Solution:

(a) $G(s) = 1/s^3$ has phase angle of -270° for all frequencies. The maximum phase lead from a compensator $D_c(s) = K\frac{s+a}{s+b}$ is 90° with $\frac{b}{a} = \infty$. In practice a lead compensator with $\frac{b}{a} = 100$ contributes phase lead of approximately 80°. Hence the closed-loop system will be unstable with PM $= -10°$. To have PM $\approx 70°$ we need, for example, a double lead compensator $D_c(s) = \frac{(s+a)^2}{(s+b)^2}$ with $\frac{b}{a} = 100$.

(b) No, this plant cannot be made "unconditionally stable" because the root locus departure angles from the three poles at the origin are $\pm$ 60°. For low enough gain, the poles are always in the right-half-plane. If we try positive feedback, one pole departs at 0° so again, one pole starts into the right-half-plane. For low-enough gain, the system will be unstable.

4. Consider the closed-loop system shown in Fig. 10.88.

   (a) What is the phase margin if $K = 70,000$?

   (b) What is the gain margin if $K = 70,000$?

   (c) What value of K will yield a phase margin of $\sim 70°$?

   (d) What value of K will yield a phase margin of $\sim 0°$?

   (e) Sketch the root locus with respect to K for the system, and determine what value of K causes the system to be on the verge of instability.

   (f) If the disturbance w is a constant and $K = 10,000$, what is the maximum allowable value for w if $y(\infty)$ is to remain less than 0.1? (Assume $r = 0$.)

   (g) Suppose the specifications require you to allow larger values of w than the value you obtained in part (f) but with the same error constraint $[|y(\infty)| < 0.1]$. Discuss what steps you could take to alleviate the problem.

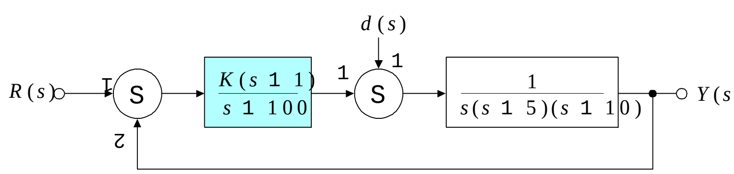


Figure 10.88: Control system for Problem 10.4.

Solution:

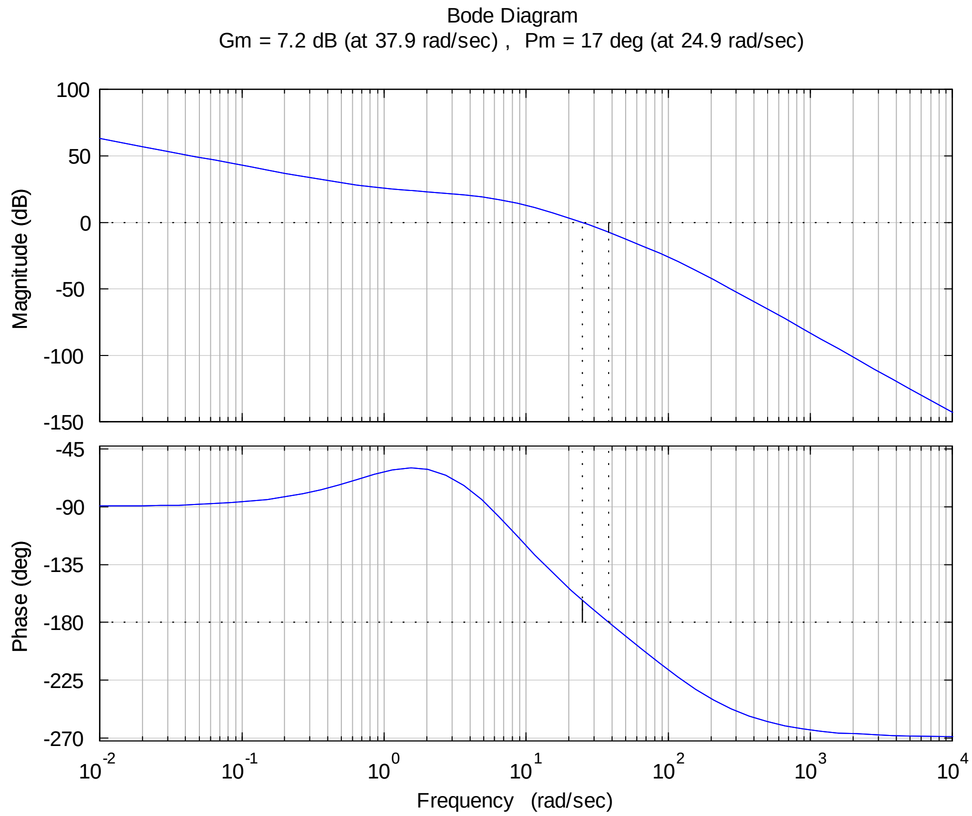
(a) To determine the phase and gain margin of the system given in Fig. 10.88, we produce the Bode plot of the loop gain shown on the next page (using MATLAB's margin command),

$$KD_c(s)G(s) = \frac{K(s+1)}{s(s+5)(s+10)(s+100)} = \frac{14(s+1)}{s(s/5+1)(s/10+1)(s/100+1)},$$

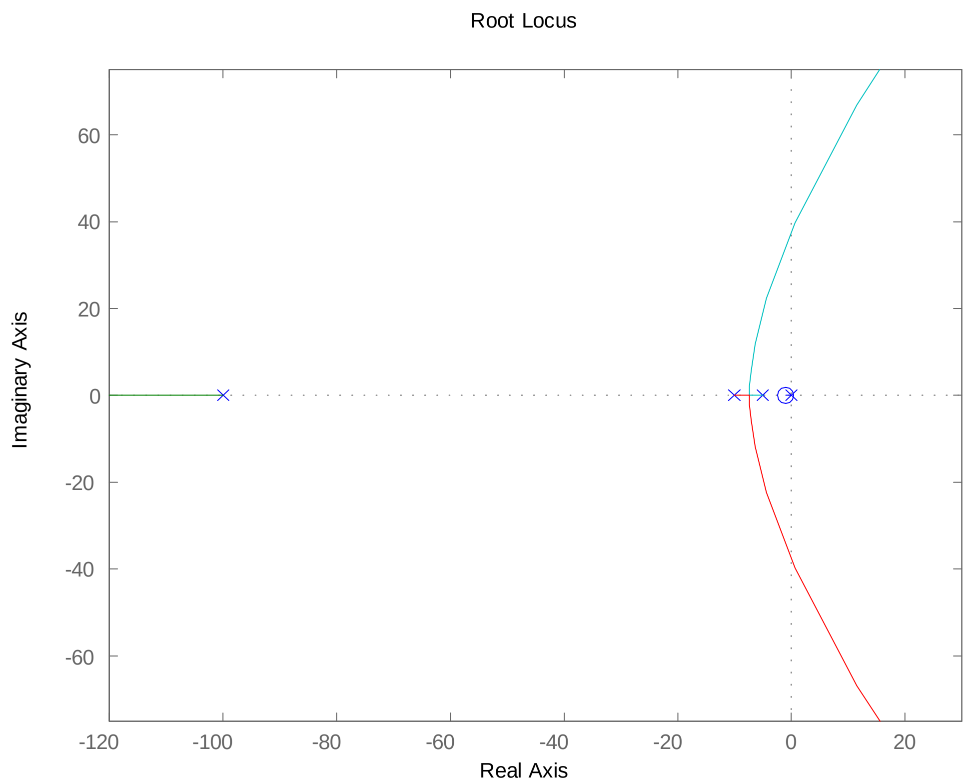
where $K = 70,000$. The Bode plot is shown on the next page along with the phase and gain margin. From the figure, the phase margin is about 17° near $\omega = 25.0$ rad/sec.

(b) The gain margin, from the figure, is approximately 7.2 db at $\omega = 38.0$ rad/sec. Therefore, the gain and phase margins are

$$\begin{aligned}\text{Gain Margin} &= 7.2 \text{ db},\\ \text{Phase Margin} &= 17.01°.\end{aligned}$$



Bode plot for Problem 10.4.



Root Locus for Problem 10.4.

(c) A phase margin of 70° requires the magnitude to cross the 0 db line near a frequency of $\omega = 8.3$ rad/sec. Hence, the magnitude frequency response must be attenuated by 15 db, or the loop gain multiplied by 0.178. Therefore,

$$K_{70^\circ} = 0.178, \;\; K = 12,500.$$

(d) A phase margin of 0° results from amplifying the gain by exactly the gain margin value found in part (b). Hence, we amplify the loop gain by 7.2 db, or 2.293.

$$K_{0^\circ} = 2.293, \quad K = 160,500.$$

(e) The root locus of the system is given (using MATLAB's rlocus command). The value of K that causes the system to be on the verge of stability is the gain where the root loci cross the $j\omega$ axis. This value of K can be calculated algebraically or can be determined by the use of the MATLAB command rlocfind. In addition, the result from part (d) can be used since zero phase and gain margin translate to the system being on the verge of instability. Hence, the range of K for stability is $0 < K < 160,500$.

(f) With $R = 0$ and the disturbance labeled as w, we can write the transfer function from $W(s)$ to $Y(s)$ to determine the steady-state output value due to a constant disturbance input.

$$
\begin{aligned}
Y(s) &= \frac{G}{1 + KD_cG}W(s), \\
y_{ss} &= \lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} s\frac{G}{1 + KD_cG}W(s).
\end{aligned}
$$

If $w(t)$ is constant, $w(t) = c$, then $W(s) = c/s$, so we have,

$$
y_{ss} = \frac{100c}{K}.
$$

(g) Therefore, with $K = 10000$ and $y < 0.1$, we have $c < 10$. Since $y_{ss} = 100c/K$, we can increase the gain K to obtain the same error specification, y_{ss}, given larger values of c. However, this will sacrifice system stability and possibly transient performance. In this case, integral control can be added to reduce the steady-state output error to zero.

5. Consider the system shown in Fig. 10.89, which represents the attitude rate control for a certain aircraft.

   (a) Design a compensator so that the dominant poles are at $-2 \pm 2j$.

   (b) Sketch the Bode plot for your design, and select the compensation so that the crossover frequency is at least $2\sqrt{2}$ rad/sec and PM $\geq 50°$.

   (c) Sketch the root locus for your design, and find the velocity constant when $\omega_n > 2\sqrt{2}$ and $\zeta \geq 0.5$.

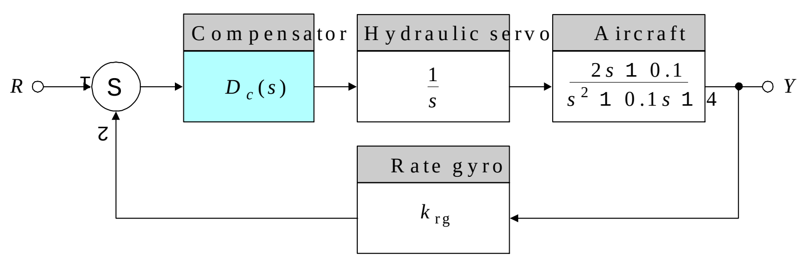


Figure 10.89: Block diagram for aircraft-attitude rate control.

Solution:

(a) With a constant gain compensator, $D_c(s) = K$, the root locus of,

$$
D_c(s)G(s) = \frac{2K(s + 0.05)}{s(s^2 + 0.1s + 4)} = \frac{num}{den}.
$$

does not pass through $-2 \pm 2j$. Therefore we need compensation of at least a lead network. Let,

$$
D_c(s) = K\frac{s + z}{s + p}.
$$

Using the angle criterion, at the closed-loop pole location $s = -2 + 2j$, we can write an expression for the angle contribution from the lead network zero, ϕ_z, and lead network pole, ϕ_p.

$$\sum \phi_{z_i} - \sum \phi_{p_i} = -180^\circ \Longrightarrow \phi_z - \phi_p + 134^\circ - 180^\circ - 135^\circ - 116^\circ = -180^\circ.$$

So we have, $\phi = \phi_z - \phi_p = 117^\circ$. In MATLAB,

$$\mathsf{PHI} = \ \mathsf{180/pi} \ * \ [\ \mathsf{angle(polyval\ (n,s)\ /polyval(d,s))} \ - \ \mathsf{pi}\ ].$$

With selection of $z = 0.4$, we get $p = 11.7$. So that our lead design is,

$$D_c(s) = K\frac{s + 0.4}{s + 11.7}.$$

To find the compensator gain, K, we can utilize the magnitude criterion at the desired dominant closed-loop pole locations. We find that,

$$|D_c(s)G(s)|_{s=-2\pm j2} = 1 \Longrightarrow K = 17.0.$$

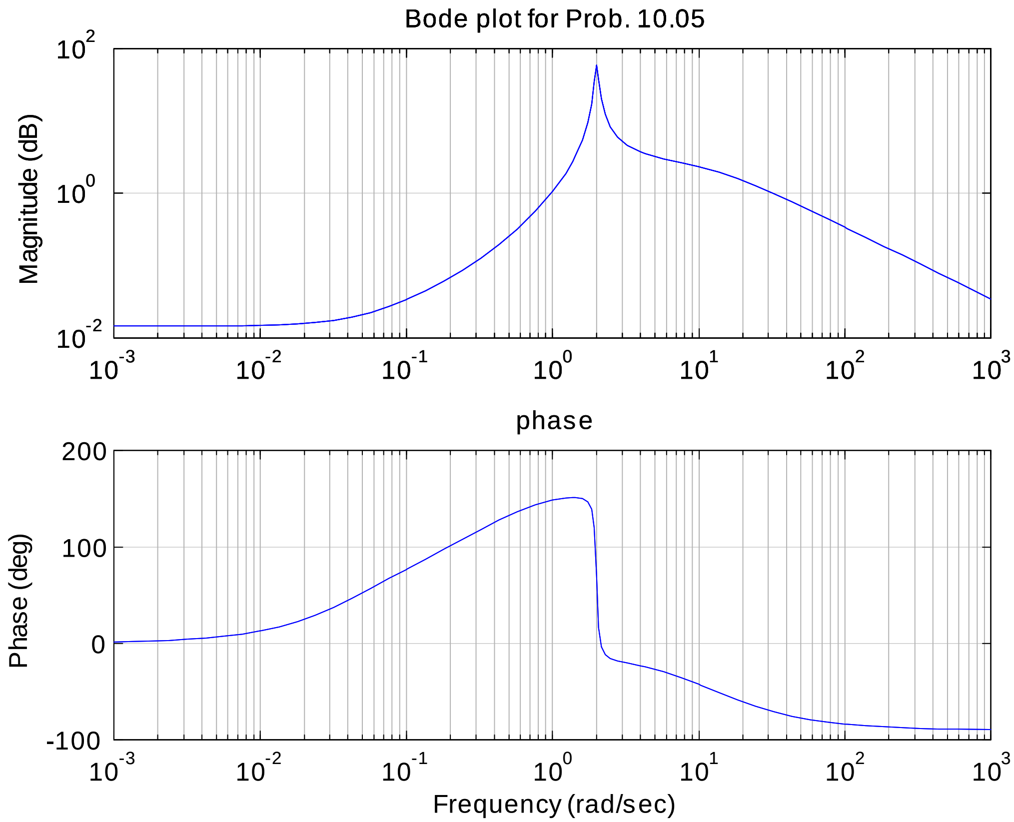
So the lead design is,

$$D_c(s) = 17\frac{s + 0.4}{s + 11.7}.$$

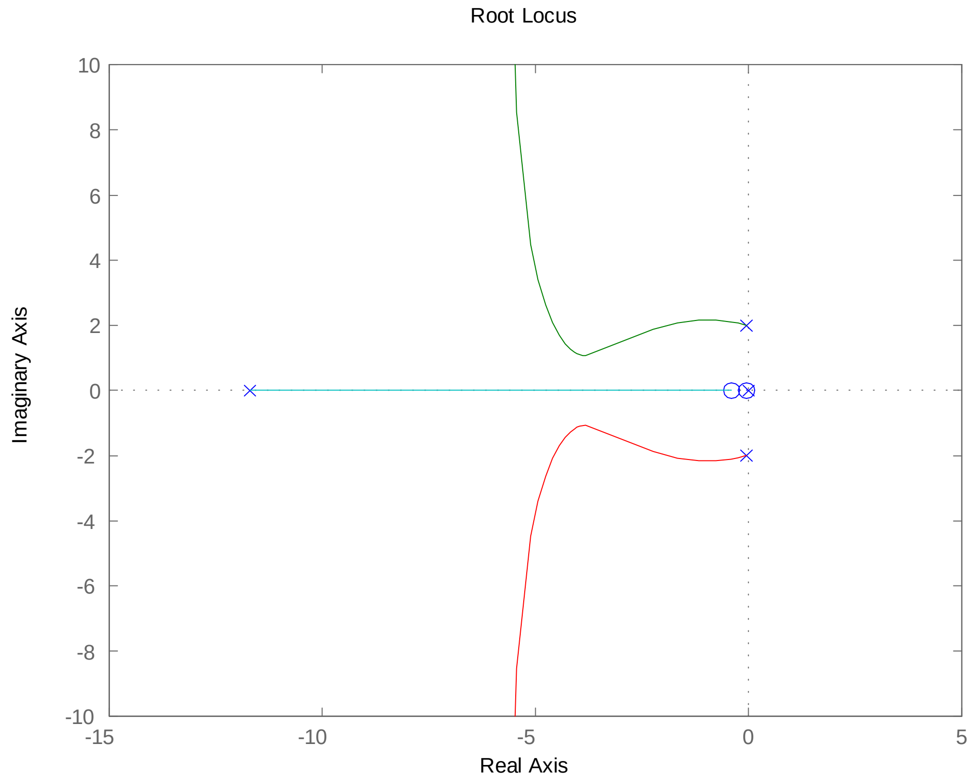
(b) The Bode plot of the system loop transfer function,

$$D_c(s)G(s) = 34\frac{(s + 0.4)(s + 0.05)}{s(s + 11.7)(s^2 + 0.1s + 4)},$$

is shown on the next page using MATLAB's Bode command. As the plot shows $\omega_c = 3$ and PM $= 67.3^\circ$. Therefore, both of the specifications are met by our design.



Problem 10.5 PD control of an aircraft: Bode plot.



Problem 10.5 PD control of an aircraft: root locus.

(c) The root locus plot is shown above using MATLAB's `rlocus` command. The velocity constant is most easily found from either the Bode plot or from,

$$K_v = \lim_{s \to 0} sD_c(s)G(s).$$

For our compensated system, $K_v = 0.0145$.

6. Consider the block diagram for the servomechanism drawn in Fig. 10.90. Which of the following claims are true?

(a) The actuator dynamics (the pole at 1000 rad/sec) must be included in an analysis to evaluate a usable maximum gain for which the control system is stable.

(b) The gain K must be negative for the system to be stable.

(c) There exists a value of K for which the control system will oscillate at a frequency between 4 and 6 rad/sec.

(d) The system is unstable if $|K| > 10$.

(e) If K must be negative for stability, the control system cannot counteract a positive disturbance.

(f) A positive constant disturbance will speed up the load, thereby making the final value of e negative.

(g) With only a positive constant command input r, the error signal e must have a final value greater than zero.

(h) For $K = -1$ the closed-loop system is stable, and the disturbance results in a speed error whose steady-state magnitude is less than 5 rad/sec.

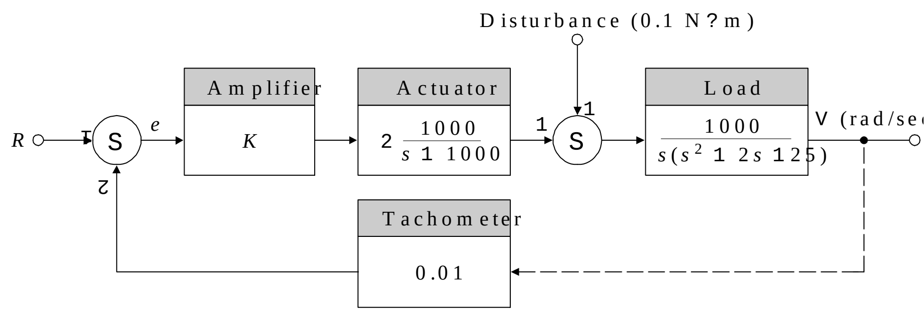


Figure 10.90: Servomechanism for Problem. 10.6.

Solution:

(a) True. Even though it is tempting to approximate the actuator dynamics as infinitely fast, and hence, not important, the actuator pole dramatically alters the root-locus plot of the system to be controlled. The root locus shown on the next page is for the system without the actuator pole. The root locus for the entire system is also shown. Note that two very different root loci result.

(b) True. On a root locus plot, the pole at $s = 0$ will immediately move into the right-half plane unless the gain is negative. The root locus of the system for negative gain K is shown on the next page.

(c) True. A gain of $K = -4.99$ produces imaginary poles at $s = \pm\, 5j$.

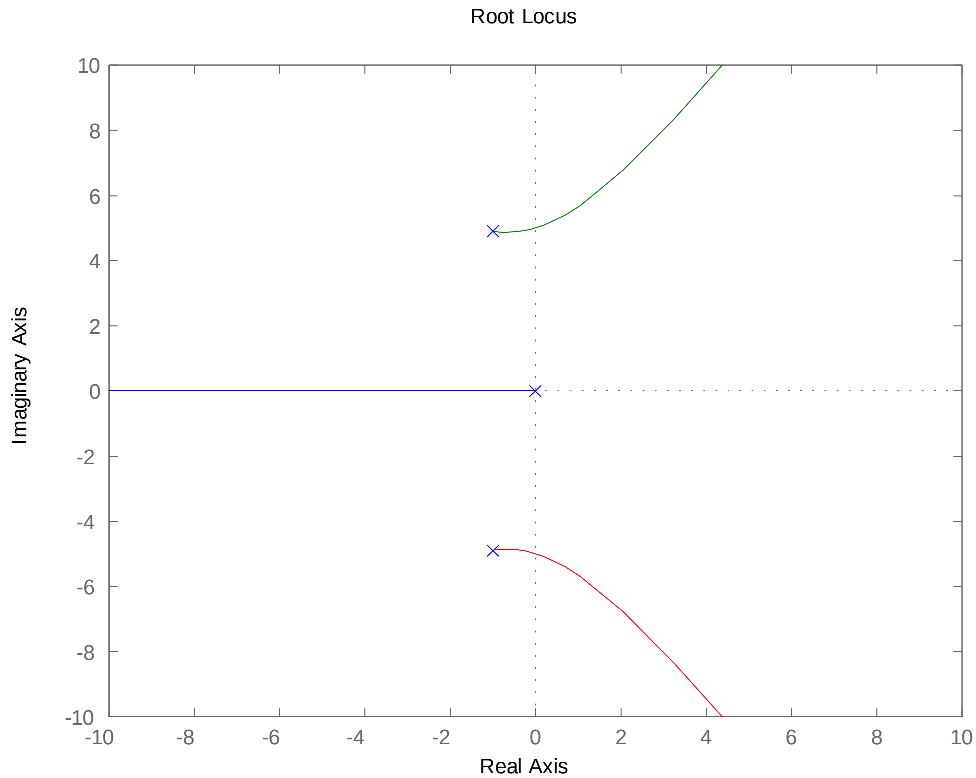
(d) True. The system is unstable for any gain $K > 0$, and is unstable for $K < -5$. Therefore, it is true that the system is unstable for $|K| > 10$.

(e) False. Since the actuator has a negative DC gain, a positive disturbance will cause a negative feedback signal to the load.

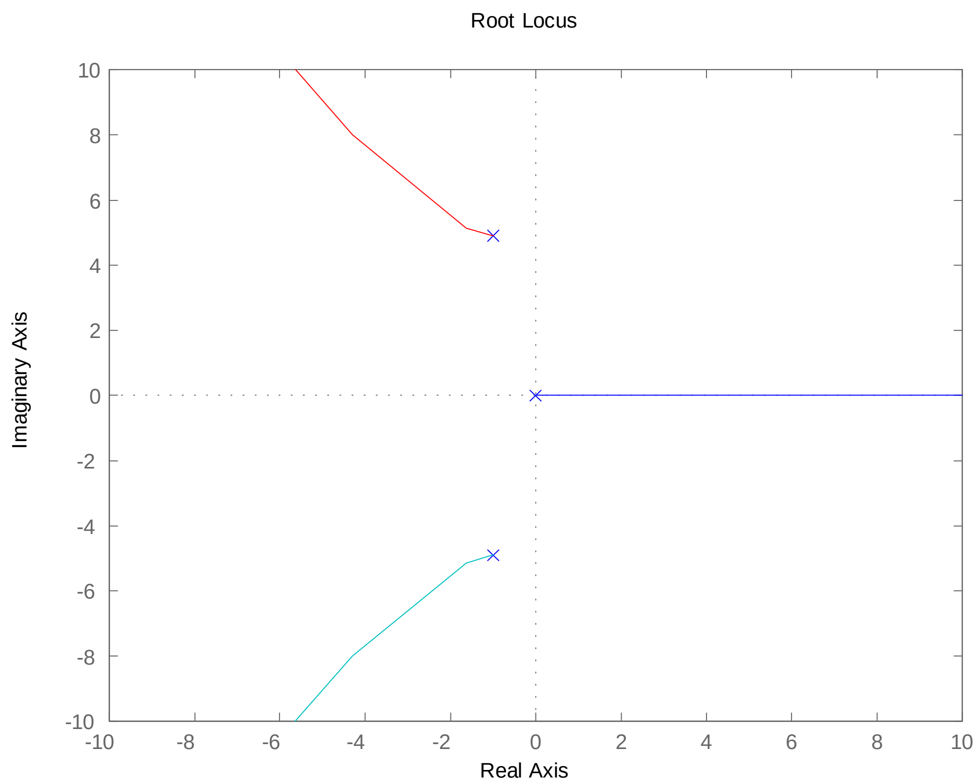
(f) True. The disturbance will speed up the load, resulting in a negative error. The closed-loop system has a DC gain from the disturbance, d, to the error signal, e, of -1. Therefore, the final value of the error due to a disturbance will be $-d$.

(g) False. The closed-loop system will result in an error signal equal to zero, if the disturbance is zero. The DC gain from the reference input to the error signal is zero. In addition, a position disturbance will cause a negative steady-state error.

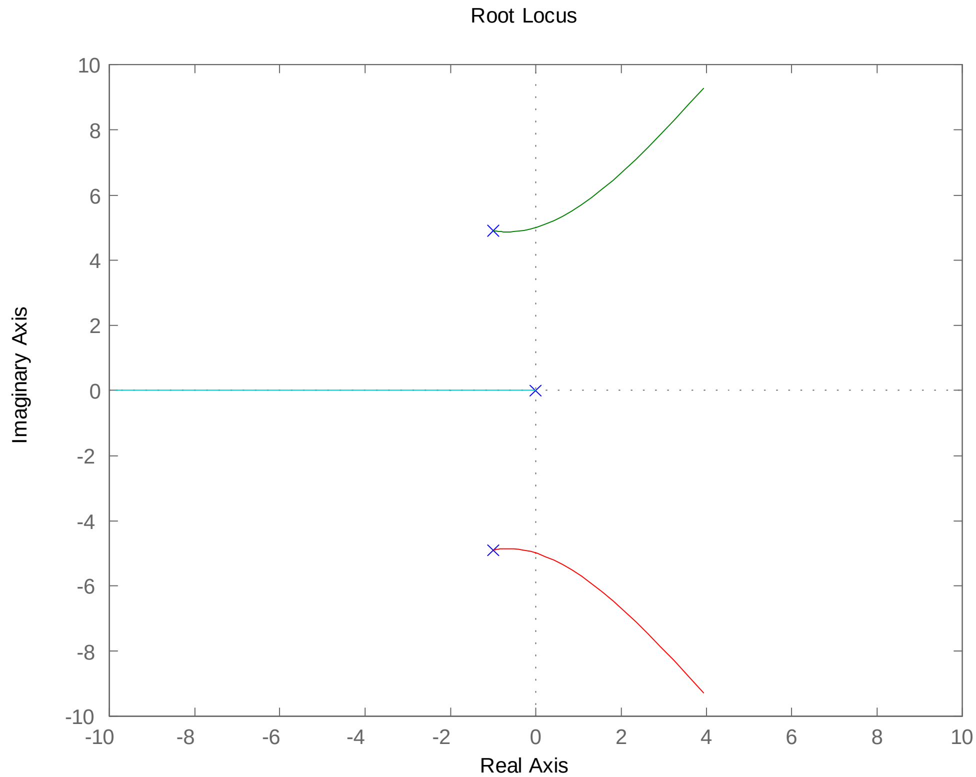
(h) False. The steady-state speed error due to the disturbance of .1, is 10 rad/sec, since the DC gain from d to y is 100. The error signal, e, is -0.1.



Problem 10.6 Servo mechanical root locus plot: without actuator.



Problem 10.6 Servo mechanical root locus plot: with actuator dynamics.



Problem 10.6 Servo mechanical root locus plot: for negative gain.

7. A stick balancer and its corresponding control block diagram are shown in Fig. 10.91. The control is a torque applied about the pivot.

   (a) Using root-locus techniques, design a compensator $D(s)$ that will place the dominant roots at $s = -5 \pm 5j$ (corresponding to $\omega_n = 7$ rad/sec, $\zeta = 0.707$).

   (b) Use Bode plotting techniques to design a compensator $D(s)$ to meet the following specifications:

   - steady-state θ displacement of less than 0.001 for a constant input torque $T_d = 1$,
   - Phase Margin $\geq 50°$,
   - Closed-loop bandwidth $\cong 7$ rad/sec.

   Solution:

   (a) To have the compensated plant root locus go through the pole location $s = -5 \pm 5j$, we employ a lead compensator,

$$D_{c1}(s) = K\frac{s+z}{s+p}.$$

Using the angle criterion,

$$\sum \phi_{z_i} - \sum \phi_{p_i} = -180°,$$

at the closed-loop pole location $s = -5 + 5j$, we can write an expression for the angle contribution from the lead network zero, ϕ_z, and lead network pole, ϕ_p. We have,

$$\phi_z - \phi_p - 59^\circ - 159^\circ = -180^\circ,$$

or,

$$\phi = \phi_z - \phi_p = 38^\circ.$$

In MATLAB,

$$\mathsf{PHI} = \mathsf{180/pi} * [\,\mathsf{angle(polyval\ (n,s)\ /polyval(d,s))} - \mathsf{pi}].$$

So we have, $\phi = \phi_z - \phi_p = 38^\circ$. With selection of $z = 10$, we get $p = 45.7$. So that our lead design is,

$$D_{c1}(s) = K\frac{s+10}{s+45.7}.$$

To find the compensator gain, K, we can utilize the magnitude criterion at the desired dominant closed-loop pole locations. We find that,

$$|D_{c1}(s)G(s)|_{s=-5\pm 5j} = 1 \Rightarrow K = 471.$$

Therefore, we have the compensator,

$$D_{c1}(s) = 471\frac{s+10}{s+45.7}.$$

The root locus plot of the compensated plant is shown using MATLAB's rlocus command.

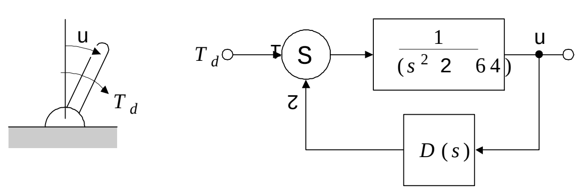
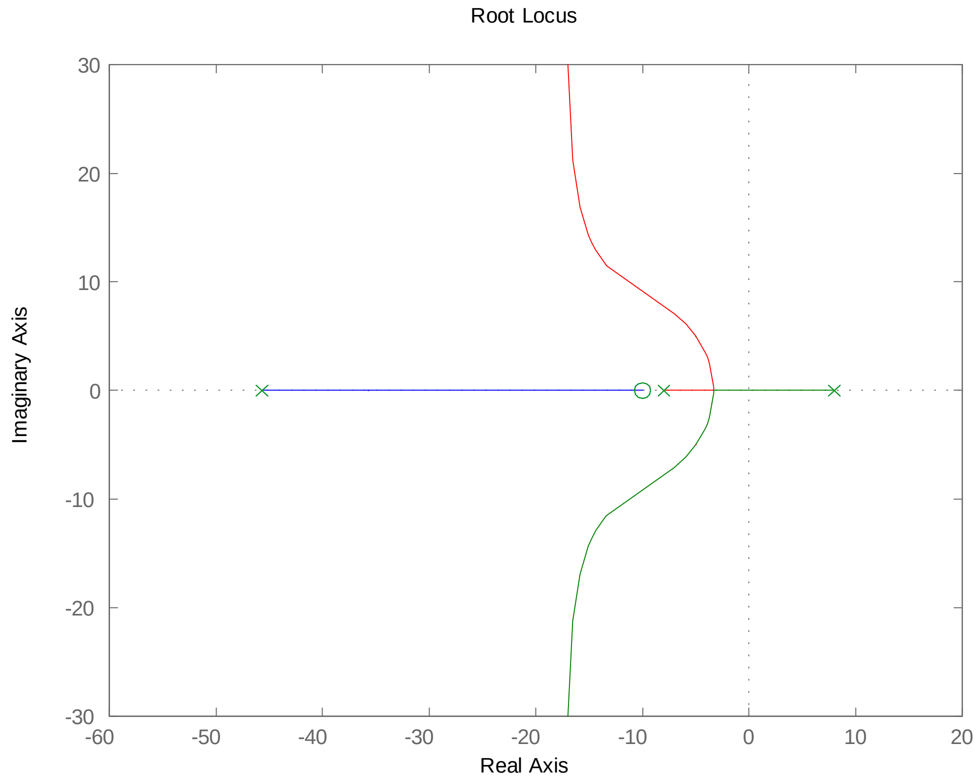


Figure 10.91: Stick balancer.



Problem 10.7: Root locus of stick balancer compensated system.

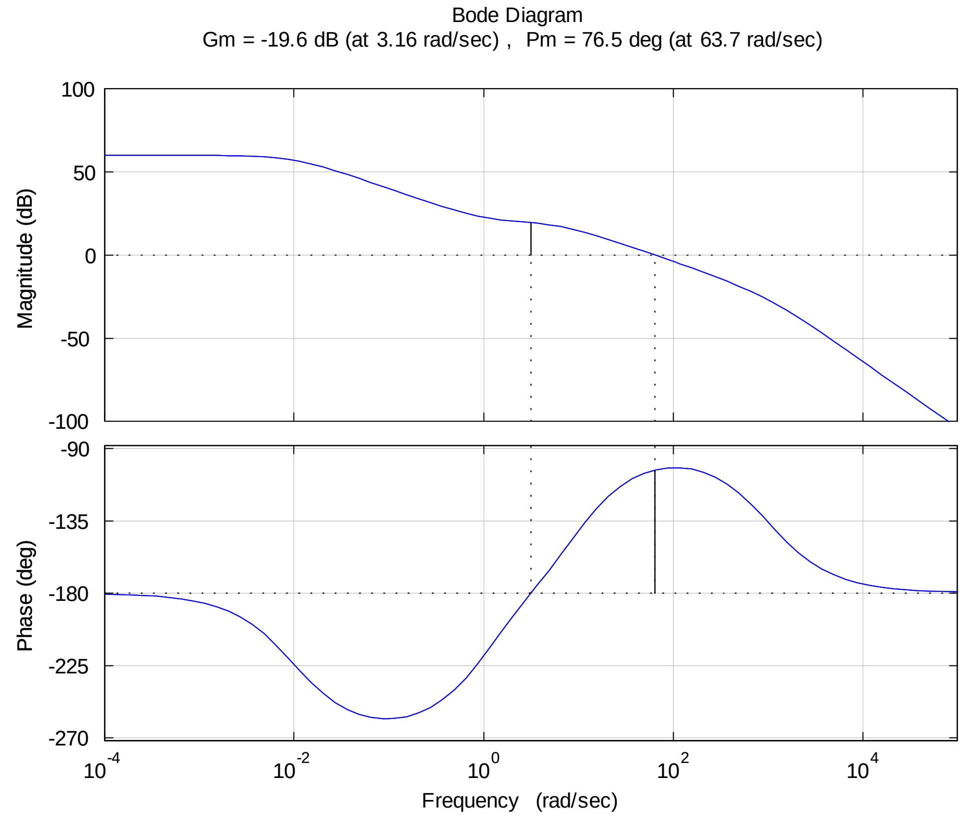
(b) We need a lag network in addition to a lead network to get the required K_p. Let,

$$D_{c2}(s) = 64000\frac{(s+1)(s/10+1)}{(s/0.01+1)(s/1000+1)}.$$

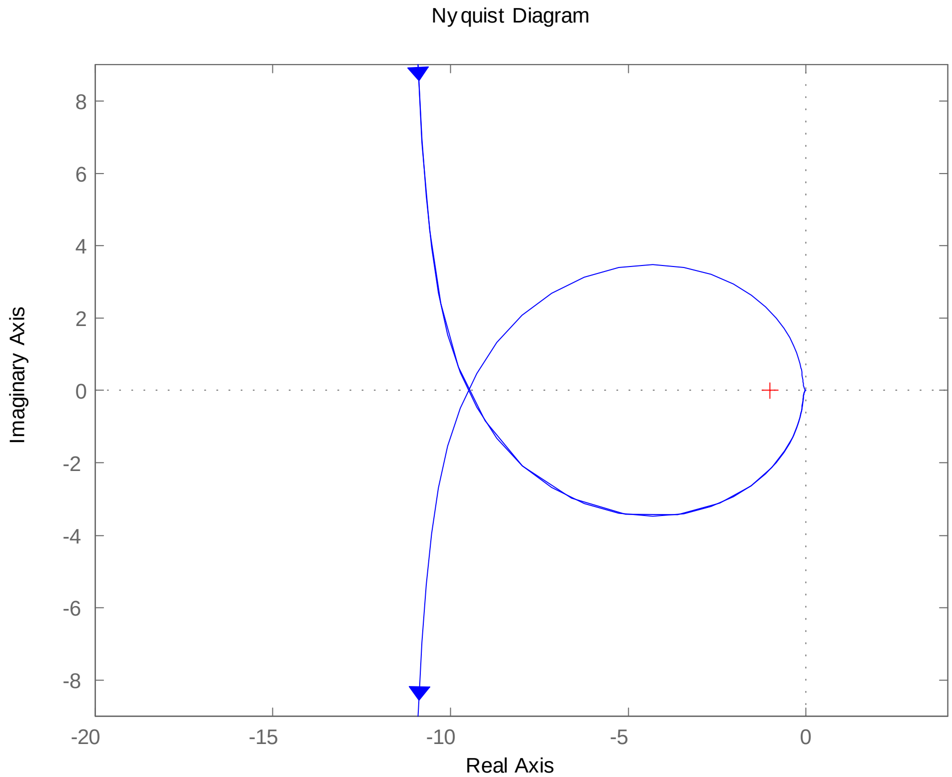
This compensator will meet our design specifications. The Bode plot of $D_{c2}(s)G(s)$ is shown on the next page using MATLAB's Bode command. Note that the phase margin is near 75 degrees. The 0 db cross-over frequency, ω_c, is approximately 64 rad/sec. Hence, the bandwidth is near 64 rad/sec. The steady-state displacement to a unity constant input torque is,

$$\theta_{ss} = \lim_{s\to 0}\frac{G(s)}{1+G(s)D_c(s)} = 1.56\times 10^{-5} < 0.001.$$

Notice that this is an *unstable* open-loop system and the Bode plot must be interpreted carefully. A Nyquist plot is useful here. One is given for this compensator and plant using MATLAB's nyquist command as shown on the next page.



Problem 10.7 Frequency design method for stick balancer: Bode plot of compensated system.



Problem 10.7 Frequency design method for stick balancer: Nyquist plot of compensated system.

8. Consider the standard feedback system drawn in Fig. 10.92.

(a) Suppose,

$$G(s) = \frac{2500\ K}{s(s+25)}.$$

Design a lead compensator so that the phase margin of the system is more than 45°; the steady-state error due to a ramp should be less than or equal to 0.01.

(b) Using the plant transfer function from part (a), design a lead compensator so that the overshoot is less than 25% and the 1% settling time is less than 0.1 sec.

(c) Suppose

$$G(s) = \frac{K}{s(1+0.1s)(1+0.2s)},$$

and let the performance specifications now be $K_v = 100$ and PM $\geq 40°$. Is the lead compensation effective for this system? Find a lag compensator, and plot the root locus of the compensated system.

(d) Using $G(s)$ from part (c), design a lag compensator such that the peak overshoot is less than 20% and $K_v = 100$.

(e) Repeat part (c) using a lead-lag compensator.

(f) Find the root locus of the compensated system in part (e), and compare your findings with those from part (c).

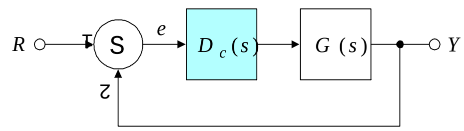


Figure 10.92: Block diagram of a standard feedback control system.

Solution:

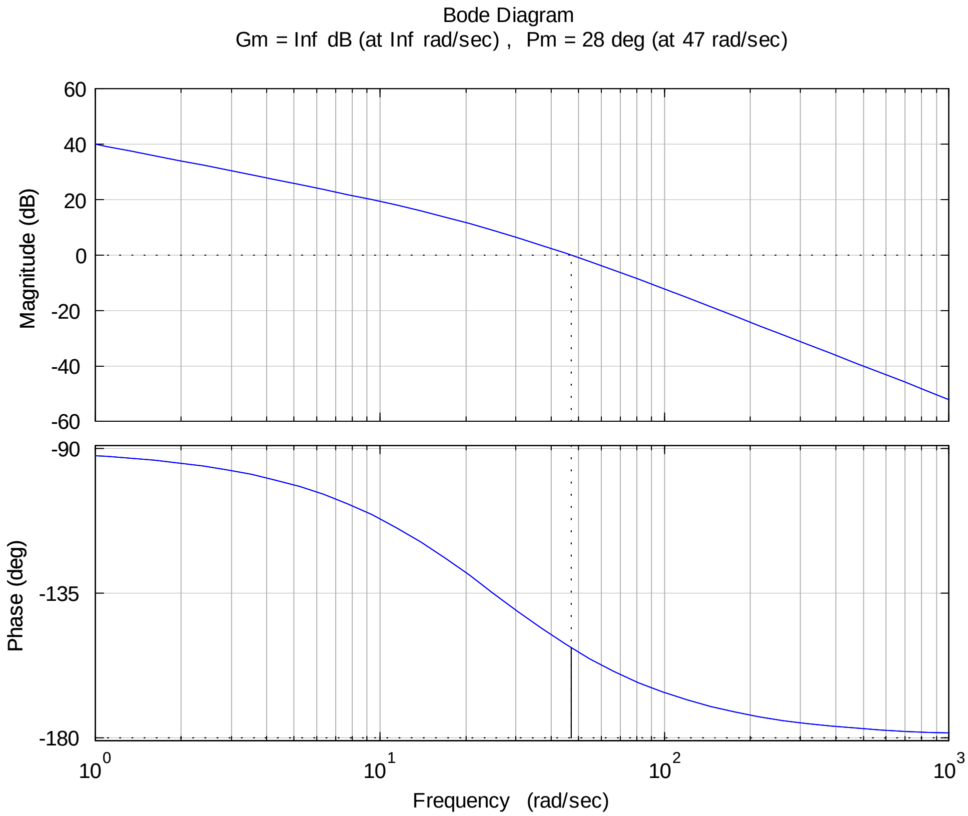
(a) The design specification of steady-state error provides information for the design of K.

$$\begin{aligned} e_\infty &= \frac{1}{K_v} = 0.01 \Longrightarrow K_v = 100. \\ K_v &= \lim_{s\to 0} sG(s) = 100K \Longrightarrow K = 1. \end{aligned}$$

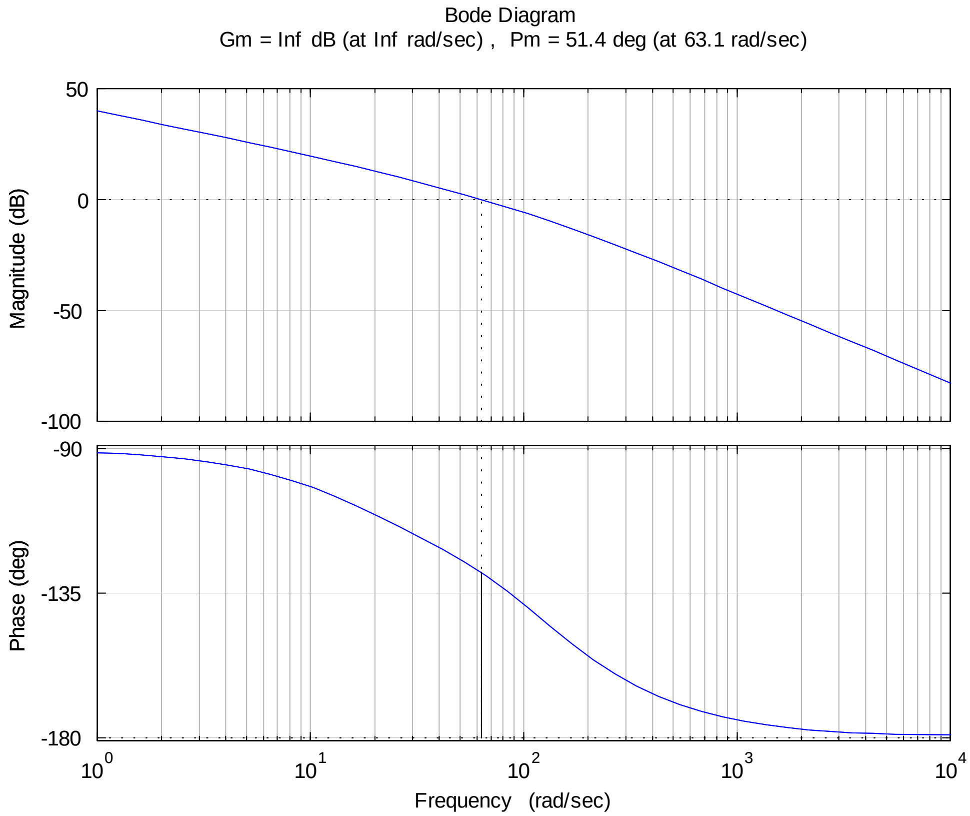
The Bode plot of,

$$G(s) = \frac{2500}{s(s+25)},$$

is given, using MATLAB's `margin` command and shows that the phase margin is approximately 30°. Therefore, we need 15° of phase lead. We select 30° of phase lead.



Problem 10.8: Frequency response of $G(s)$.



Problem 10.8: Frequency response of $D_c(s)G(s)$.

From text Fig. 6.53 of the text, we have $\frac{1}{\alpha} = 3$. Now, we need to find the frequency such that $|G(j\omega)| = \sqrt{\alpha} = 0.58$. From the Bode plot of $G(s)$, this results in $\omega = 63.5$ rad/sec. This frequency will be the crossover frequency of $D_c(s)G(s)$, i.e., $\omega_c = 63.5$ rad/sec. So the lead compensator is,

$$D_c(s) = \frac{\frac{s}{\omega} + 1}{\frac{s}{\frac{\omega}{\alpha}} + 1} = \frac{\frac{s}{z} + 1}{\frac{s}{p} + 1},$$

such that $\omega = \omega_c\sqrt{\alpha} = z \simeq 37$ and $\omega/\alpha = p \simeq 110$. Therefore, we have,

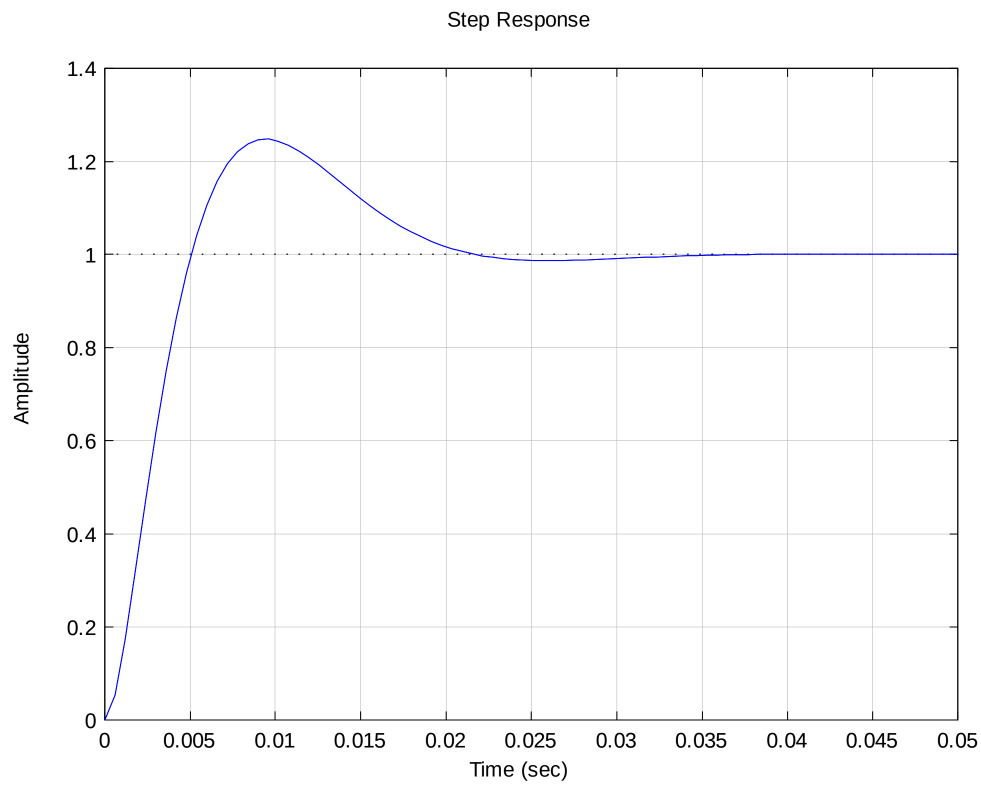
$$D_c(s) = \frac{s/37 + 1}{s/110 + 1}.$$

The Bode plot of the compensated system is shown on the previous page. The phase margin is 51° and $\omega_c = 63.2$ rad/sec.

(b) For $M_p = 25\%$, let $\zeta = 0.4$. For $t_s < 0.1$, let $\zeta\omega_n \approx 4.6/0.1 = 46$. Thus, $\omega_n = 115$ rad/sec, and $s = -46 \pm j105$. We set the lead zero at $s = 1.5 * \mathsf{abs}(s) = -172$ and compute the pole to be at $s = -1284$ using the angle criterion. The Bode plot and step response show the specifications are met with an additional gain of 20. Therefore, the compensator is,

$$D_c(s) = 74.65\frac{s + 172}{s + 1284}.$$

The closed-loop step response is shown below (using MATLAB's `step` command).



Step response of the closed-loop system for Problem 10.8 (b).

(c) The design specification of steady-state error provides information for the design of K.

$$K_v = \lim_{s \to 0} sG(s) = K \Longrightarrow K = 100.$$

The Bode plot of,

$$G(s) = \frac{100}{s(s/5+1)(s/10+1)},$$

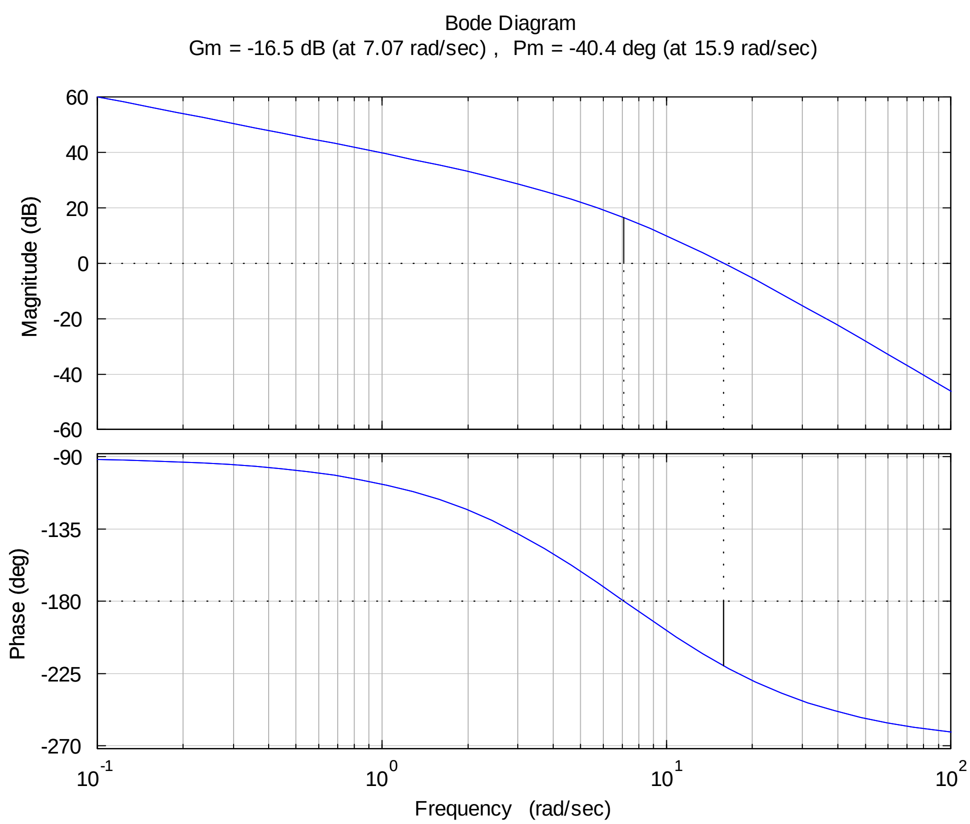
shows that the phase margin (using MATLAB's `margin`) is $-40°$. This is shown below. Therefore, we need a phase lead of greater than $80°$. A lead compensation $D_c(s) = \frac{(s+a)}{(s+b)}$ can not achieve this phase margin requirement. Hence, we try a lag network. We find the frequency such that phase margin of $G(j\omega)$ is our phase margin specification plus $10°$, or phase margin equals $50°$. At $\omega = 2.5$ rad/sec, the phase of $G(j\omega)$ is $-130°$ and $|G(j\omega)| = \alpha = 34.7$. This ω will be crossover frequency of $D_c(j\omega)G(j\omega)$, $\omega_c = 2.5$. Now, select the zero of $D_c(s)$ one decade below ω_c, which implies $\omega = 0.25$ rad/sec. This results in $\frac{\omega}{\alpha} = 0.25/34.7 = 0.0072$. The lag network is thus,

$$D_c(s) = \frac{s/0.25+1}{s/0.0072+1},$$

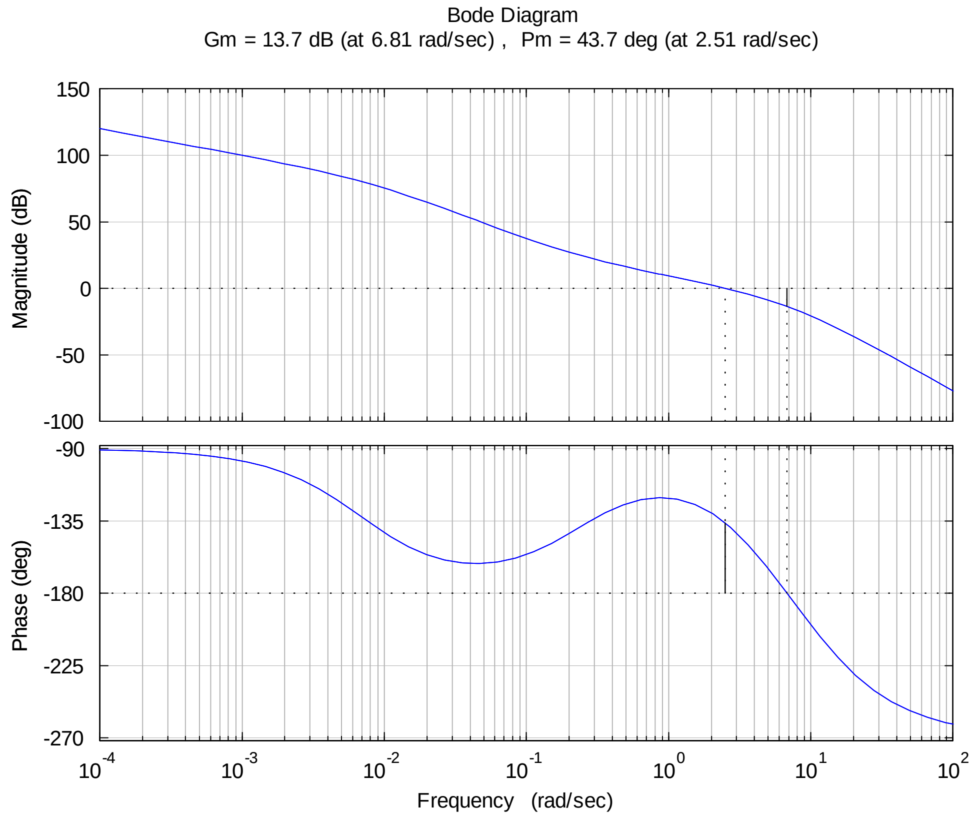
and the loop gain is,

$$D_c(s)G(s) = \frac{100(s/0.25+1)}{s(s/0.0072+1)(s/5+1)(s/10+1)}.$$

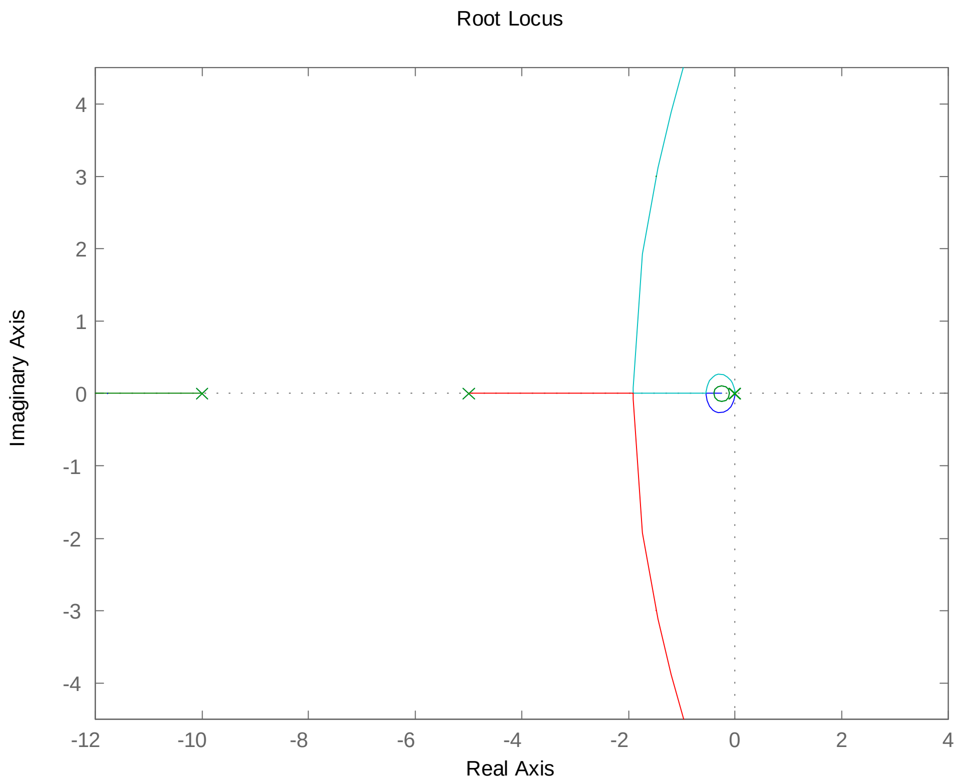
The Bode plot, root locus, and step responses are given on the next two pages (using MATLAB's bode, rlocus, step commands). Note that the design produces a phase margin of 43.7°.



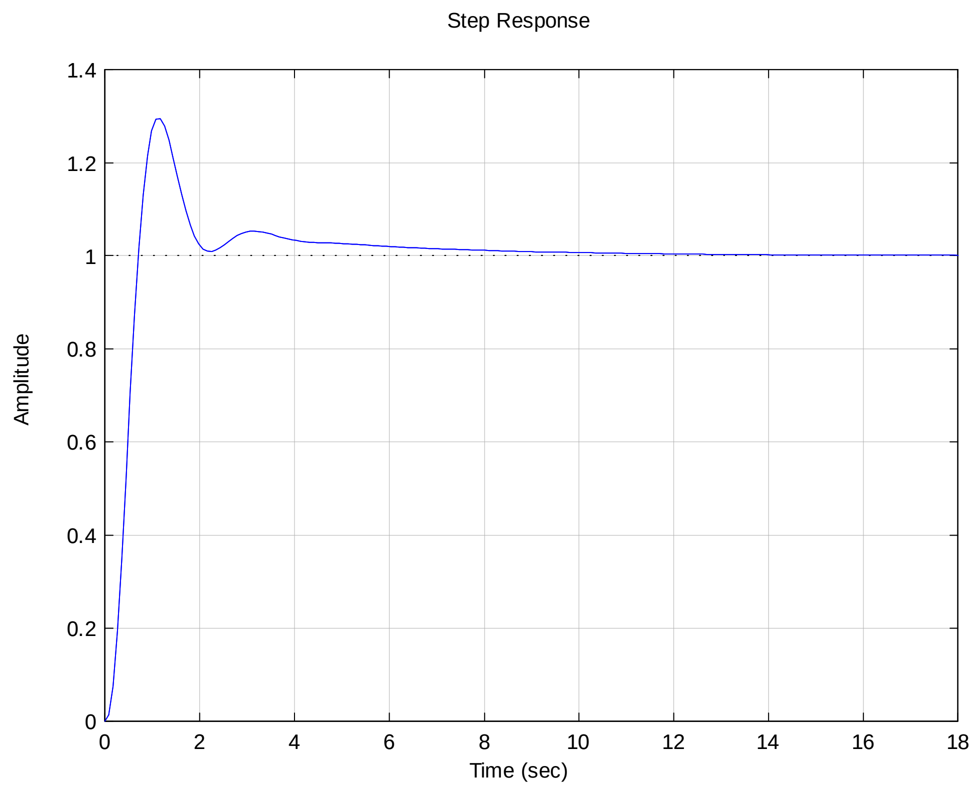
Bode plot of $G(s)$ for Problem 10.8 (c).



Bode plot of $D_c(s)G(s)$ for Problem 10.8 (c).



Root locus of $D_c(s)G(s)$ for Problem 10.8 (c).



Step response of closed-loop system for Problem 10.8 (c).

(d) We can design a lag compensator using root locus methods. The velocity constant requires the plant gain to be equal to 100, since,

$$K_v = \lim_{s \to 0} sG(s) = K \Longrightarrow K = 100.$$

Therefore,

$$G(s) = \frac{100}{s(1+0.1s)(1+0.2s)} = \frac{5000}{s(s+5)(s+10)}.$$

The root locus plot of $G(s)$ is shown below using MATLAB's rlocus command. For an overshoot specification of $M_p = 20\%$, we chose $\zeta = 0.46$. We can find the desired closed-loop pole locations by finding the intersection of the root locus shown with the constant damping line for $\zeta = 0.46$. This results in desired dominant poles at $s = -1.61 \pm 3.11j$. However, $K_v = K$ of $G(s)$ at these pole locations is 2.875, since,

$$\left| \frac{K}{s(s/5+1)(s/10+1)} \right|_{s=-1.61 \pm j3.11} = 1 \Longrightarrow K = 2.875.$$

Therefore, we need to raise K_v to 100. This implies using a lag compensator with $\alpha = \frac{100}{2.875} = 34.8$. If we select the compensator zero at $s = 0.1$, the pole location is $s = \frac{0.1}{\alpha} =$

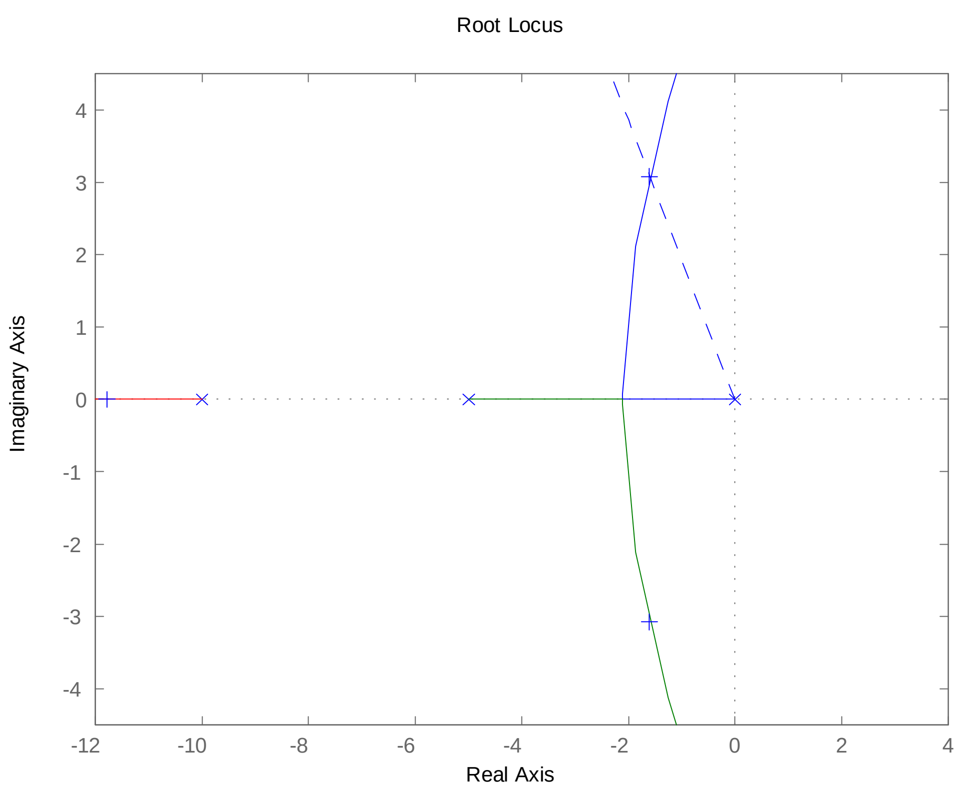
0.003. Hence,

$$D_c(s) = \frac{s/0.1+1}{s/0.003+1},$$

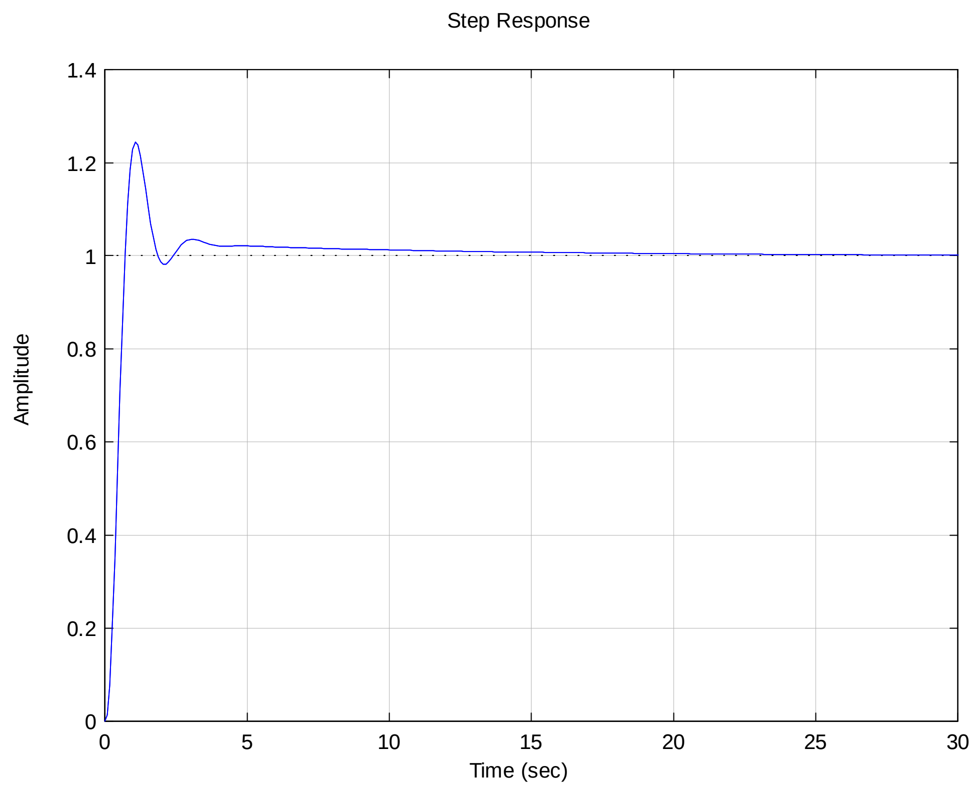
and the loop gain is,

$$D_c(s)G(s) = \frac{100(s/0.1+1)}{s(s/.003+1)(s/5+1)(s/10+1)}.$$

The step response of the closed-loop system is given on the next page (using MATLAB's `step` command). Note the small slow transient in the step response from the lag compensator.



Root locus for Problem 10.8 (d).



Step response of closed-loop system for Problem 10.8 (d).

(e) Again, the design specification of steady-state error provides information for the design of K.

$$K_v = 100 \Longrightarrow K = 100.$$

As mentioned in part (c), the phase margin for,

$$G(s) = \frac{100}{s(s/5+1)(s/10+1)},$$

is $-40°$. First, we select the cross-over frequency, ω_c. From the Bode plot of $G(s)$ given,

$$\angle G(j\omega) = -180° \Longrightarrow \omega_c = 7.0.$$

With $\omega_c = 7.0$ we need $40°$ more lead. From Fig. 6.52 in the text, an $\alpha = 0.1$ will provide $55°$of lead. We select the lead such that zero location is $s = \omega = \omega_c\sqrt{\alpha} = 2.21$. The lead pole location is $s = \frac{\omega}{\alpha}$. So we have:

$$D_{lead}(s) = \frac{s/2.21+1}{s/22.1+1}.$$

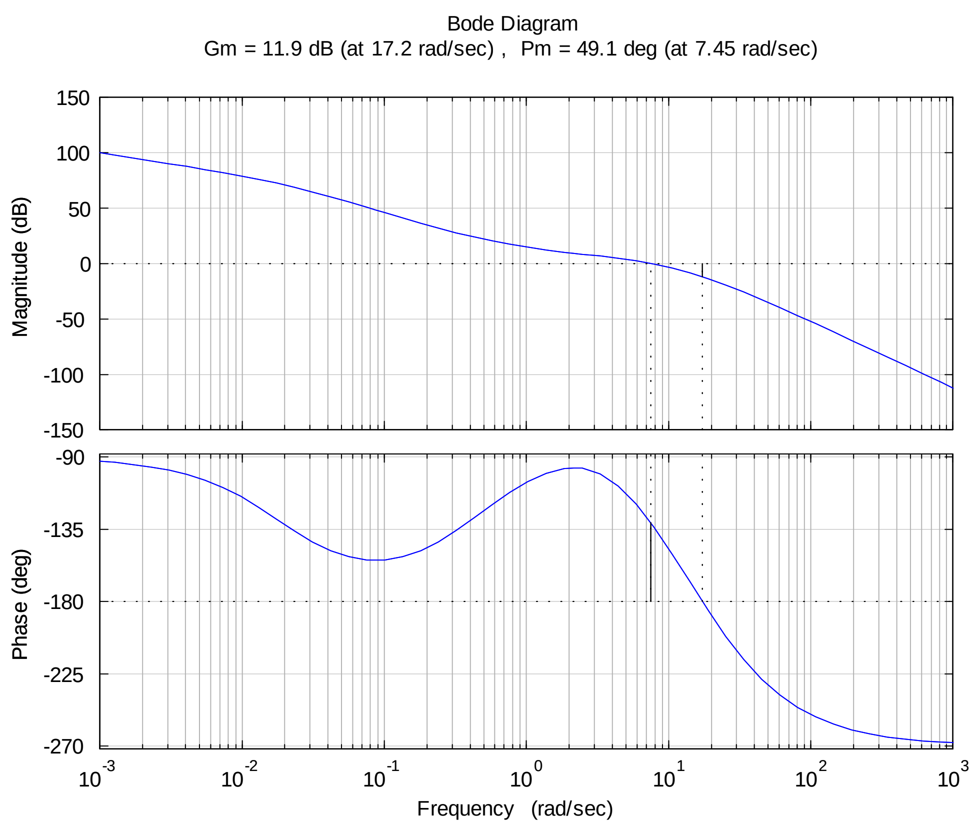
Now, we select the zero of the lag at least one decade lower than ω_c. With α of the lag equal to 20, we have,

$$D_{lag}(s) = \frac{s/0.7+1}{s/0.035+1}.$$

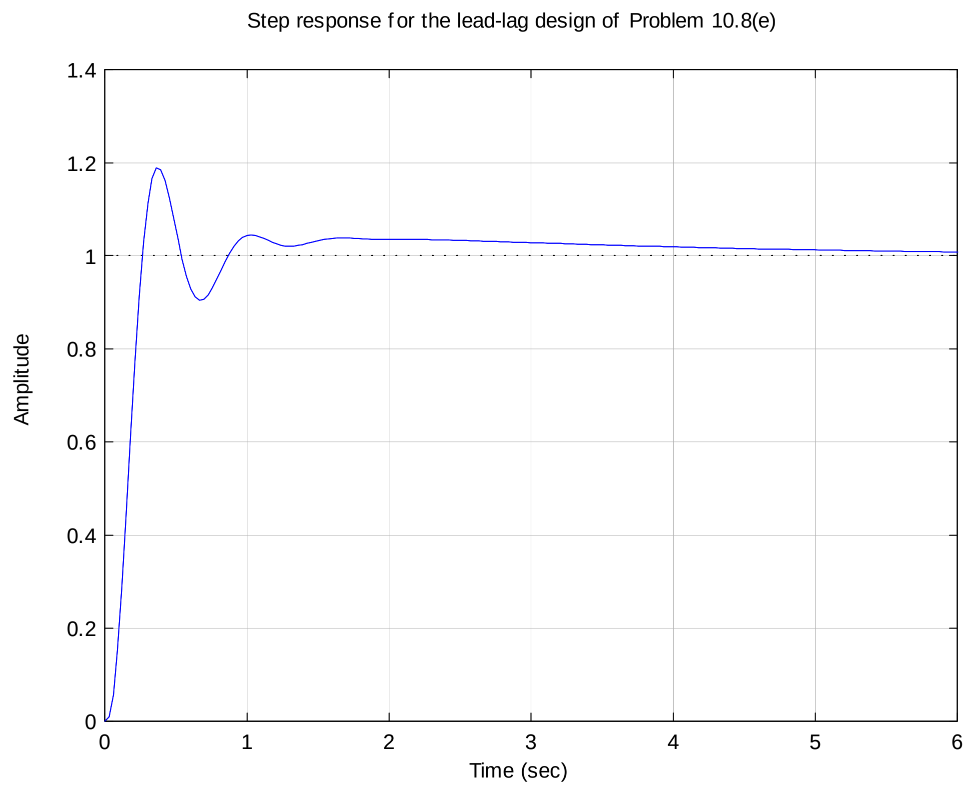
The lead-lag compensator is,

$$D_c(s) = \frac{(s/0.7+1)(s/2.21+1)}{(s/0.035+1)(s/22.1+1)}.$$

The system Bode plot and step response appear on the next page.

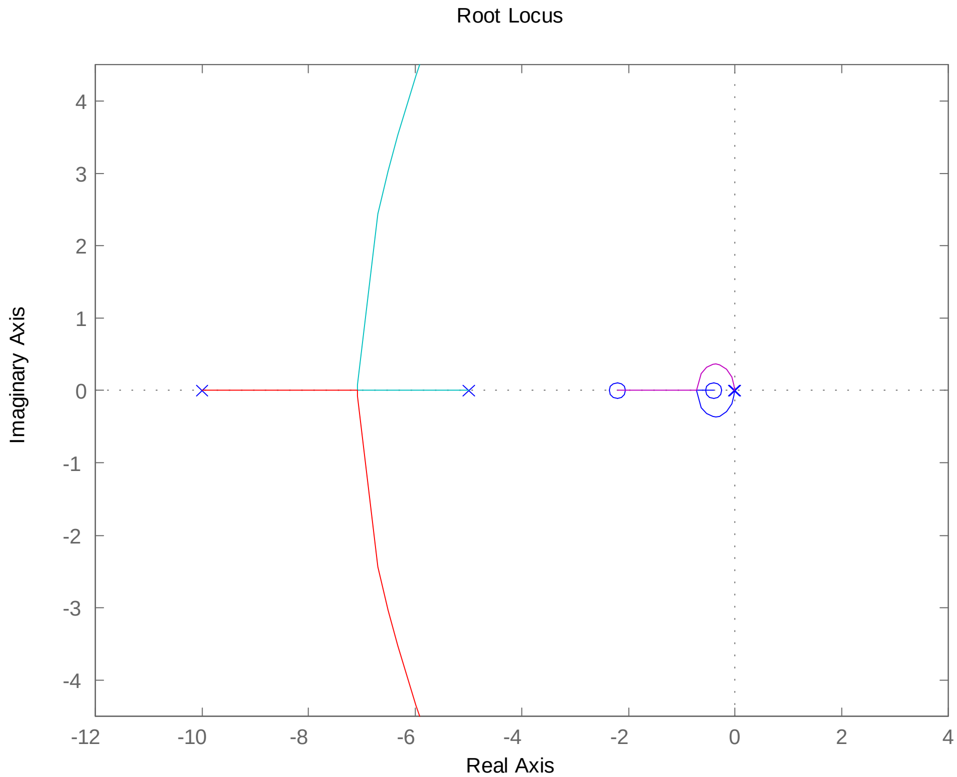


Bode plot of $G(s)D_c(s)$. for Problem 10.8 (e).



Step response for Problem 10.8 (e).

(f) The root locus plot of $D_c(s)G(s)$ from part (d) is shown on the next page.



Root locus for Problem 10.8 (e).

The main difference between the designs of part (c) and part (e) is that with lead-lag we have higher ω_c, and hence higher bandwidth, and also lower rise time and lower overshoot.

9. Consider the system in Fig. 10.92, where

$$G(s) = \frac{300}{s(s+0.225)(s+4)(s+180)}.$$

The compensator $D_c(s)$ is to be designed so that the closed-loop system satisfies the following specifications:

1.
   - zero steady-state error for step inputs,
   - PM $= 55°$, GM ≥ 6 db,
   - gain crossover frequency is not smaller than that of the uncompensated plant.

   (a) What kind of compensation should be used and why?

   (b) Design a suitable compensator $D_c(s)$ to meet the specifications.
   Solution:

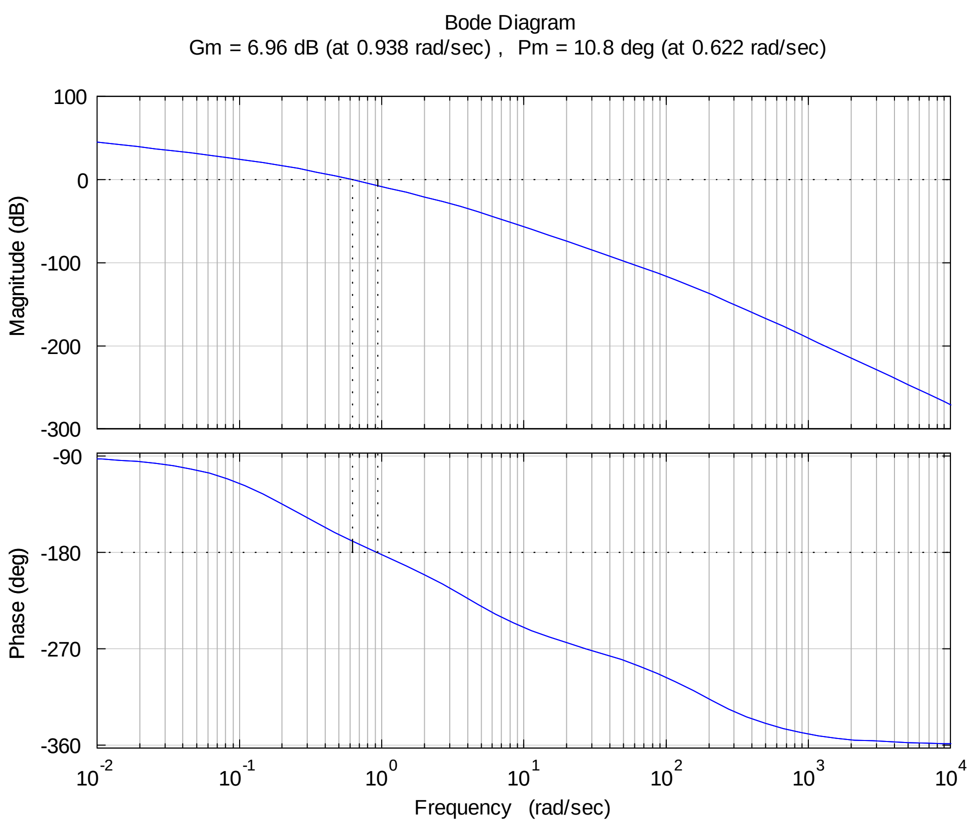
The Bode plot of $G(s)$ is shown on the next page. From the figure, the phase margin is $10.8°$ and $\omega_c = 0.623$.

1. (a) Since we need $55° - 10° = 45°$ of phase lead, a single lead network will do the job.

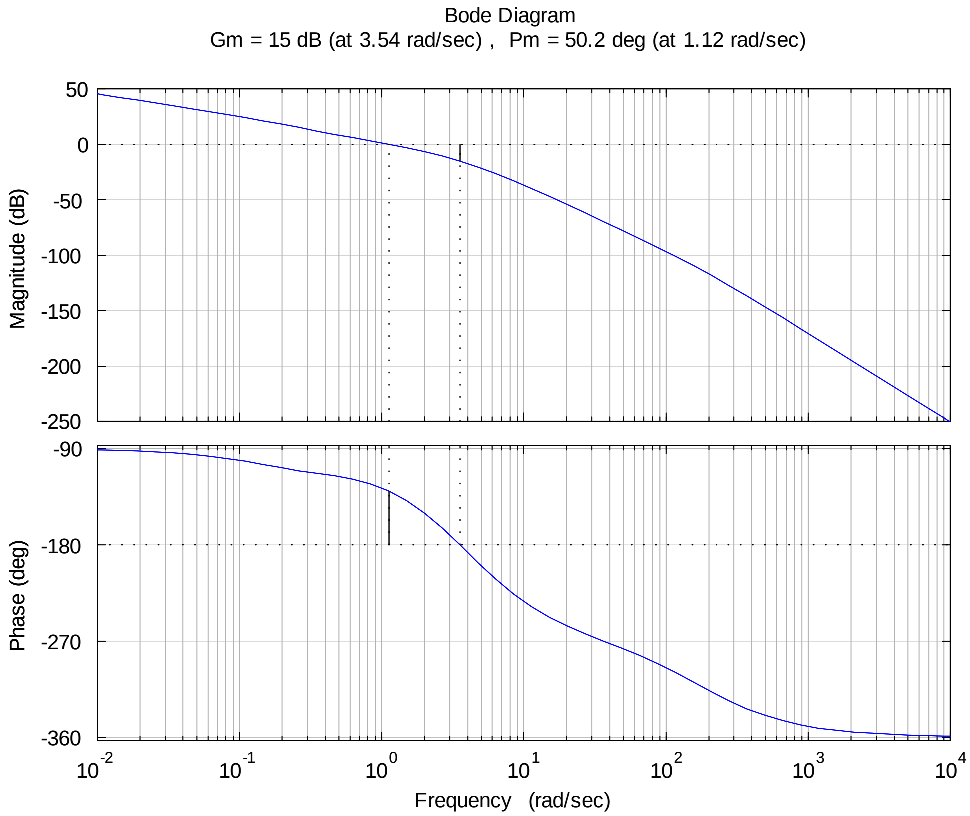
   (b) From a phase lead requirement of $45°$, we have $\frac{1}{\alpha} \approx 10$. Note that you can use either Fig. 6.52 of the text, or $\sin(\phi) = \frac{1-\alpha}{1+\alpha}$ where ϕ is the required phase lead in radians. Now we find the frequency, ω, of $G(j\omega)$ such that $|G(j\omega)| = \sqrt{\alpha} = 0.32$. We find $\omega = 1.11$ which will be the ω_c of the compensated system. The zero of lead network is chosen as $s = \omega_c\sqrt{\alpha} = 0.35$. The pole location is located at $s = \frac{\omega_c}{\alpha} = 3.5$. Hence, the compensator and the loop gain are,

$$\begin{aligned} D_c(s) &= \frac{s/0.35+1}{s/3.5+1}, \\ D_c(s)G(s) &= \frac{1.8519(s/0.35+1)}{s(s/3.5+1)(s/0.225+1)(s/4+1)(s/180+1)}. \end{aligned}$$

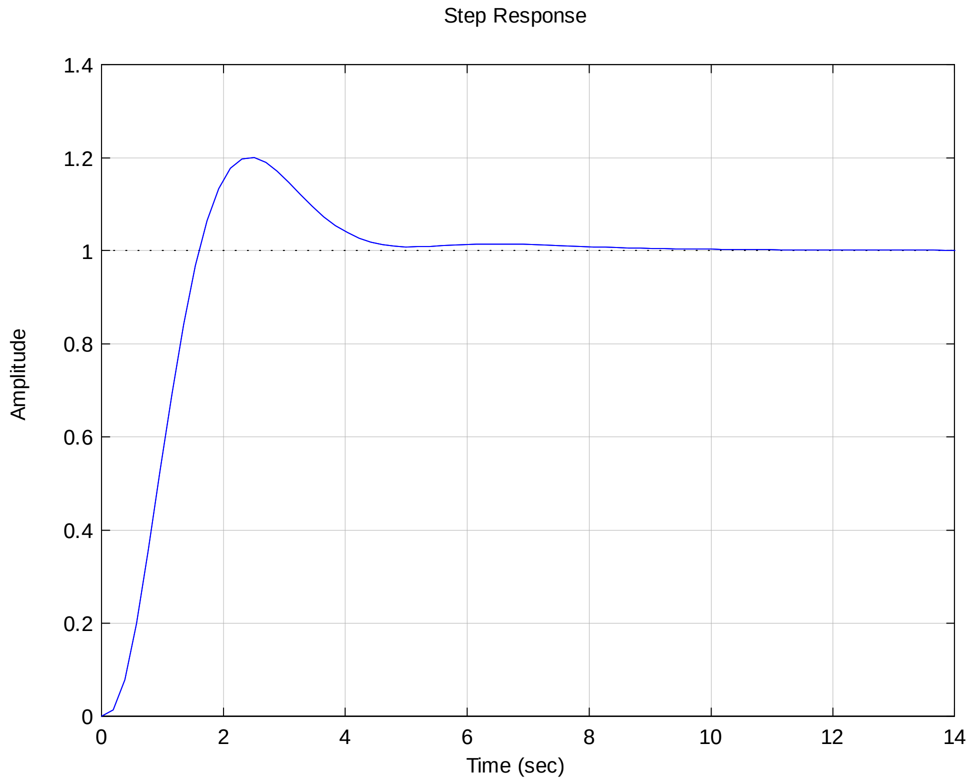
   The Bode plot of $D_c(s)G(s)$, the compensated system is shown on the next page using MATLAB's `margin` command. As the figure shows, $\omega_c = 1.1$, which is larger than the crossover frequency of the uncompensated plant, $G(s)$. The Bode plot shows a phase margin of $55°$ and a gain margin of 15 db. Both specifications meet the requirements. Finally, since DG is a type 1 system, the steady-state error, e_∞, due to a step function is zero as shown on the next page.



Bode plot of $G(s)$ for Problem 10.9.



Lead design for Problem 10.9: Bode plot of the compensated system.



Step response of closed-loop system for Problem 10.9.

10. We have discussed three design methods: the root-locus method of Evans, the frequency-response method of Bode, and the state-variable pole-assignment method. Explain which of these methods is *best* described by the following statements (if you feel more than one method fits a given statement equally well, say so and explain why):

1. (a) This method is the one most commonly used when the plant description must be obtained from experimental data.

   (b) This method provides the most direct control over dynamic response characteristics such as rise time, percent overshoot, and settling time.

   (c) This method lends itself most easily to an automated (computer) implementation.

   (d) This method provides the most direct control over the steady-state error constants K_p and K_v.

   (e) This method is most likely to lead to the *least complex* controller capable of meeting the dynamic and static accuracy specifications.

   (f) This method allows the designer to guarantee that the final design will be unconditionally stable.

   (g) This method can be used without modification for plants that include transportation lag terms, for example,

$$G(s) = \frac{e^{-2s}}{(s+3)^2}.$$

This method is the one most commonly used when the plant description must be obtained from experimental data.

Solution:

(a) Frequency response method is the most convenient for experimental data because the sinusoidal steady-state records can be obtained directly in the laboratory. Either the root locus or state variable design generally requires a separate system identification effort between the experimental data and the construction of a model suitable for the design method.

(b) Either the root-locus or state variable pole assignment are the most direct for control over dynamic response. The pole-zero characteristics are the items of concentration in these two design methods.

(c) The state variable pole-assignment is most easily programmed because, once the specifications are given, the design is completely algorithmic. In the other methods, a trial and error cycle is required and while the analysis may be done by a computer the design is not easily implemented.

(d) The frequency response method of Bode shows the error constant (either K_p or K_v) directly on the graph. State variable or root locus require a separate calculation for these numbers. (Using Truxal's formula, however, the state variable pole-assignment method can be used to give a specific control over K_p or K_v).

(e) The root locus or Bode method will give the least complex controller. These techniques begin with gain alone and then add network compensation only as necessary to meet the specifications; whereas the state variable technique requires a controller of complexity comparable to that of the plant right from the start.

(f) Either the root locus, whereby the locus is required to be entirely in the left half plane up to the operating gain, or the Bode method whereby the phase margin is required to be positive for all frequencies below crossover to allow the designer to guarantee unconditionally stable behavior. The state variable design technique does not permit this guarantee.

(g) The frequency response technique can be used immediately for transportation lag, while the root locus requires a small modification and the state variable design method requires an approximation.

11. Lead and lag networks are typically employed in designs based on frequency response (Bode) methods. Assuming a type 1 system, indicate the effect of these compensation networks on each of the listed performance specifications. In each case, indicate the effect as "an increase," "substantially unchanged," or "a decrease." Use the second-order plant $G(s) = K/[s(s+1)]$ to illustrate your conclusions.

(a) K_v

(b) Phase margin

(c) Closed-loop bandwidth

(d) Percent overshoot

Figure 10.93: Spirit of Freedom balloon

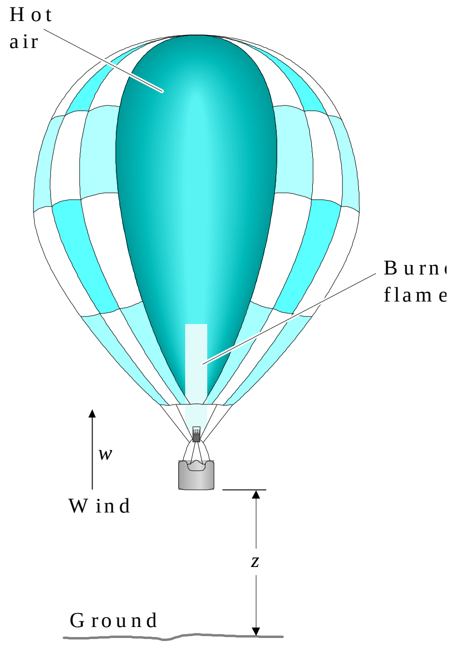


Figure 10.94: Hot-air balloon

(e) Settling time

Solution:

	Lead	Lag
K_v	Unchanged	Increased
Phase margin	Increased	Unchanged
Closed loop bandwidth	Increased	Unchanged
Percent overshoot	Decreased	Unchanged
Settling time	Decreased	Unchanged

1. 12. *Altitude Control of a Hot-air Balloon*: American solo balloonist Steve Fossett landed in the Australian outback aboard *Spirit of Freedom* on July 3rd, 2002, becoming the first solo balloonist to circumnavigate the globe (see Fig. 10.93). The equations of vertical motion for a hot-air balloon (Fig. 10.94), linearized about vertical equilibrium are

$$\delta \dot{T} + \frac{1}{\tau_1}\delta T = \delta q,$$
$$\tau_2 \ddot{z} + \dot{z} = a\delta T + w,$$

where

δT = deviation of the hot − air temperature from the equilibrium temperature where buoyant force = weight,

z = altitude of the balloon,

δq = deviation in the burner heating rate from the equilibrium rate (normalized by the thermal capacity of the hot air),

w = vertical component of wind − velocity,

τ_1, τ_2, a = parameters of the equations.

An altitude-hold autopilot is to be designed for a balloon whose parameters are

$$\tau_1 = 250 \text{ sec}, \quad \tau_2 = 25 \text{ sec}, \quad a = 0.3 \text{ m/(sec} \cdot {}^\circ\text{C)}.$$

Only altitude is sensed, so a control law of the form

$$\delta q(s) = D(s)[z_d(s) - z(s)],$$

will be used, where z_d is the desired (commanded) altitude.

(a) Sketch a root locus of the closed-loop eigenvalues with respect to the gain K for a proportional feedback controller, $\delta q = -K(z - z_d)$. Use Routh's criterion (or let $s = j\omega$ and find the roots of the characteristic polynomial) to determine the value of the gain and the associated frequency at which the system is marginally stable.

(b) Our intuition and the results of part (a) indicate that a relatively large amount of lead compensation is required to produce a satisfactory autopilot. Because Steve Fossett is a millionaire, he can afford a more complex controller implementation. Sketch a root locus of the closed-loop eigenvalues with respect to the gain K for a double-lead compensator, $\delta q = D(s)(z_d - z)$, where,

$$D(s) = K\left(\frac{s + 0.03}{s + 0.12}\right)^2.$$

(c) Select a gain K for the lead-compensated system to give a crossover frequency of 0.06 rad/sec.

(d) Sketch the magnitude portions of the Bode plots (straight-line asymptotes only) for the open-loop transfer functions of the proportional feedback and lead-compensated systems.

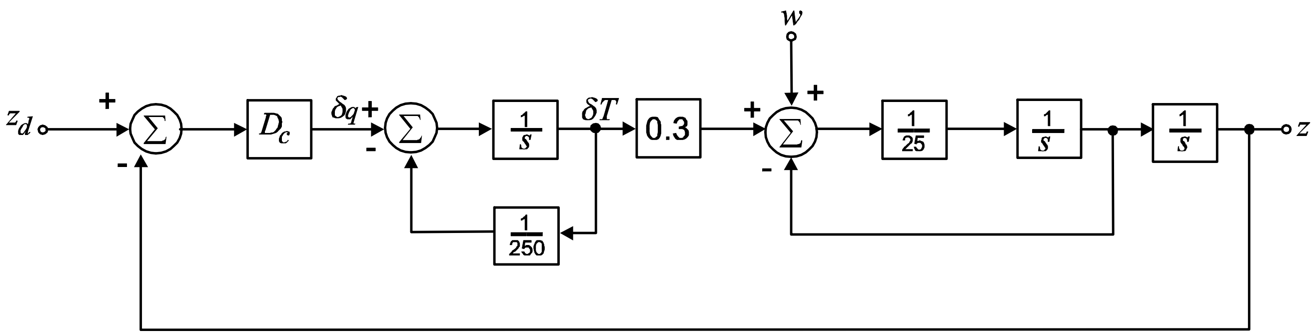
(e) With the gain selected in part (d), what is the steady-state error in altitude for a steady vertical wind of 1 m/sec? (Be careful: First find the closed-loop transfer function from w to the error.)

(f) If the error in part (e) is too large, how would you modify the compensation to give higher low-frequency gain? (Give a qualitative answer only.)

Solution:

$$\begin{aligned}
\delta\dot{T} + \frac{1}{\tau_1}\delta T &= \delta q \Longrightarrow \delta T = \frac{1}{s + \frac{1}{\tau_1}}\delta q = \frac{\tau_1}{\tau_1 s + 1}\delta q = G_1(s)\delta q, \\
\tau_2 \ddot{z} + \dot{z} &= a\delta T + w \Longrightarrow z = \frac{1}{s(\tau_2 s + 1)}(a\delta T + w) = G_2(s)\delta T + G_3(s)w.
\end{aligned}$$

The block diagram of the system is shown below.



Problem 10.13: Block diagram for balloon problem with only altitude measurement.

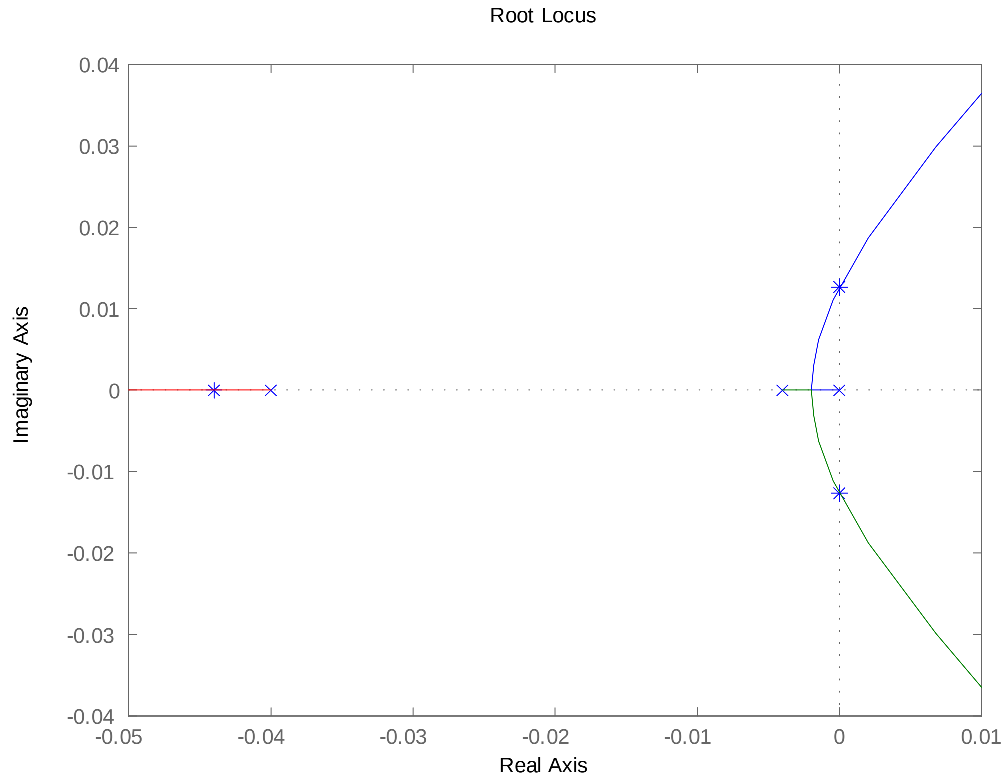
1. (a) With $D(s) = K$, the open-loop transfer function is,

$$DG_1G_2 = K\left(\frac{\tau_1}{\tau_1 s + 1}\right)\left(\frac{a}{s(\tau_2 s + 1)}\right) = \frac{75K}{s(250s+1)(25s+1)}.$$

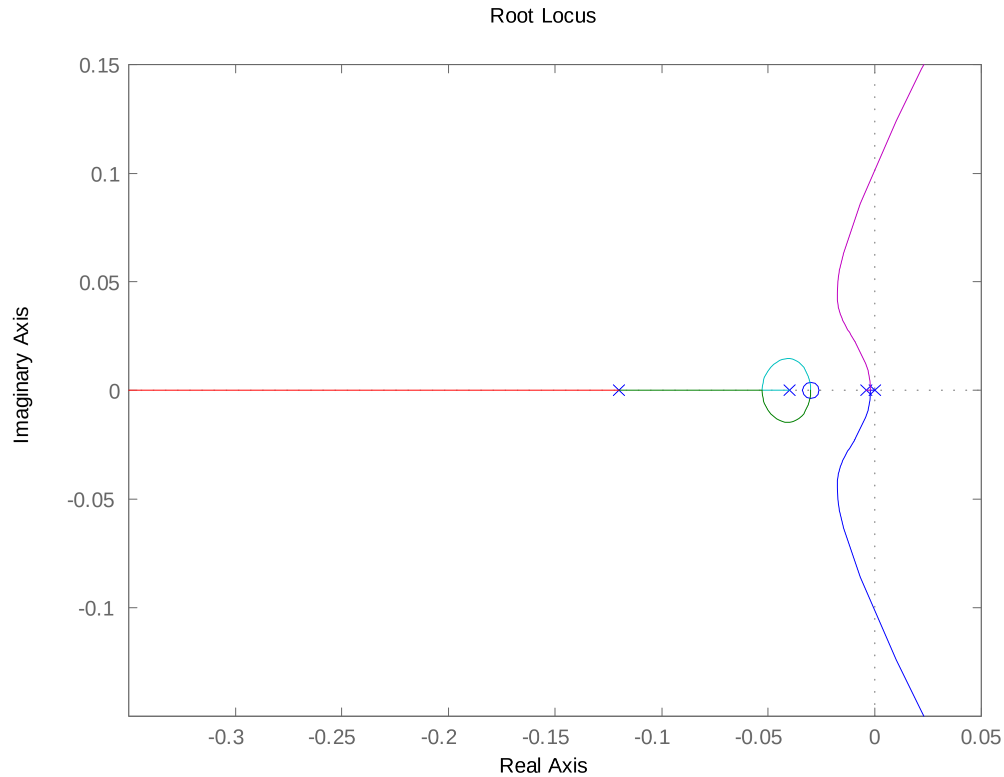
The closed-loop system roots are found from the numerator of the equation $1+DG_1G_2 = 0$. We can find the closed-loop roots which are on the imaginary axis by setting $s = j\omega$ (i.e., constrain the solution to lie on the $j\omega$ axis) and then equating the real and imaginary parts to zero. We find,

$$\begin{aligned}
\tau_1\tau_2 s^3 + (\tau_1+\tau_2)s^2 + s + Ka\tau_1 &= 0, \\
&\Longrightarrow Ka\tau_1 - (\tau_1+\tau_2)^2 = 0, \\
\omega - \tau_1\tau_2\omega^3 &= 0.
\end{aligned}$$

The result is $K = 5.87 \times 10^{-4}$ and $\omega = 0.01265$. Note that the system is unstable for $K > 5.87 \times 10^{-4}$. The next figure shows the root locus plot of DG_1G_2.



Problem 10.13: Root locus for balloon altitude control system with $D(s) = K$.



Problem 10.13: Root locus for balloon altitude control system with double-lead compensation.

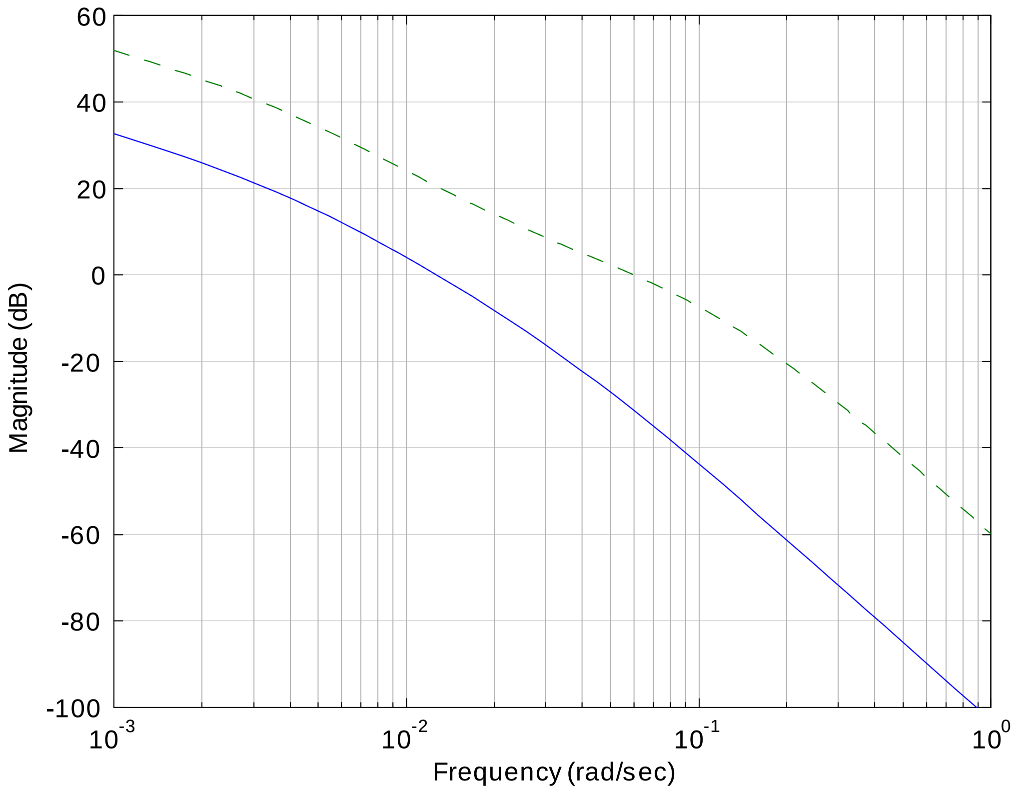
(b) The root locus using a double lead compensator is shown above. The open-loop transfer function used is,

$$DG_1G_2 = K\left(\frac{s+0.03}{s+0.12}\right)^2\left(\frac{\tau_1}{\tau_1 s+1}\right)\left(\frac{a}{s(\tau_2 s+1)}\right).$$

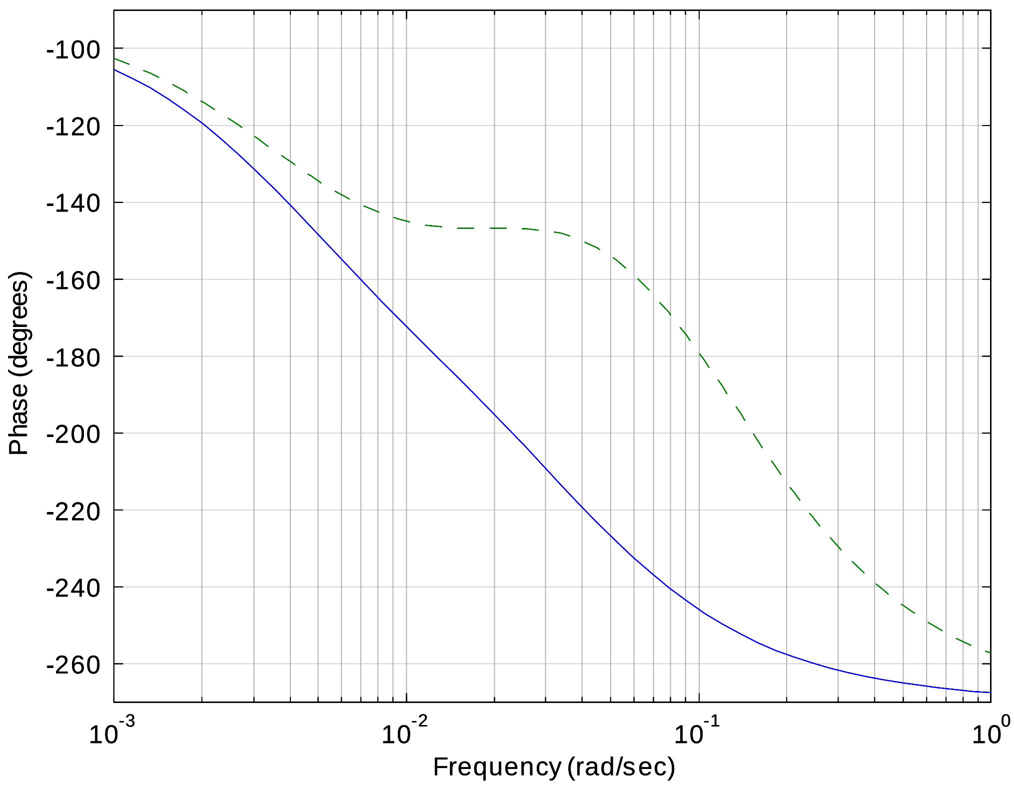
(c) To find K such that $\omega_c = 0.06$,

$$|DG_1G_2|_{\omega=0.06} = 1 \Longrightarrow K = 0.0867.$$

(d) In order to plot the Bode plots, we need to specify which values for K we are going to use. For the Bode plot of the proportional compensator, we use $K = 5.87\times10^{-4}$ from part (b) (the case where the closed-loop system is marginally stable). For the Bode plot of the double lead compensator, we use $K = 0.0867$ from part (d) (the gain when the crossover frequency is 0.06 rad/sec). The following figures show Bode magnitude and phase plots for the balloon control system for both cases. The solid line (blue) corresponds to the proportional compensator and the dashed line (green) corresponds to the double lead compensator.



Problem 10.13: Bode magnitude plots for proportional control (blue), and double lead compensation (green).



Problem 10.13: Bode phase plots for proportional control (blue), and double lead compensation (green).

Using the notation from part (a), we have (suppressing the Laplace variable s),

$$\begin{aligned} Z &= G_3W + DG_1G_2E, \\ E &= Z_d - Z = Z_d - G_3W - DG_1G_2E, \\ &\Longrightarrow E(1 + DG_1G_2) = Z_d - G_3W, \\ &\Longrightarrow E = (1 + DG_1G_2)^{-1}(Z_d - G_3W). \end{aligned}$$

Using a unit step on $w(t)$, i.e., $W(s) = 1/s$, and ignoring z_d because it is not involved in the transfer function from w to e, we have,

$$e_\infty = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{-sG_3}{1 + DG_1G_2} W = -2.46\ m.$$

(e) We can add a lag network at low frequency to boost the K_v ($e_\infty = 1/K_v$). This will not affect the crossover frequency, $\omega_c = 0.06$ rad/sec. For example,

$$D = \left(\frac{s + 0.02}{s + 0.002}\right)^2 \left(\frac{s + 0.03}{s + 0.12}\right)^2,$$

will increase K_v by a factor of 100 or equivalently reduce the error by factor of 0.01, which implies $e_\infty = -0.0246$ m.

13. Satellite-attitude control systems often use a reaction wheel to provide angular motion. The equations of motion for such a system are

$$\begin{aligned} \text{Satellite}: \quad & I\ddot{\phi} = T_c + T_{\text{ex}}, \\ \text{Wheel}: \quad & J\dot{r} = -T_c, \\ \text{Measurement}: \quad & \dot{Z} = \dot{\phi} - aZ, \\ \text{Control}: \quad & T_c = -D(s)(Z - Z_d), \end{aligned}$$

where,

$$\begin{aligned} J &= \text{moment of inertia of the wheel}, \\ r &= \text{wheel speed}, \\ T_c &= \text{control torque}, \\ T_{\text{ex}} &= \text{disturbance torque}, \\ \phi &= \text{angle to be controlled}, \\ Z &= \text{measurement from the sensor}, \\ Z_d &= \text{reference angle}, \\ I &= \text{satellite inertia } (1000 \text{ kg/m}^2), \\ a &= \text{sensor constant } (1 \text{ rad/sec}), \\ D(s) &= \text{compensation}. \end{aligned}$$

(a) Suppose $D(s) = K_0$, a constant. Draw the root locus with respect to K_0 for the resulting closed-loop system.

(b) For what range of K_0 is the closed-loop system stable?

(c) Add a lead network with a pole at $s = -1$ so that the closed-loop system has a bandwidth $\omega_{\text{BW}} = 0.04$ rad/sec, a damping ratio $\zeta = 0.5$, and compensation given by,

$$D(s) = K_1 \frac{s + z}{s + 1}.$$

Where should the zero of the lead network be located? Draw the root locus of the compensated system, and give the value of K_1 that allows the specifications to be met.

(d) For what range of K_1 is the system stable?

(e) What is the steady-state error (the difference between Z and some reference input Z_d) to a constant disturbance torque T_{ex} for the design of part (c)?

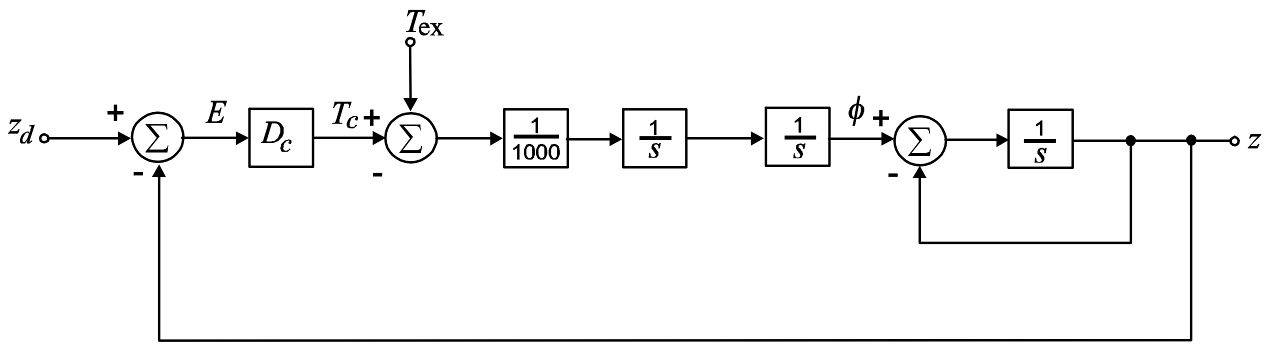
(f) What is the type of this system with respect to rejection of T_{ex}?

(g) Draw the Bode plot asymptotes of the *open-loop* system, with the gain adjusted for the value of K_1 computed in part (c). Add the compensation of part (c), and compute the phase margin of the closed-loop system.

(h) Write state equations for the open-loop system, using the state variables ϕ, $\dot{\phi}$, and Z. Select the gains of a state-feedback controller $T_c = -K_\phi \phi - K_{\dot{\phi}} \dot{\phi}$ to locate the closed-loop poles at $s = -0.02 \pm 0.02j\sqrt{3}$.

Solution:

The block diagram is shown below.



Problem 10.13: Block diagram for satellite attitude control problem.

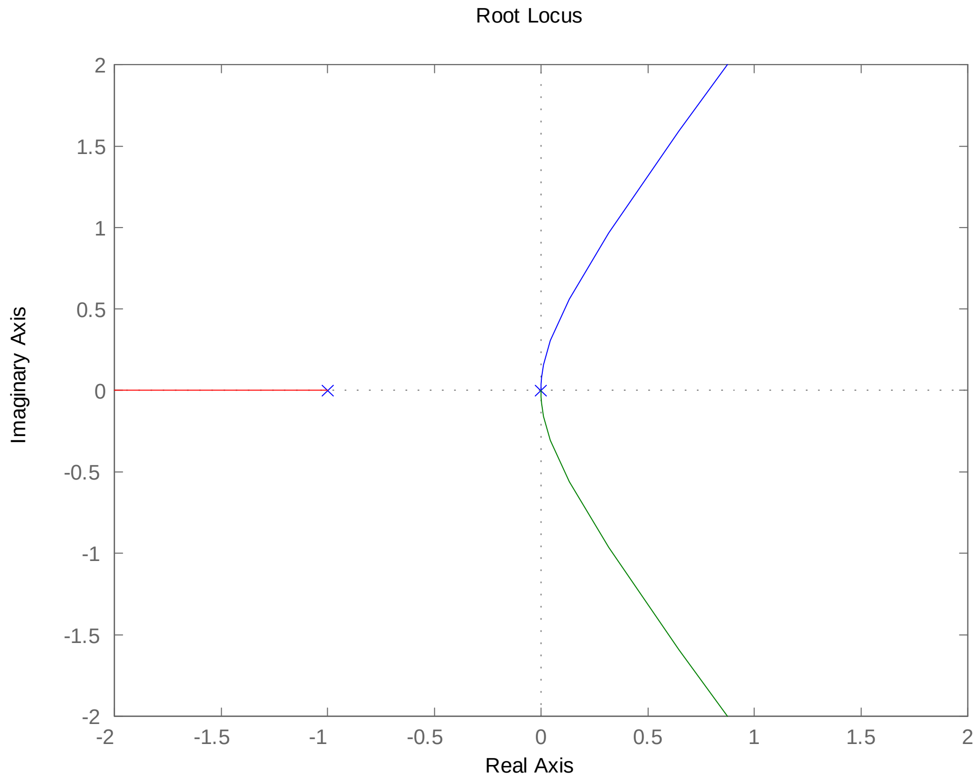
1. (a) With the transfer function from the measurement to the satellite's angle is,

$$\frac{\Phi}{Z} = \frac{DG}{1 + DGH}.$$

To form the root locus, we use,

$$DGH = \frac{K_0/I}{s^2(s+1)} = \frac{0.001K_0}{s^2(s+1)}.$$

The root locus is shown on the next page.



Problem 10.13: Root locus for satellite problem.

(b) From the root locus, using MATLAB's `rlocus` command, the system is unstable for any value of K_0.

(c) With $\omega_n = 0.04$ and $\zeta = 0.5$, the closed-loop poles are at $s = -0.02 \pm 0.02\sqrt{3}j$. Using the phase angle criterion,

$$\begin{aligned}
\sum \phi_{z_i} - \sum \phi_{p_i} &= -180^\circ, \\
\phi_z - 120^\circ - 120^\circ - 2^\circ - 2^\circ &= -180^\circ, \\
\Longrightarrow & \quad \phi_z = 64^\circ.
\end{aligned}$$

We can now calculate the location of the zero,

$$z = \frac{0.02\sqrt{3}}{\tan\phi_z} + 0.02 \Longrightarrow z = 0.0369.$$

So the compensator is,

$$D(s) = K_1 \frac{s + 0.0369}{s + 1}.$$

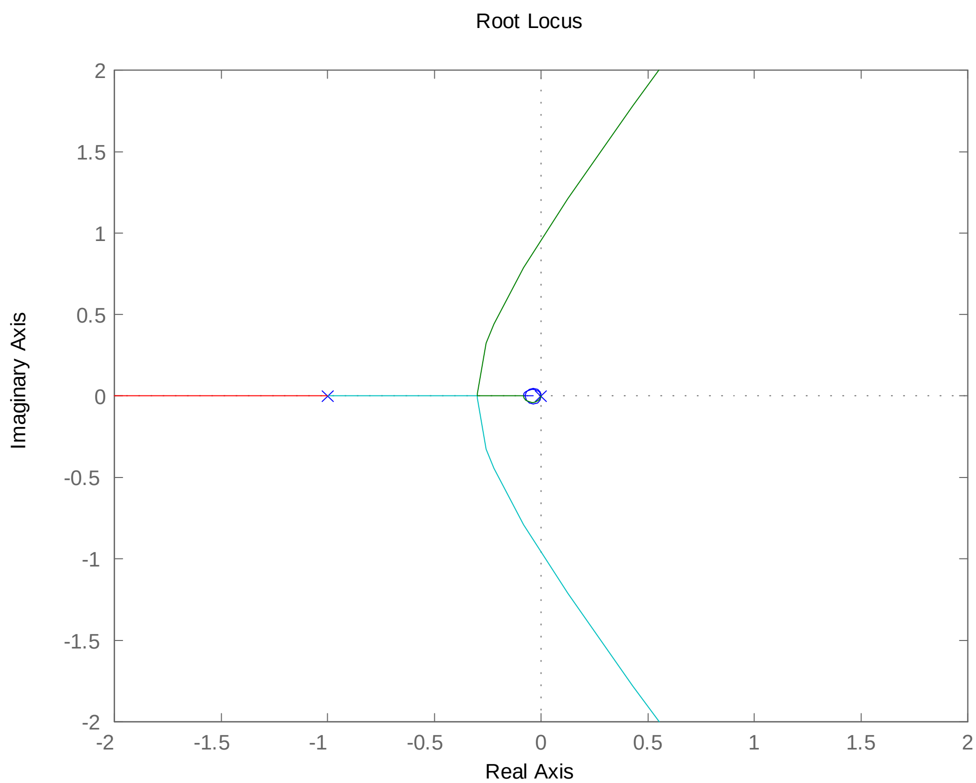
To find value of K_1 at $s = -0.02 \pm 0.02\sqrt{3}$, we set,

$$|DGH|_{s=-0.02+j0.02\sqrt{3}} = 1.$$

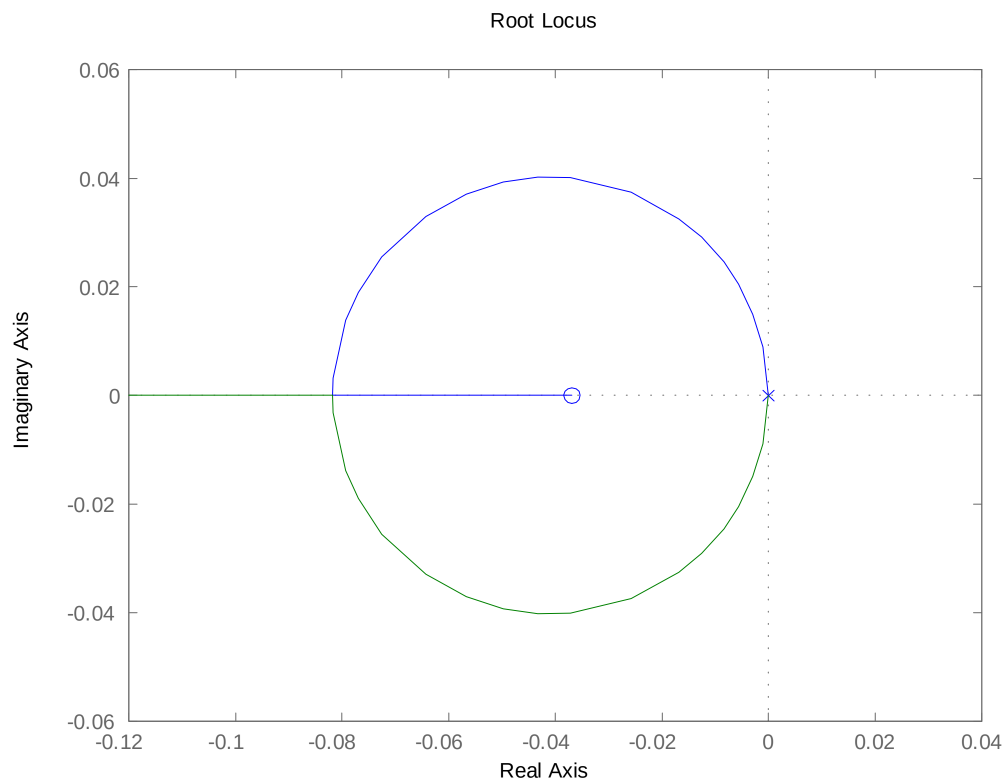
Solving for K_1 yields $K_1 = 39.92$. We plot the root locus of,

$$DGH = \frac{\frac{K_1}{I}(s + 0.0369)}{s^2(s+1)^2},$$

using MATLAB's `rlocus` command as shown on the next page.



Problem 10.13: Root locus for satellite problem with lead network.



Problem 10.13: Root locus for satellite problem with lead network: detailed view.

(d) To find the range of K_1 for which the system is stable, we use Routh's method on the numerator of $1 + DGH = 0$, i.e.,

$$s^4 + 2s^3 + s^2 + 0.001K_1 s + 3.69 \times 10^{-5} K_1 = 0.$$

This leads to the stable region $0 < K_1 < 1852.4$.

(e) We need to find the transfer function from T_{ex} to e. In the Laplace domain (suppressing the s for clarity),

$$\begin{aligned}
\Phi &= G(T_{ex} + DE), \\
E &= Z_d - Z = Z_d - H\Phi = Z_d - HGT_{ex} - HGDE, \\
&\Longrightarrow (1 + HGD)E = Z_d - HGT_{ex}, \\
&\Longrightarrow E = (1 + HGD)^{-1} Z_d - (1 + HGD)^{-1} HGT_{ex}.
\end{aligned}$$

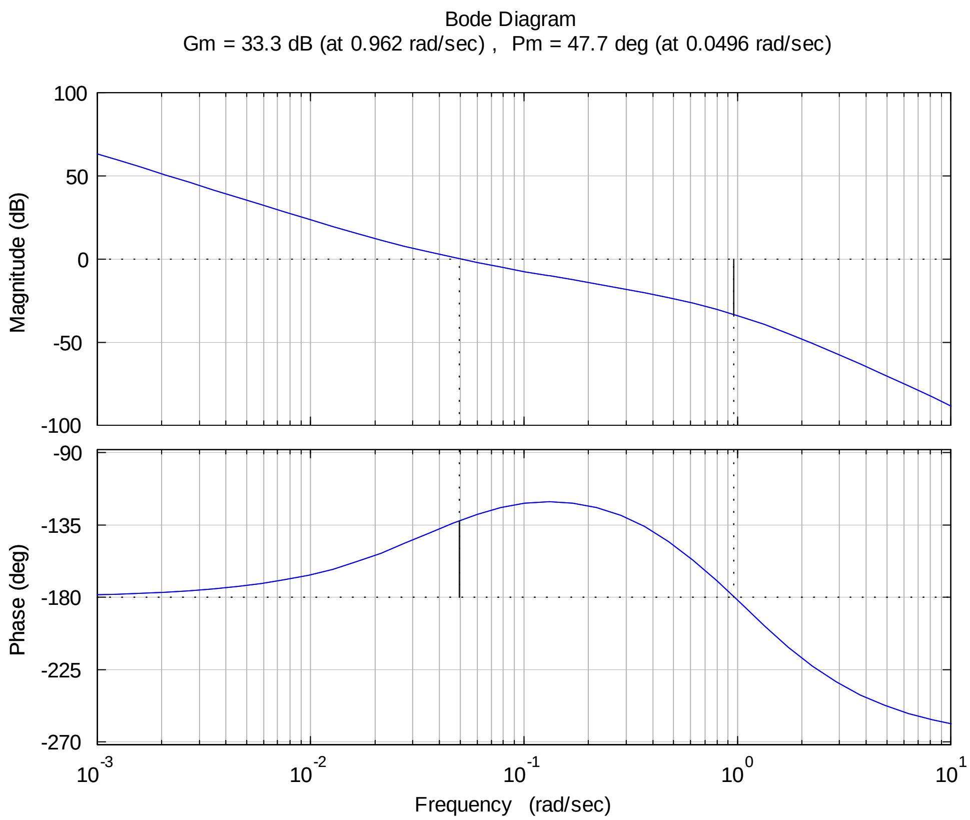
Thus the steady-state error from a unit step input on T_{ex} can be calculated using the Final Value Theorem. With Z_d and $T_{ex} = 1/s$, we find,

$$\begin{aligned}
e_\infty &= \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} -\frac{sHGT_{ex}}{1 + HGD} \\
&= \lim_{s\to 0} -\frac{HG}{1 + HGD} = -\frac{1}{K_1 z} = -0.679.
\end{aligned}$$

Because the system is linear, the steady-state error for any other size step input can be determined by simply scaling this result.

(f) Type 0.

(g) The Bode plot of DGH is shown below using MATLAB's `margin` command. The phase margin is approximately 50° at $\omega_c = 0.05$ rad/sec and the gain margin is approximately 33 db at $\omega = 1$ rad/sec.



Problem 10.13: Bode plot for satellite problem.

(h) Taking $\mathbf{x} = [x_1 \; x_2 \; x_3]^T = [\phi \; \dot{\phi} \; z]^T$, $u = T_c$, and $w = T_{ex}$, we have,

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{F}\mathbf{x} + \mathbf{G}u + \mathbf{G}_1 w, \\ y &= \mathbf{H}\mathbf{x}, \end{aligned}$$

where,

$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \; \mathbf{G} = \begin{bmatrix} 0 \\ \frac{1}{1000} \\ 0 \end{bmatrix}, \; \mathbf{G}_1 = \begin{bmatrix} 0 \\ \frac{1}{1000} \\ 0 \end{bmatrix}, \; \mathbf{H} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Since state feedback will only use ϕ and $\dot{\phi}$, we have $K_z = 0$. Thus, we can only expect to place arbitrary at most two of the control poles,

$$\det(s\mathbf{I}-\mathbf{F}+\mathbf{GK}) = \begin{bmatrix} s & -1 & 0 \\ K_\phi/1000 & s+K_{\dot\phi}/1000 & 0 \\ -1 & 0 & s+1 \end{bmatrix}$$
$$= (s+1)(s^2+K_{\dot\phi}/1000s+K_\phi/1000).$$

So in order to get the desired closed-loop roots we need,

$$\alpha_c(s) = s^2+2\zeta\omega_n s+\omega_n^2 = s^2+0.04s+0.0016.$$

Equating coefficients gives $K_\phi = 1.6$, and $K_{\dot\phi} = 40$. We can also use MATLAB's `place` command.

14. Three alternative designs are sketched in Fig. 10.95 for the closed-loop control of a system with the plant transfer function $G(s) = 1/s(s+1)$. The signal w is the plant noise and may be analyzed as if it were a step; the signal v is the sensor noise and may be analyzed as if it contained power to very high frequencies.

1. (a) Compute values for the parameters K_1, a, K_2, K_T, K_3, d, and K_D so that in each case (assuming $w=0$ and $v=0$),
$$\frac{Y}{R} = \frac{16}{s^2+4s+16}.$$
Note that in system III, a pole is to be placed at $s=-4$.

(b) Complete the following table, expressing the last entries as A/s^k to show how fast noise from v is attenuated at high frequencies:

System	K_v	$\frac{y}{w}\big\vert_{s=0}$	$\frac{y}{v}\big\vert_{s\to\infty}$
I			
II			
III			

(c) Rank the three designs according to the following characteristics (the best as "1," the poorest as "3"):

	I	II	III
Tracking			
Plant-noise rejection			
Sensor-noise rejection			

Solution:

(a)

$$I \quad : \quad \frac{Y}{R} = \frac{K_1}{s^2+as+K_1},$$
$$\Longrightarrow \quad K_1 = 16,\ a = 4.$$

$$II \quad : \quad \frac{Y}{R} = \frac{K_2}{s^2+(1+K_T)s+K_2},$$
$$\Longrightarrow \quad K_2 = 16,\ K_T = 3.$$

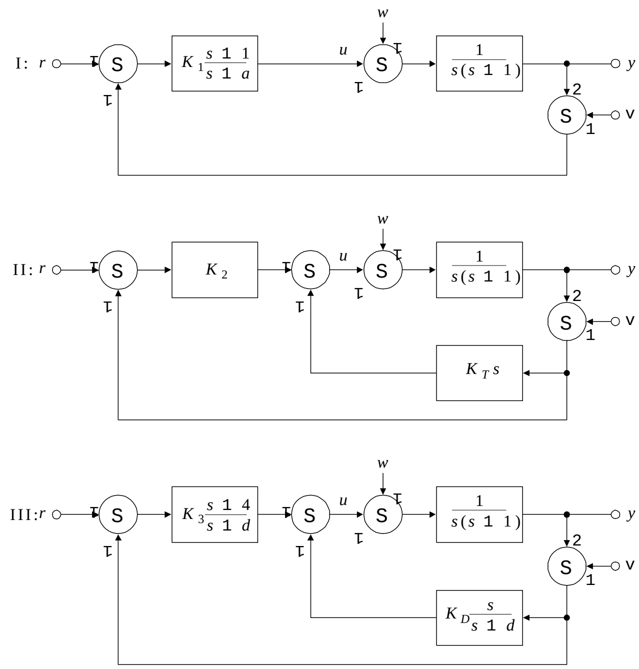


Figure 10.95: Alternative feedback structures for Problem 10.14.

$$
\begin{aligned}
III \quad &: \quad \frac{Y}{R} = \frac{K_3(s+4)}{s^3+(1+d)s^2+(K_D+d+K_3)s+4K_3}, \\
\frac{Y}{R} &= \frac{K_3(s+4)}{(s+4)(s^2+\frac{d+K_D}{4}s+K_3)}, \text{ and } K_D = 3(d-4) \\
&\Longrightarrow K_3 = 16, d = 7, K_D = 9.
\end{aligned}
$$

(b) K_v:

$$
\begin{aligned}
E(s) &= R - Y = R - \frac{16}{s^2+4s+16}R = \frac{s^2+4s}{s^2+4s+16}R, \ R(s) = \frac{1}{s^2}, \\
e_\infty &= \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \frac{1}{4}, \\
K_v &= \frac{1}{e_\infty} \Longrightarrow K_v = 4 \text{ for all the designs.}
\end{aligned}
$$

$\frac{Y}{W}|_{s=0}$:

$$
\begin{aligned}
I \quad &: \quad \frac{Y}{W}|_{s=0} = \frac{a}{K_1} = \frac{1}{4}. \\
II \quad &: \quad \frac{Y}{W}|_{s=0} = \frac{1}{K_2} = \frac{1}{16}. \\
III \quad &: \quad \frac{Y}{W}|_{s=0} = \frac{d}{4K_3} = \frac{7}{64}.
\end{aligned}
$$

$\frac{Y}{V}|_{s\to\infty}$:

$$I \quad : \quad \frac{Y}{V}|_{s\to\infty} = \frac{K_1}{s(s+a)+K_1}|_{s\to\infty} \simeq \frac{K_1}{s^2} = \frac{16}{s^2}.$$

$$II \quad : \quad \frac{Y}{V}|_{s\to\infty} = \frac{K_2+K_T s}{s^2+(1+K_T)s+K_2}|_{s\to\infty} \simeq \frac{K_T}{s} = \frac{3}{s}.$$

$$III \quad : \quad \frac{Y}{V}|_{s\to\infty} = \frac{K_3(s+4)+K_D s}{s(s+d)(s+1)+K_3(s+4)+K_D s}|_{s\to\infty} = \frac{K_3+K_D}{s^2} = \frac{25}{s^2}.$$

Filling the table, and ranking the three designs:

System	K_v	$\frac{Y}{W}\vert_{s=0}$	$\frac{Y}{V}\vert_{s\to\infty}$	tracking	Plant noise rejection	Sensor noise rejection
I	4	1/4	$16/s^2$	Same	3	1
II	4	1/16	$3/s$	Same	1	3
III	4	7/65	$25/s^2$	Same	2	2

15. The equations of motion for a cart-stick balancer with state variables of stick angle, stick angular velocity, and cart velocity are

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 31.33 & 0 & 0.016 \\ -31.33 & 0 & -0.216 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ -0.649 \\ 8.649 \end{bmatrix} u,$$
$$y = [\ 10 \quad 0 \quad 0\ ]\mathbf{x},$$

where the output is stick angle, and the control input is voltage on the motor that drives the cart wheels.

1. (a) Compute the transfer function from u to y, and determine the poles and zeros.
   (b) Determine the feedback gain $\mathbf{K}$ necessary to move the poles of the system to the locations -2.832 and $-0.521 \pm 1.068j$, with $\omega_n = 4$ rad/sec.
   (c) Determine the estimator gain $\mathbf{L}$ needed to place the three estimator poles at -10.
   (d) Determine the transfer function of the estimated-state-feedback compensator defined by the gains computed in parts (b) and (c).
   (e) Suppose we use a reduced-order estimator with poles at -10, and -10. What is the required estimator gain?
   (f) Repeat part (d) using the reduced-order estimator.
   (g) Compute the frequency response of the two compensators.

   Solution:

   (a) The transfer function (using MATLAB's tf) is,

$$G(s) = \frac{-0.649(s+0.0028)}{(s-5.59)(s+5.606)(s+0.2)}.$$

   (b) With $\alpha_c = (s+2.832)(s+2.084 \pm 4.272j)$, the feedback gains are calculated using the Ackermann's formula or equating α_c with $\det(s\mathbf{I}-\mathbf{F}+\mathbf{GK})$. The result using MATLAB's place command is,

$$\mathbf{K} = [-101.2 \quad 14.18 \quad -0.2796].$$

(c) The estimator gains with $\alpha_e(s) = (s+10)^3$ are calculated using the Ackermann's formula or equating α_e with $\det(s\mathbf{I} - \mathbf{F} + \mathbf{LH})$. The result is using MATLAB's acker command,

$$\mathbf{L} = [2.98 \quad 32.5 \quad 5850.6]^T.$$

(d) The compensator transfer function can be obtained from (using MATLAB's ss2tf),

$$D_c(s) = -\mathbf{K}(s\mathbf{I} - \mathbf{F} + \mathbf{GK} + \mathbf{LH})^{-1}\mathbf{L} = \frac{0.2398(s+5.60)(s-3.06)}{(s+23.4 \pm j22.1)(s-9.98)}.$$

Notice that the compensator is *unstable*.

(e) For reducing order estimator using and matching coefficients of $det(s\mathbf{I} - \mathbf{F}_{bb} + \mathbf{LF}_{ab}) = 0$ where,

$$\mathbf{F} = \begin{bmatrix} 0 & 1 & 0 \\ 31.33 & 0 & 0.016 \\ -31.33 & 0 & -0.216 \end{bmatrix}, \mathbf{G} = \begin{bmatrix} 0 \\ -0.649 \\ 8.649 \end{bmatrix},$$

and coefficients of $\alpha_e(s) = (s+10)^2$ will yield (or using MATLAB's acker command),

$$\mathbf{L} = [19.8 \quad 5983]^T.$$

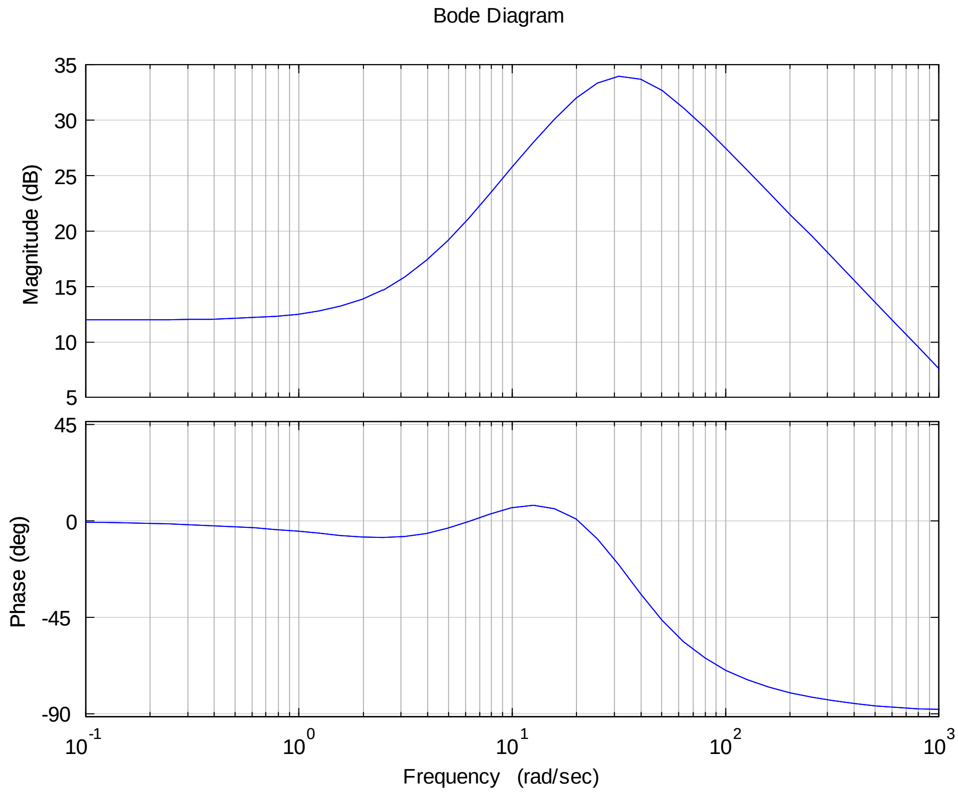
(f) The reduced order compensator is,

$$\begin{aligned} \mathbf{A}_r &= \mathbf{F}_{bb} - \mathbf{LF}_{ab} - (\mathbf{G}_b - \mathbf{LG}_a)\mathbf{K}_b, \\ \mathbf{B}_r &= \mathbf{A}_r\mathbf{L} + \mathbf{F}_{ba} - \mathbf{LF}_{aa} - (\mathbf{G}_b - \mathbf{LG}_a)\mathbf{K}_a, \\ \mathbf{C}_r &= -\mathbf{K}_b, \\ D_r &= -\mathbf{K}_a - \mathbf{K}_b\mathbf{L}, \\ D_{cr}(s) &= \mathbf{C}_r(s\mathbf{I} - \mathbf{A}_r)^{-1}\mathbf{B}_r + D_r. \end{aligned}$$

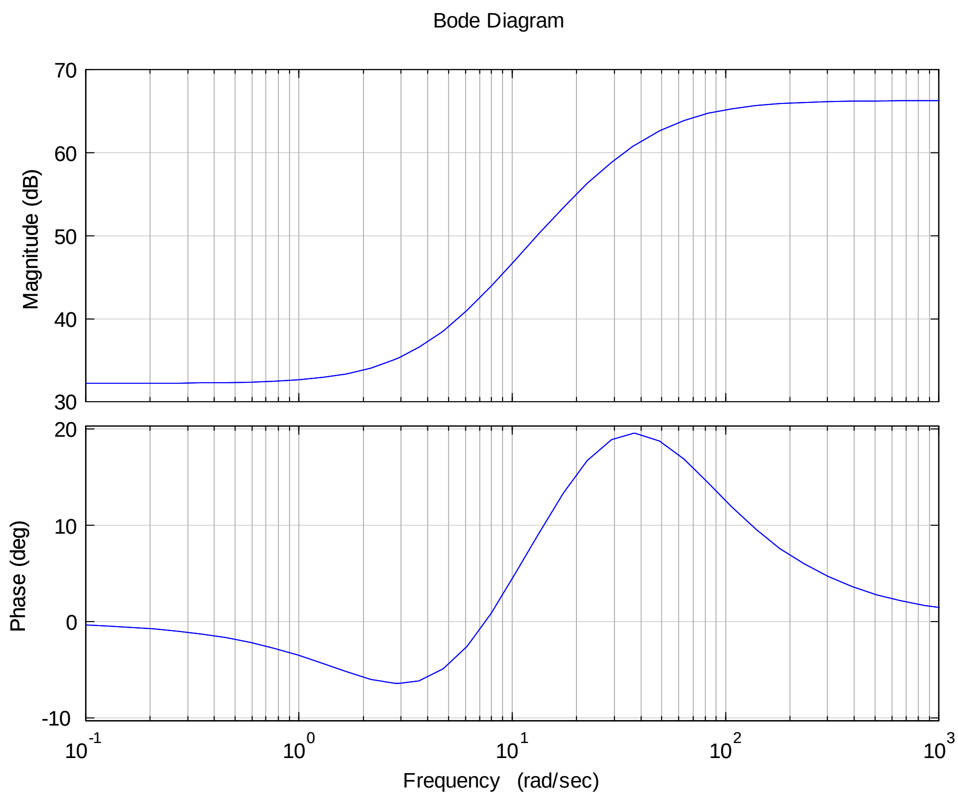
These calculations yield (using MATLAB's ss2tf),

$$D_{cr}(s) = \frac{2055(s+5.58)(s-3.69)}{(s+48.2)(s-21.4)}.$$

(g) The frequency responses of the two compensators follow.



Frequency response of the compensator.



Frequency response of the reduced order compensator.

16. A 282-ton Boeing 747 is on landing approach at sea level. If we use the state given in the case study (Section 10.3) and assume a velocity of 221 ft/sec (Mach 0.198), then the lateral-direction perturbation equations are,

$$\begin{bmatrix} \dot{\beta} \\ \dot{r} \\ \dot{p} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} -0.0890 & -0.989 & 0.1478 & 0.1441 \\ 0.168 & -0.217 & -0.166 & 0 \\ -1.33 & 0.327 & -0.975 & 0 \\ 0 & 0.149 & 1 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ r \\ p \\ \phi \end{bmatrix} + \begin{bmatrix} 0.0148 \\ -0.151 \\ 0.0636 \\ 0 \end{bmatrix} \delta r,$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ r \\ p \\ \phi \end{bmatrix}.$$

The corresponding transfer function is (using MATLAB's ss2tf),

$$G(s) = \frac{r(s)}{\delta r(s)} = \frac{-0.151(s+1.05)(s+0.0328 \pm 0.414j)}{(s+1.109)(s+0.0425)(s+0.0646 \pm 0.731j)}.$$

(a) Draw the uncompensated root locus [for $1 + KG(s)$] and the frequency response of the system. What type of classical controller could be used for this system?

(b) Try a state-variable design approach by drawing a symmetric root locus for the system. Choose the closed-loop poles of the system on the SRL to be

$$\alpha_c(s) = (s + 1.12)(s + 0.165)(s + 0.162 \pm 0.681j),$$

and choose the estimator poles to be five times faster at

$$\alpha_e(s) = (s + 5.58)(s + 0.825)(s + 0.812 \pm 3.40j).$$

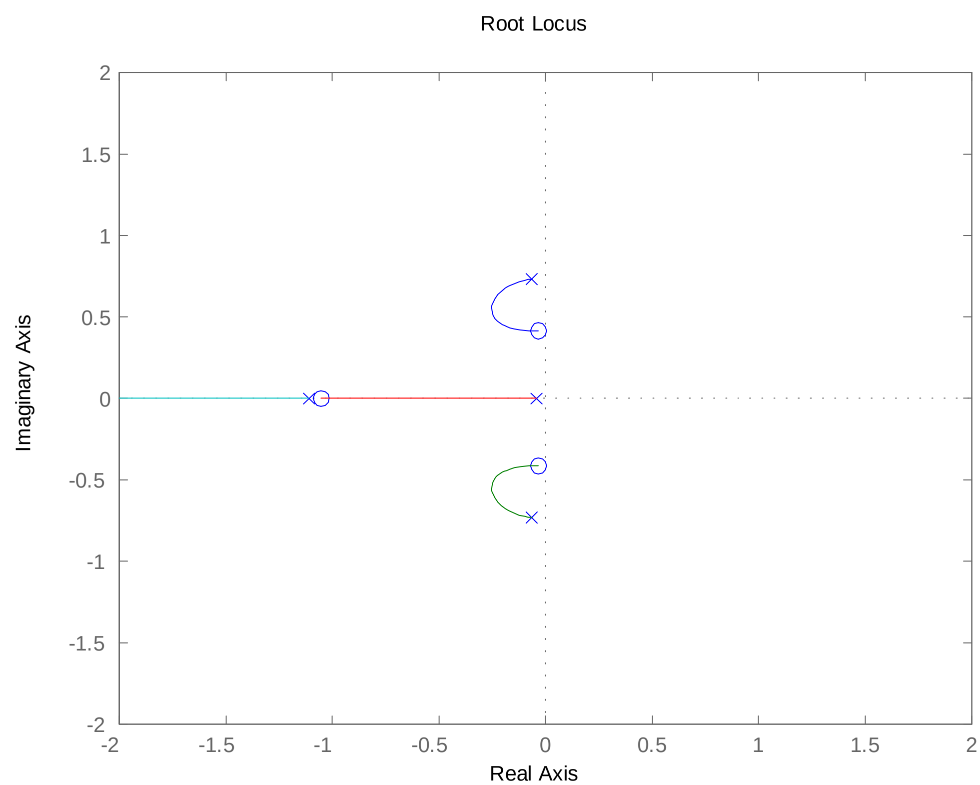
(c) Compute the transfer function of the SRL compensator.

(d) Discuss the robustness properties of the system with respect to parameter variations and unmodeled dynamics.

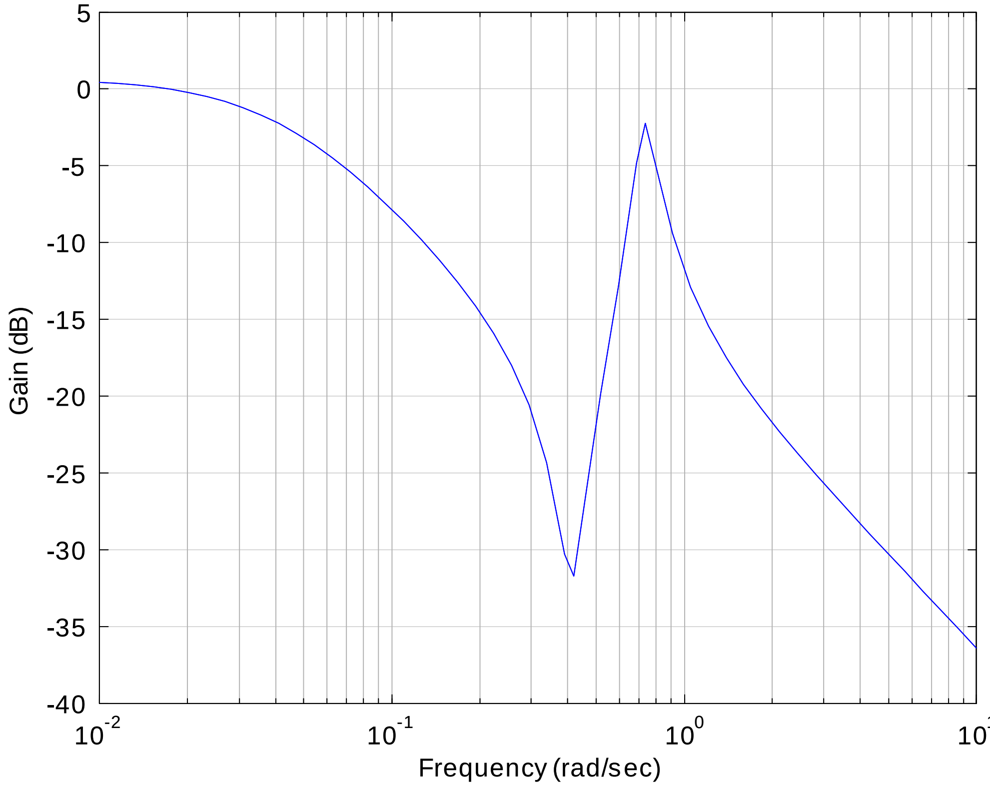
(e) Note the similarity of this design to the one developed for different flight conditions earlier in the chapter. What does this suggest about providing a continuous (nonlinear) control throughout the operating envelope?

Solution:

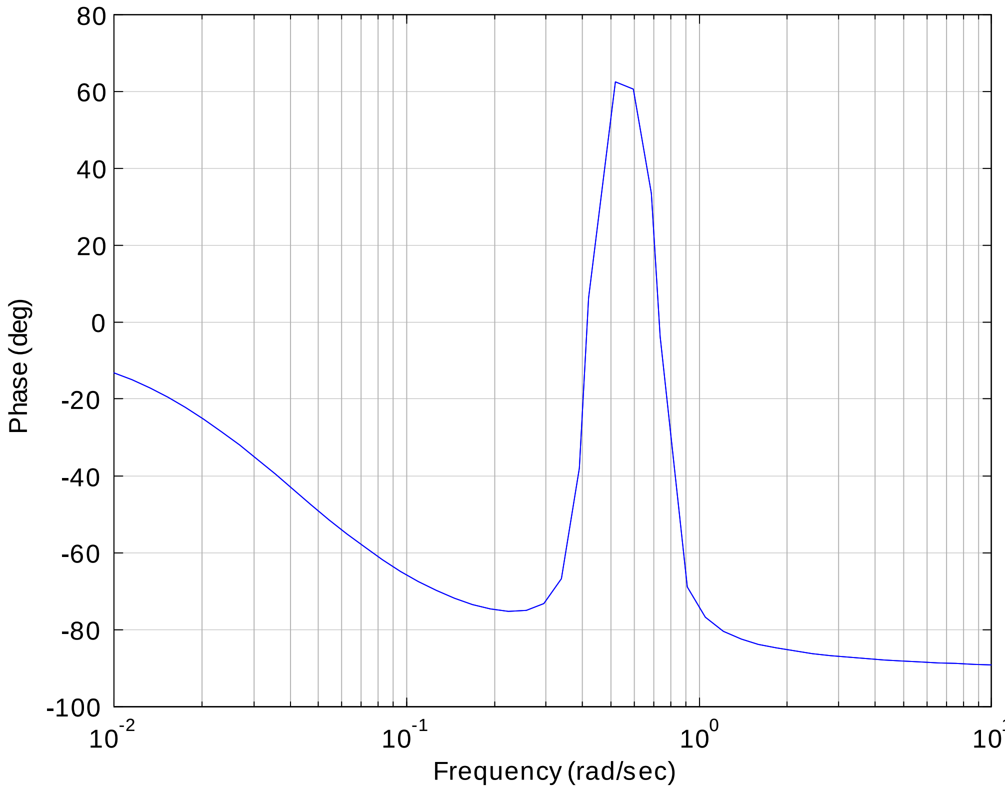
(a) The root locus (using MATLAB's rlocus command) and Bode plots (using MATLAB's bode command) are shown on the next two pages. From the figures, we see that a classical lag network could be used to lower the resonant gain.



Problem 10.16: Root locus for Boeing 747 problem.

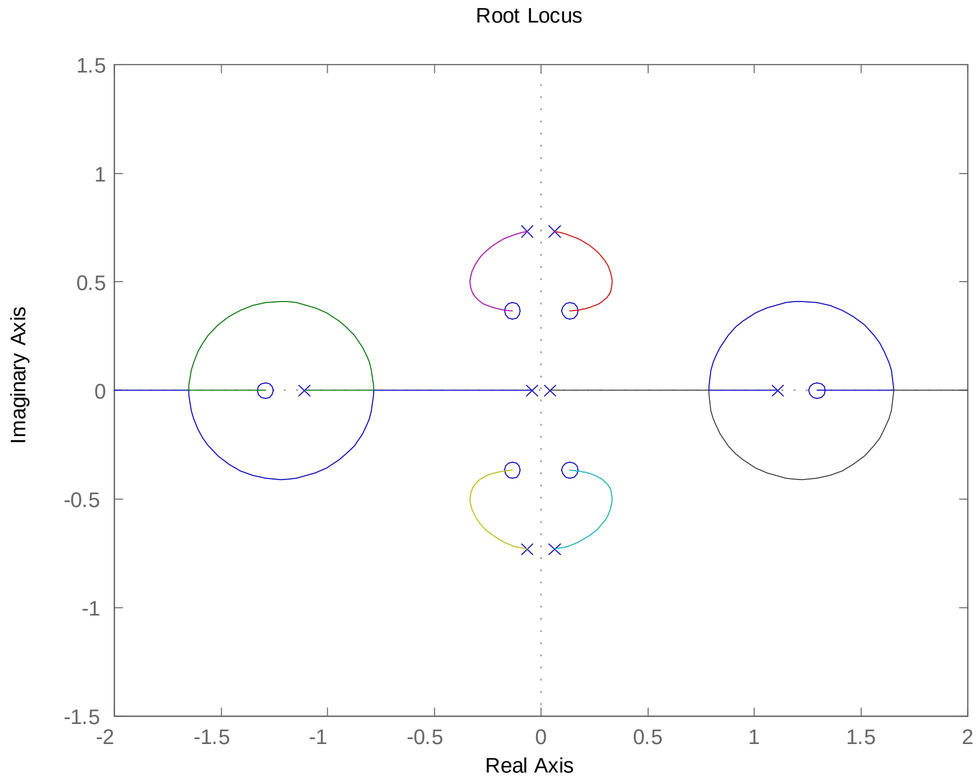


Problem 10.16: Bode magnitude plot for Boeing 747 problem.



Problem 10.16: Bode phase plot for Boeing 747 problem.

(b) The symmetric root locus $1 + kG(s)G(-s) = 0$ is shown below using MATLAB's rlocus command.



Problem 10.16: Symmetric root locus for Boeing 747 problem.

With the closed-loop poles of the system on the symmetric root locus at $\alpha_c(s) = (s + 1.12)(s + 0.165)(s + 0.162 \pm j0.681)$, the controller feedback gains are (using Ackermann's formula or matching the coefficients or using MATLAB's `place` command),

$$\mathbf{K} = [0.0308 \quad -2.122 \quad 0.112 \quad -0.034].$$

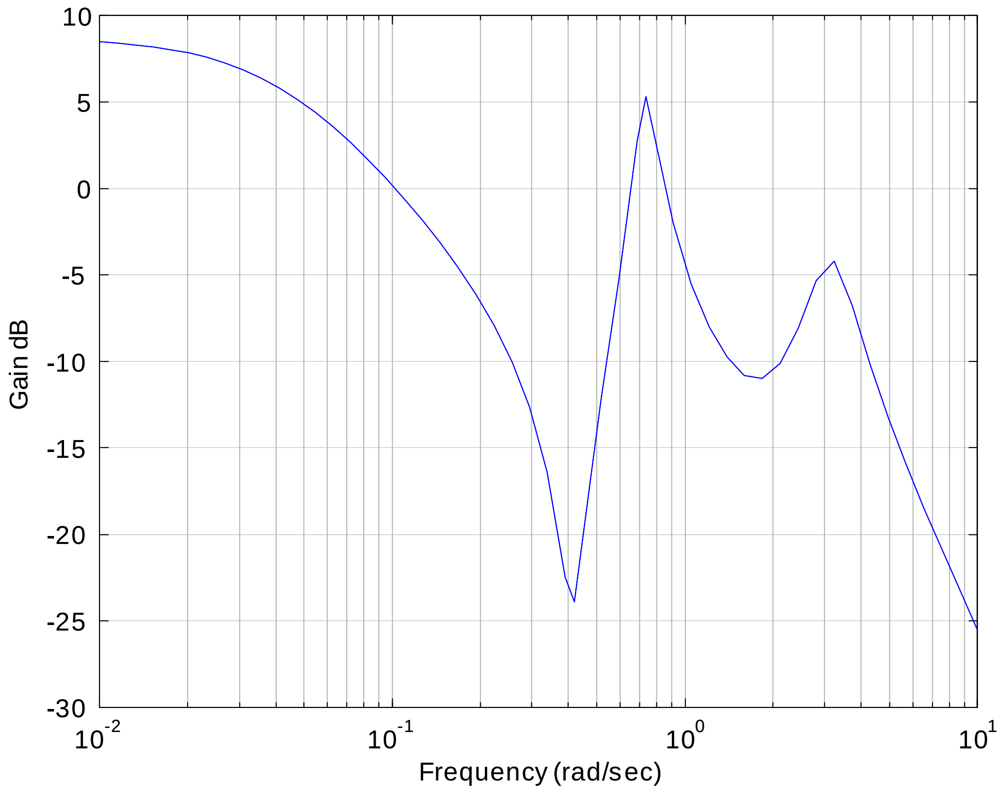
Similarly, the estimator gains with the estimator poles at $\alpha_e(s) = (s + 5.58)(s + 0.825)(s + 0.812 \pm j3.4)$ are found using Ackermann's formula or matching the coefficients or using MATLAB's `place` command. The estimator gains are,

$$\mathbf{L} = [154 \quad 6.75 \quad 39.53 \quad 973.98]^T.$$

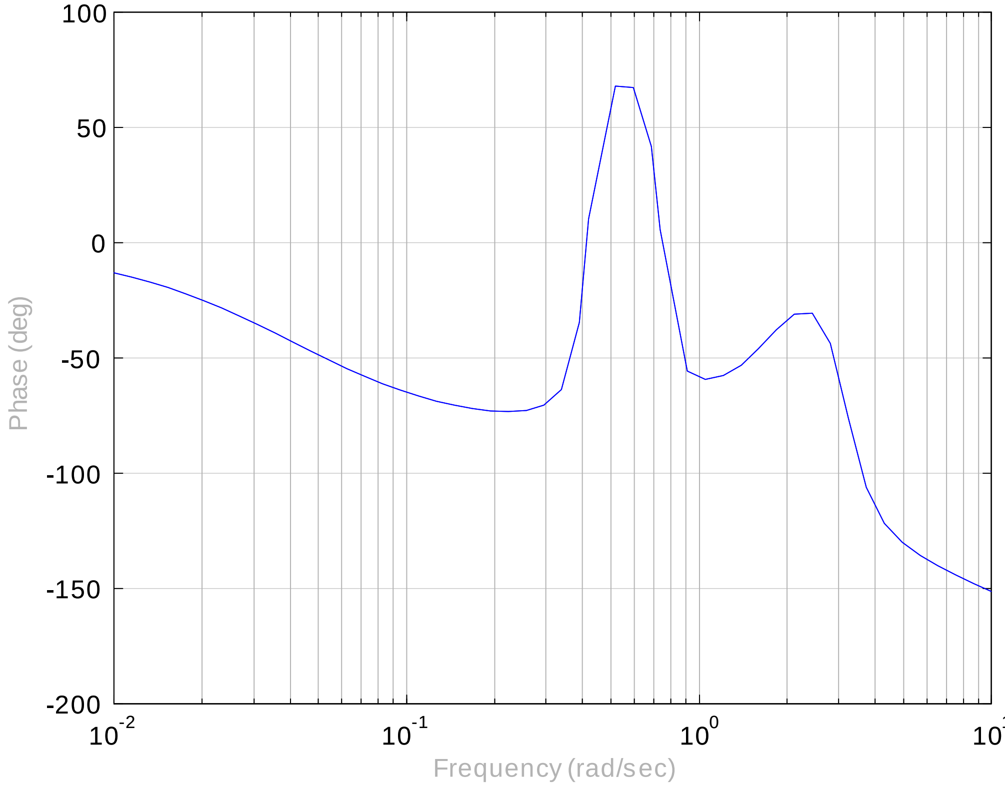
(c) The compensator transfer function is given by,

$$\begin{aligned} D_c(s) &= \frac{-38.25s^3 - 111.5s^2 - 215.1s - 136}{s^4 + 8.36s^3 + 24.02s^2 + 78.17s + 53.80} \\ &= \frac{-38.247(s + 0.94479)(s + 0.9851 \pm j1.6713)}{(s + 6.2987)(s + 0.85187)(s + 0.60319 \pm j3.1086)}. \end{aligned}$$

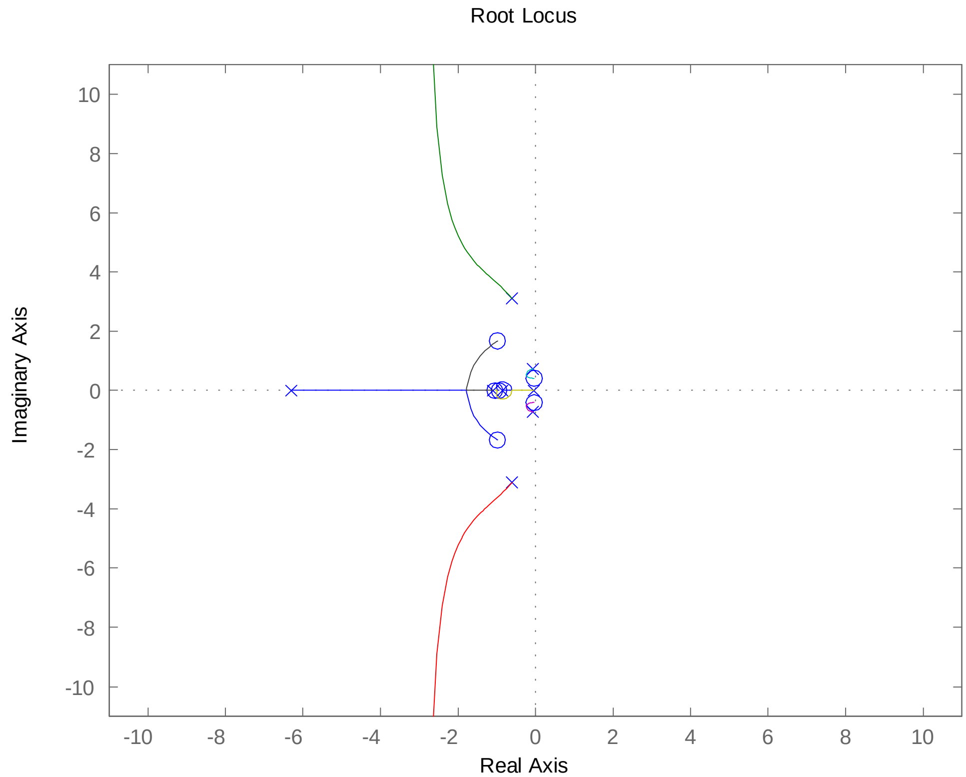
(d) The compensated Bode plot is shown below using MATLAB's `bode` command. Because the phase is always less than $-180°$, we would expect the system to be very robust with respect to gain changes. The phase margin also indicates good robustness.



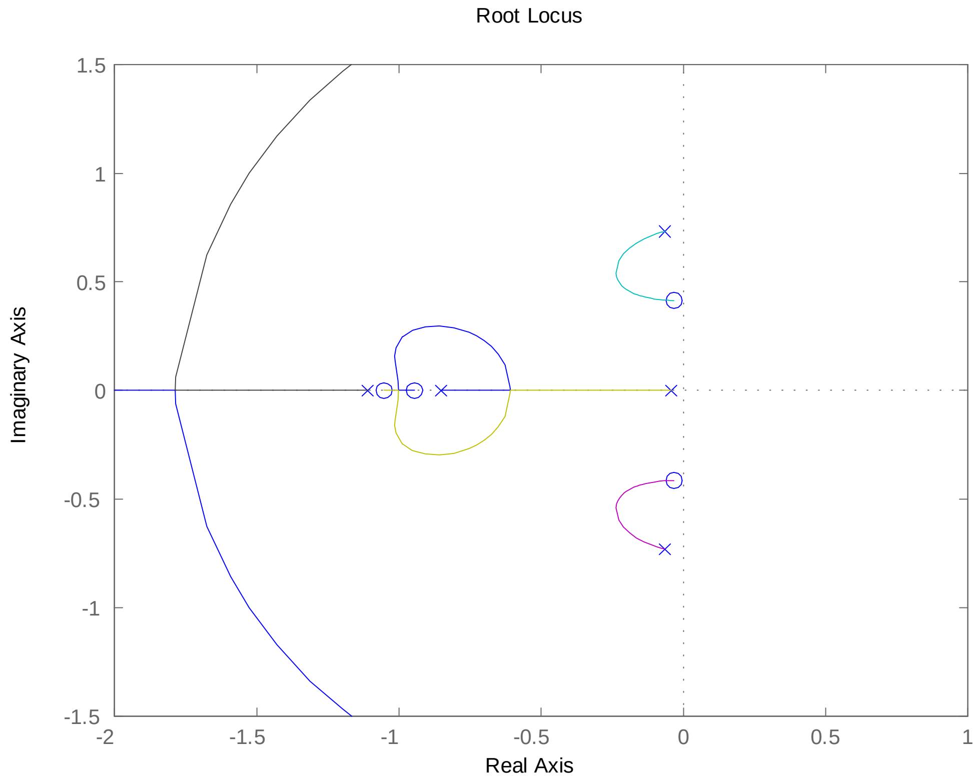
Problem 10.16: Bode magnitude for compensated system.



Problem 10.16: Bode phase plot for compensated system.



Problem 10.16: Root locus of compensated system for Boeing 747.



Problem 10.16 Root locus of compensated system for Boeing 747: Detailed view.

(e) Because a similar compensator stabilizes both cases, we would expect one compensator to be satisfactory over a wide range of flight conditions.

17. (Contributed by Prof. L. Swindlehurst) The feedback control system shown in Fig. 10.96 is proposed as a position control system. A key component of this system is an armature-controlled DC motor. The input potentiometer produces a voltage E_i that is proportional to the desired shaft position: $E_i = K_p\theta_i$. Similarly, the output potentiometer produces a voltage E_0 that is proportional to the actual shaft position: $E_0 = K_p\theta_0$. Note that we have assumed that both potentiometers have the same proportionality constant. The error signal $E_i - E_0$ drives a compensator, which in turn produces an armature voltage that drives the motor. The motor has an armature resistance R_a, an armature inductance L_a, a torque constant K_t, and a back-emf constant K_e. The moment of inertia of the motor shaft is J_m, and the rotational damping due to bearing friction is B_m. Finally, the gear ratio is $N : 1$, the moment of inertia of the load is J_L, and the load damping is B_L.

1. (a) Write the differential equations that describe the operation of this feedback system.

   (b) Find the transfer function relating $\theta_0(s)$ and $\theta_i(s)$ for a general compensator $D_c(s)$.

   (c) The open-loop frequency-response data shown in Table 10.2 were taken using the armature voltage v_a of the motor as an input and the output potentiometer voltage E_0 as the output. Assuming that the motor is linear and minimum-phase, make an estimate of the transfer

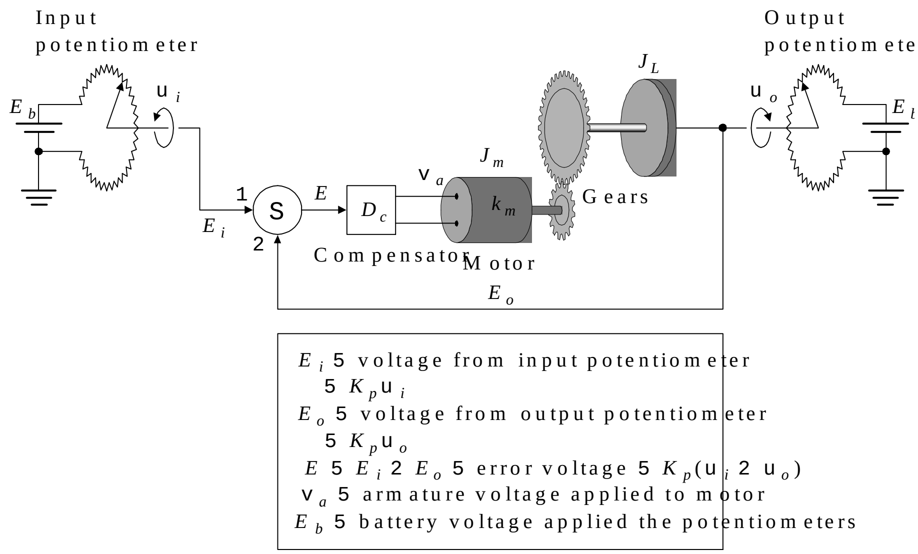


Figure 10.96: A servomechanism with gears on the motor shaft and potentiometer sensors.

function of the motor,

$$G(s) = \frac{\theta_m(s)}{V_a(s)},$$

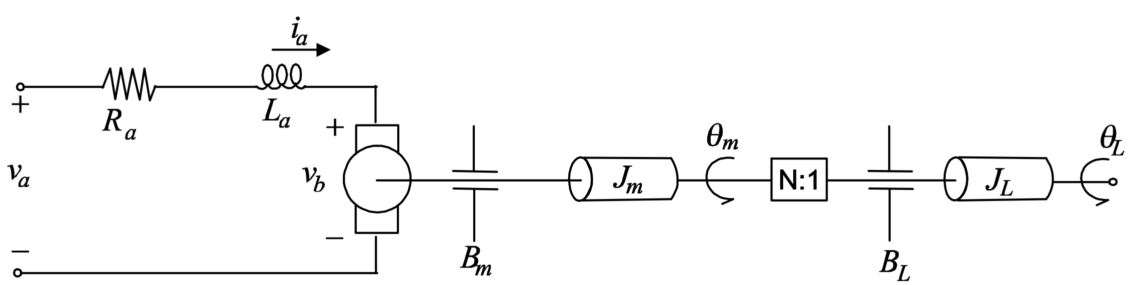
where θ_m is the angular position of the motor shaft.

(d) Determine a set of performance specifications that are appropriate for a position control system and will yield good performance. Design $D_c(s)$ to meet these specifications.

(e) Verify your design through analysis and simulation using MATLAB.

Solution:

(a) First of all, we describe the motor dynamics in more detail. This is illustrated below.



Problem 10.17: DC motor.

The figure defines a few additional variables not mentioned in the problem statement: i_a is the armature current, v_b is the back emf, and θ_m is the angular position of the motor. Using Kirchhoff's voltage laws we can write,

$$v_a - v_b = R_a i_a + L_a \frac{di_a}{dt}. \qquad (1)$$

Table 10.2: Frequency-response Data for Problem 10.17.

Frequency (rad/sec)	$\left\|\frac{E_0(s)}{V_a(s)}\right\|$ (db)	*Frequency* (rad/sec)	$\left\|\frac{E_0(s)}{V_a(s)}\right\|$ (db)
0.1	60.0	10.0	14.0
0.2	54.0	20.0	2.0
0.3	50.0	40.0	−10.0
0.5	46.0	60.0	−20.0
0.8	42.0	65.0	−21.0
1.0	40.0	80.0	−24.0
2.0	34.0	100.0	−30.0
3.0	30.5	200.0	−48.0
4.0	27.0	300.0	−59.0
5.0	23.0	500.0	−72.0
7.0	19.5		

The torque of the motor, T is proportional to the armature current. Thus,

$$T = K_t i_a. \qquad (2)$$

The back emf, v_b, is proportional to the angular speed. Hence,

$$v_b = K_e \frac{d\theta_m}{dt}. \qquad (3)$$

At the point of contact of the gears, we assign an equal and oppositely directed force F. (Since we do not know this force, we will eliminate it momentarily). Using Newton's law of motion, we have

$$\begin{aligned} J_m\ddot{\theta}_m + B_m\dot{\theta}_m &= T - Fr_m, & (4)\\ J_L\ddot{\theta}_o + B_L\dot{\theta}_o &= -Fr_o, & (5) \end{aligned}$$

where r_m is the radius of the gear connected to the motor shaft and r_o is the radius of the gear connected to the output shaft. The minus sign on both terms arises because of the defined directions for θ_m and θ_o. And from the gear ratio information, we have

$$r_o = Nr_m \Longrightarrow \theta_m = -N\theta_o \qquad (6)$$

(b) First, we will find the transfer function from v_a to θ_o (the plant) and then we will find the closed loop transfer function. From the gear ratio information, we can combine Eq. (4) and (5) to eliminate θ_m.

$$\begin{aligned} -NT &= -NK_t i_a = (N^2J_m + J_L)\ddot{\theta}_o + (N^2B_m + B_L)\dot{\theta}_o, & (7)\\ \ddot{\theta}_o &= -\underbrace{\frac{(N^2B_m + B_L)}{(N^2J_m + J_L)}}_{=\alpha}\dot{\theta}_o - \underbrace{\frac{NK_t}{(N^2J_m + J_L)}}_{=\beta} i_a. & (8) \end{aligned}$$

Next, we combine Eq. (1) and (3) to get,

$$\frac{di_a}{dt} = -\frac{R_a}{L}i_a + \frac{NK_e}{L}\dot{\theta}_o + \frac{1}{L}v_a. \qquad (9)$$

Thus, if we define that state $\mathbf{x} = [\theta_o\ \dot{\theta}_o\ i_a]^T$, we have,

$$\begin{bmatrix} \dot{\theta}_o \\ \ddot{\theta}_o \\ \dot{i}_a \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\alpha & -\beta \\ 0 & \frac{NK_e}{L_a} & -\frac{R_a}{L_a} \end{bmatrix} \begin{bmatrix} \theta_o \\ \dot{\theta}_o \\ i_a \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix} v_a.$$

The output equation is $y = [1\ 0\ 0]\mathbf{x}$. This state space realization can then be converted to transfer function form. We find,

$$G(s) = \frac{\Theta(s)}{V_a(s)} = \frac{-\beta/L_a}{s\left[s^2 + (\alpha + R_a/L_a)s + (\frac{R_a\alpha}{L_a} + \frac{\beta NK_e}{L_a})\right]}.$$

And so the closed-loop transfer function is,

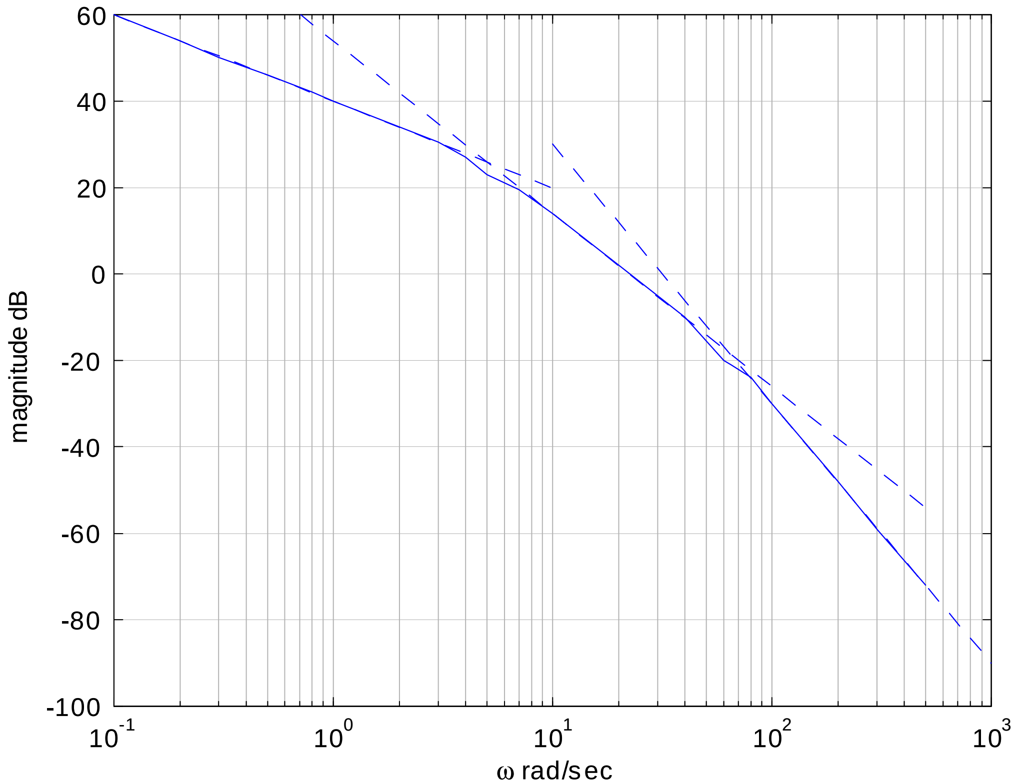
$$\frac{\Theta_o(s)}{\Theta_i(s)} = \frac{K_pG(s)D_c(s)}{1 + K_pG(s)D_c(s)}.$$

(c) The figure on the next page shows three straight lines fit through the frequency response data. From this information, we can estimate the transfer function of the plant, $G(s)$. From the figure, the poles appear to be located at $\omega = 5$ rad/sec and $\omega = 70$ rad/sec. Keeping the sign convention from part (b), we have,

$$G(s) = \frac{-\beta/L_a}{s(s+5)(s+70)}.$$

The gain, β/L_a, is determined by picking a particular value of ω, say $\omega = 1$, comparing the calculated transfer function with the frequency response data. We find,

$$G(s) = \frac{-35700}{s(s+5)(s+70)}.$$



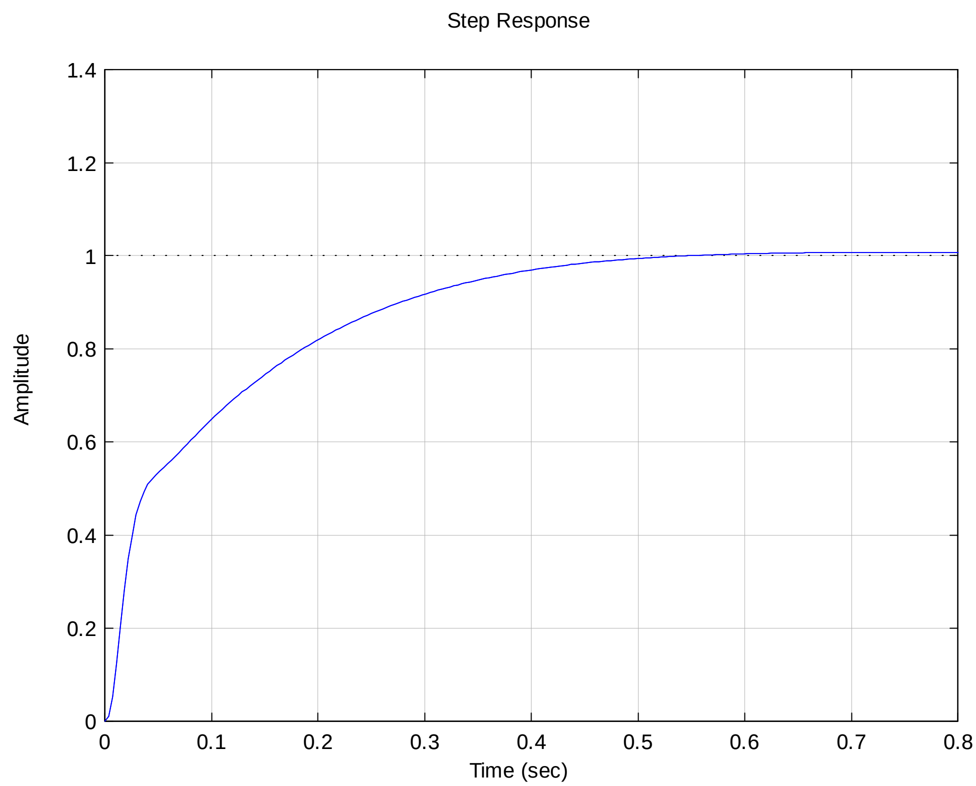
Problem 10.17: Straight line fits to frequency response.

(d) For a positioning system, we would like to keep the overshoot small, less than 1% (say). And we also like a reasonably fast rise time. For this plant, let's try to obtain $\omega_n = 6$ rad/sec (for the dominant roots). Which translates, using the rule of thumb for a dominant second order system, to $t_r = 1.8/6 = 0.3$ sec.

(e) Both of the time domain specifications are met using a double lead compensator,

$$G(s) = 50\frac{(s+9)^2}{(s+200)^2}.$$

The corresponding step response is shown on the next page using MATLAB's `step` command. It has an overshoot of 0.68% and a rise time of 0.2681 sec.



Problem 10.17: Step response of position control system.

18. Design and construct a device to keep a ball centered on a freely swinging beam. An example of such a device is shown in Fig. 10.97. It uses coils surrounding permanent magnets as the actuator to move the beam, solar cells to sense the ball position, and a hall-effect device to sense the beam position. Research other possible actuators and sensors as part of your design effort. Compare the quality of the control achievable for ball-position-feedback only with that of multiple-loop feedback of both ball and beam position.

Solution:

See Text Figure 10.97.

19. Design and construct the magnetic levitation device shown in Figure 9.2. You may wish to use LEGO components in your design.

1. **Solution:** see K. A. Lilienkamp and K. Lundberg,"Low-cost magnetic levitation project kits for teaching feedback system design," *Proceeding of the American Control Conference*, pp. 1308-1313, 2004.

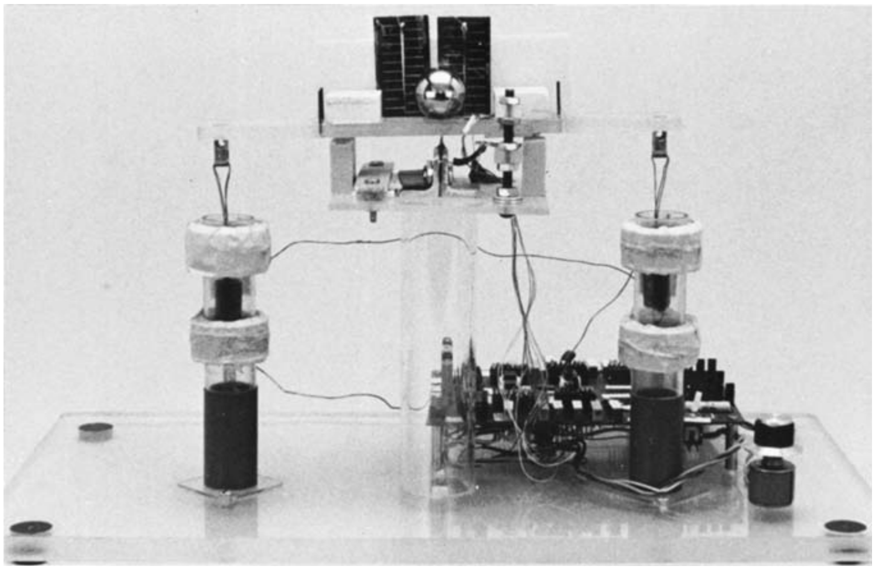
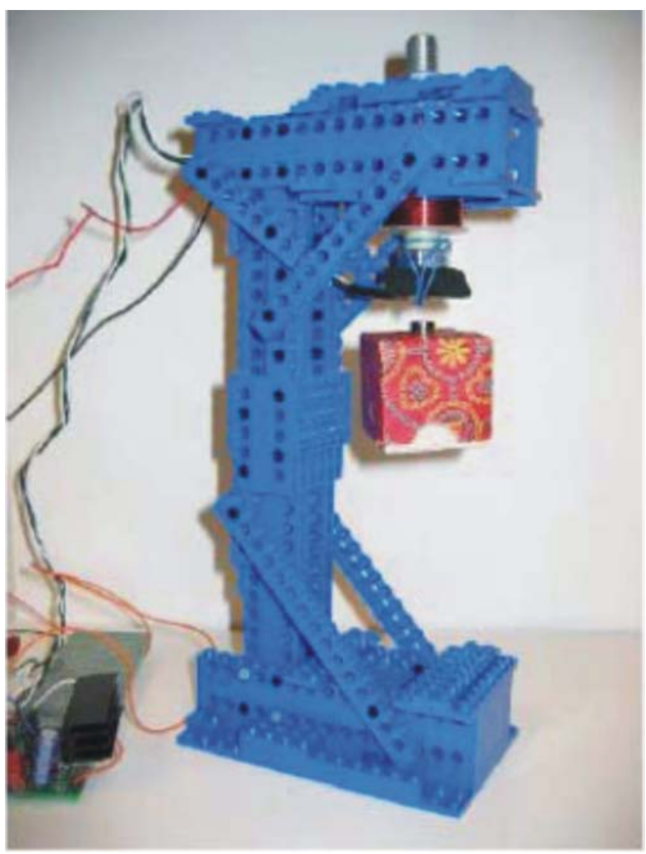


Figure 10.97: Ball-balancer design example.



Magnetic levitation system [see above Reference by K. A. Lilienkamp and K. Lundberg].

20. *Run-to-Run Control*: Consider the rapid thermal processing (RTP) system shown in Fig. 10.98. We wish to heat up a semiconductor wafer, and control the wafer surface temperature accurately using rings of tungsten halogen lamps. The output of the system is temperature T as a function of time, $y = T(t)$. The system reference input R is a desired step in temperature ($700^{\circ}C$) and the control input is lamp power. A pyrometer is used to measure the wafer center temperature. The model of the system is first order and an integral controller is used as shown in Figure 10.98. Normally, there is not a sensor bias ($b = 0$).

a. Suppose the system suddenly develops a sensor bias $b \neq 0$, where b is known. What can be done to ensure zero steady-state tracking of temperature command R despite the presence of the sensor bias?

1. (a) Now assume $b = 0$. In reality, we are trying to control the thickness of the oxide film grown (Ox) on the wafer and not the temperature. At present no sensor can measure Ox in real time. The semiconductor process engineer must use an off-line equipment (called *metrology*) to measure the thickness of the oxide film grown on the wafer. The relationship between the system output temperature and Ox is nonlinear and given by

$$\text{Oxide thickness} = \int_0^{t_f} p e^{-\frac{c}{T(t)}} dt,$$

where t_f is the process duration, and p and c are known constants. Suggest a scheme in which the center wafer oxide thickness Ox can be controlled to a desired value (say, $Ox = 5000$ Å) by employing the temperature controller and the output of the metrology.

Solution:

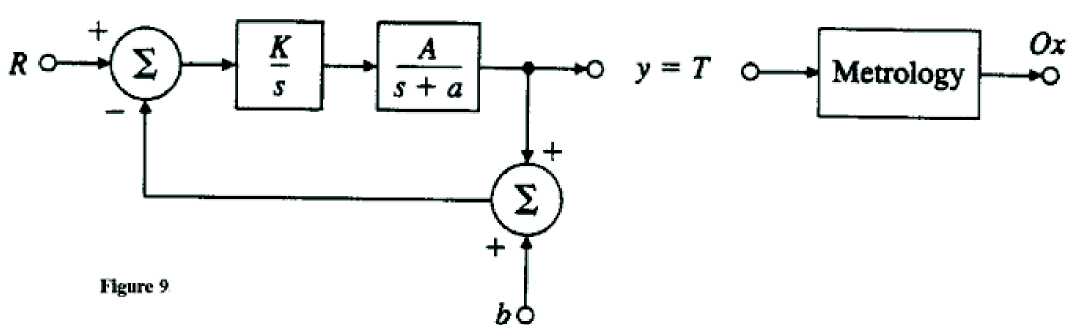


Figure 10.98: RTP system.

(a) We just increase R by $+b$, i.e., replace R by $(R + b)$ to cancel the sensor bias.

(b) Since there is a direct relationship between the temperature and oxide thickness, we could use the results of metrology to adjust the reference temperature until a desired thickness is obtained. We can do one "run" and measure the oxide thickness. Let us say, the metrology yields an oxide thickness of 5050 Angstrom (50 Å higher than desired). We would then lower the temperature, R, and try again. This is called "run-to-run" control and a linear static model can be used to provide the control adjustments. In effect, we will be "closing" the loop on metrology with a discrete integrator [1]. The recipe is adjusted from "run-to-run" using the following simple algorithm based on the attributes of the product produced in the previous run or runs [1]. Let $k = 1, 2, ...,$ denote the run number, r_k the recipe variable used during run k, y_k the product quality attribute (oxide thickness) produced at the end of run k, and e_k the normalized product quality error, defined as,

$$\begin{aligned} e_k &= y_k(i_{center}) - y_{des}(i_{center}), \\ e_k(i_{center)} &= Ox(i_{center}) - 5000. \end{aligned} \tag{1}$$

where $y_{des}(i_{center})$ is the desired center oxide thickness at the center wafer node, i_{center}. The simplest choice for run-to-run control is to correct the previous recipe by an amount proportional to the current error. Thus, for run $k = 1, 2, ...,$ adjust the recipe according to,

$$\begin{aligned} r_k &= r_{nom} + u_k \\ u_k &= u_{k-1} - \Gamma e_{k-1} \qquad u_0 = 0. \end{aligned} \tag{2}$$

where r_{nom} is the nominal recipe, u_k is the correction to the nominal recipe for run k, and Γ is the control design gain. Γ is determined experimentally as follows. We would step up R and measure the associated change in Ox, i.e., perturb R by δR and measure the associated output perturbation δOx. Therefore,

$$\Gamma = \frac{\delta R}{\delta Ox}$$

is the control design gain. It is important to emphasize that (1)-(2) constitute the complete run-to-run algorithm. Also (2) has the same form as a gradient descent optimization algorithm. It is possible to choose the run-to-run control gain matrix Γ and to analyze the algorithm under a variety of assumptions about how u_k effects e_k [1]. It can be shown that most of the widely used run-to-run algorithms are in the form of (2) for different choices of Γ. For more details see Reference [1].

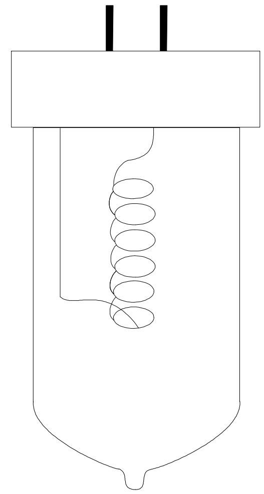
Reference:

[1] R. L. Kosut, D. de Roover, A. Emami-Naeini, J. L. Ebert, "Run-to-Run Control of Static Systems," in P*roc. 37th IEEE Conf. Decision Control*, pp. 695-700, December 1998.

21. Develop a nonlinear model for a tungsten halogen lamp and simulate it in Simulink.

Solution:

Discovered in 1959, a tungsten halogen bulb is similar to an ordinary incandescent bulb with the filament made from tungsten but the fill gas is a halogen compound, usually iodine or bromine. A schematic of the lamp is shown below. The idea behind the use of the halogen is to re-deposit the evaporated tungsten molecules back onto the filament. The Tungsten atoms evaporate off of the hot filament and condense onto the cooler inside wall of the bulb. However, halogen reacts with the tungsten and re-evaporates the deposited tungsten which reaches the hot filament again. This process is known as the "halogen cycle", and extends the lifetime of the bulb. In order for the halogen cycle to work, the bulb surface must be very hot, generally over 250°C. The bulb is made from quartz. Tungsten halogen lamps are now commonly used in rapid thermal processing (RTP) in semiconductor manufacturing [1].



Problem 10.21: Schematic of a tungsten halogen lamp.

Consider the following physical parameters for the lamp,

$$\begin{aligned}
\text{total emissivity } \epsilon &\cong 0.4, \\
\text{density } \rho &= 19300\ [kg/m^3], \\
\text{specific heat } c &\cong 150\ [J/kgK]\ (\text{at } T = 1000K), \\
\text{electrical resistivity } \rho_e &= 21.9 \times 10^{-8}\ [\Omega m]\ (\text{at } T = 1000K), \\
\text{Stefan-Boltzman constant } \sigma &= 5.67 \times 10^{-8}\ [W/m^2K^4].
\end{aligned}$$

Lamp design is based on a specification of maximum temperature, maximum applied voltage, and maximum delivered power. Using the following notation,

$$\begin{aligned}
d &= \text{filament diameter}, [mm], \\
L &= \text{filament length}, [m], \\
T &= \text{filament temperature}, [K], \\
T_{\max} &= \text{maximum filament temperature}, [K], \\
T_{\infty} &= \text{ambient (room) temperature}, [K], \\
T_{e0} &= \text{temperature at which resistivity } \rho_e \text{ is specified}, [K], \\
V &= \text{normalized lamp voltage } (0 \leq V \leq 1), \\
V_{\max} &= \text{maximum applied voltage}, [V], \\
P_{\max} &= \text{maximum power}, [W], \\
I &= \text{current}, [A],
\end{aligned}$$

the electrical resistivity is given by the relationship [3],

$$\rho_e = \rho_{e0} \left(\frac{T}{T_{e0}} \right)^{1.2}. \tag{1}$$

For example at $T_{e0} = 1000K$, we find $\rho_{e0} = 21.9 \times 10^{-8}\ \Omega m$. Employing the energy balance technique [1] leads to the equation,

$$\frac{1}{4}\pi d^2 L\rho c\,\dot{T} = -\,\epsilon\sigma\pi dL(T^4 - T_\infty^4) + \frac{\pi d^2 V_{\max}^2}{4\rho_{e0}L\left(\frac{T}{T_{e0}}\right)^{1.2}}V^2. \tag{2}$$

It is interesting to note that if we specify the maximum radiant power desired, $P_{\max}$, then the filament diameter, d, and length L, of the filament are specified for a given $T_{\max}$ and $V_{\max}$. From Eq. (2), in the steady-state, $\dot{T} = 0$, and with $V = 1$,

$$P_{\max} = \frac{\pi d^2 V_{\max}^2}{4\rho_{e0}L\left(\frac{T_{\max}}{T_{e0}}\right)^{1.2}} = \epsilon\sigma\pi dL(T_{\max}^4 - T_\infty^4), \tag{3}$$

and we can solve Eqs. (2) and (3), for d and L as follows. Equation (3) can be written as,

$$\alpha\frac{d^2}{L} = \beta dL, \tag{4}$$

where,

$$\alpha = \frac{\pi V_{\max}^2}{4\rho_{e0}\left(\frac{T_{\max}}{T_{e0}}\right)^{1.2}}, \tag{5}$$

and,

$$\beta = \epsilon\sigma\pi\left(T_{\max}^4 - T_\infty^4\right), \tag{6}$$

are prescribed. Then, the filament diameter, d, is simply related to filament length, L, by,

$$d = \frac{\beta}{\alpha}L^2. \tag{7}$$

Using Eq. (3), (4), and (7), and the maximum radiant power is related to the filament length by,

$$P_{\max} = \frac{\beta^2}{\alpha}L^3. \tag{8}$$

Using Eq. (8) together with Eqs. (5) and (6), we can then solve for the filament length , L, as,

$$L = \left(\frac{P_{\max}\alpha}{\beta^2}\right)^{\frac{1}{3}} = \left[\frac{P_{\max}\pi V_{\max}^2}{4\rho_{e0}\left(\frac{T_{\max}}{T_{e0}}\right)^{1.2}\pi^2\epsilon^2\sigma^2\left(T_{\max}^4 - T_\infty^4\right)^2}\right]^{\frac{1}{3}}. \tag{9}$$

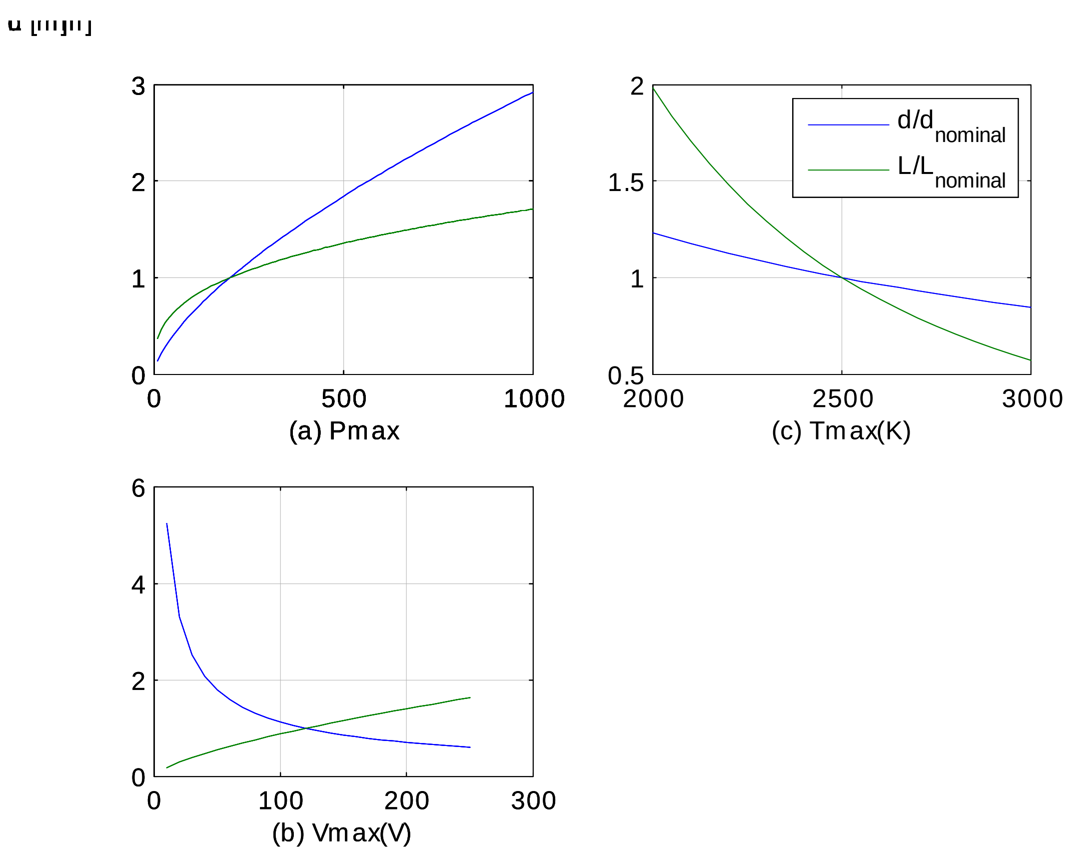
This can be simplified further by assuming that the maximum filament temperature is much higher than the ambient temperature, $T_{\max}^4 \gg T_\infty^4$, so that,

$$L \cong \left[\frac{P_{\max}\pi V_{\max}^2}{4\rho_{e0}\left(\frac{T_{\max}}{T_{e0}}\right)^{1.2}\pi^2\epsilon^2\sigma^2 T_{\max}^8}\right]^{\frac{1}{3}}. \tag{10}$$

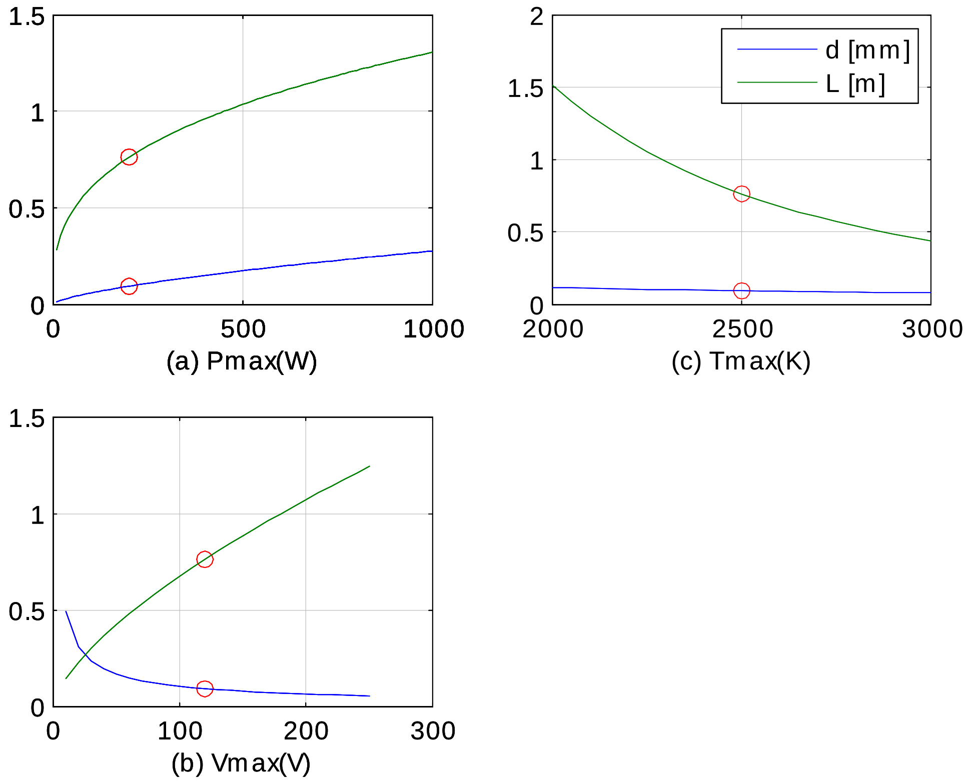
We can solve for the filament length and diameter in terms of the given quantities $P_{\max}$, $V_{\max}$, and $T_{\max}$ as well,

$$\begin{aligned} L &\propto P_{\max}^{\frac{1}{3}}V_{\max}^{\frac{2}{3}}T_{\max}^{-3.07}, \\ d &\propto P_{\max}^{\frac{2}{3}}V_{\max}^{-\frac{2}{3}}T_{\max}^{-0.93}. \end{aligned} \tag{11}$$

Figures (a), (b), and (c) on top of the next page show plots of the functional relationship between filament diameter and length as a function of the maximum power, $P_{\max}$, maximum temperature, $T_{\max}$, and maximum voltage, $V_{\max}$, respectively as well as the nominal operating point corresponding to $P_{\max} = 200W$, $T_{\max} = 3000K$, and , $V_{\max} = 120V$. Figure (a) shows that increasing $P_{\max}$ requires increases in both the filament diameter and length. As seen from Figure (b), increasing the maximum temperature, $T_{\max}$, requires decreasing the filament length but is relatively insensitive to the filament diameter. Figure (c) shows that increasing the maximum voltage, $V_{\max}$, requires increasing the filament length but decreasing the filament diameter. Figures (a), (b), and (c) on the bottom of the next page show the same relationships as in the top figure but in terms of the normalized diameter, $\frac{d}{d_{no\min al}}$, and normalized filament length, $\frac{L}{L_{no\min al}}$.



Problem 10.20: Lamp design parametrs.



Problem 10.21: Lamp design parameters: normalized.

Assume the nominal operating temperature is denoted by T_0. Re-writing Equation (2) in terms of the normalized temperature, we have

$$\left(\frac{\dot{T}}{T_0}\right) = -\frac{4\epsilon\sigma T_0^3}{\rho c d}\left[\left(\frac{T}{T_0}\right)^4 - \left(\frac{T_\infty}{T_0}\right)^4\right] + \frac{V_{\max}^2}{4\rho_{e0}\rho c L^2 T_0}\frac{V^2}{\left(\frac{T}{T_{e0}}\right)^{1.2}}. \tag{12}$$

Let us define the normalized temperature, $x = \frac{T}{T_0}$,and re-write Eq. (12) as the nonlinear first-order system,

$$\dot{x} = -A(x^4 - x_\infty^4) + B\frac{V^2}{x^{1.2}}, \tag{13}$$

where,

$$\begin{aligned} A &= \frac{4\epsilon\sigma T_0^3}{\rho c d}, \\ B &= \frac{V_{\max}^2}{4\rho_{e0}\rho c L^2 T_0}. \end{aligned}$$

Linearizing Equation (13) about the nominal (normalized) temperature, we find the first-order linear dynamic model for the lamp as,

$$\dot{x} = -4Ax_0^3 x - B\frac{1.2}{x_0^{2.2}}V^2, \tag{14}$$

which means that the lamp time constant is,

$$\tau = \frac{1}{4Ax_0^3} \cong \frac{\rho c d}{16\epsilon\sigma T_0^3 x_0^3}. \tag{15}$$

In terms of lamp current, we have,

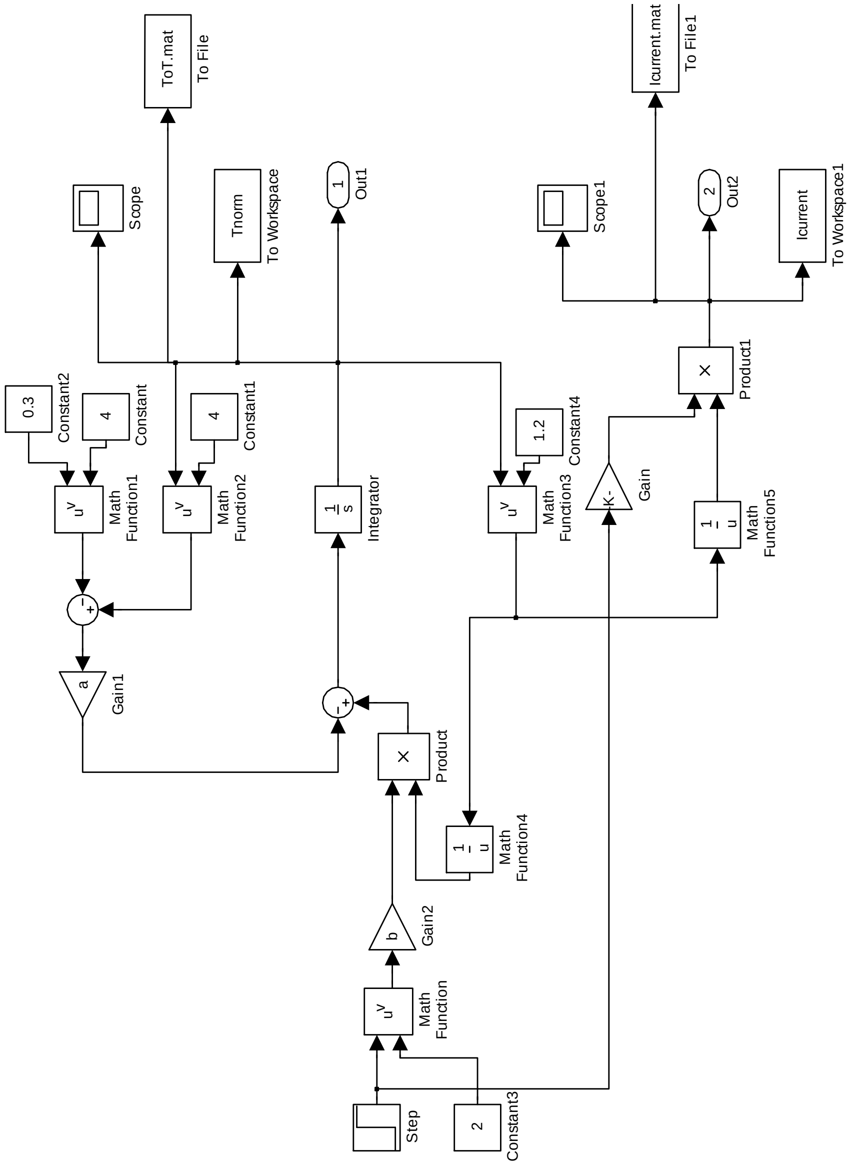
$$I = \frac{\pi d^2 V_{\max}}{4\rho_{e0}L\left(\frac{T}{T_{e0}}\right)^{1.2}}V. \tag{16}$$

Equation (15) implies that fast lamp response requires high filament temperature and low filament diameter. Typical values for the lamp filament time constant range from 0.5 to 2 seconds.

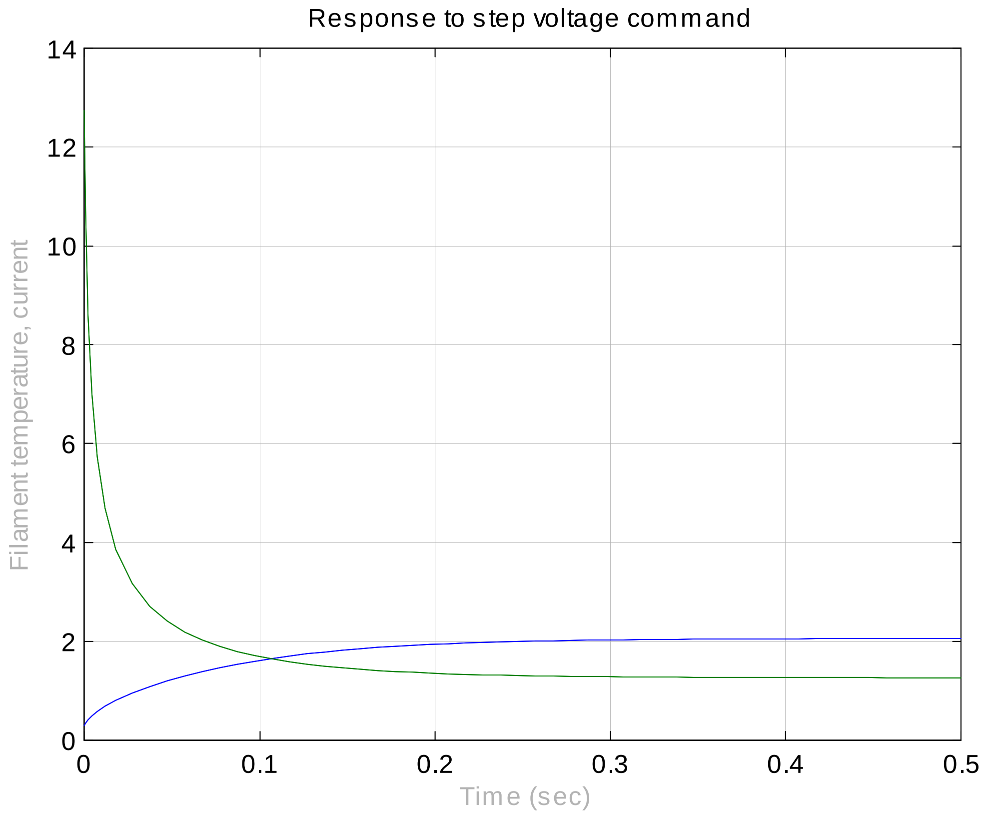
The output of the lamp may be considered to be current, normalized filament temperature, or radiative power,

$$y_{lamp} = \begin{bmatrix} I \\ x \\ P_{\max} \end{bmatrix}. \tag{17}$$

A nonlinear simulation for the lamp model may be implemented in Simulink as shown on the next page. The results of the simulation show the temperature and current response of the lamp to a step voltage command input as shown. The figure shows the fast lamp filament temperature response with a time constant of 0.07 seconds, and also shows that the current initially surges but quickly drops to a steady-state within approximately 0.3 seconds.



Problem 10.21: Simulink diagram for lamp model.



Problem 10.21: Lamp response to a step voltage command.

References:

[1] Reynolds, W. C., and H. C. Perkins, *Engineering Thermodynamics*, McGraw-Hill, 1977.

[2] A. Emami-Naeini, et al., "Modeling and Control of Distributed Thermal Systems," *IEEE Tran. Contrl. Syst. Tech.*, pp. 668-683, September 2003.

[3] Lide, D. R., Ed., *Handbook of Chemistry and Physics*, CRC Press, 1993-1994.

[4] Modest, M. F., *Radiative Heat Transfer*, McGraw-Hill, 1993.

22. Develop a nonlinear model for a pyrometer. Show how temperature can be deduced from the model.

Solution:

Temperature measurement can be done by a variety of methods including thermocouples, resistive temperature detectors (RTDs), and pyrometers [1]. A pyrometer is a non-contact temperature sensor and measures the Infrared (IR) radiation which is directly a function of the temperature. It is known that objects emit radiant energy proportional to T^4 where T is the temperature of the object. Among the advantages of pyrometers are that they have very fast response time, can be used to measure the temperature of moving objects (e.g., a rotating semiconductor wafer), and in vacuum for semiconductor manufacturing.

The single-wavelength pyrometer measures the total energy emitted from a surface at a given wavelength. To understand the operation of a pyrometer, we need to review some concepts from radiation heat transfer [2-3]. The emissivity of an object, ϵ, is defined as the ratio of the energy flux emitted by a surface to that from a black body at the same temperature:

$$\epsilon = \frac{\int \epsilon_\lambda I_{b\lambda}(T) d\lambda}{\int I_{b\lambda}(T) d\lambda}, \tag{18}$$

where,

$$\begin{aligned}
\epsilon &= \text{total emissivity}, \\
\epsilon_\lambda &= \text{spectral emissivity}, \\
T &= \text{absolute temperature}, K, \\
\lambda &= \text{wavelength of radiation}, \mu m, \\
I_{b\lambda} &= \text{spectral black body intensity}.
\end{aligned}$$

The frequency, ν, is given by,

$$\nu = \frac{c_0}{\lambda_0},$$

where,

$$\begin{aligned}
c_0 &= \text{speed of light in vacuum} = 2.998 \times 10^8 m/s, \\
\lambda_0 &= \text{wavelength of light in vacuum}.
\end{aligned}$$

The Plank's law of radiation states that the spectral radiance of a blackbody, or spectral intensity, $I_{b\nu}$, in a dielectric medium as a function of the wavelength and temperature is,

$$I_{b\nu}(T) = \frac{2h\nu^3 n^2}{c_o^2 (e^{\frac{h\nu}{kT}} - 1)}, \tag{19}$$

where,

$$\begin{aligned}
h &= \text{Planck's constant} = 6.626 \times 10^{-34} Js, \\
k &= \text{Boltzmann's constant} = 1.3806 \times 10^{-23} J/K, \\
n &= \text{real refractive index } (n = 1 \text{ for most gases}),
\end{aligned}$$

and the frequency and wavelength are related by,

$$\nu = \frac{c}{\lambda}, \tag{20}$$

where c is the speed of light in the medium and is given by,

$$c = \frac{c_0}{n}.$$

After some manipulation, we can re-write Eq. (19) as [2],

$$I_{b\lambda}(T) = \frac{2hc_0^2}{n^2 \lambda^5 (e^{\frac{hc_0}{n\lambda kT}} - 1)}. \tag{21}$$

The total black-body radiation intensity, I_b, is obtained by integrating over all frequencies or wavelength [2]

$$\begin{aligned} I_b(T) &= \int I_{\nu b}(T) d\nu \\ &= \frac{n^2 \sigma T^4}{\pi}. \end{aligned} \tag{22}$$

The black body emissive flux is given by,

$$q_{\lambda b}(T) = \frac{C_1}{n^2 \lambda^5 (e^{\frac{C_2}{n\lambda T}} - 1)}, \tag{23}$$

where,

$$\begin{aligned} C_1 &= 2\pi h c_0^2 = 3.7419 \times 10^{-16},\ W/m^2, \\ C_2 &= \frac{h c_0}{k} = 14388\ \mu m K. \end{aligned} \tag{24}$$

Integrating over all wavelengths λ, we obtain the total black body emissive flux [2],

$$\begin{aligned} q_b(T) &= \int_0^\infty q_{\lambda b}(T) d\lambda \\ &= n^2 \sigma T^4, \end{aligned} \tag{25}$$

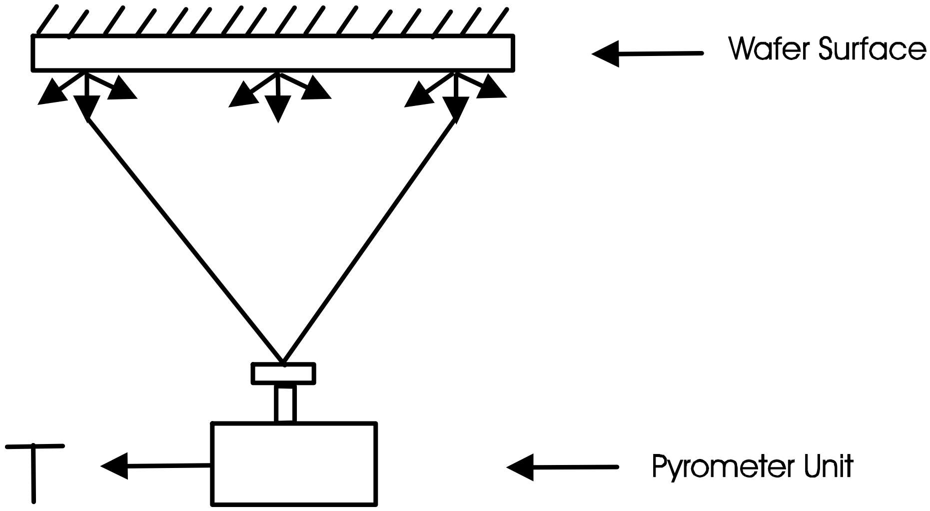
where σ is the Stefan-Boltzman constant, $\sigma = 5.67 \times 10^{-8}\ W/(m^2 K^4)$.

The temperature may be determined from Eq. (19) as,

$$T = \frac{C_2}{\lambda} \frac{1}{\ln(1 + \frac{\epsilon_\lambda C}{I})}, \tag{26}$$

where,

$$C = \frac{C_1}{\lambda^5}. \tag{27}$$



Problem 10.22: Schematic of temperature measurement using pyrometry.

The figure on the bottom of the previous page shows the schematic of temperature measurement of a semiconductor wafer using pyrometry where, for this particular application, response time and view angle are very important. A two color pyrometer is also used for applications where absolute temperature measurement is important. The measurement can then be used for feedback control purposes, e.g., pyrometers are now routinely used in control of rapid thermal processing (RTP) systems in semiconductor manufacturing [4].

References:

[1] Fraden, J., *Handbook of Modern sensors: Physics, Designs, and Applications*, Springer, 1996.

[2] Ozisik, M. N., *Radiative Transfer and Interactions with Conduction and Convection*, Wiley-Interscience, 1973.

[3] Siegel, R. and J. R. Howell, *Thermal Radiation Heat Transfer*, Second Ed., Hemisphere Publishing Corp., 1981.

[4] A. Emami-Naeini, et al., "Modeling and Control of Distributed Thermal Systems," *IEEE Tran. Contrl. Syst. Tech.*, pp. 668-683, September 2003.

23. Repeat the RTP case study design by summing the three sensors to form a single signal to control the average temperature. Demonstrate the performance of the linear design, and validate the performance on the nonlinear Simulink simulation.

Solution:

A linear model for the system was derived in the text as,

$$\begin{aligned} \dot{\mathbf{T}} &= \mathbf{F}_3\ \mathbf{T} + \mathbf{G}_3\ \mathbf{u}, \\ \mathbf{y} &= \mathbf{H}\ _3\mathbf{T} + \mathbf{J}_3\ \mathbf{u}, \end{aligned} \tag{28}$$

where $\mathbf{y} = [T_{y1}\ T_{y2}\ T_{y3}]^T$ and,

$$\mathbf{F}_3 = \begin{bmatrix} -0.0682 & 0.0149 & 0.0000 \\ 0.0458 & -0.1181 & 0.0218 \\ 0.0000 & 0.04683 & -0.1008 \end{bmatrix}, \quad \mathbf{G}_3 = \begin{bmatrix} 0.3787 & 0.1105 & 0.0229 \\ 0.0000 & 0.4490 & 0.0735 \\ 0.0000 & 0.0007 & 0.4177 \end{bmatrix},$$

$$\mathbf{H}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{J}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The three open-loop poles are computed from MATLAB and are located at $-0.0527, -0.0863$, and -0.1482. Since we tied the three lamps into one actuator and are only using the *average* temperature for feedback, the linear model is then:

$$\mathbf{F} = \begin{bmatrix} -0.0682 & 0.0149 & 0.0000 \\ 0.0458 & -0.1181 & 0.0218 \\ 0.0000 & 0.04683 & -0.1008 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0.5122 \\ 0.5226 \\ 0.4185 \end{bmatrix},$$

$$\mathbf{H}_{avg} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \ J = [0],$$

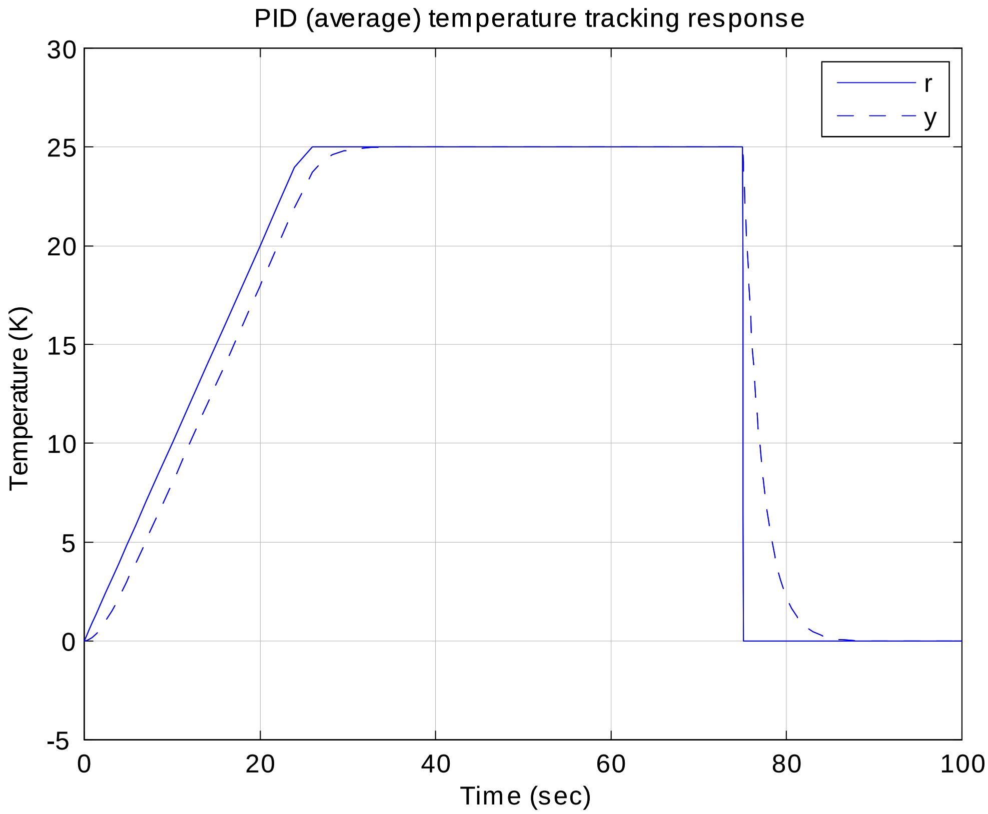
resulting in the transfer function,

$$G(s) = \frac{T_{yavg}(s)}{V_{cmd}(s)} = \frac{0.4844(s + 0.0878)(s + 0.1485)}{(s + 0.1482)(s + 0.0527)(s + 0.0863)}.$$

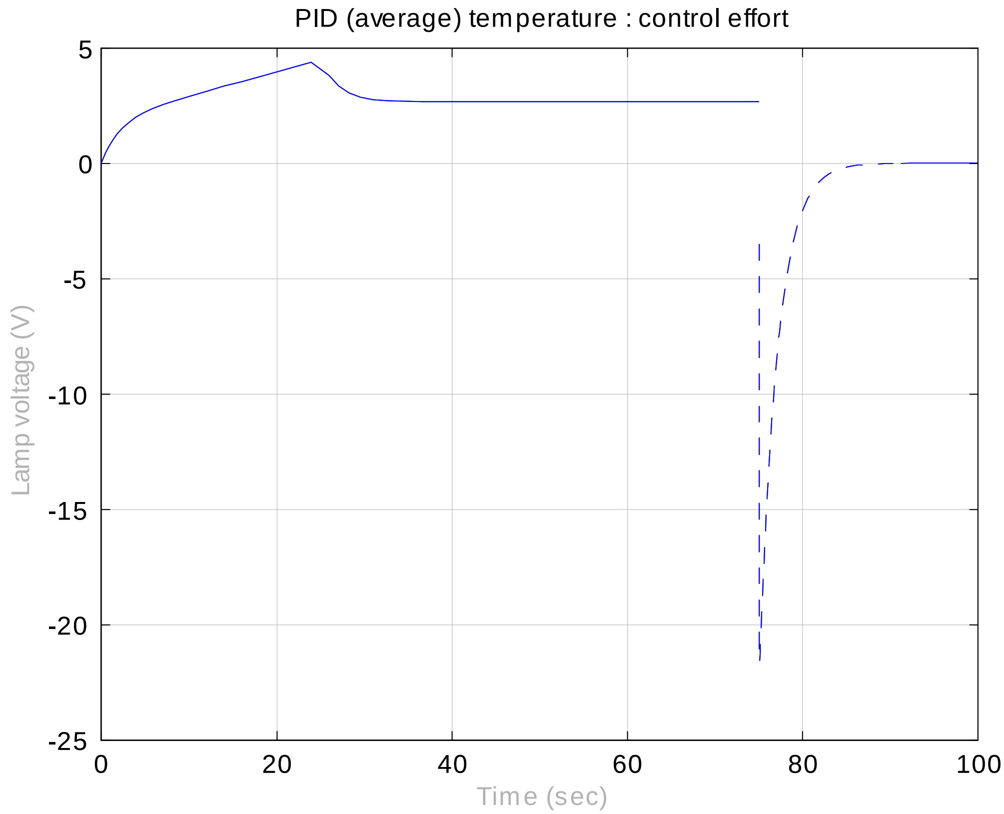
We may try a simple PI controller of the form,

$$D_c(s) = \frac{(s + 0.0527)}{s},$$

so as to cancel the effect of the slower pole. The linear closed-loop response is shown as well as the associated control effort The system response follows the commanded trajectory with a time delay of approximately 2 sec and no overshoot. The lamp has its normal response until 75 sec and goes negative (shown in dashed) to try to follow the sharp drop in commanded temperature. As mentioned in the text, this behavior is not possible in the system as there is no means of active cooling and the lamps do saturate low. There is no explicit means of controlling the temperature nonuniformity using the PI controller.



Problem 10.23: Linear closed-loop RTP response for PI controller.



Problem 10.23: RTP Linear response for PI: control effort.

Next we design a state-space based controller. As in the text, we use the error space approach for inclusion of integral control and employ the linear quadratic gaussian technique of Chapter 7. The error system is,

$$\left[\begin{array}{c} \dot{e} \\ \dot{\boldsymbol{\xi}} \end{array}\right] = \left[\begin{array}{cc} 0 & \mathbf{H}_{avg} \\ \mathbf{0} & \mathbf{F} \end{array}\right] \left[\begin{array}{c} e \\ \boldsymbol{\xi} \end{array}\right] + \left[\begin{array}{c} J \\ \mathbf{G} \end{array}\right] \mu, \tag{29}$$

where,

$$\mathbf{A} = \left[\begin{array}{cc} 0 & \mathbf{H}_{avg} \\ \mathbf{0} & \mathbf{F} \end{array}\right], \mathbf{B} = \left[\begin{array}{c} J \\ \mathbf{G} \end{array}\right].$$

and $e = y - r$, $\boldsymbol{\xi} = \dot{\mathbf{T}}$ with $\mu = \dot{u}$. For state feedback design, the LQR formulation of Chapter 7 is used

$$\mathcal{J} = \int_0^\infty \{\mathbf{z}^T \mathbf{Q}\ \mathbf{z} + \rho\mu^2\} dt,$$

where $\mathbf{z} = [e\ \boldsymbol{\xi}^T]^T$. Note that $\mathcal{J}$ has been chosen in such a way as to penalize the tracking error, e, the control, u, as well as the differences in the three temperatures– therefore, the performance index should include a term of the form,

$$10\left\{(T_1 - T_2)^2 + (T_1 - T_3)^2 + (T_2 - T_3)^2\right\},$$

and hence minimizes the *temperature non-uniformity*. As in the text, the factor of ten is used as the relative weighting between the error state and the plant state. The state and control weighting matrices, $\mathbf{Q}$ and R, are then,

$$\mathbf{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 20 & -10 & -10 \\ 0 & -10 & 20 & -10 \\ 0 & -10 & -10 & 20 \end{bmatrix}, \quad R = \rho = 1.$$

The following MATLAB command is used to design the feedback gain,

[K]=lqr(A,B,Q,R).

The resulting feedback gain matrix computed from MATLAB is,

$$\mathbf{K} = [K_1 \quad : \mathbf{K}_0],$$

where,

$$K_1 \quad = 1, \; \mathbf{K}_0 \quad = \left[\begin{array}{ccc} 0.7344 & 0.9344 & 0.3921 \end{array} \right],$$

which results in the internal model controller of the form,

$$\begin{aligned} \dot{x}_c &= B_c\, e, \\ u &= C_c\, x_c - \mathbf{K}_0 \quad \mathbf{T}, \end{aligned} \tag{30}$$

with x_c denoting the controller state and,

$$B_c = -K_1 = -1, \; C_c = 1.$$

The resulting state-feedback closed-loop poles computed using MATLAB's eig command are at $-0.5395 \pm 0.4373j, -0.1490$, and -0.0879. The full-order estimator was designed with the same process and sensor noise intensities used in the text as the estimator design knobs,

$$R_w = 1, \; R_v = 0.001.$$

The following MATLAB command is used to design the estimator,

[L]=lqe(F,G,H,Rw,Rv).

The resulting estimator gain matrix is,

$$\mathbf{L} = \begin{bmatrix} 16.142 \\ 16.4667 \\ 13.1975 \end{bmatrix},$$

with estimator error poles at $-15.3197, -0.1485$, and -0.0878. The estimator equation is,

$$\dot{\hat{\mathbf{T}}} = \mathbf{F}\hat{\mathbf{T}} + \mathbf{G}u + \mathbf{L}(y - \mathbf{H}\hat{\mathbf{T}}). \tag{31}$$

With the estimator, the internal model controller equation is modified as in the text

$$\begin{aligned} \dot{x}_c &= B_c\, e, \\ u &= C_c\, x_c - \mathbf{K}_0 \quad \hat{\mathbf{T}}. \end{aligned} \tag{32}$$

The closed-loop system equations are given in the text,

$$\begin{aligned} \dot{\mathbf{x}}_{cl} &= \mathbf{A}_{cl}\ \mathbf{x}_{cl} + \mathbf{B}_{cl}\ r, \\ y &= \mathbf{C}_{cl}\ \mathbf{x}_{cl} + \mathbf{D}_{cl}\ r, \end{aligned} \tag{33}$$

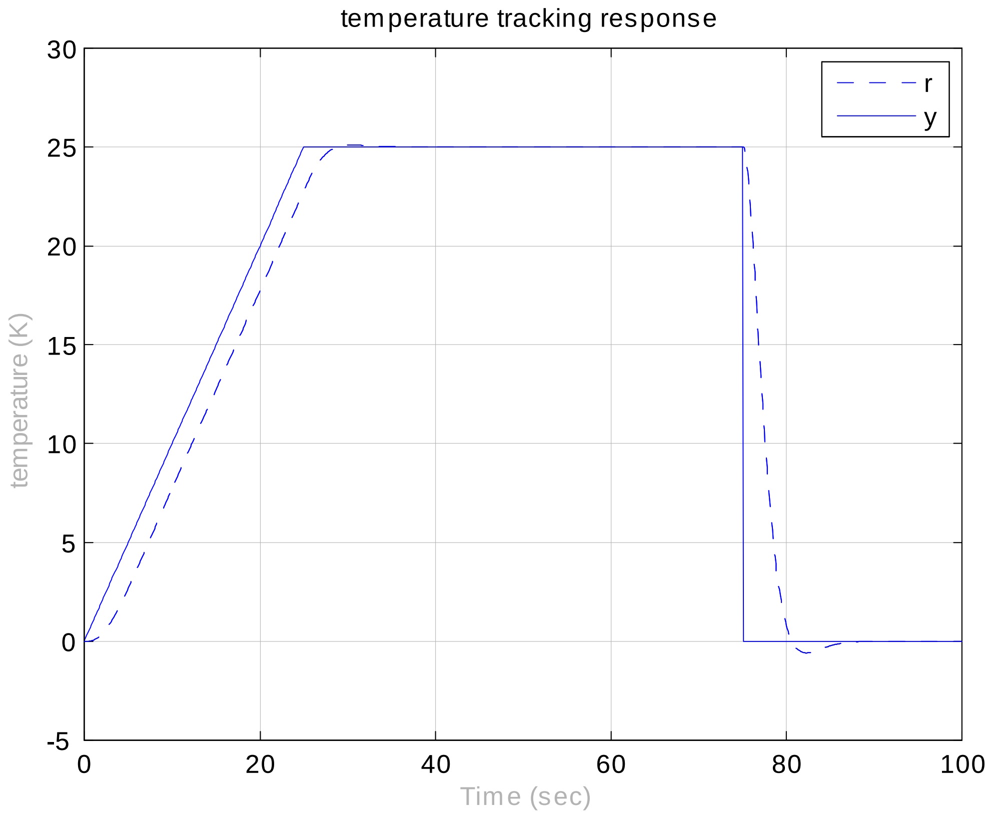
where r is the reference input temperature trajectory, the closed-loop state vector is $\mathbf{x}_{cl} = [\mathbf{T}^T\ x_c^T\ \hat{\mathbf{T}}^T]^T$ and the system matrices are,

$$\begin{aligned} \mathbf{A}_{cl} &= \begin{bmatrix} \mathbf{F} & \mathbf{G}C_c & -\mathbf{G}\mathbf{K}_0 \\ B_c\mathbf{H} & \mathbf{0} & \mathbf{0} \\ \mathbf{L}\mathbf{H} & \mathbf{G}C_c & \mathbf{F}-\mathbf{G}\mathbf{K}_0-\mathbf{L}\mathbf{H} \end{bmatrix}, \quad \mathbf{B}_{cl} = \begin{bmatrix} \mathbf{0} \\ -B_c \\ \mathbf{0} \end{bmatrix}, \\ \mathbf{C}_{cl} &= \begin{bmatrix} \mathbf{H} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{D}_{cl} = [0], \end{aligned}$$

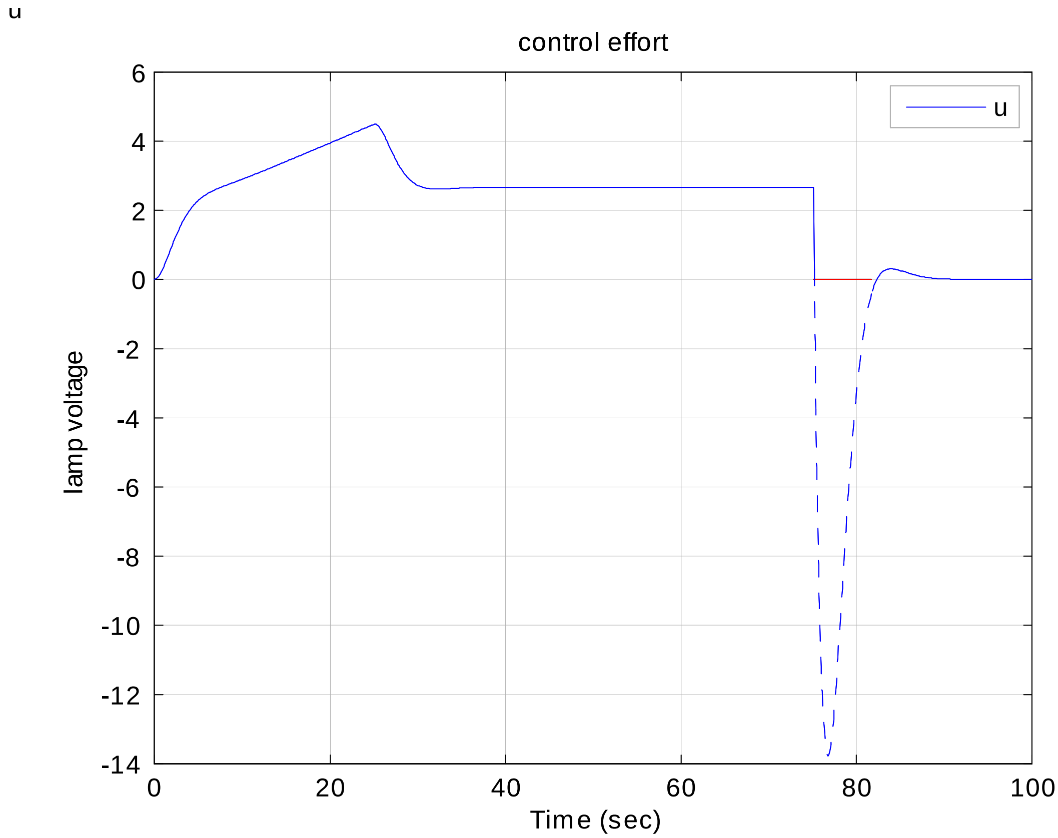
with closed-loop poles (computed with MATLAB) located at $-0.5395 \pm 0.4373j, -0.1490,\ -0.0879,$

$-15.3197, -0.1485$ and -0.0879 as expected. The closed-loop control structure is as shown in text Figure 10.85.

The linear closed-loop response and the associated control effort are shown. The commanded temperature trajectory, r, is a ramp from 0°C to 25°C with a 1°C/sec slope followed by 50 sec soak time and drop back to 0°C. The system tracks the commanded temperature trajectory – albeit with a time delay of approximately 2 seconds for the ramp and a maximum of 0.0216°C overshoot. As expected the system tracks a constant input asymptotically with zero steady-state error. The lamp command increases as expected to allow for tracking the ramp input, reaches a maximum value at 25 sec and then drops to a steady-state value around 35 sec. The normal response of the lamp is seen from 0 to 75 sec followed by negative commanded voltage for a few seconds corresponding to fast cooling. Again, the negative control effort voltage (shown in dashed lines) is physically impossible as there is no active cooling in the system. Hence in the nonlinear simulations, commanded lamp power must be constrained to be strictly non-negative. Note that the response from $75-100$ sec is that of the (negative) step response of the system.

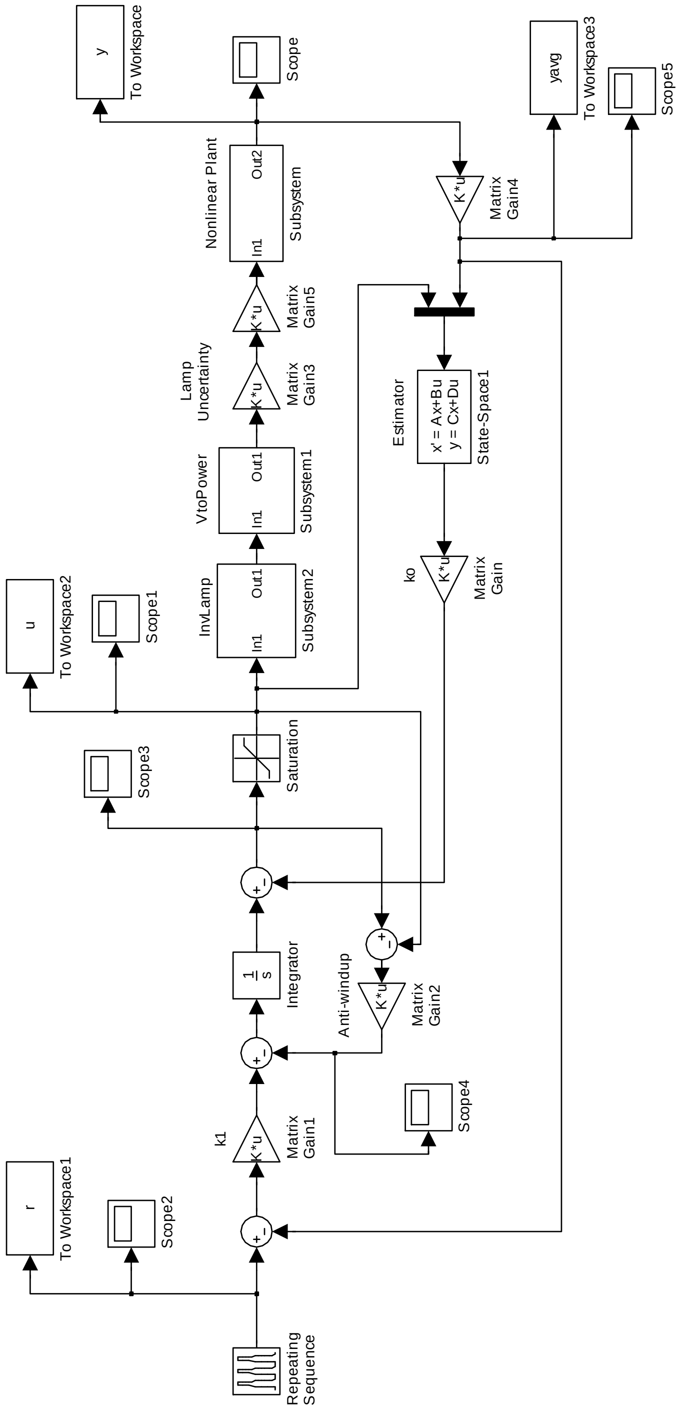


Problem 10.23: RTP linear (average) temperature tracking response.



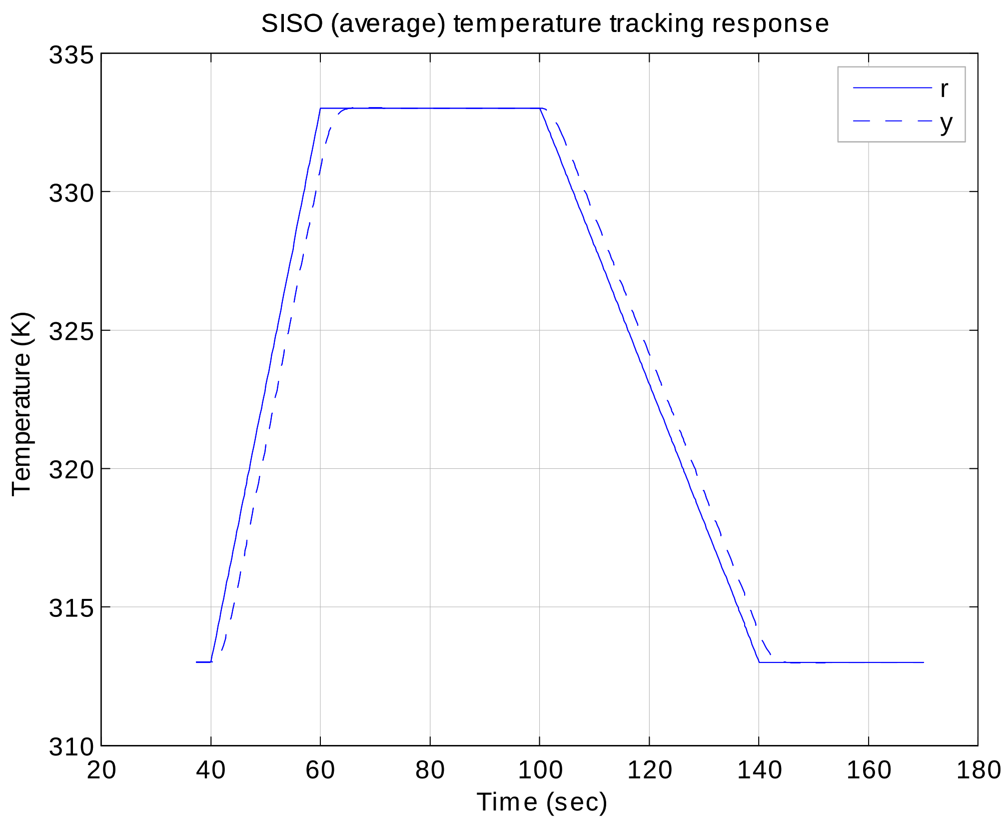
Problem 10.23 RTP linear (average) : control effort.

The nonlinear closed-loop system was simulated in Simulink as shown on the next page. In the diagram, $\mathsf{Gain4} = \left[\begin{array}{ccc} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right]$. As in the text, the model was implemented in temperature units of degrees Kelvin and the ambient temperature is 301K. The nonlinear plant model is the implementation of text Eq. 10.48. The voltage range for system operation is between 1 to 4 volts as seen from the diagram. As in the text, a saturation nonlinearity is included for the lamp as well as integrator anti-windup logic to deal with lamp saturation. The nonlinear dynamic response and the control effort are shown. Note that the nonlinear response is in general agreement with the linear response.

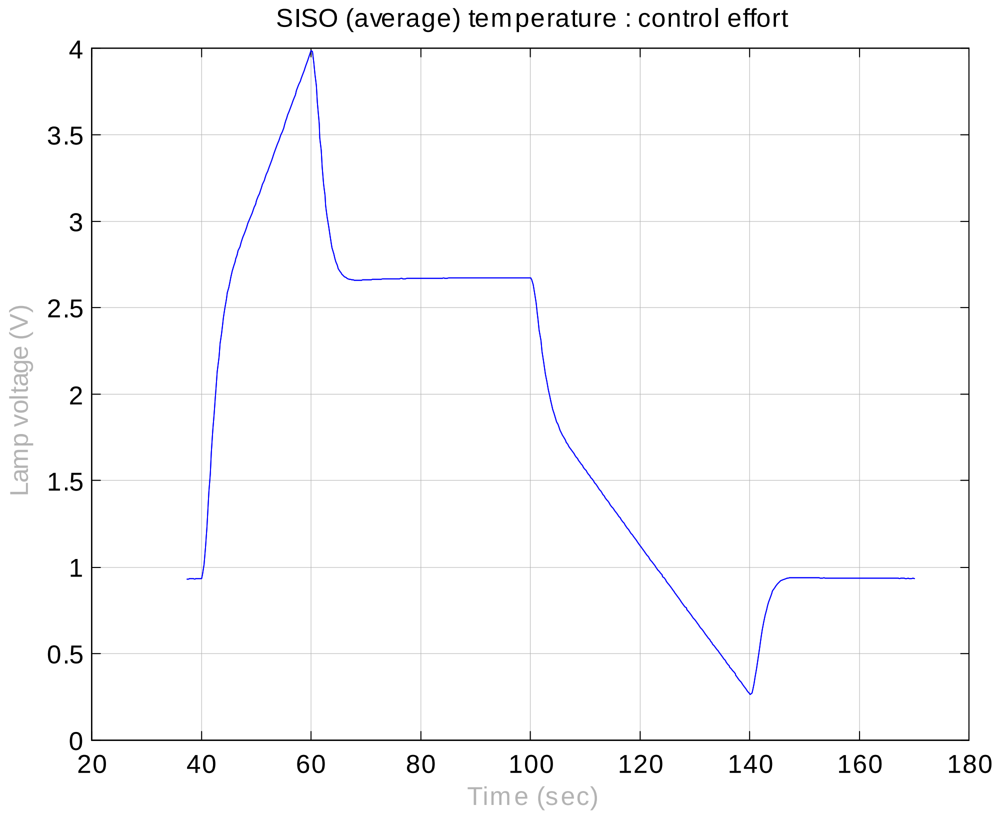


Problem 10.23: Simulink diagram for nonlinear closed-loop RTP system to control average temperature.

y



Problem 10.23: Nonlinear closed-loop response.



Problem 10.23: Nonlinear closed-loop response: control effort.

24. One of the steps in semiconductor wafer manufacturing during photolithography is performed by placement of the wafer on a heated plate for a certain period of time. Laboratory experiments have shown that the transfer function from the heater power, u, to the wafer temperature, y, is given by

$$\frac{y(s)}{u(s)} = G(s) = \frac{0.09}{(s+0.19)(s+0.78)(s+0.00018)}$$

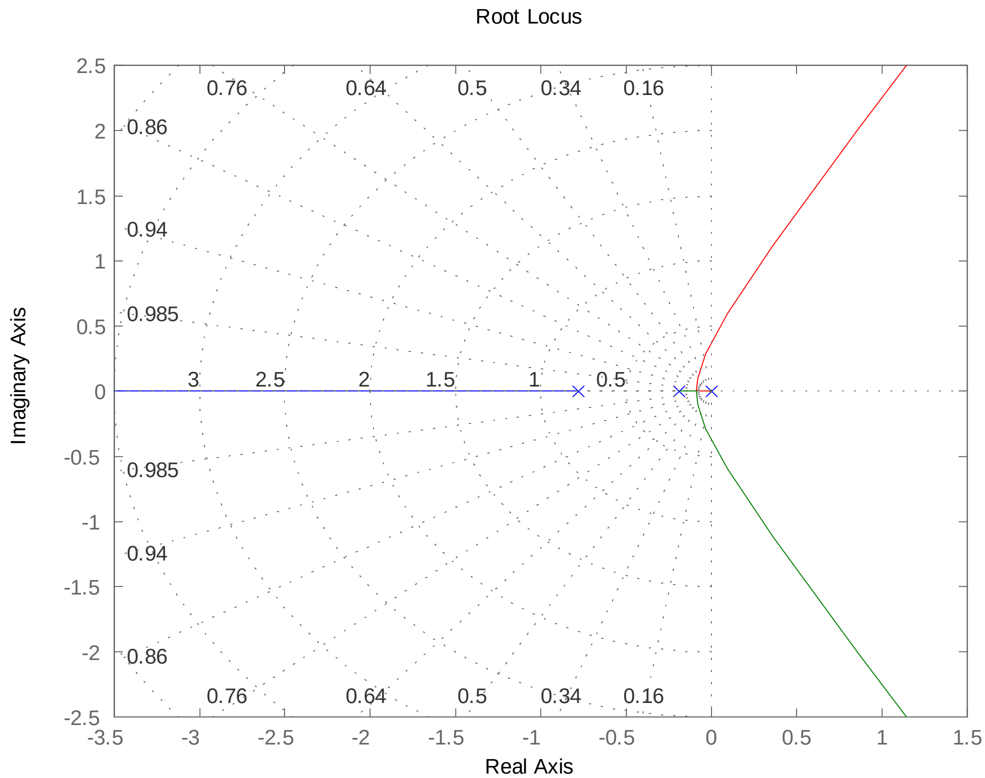
1. (a) Sketch the 180° root locus for the uncompensated system.

   (b) Using the root locus design techniques, design a *dynamic* compensator, $D(s)$, such that the system meets the following time-domain specifications

   i. $M_p \leq 5\%$
   ii. $t_r \leq 20\,\text{sec}$
   iii. $t_s \leq 60\,\text{sec}$
   iv. Steady-state error to a 1°C step input command < 0.1°C.
   Draw the 180° root locus for the compensated system.

Solution:

The uncompensated root locus is shown in the following figure.



Uncompensated root locus.

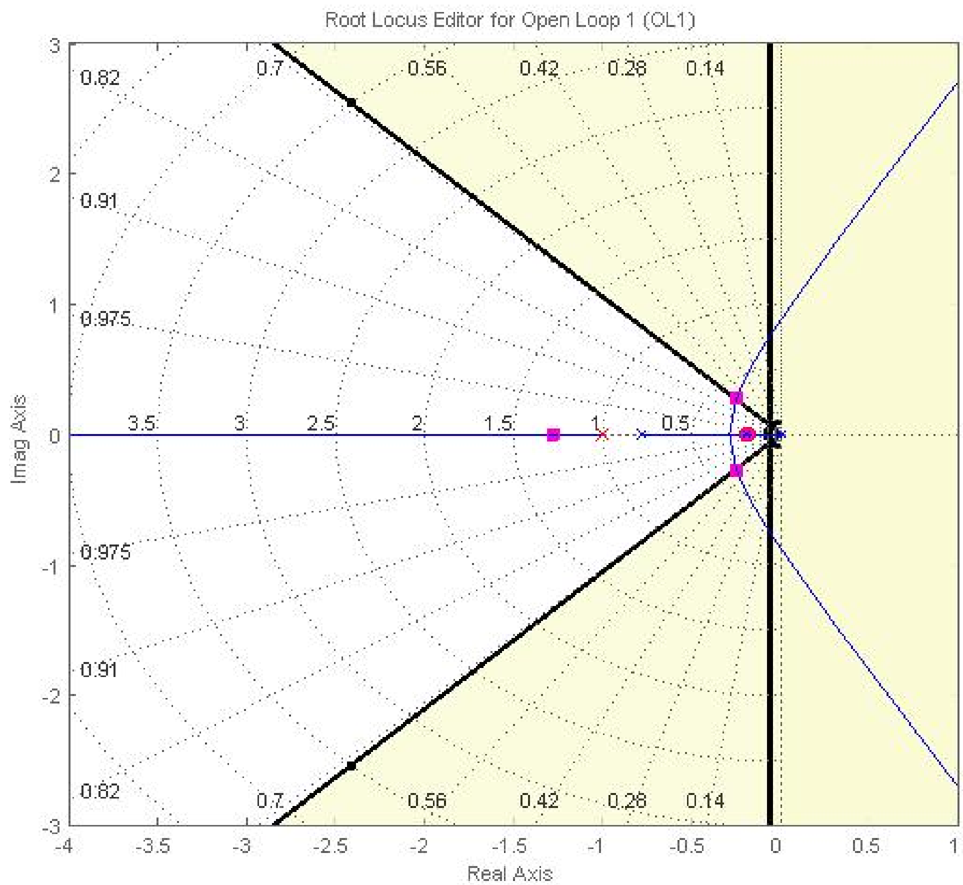
(a) First, convert the time domain specifications to s-plane specifications:

$$\begin{aligned} M_p &\leq 5\% \Rightarrow \varsigma \geqslant 0.707 \\ t_r &\leq 20\,\text{sec} \Rightarrow \omega_n \geqslant 0.09 \\ t_s &\leq 60\,\text{sec} \Rightarrow \sigma \geqslant 0.0767 \end{aligned}$$

(b) At this point there are several design methods one can use for this problem. **Method I**: We know that a pure integrator will improve steady-state behavior and we nearly have a pure integrator with the pole at -0.00018. Thus, we can cancel another (stable) pole with a lead network zero, and place a fast lead pole to appropriately shape the root locus. One possible lead compensator is

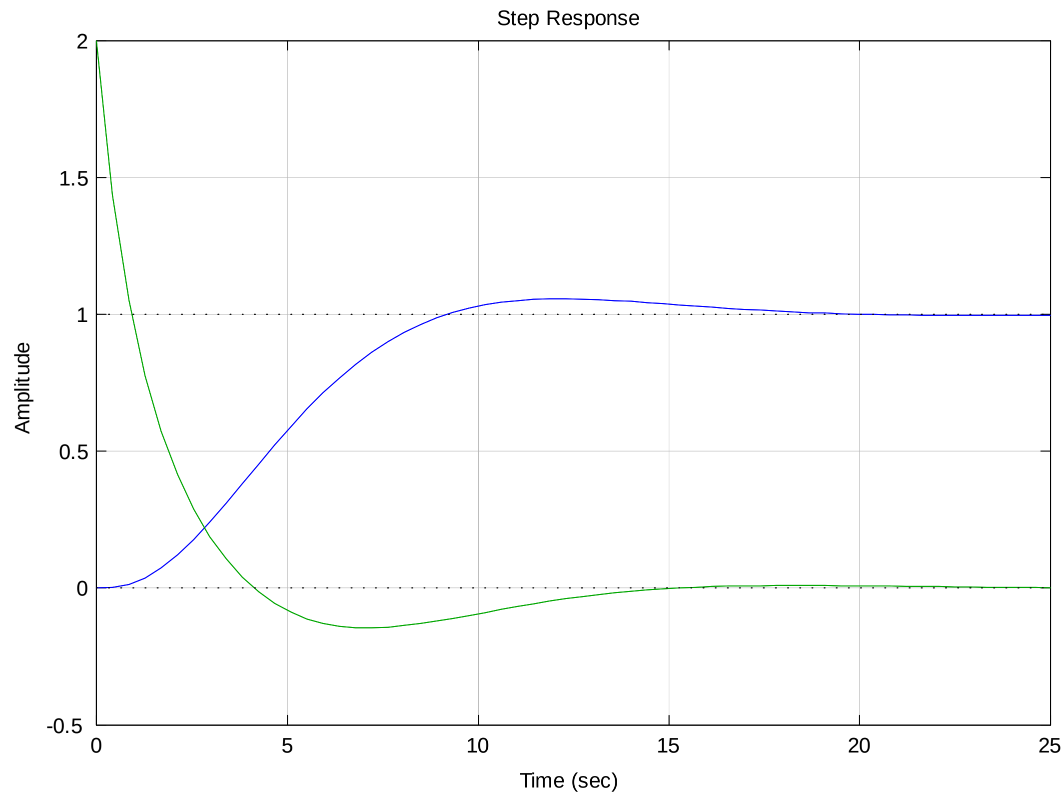
$$D(s) = K_c \frac{s + 0.19}{s + 1}$$

We can display the compensated root locus and s-plane regions in RLTool in the following figure.



RLTool: Compensator I.

Choosing $K_c = 2$ gives us the closed-loop poles within the specifications at $s = -0.24 \pm j0.28$. The results from RLTool follows.



RLTool: Step response (y) and control effort (u) for Compensator I.

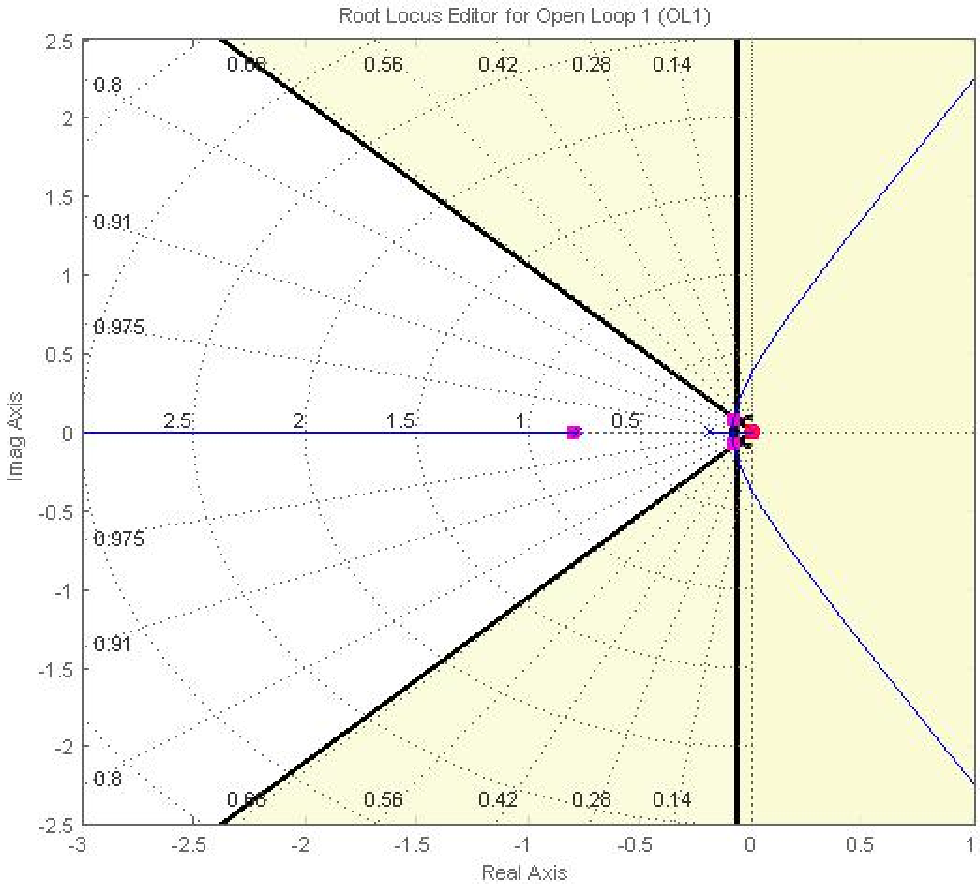
Method II: We can cancel out the nearly pure integrator with our compensator zero, and add a pure integrator. It is possible to achieve the time domain specifications with this compensator. The compensator becomes

$$D(s) = K\frac{s + 0.00018}{s}$$

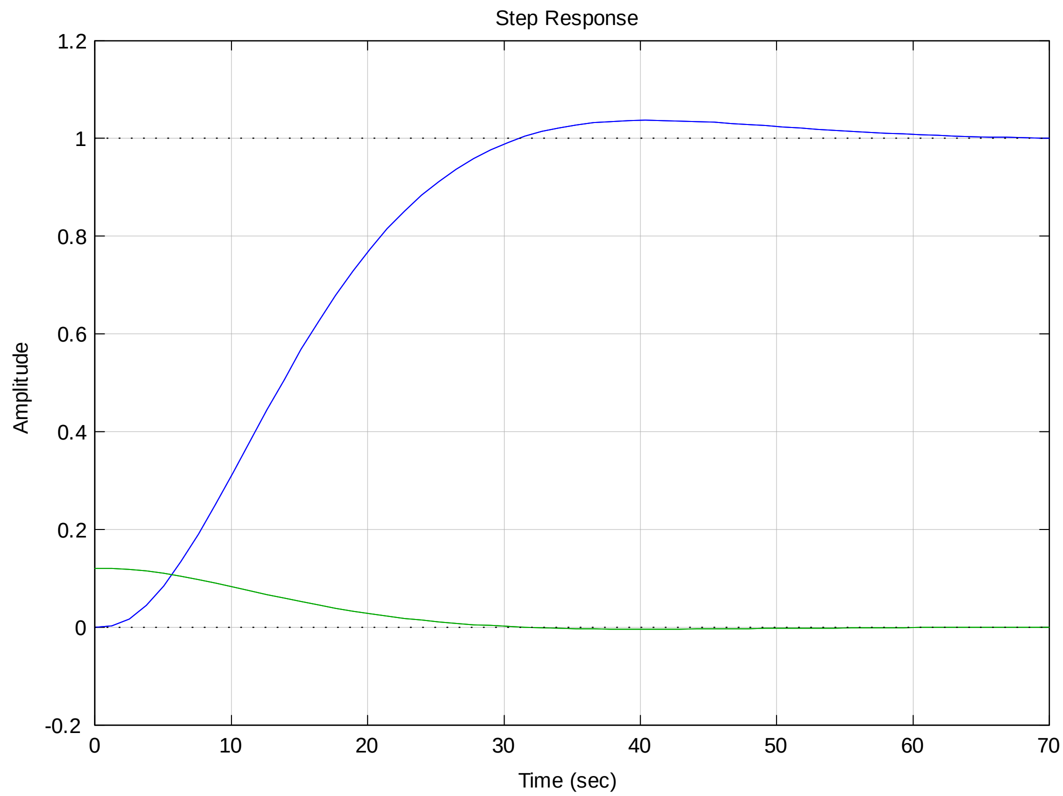
Applying the magnitude condition with the desired poles at $s = -008 \pm j0.08$, we get

$$K = 0.12$$

We can display the compensated root locus and s-plane regions in RLTool in the following figure.



RLTool: Compensator II.



RLTool: Step response (y) and control effort (u) for Compensator II.

25. *Excitation-Inhibition Model from Systems Biology* (Yang and Iglesias, 2005): In *Dictyostelium* cells, the activation of key signaling molecules involved in chemoattractant sensing can be modeled by the following third order linearized model. The external disturbance to the output transfer function is:

$$\frac{y(s)}{w(s)} = S(s) = \frac{(1-\alpha)s}{(s+\alpha)(s+1)(s+\gamma)}$$

where, w is the external disturbance signal proportional to chemoattractant concentration, and y is the output which is the fraction of active response regulators. Show that there is an alternate representation of the system with the "plant" transfer function

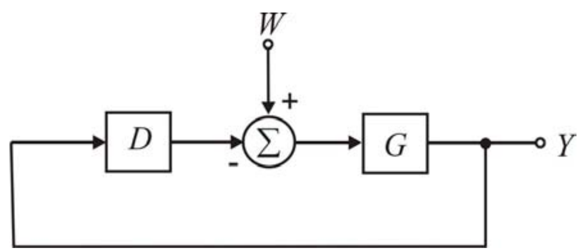
$$G(s) = \frac{(1-\alpha)}{s^2 + (1+\alpha+\gamma)s + (\alpha+\gamma+\alpha\gamma)}$$

and the "feedback regulator"

$$D(s) = \frac{\alpha\gamma}{(1-\alpha)s}$$

It is known that $\alpha \neq 1$ for this version of the model. Draw the feedback block diagram of the system showing the locations of the disturbance input and the output. What is the significance of this particular representation of the system? What hidden system property does it reveal? Is the disturbance rejection a robust property for this system? Plot the disturbance rejection response of the system for a unit step disturbance input. Assume the system parameter values are $\alpha = 0.5$ and $\gamma = 0.2$.

1. **Solution:** From the following figure we have



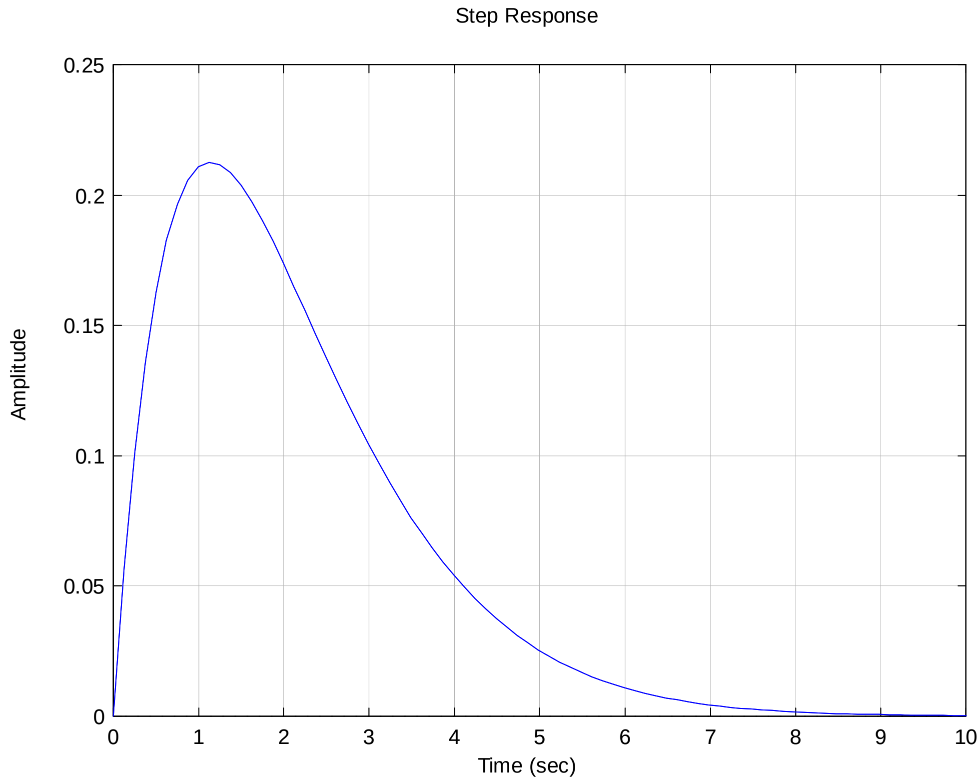
Feedback loop representation.

$$\begin{aligned}\frac{Y(s)}{W(s)} &= \frac{\frac{(1-\alpha)}{s^2+(1+\alpha+\gamma)s+(\alpha+\gamma+\alpha\gamma)}}{1+\frac{(1-\alpha)}{s^2+(1+\alpha+\gamma)s+(\alpha+\gamma+\alpha\gamma)}\frac{\alpha\gamma}{(1-\alpha)s}} \\ &= \frac{(1-\alpha)s}{(s+\alpha)(s+1)(s+\gamma)}\end{aligned}$$

The significance of this particular representation is that it reveals the internal model, namely the pure *integrator*. Hence the system is Type I with respect to disturbance rejection. It rejects constant disturbances in a robust fashion. For $\alpha = 0.5$ and $\gamma = 0.2$, we have

$$G(s) = \frac{0.5s}{s^2 + 1.7s + 0.8}$$

The disturbance response is shown in the following figure.



Disturbance rejection response for a unit step disturbance.